Riemann bilinear relations

Ching-Li Chai*

The **Riemann bilinear relations**, also called the **Riemann period relations**, is a set of quadratic relations for **period matrices**. The ones considered by Riemann are of two sorts: (a) periods of holomorphic one-forms on a compact Riemann surface, and (b) periods of holomorphic one-forms on an abelian variety.

§1. Period relations for abelian integrals of the first kind

The statements (1.2 a) and (1.2 b) in Theorem 1.2 are the *Riemann bilinear relations* for the period integrals of differentials of the first kind on a a compact Riemann surface.

(1.1) Notation and terminology

- Let S be a compact connected Riemann surface of genus $g \geq 1$.
- Let $\omega_1, \ldots, \omega_g$ be a $\mathbb{C}$-basis of the space $\Gamma(S, K_S)$ of holomorphic differential one-forms on S.
- Let $\gamma_1, \ldots, \gamma_{2g}$ be a $\mathbb{Z}$-basis of the first Betti homology group $H_1(S, Z)$.
- Let $J = J_{2g}$ be the $2g \times 2g$ matrix $\begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$, where I_g is the $g \times g$ identity matrix, and $0_g = 0 \cdot I_g$.
- For any i, j with $1 \leq i, j \leq 2g$, let $\Delta_{ij} = \gamma_i \frown \gamma_j \in \mathbb{Z}$ be the intersection product of γ_i with γ_j. Let $\Delta = \Delta(\gamma_1, \ldots, \gamma_{2g})$ be the $2g \times 2g$ skew-symmetric matrix with entries Δ_{ij}.
- $\gamma_1, \ldots, \gamma_{2g}$ is said to be a *canonical basis* of $H_1(S, Z)$ if $\Delta(\gamma_1, \ldots, \gamma_{2g}) = J_{2g}$.
- It is well-known that $H_1(S, Z)$ admits a canonical basis. In other words there exists an element $C \in \mathrm{GL}_{2g}(\mathbb{Z})$ such that ${}^t C \cdot \Delta \cdot C = J_{2g}$.
- The $g \times 2g$ matrix $P = P(\omega_1, \ldots, \omega_g; \gamma_1, \ldots, \gamma_{2g})$ whose (r, i)-th entry is $\int_{\gamma_i} \omega_r$ for every $r = 1, \ldots, g$ and every $i = 1, \ldots, 2g$ is called the **period matrix** defined by the one-cycles $\gamma_1, \ldots, \gamma_{2g}$ and the holomorphic one-forms $\omega_1, \ldots, \omega_g$.

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(1.2) THEOREM. *Let $P = P(\omega_1, \ldots, \omega_g; \gamma_1, \ldots, \gamma_{2g})$ be the period matrix for a a $\mathbb{C}$-basis of the space $\Gamma(S, K_S)$ of holomorphic one-forms on S and a $\mathbb{Z}$-basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(S, \mathbb{Z})$. We have*

$$P \cdot \Delta(\gamma_1, \ldots, \gamma_{2g})^{-1} \cdot {}^tP = 0_g \tag{1.2 a}$$

and

$$-\sqrt{-1} \cdot P \cdot \Delta(\gamma_1, \ldots, \gamma_{2g})^{-1} \cdot {}^t\overline{P} > 0_g \tag{1.2 b}$$

in the sense that $-\sqrt{-1} \cdot P \cdot \Delta(\gamma_1, \ldots, \gamma_{2g})^{-1} \cdot {}^t\overline{P}$ is a $g \times g$ hermitian positive definite *matrix.*

Note that the validity of the statements (1.2) and (1.2) are independent of the choice of the $\mathbb{Z}$-basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(S, \mathbb{R})$: $\Delta(\gamma_1, \ldots, \gamma_{2g})^{-1}$ is the skew-symmetric real matrix $\left(\gamma_i^\vee \frown \gamma_j^\vee\right)_{1 \le i,j \le 2g}$, where the intersection pairing $\frown$ on $H_1(S, \mathbb{Z})$ has been $\mathbb{R}$-linearly extended to $H_1(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\gamma_1^\vee, \ldots, \gamma_{2g}^\vee \in H_1(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ are characterized by $\gamma_k^\vee \frown \gamma_j = \delta_{kj}$ for all $j, k = 1, \ldots, 2g$.

(1.3) COROLLARY. *Suppose that $\Delta(\gamma_1, \ldots, \gamma_{2g}) = J_{2g}$. Write the period matrix P in block form as $P = (P_1\ P_2)$, where P_1 (respectively P_2) is the $g \times g$ matrices whose entries are period integrals of $\omega_1, \ldots, \omega_g$ with respect to $\gamma_1, \ldots, \gamma_g$ (respectively $\gamma_{g+1}, \ldots, \gamma_{2g}$).*

(i) *The $g \times g$ matrix P_1 is* non-singular *i.e.* $\det(P_1) \neq 0$.

