## Section X—Construction of the velocity-diagram when the bow is applied at a node

The discussion of the vibrational modes given in sections VI to IX proceeded on the assumption that the bow is applied at a point of irrational