(ii) *Let $\Omega := P_1^{-1} \cdot P_2$. Then Ω is a* symmetric *$g \times g$ matrix and its imaginary part* $\mathrm{Im}(\Omega)$ *of Ω is* positive definite.

(1.4) The basic idea for the proof of Theorem 1.2 is as follows. First one "cuts open" the Riemann surface S along $2g$ oriented simple closed paths $C_1, \ldots, C_{2g}$ in S with a common base point so that the properties (a)–(d) below hold.

(a) For any pair $i \neq j$, C_i meets C_j only at the base point.

(b) The image of $C_1, \ldots, C_{2g}$ in $H_1(S, \mathbb{Z})$ is a canonical basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(S, \mathbb{Z})$.

(c) The "remaining part" $S \setminus (C_1 \cup \cdots \cup C_{2g})$ is 2-cell S_0.

(d) The boundary ∂S_0 of S_0 (in the sense of homotopy theory) consists of

$$C_1, C_{g+1}, C_1^{-1}, C_{g+1}^{-1}, C_2, C_{g+2}, C_2^{-1}, C_{g+2}^{-1}, \ldots, C_g, C_{2g}, C_g^{-1}, C_{2g}^{-1}$$

oriented cyclically.

For any non-zero holomorphic one-form ω on S, there exists a holomorphic function f on the simply connected domain S_0 such that $df = \omega$. Then for every holomorphic one-form η on S, we have

$$\int_{\partial S_0} f \cdot \eta = \int_{S_0} d(f \cdot \eta) = 0$$

and also

$$-\sqrt{-1} \cdot \int_{\partial S_0} \bar{f} \cdot \omega = -\sqrt{-1} \cdot \int_{S_0} d\bar{f} \wedge f > 0,$$

by Green's theorem. The bilinear relations (1.2) and (1.2) follow. Details of this are carried out in [13, Ch. 3 §3], [3, pp. 231–232] and [8, pp. 139–141].

As remarked by Siegel on page 113 of [13], these two bilinear relations were discovered and proved by Riemann, using the argument sketched in the previous paragraph. It is remarkable that Riemann's original proof is still the optimal one 150 years later. The readers are encouraged to consult Riemann's famous memoir [9], especially §§20–21.

§2. Riemann bilinear relations for abelian functions

The Riemann bilinear relations provide a necessary and sufficient condition for a set of $2g$ $\mathbb{R}$-linearly independent vectors in $\mathbb{C}^g$ to be the periods of g holomorphic differentials on a g-dimensional abelian variety.

(2.1) DEFINITION. (a) An **abelian function** on a complex vector space V is a meromorphic function f on V such that there exists a lattice $\Lambda \subset V$ with the property that $f(z+\xi) = f(\xi)$ for all $z \in V$ and all $\xi \in \Lambda$.[1]

(b) An abelian function f on a g-dimensional vector space V over $\mathbb{C}$ is **non-degenerate** if its period group

$$\text{Periods}(f) := \{\, \eta \in V \mid f(z+\eta) = f(z) \ \ \forall z \in V \,\}$$

is *not* a lattice in V. (Then $\text{Periods}(f)$ contains a positive dimensional $\mathbb{R}$-vector subspace of V, and in fact also a positive dimensional $\mathbb{C}$-vector subspace of V.)

(2.2) DEFINITION. Let $g \geq 1$ be a positive integer.

(a) A $g \times 2g$ matrix P with entries in $\mathbb{C}$ is a **Riemann matrix** if there exists a skew symmetric integral $2g \times 2g$ matrix E with $\det(E) \neq 0$ satisfying the two conditions (2.2 a), (2.2 b) below.

(2.2 a) $$Q \cdot E^{-1} \cdot {}^tQ = 0_g$$

and

(2.2 b) $$\sqrt{-1} \cdot Q \cdot E^{-1} \cdot {}^t\overline{Q} > 0_g.$$

Such an integral matrix E is called a *principal part* of P.

(b) The **Siegel upper-half space** $\mathfrak{H}_g$ of genus g is the set of all symmetric $g \times g$ complex matrix Ω such that $(\Omega,\ I_g)$ is a Riemann matrix with principal part $\begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$, or equivalently Ω is symmetric and the imaginary part $\text{Im}(\Omega)$ of Ω is positive definite.[2]

(2.3) THEOREM. *Let Q be a $g \times 2g$ matrix with entries in $\mathbb{C}^g$ such that the subgroup Λ of $\mathbb{C}^g$ generated by the $2g$ columns of Q is a lattice in $\mathbb{C}^g$. There exists a non-degenerate abelian function f on $\mathbb{C}^g$ whose period group is equal to Λ if and only if Q is a Riemann period matrix.*

A proof of theorem 2.3 can be found in [6, Ch. 1] and also in [3, Ch.frm-e §6]. For a classical treatment of theorem 2.3, chapter 5 §§ 9–11 of [14] is highly recommended.

[1] Recall that a *lattice* in a finite dimensional vector space V over $\mathbb{C}$ a discrete free abelian subgroup of V rank $2\dim_{\mathbb{C}}(V)$, equivalently V/Λ is a compact torus.

[2] Elements of $\mathfrak{H}_g$ are also called "Riemann matrices" by some authors. We do not do so here.

Recall that an *abelian variety* over $\mathbb{C}$ is a complex projective variety with a an algebraic group law, or equivalently a compact complex torus which admits an holomorphic embedding to a complex projective space. It is a basic fact that the existence of a non-degenerate abelian function on $\mathbb{C}^g$ with respect to the lattice $Q\cdot\mathbb{Z}^{2g}\subset\mathbb{C}^g$. So an equivalent statement of Theorem 2.3 is:

(2.3.1) THEOREM. *A compact complex torus of the form $\mathbb{C}^g/(Q\cdot\mathbb{Z}^{2g})$ for a $g\times 2g$ complex matrix Q is an abelian variety if and only if Q is a Riemann period matrix.*

It is easy to see that g-dimensional compact complex tori vary in a g^2-dimensional analytic family. Theorem 2.3.1 says that g-dimensional abelian varieties vary in (countable union of) $g(g+1)/2$-dimensional analytic families. More precisely all g-dimensional abelian varieties with a fixed principal part E is parametrized by the Siegel upper-half space $\mathfrak{H}_g$.

(2.4) Historical Remarks. The statement of theorem 2.3 did not appear in Riemann's published papers, but Riemann's was certainly aware of it. On page 75 of [14] Siegel wrote:

> *Riemann was the first to recognize that the period relations are necessary and sufficient for the existence of non-degenerate abelian functions. However, his formulation was incomplete and he did not supply a proof. Later, Weierstrass also failed to establish a complete proof despite his many efforts in this direction. Complete proofs were finally attained by Appell for the case $g=2$ and by Poincaré for arbitrary g.*

Krazer's comments on page 120 of [5] are similar but more polite. He also said that Riemann communicated his discovery to Hermite in 1860, citing (the German translation of) [4] One can feel the excitement brought by Riemann's letter[3] in Hermite's exposition of Riemann's "extremely remarkable discovery" of the symmetry conditions on a period lattice $\Lambda\subset\mathbb{C}^n$, necessary for the existence of (non-degenerate) abelian functions, see pages 148–150 of [4]. Reference for subsequent works by Weierstrass, Hurwitz, Poincaré, Picard, Appell and Frobenius can be found on page 120 of [5].

(2.4.1) Let S be a Riemann surface, let $\omega_1,\dots,\omega_g$ be a $\mathbb{C}$-basis of holomorphic differentials on S, and let $\gamma_1,\dots,\gamma_{2g}$ be a $\mathbb{Z}$-basis of the first homology group $H_1(S,\mathbb{Z})$. Theorem 1.2 says that the $P(\omega_1,\dots,\omega_g;\gamma_1,\dots,\gamma_{2g})$ is a Riemann matrix with principle part $-\Delta(\gamma_1,\dots,\gamma_{2g})$, and theorem 2.3 tells us that the quotient of $\mathbb{C}^g$ by the lattice $P(\omega_1,\dots,\omega_g;\gamma_1,\dots,\gamma_{2g})\cdot\mathbb{Z}^{2g}$ is an abelian variety. This abelian variety is called the *Jacobian variety* $\mathrm{Jac}(S)$ of the Riemann surface S. Two lines of investigation open up immediately.

A. Choose and fix a base point x_0 on S. Considering abelian integrals from x_0 to a variable point $x\in S$, one get the a map

$$X\in x\ \mapsto\ \begin{pmatrix}\int_{x_0}^{x}\omega_1\\ \vdots\\ \int_{x_0}^{x}\omega_1\end{pmatrix}\ \mathrm{mod}\ P(\omega_1,\dots,\omega_g;\gamma_1,\dots,\gamma_{2g})\cdot\mathbb{Z}^{2g}\ \in\mathrm{Jac}(S).$$

from S to $\mathrm{Jac}(S)$. Through this Abel-Jacobi one can analyze further geometric properties of the Riemann surface S.

[3] This letter wasn't mentioned in [4].

B. As the Riemann surface S varies in its moduli space, so does the corresponding Jacobian variety $\mathrm{Jac}(S)$. A natural question is: which abelian varieties arise this way? Can we characterize the Jacobian locus either analytically or algebraically, as a subvariety of the moduli space of abelian varieties?

The best introduction to this circle of ideas is [7], which also contains a nice "guide to the literature of references". See also [2], [3, Ch. 2 §7] and [8, Ch. 2 §§2–3] for Jacobian varieties and the Abel-Jacobi map. Further information can be found in survey articles in [1].

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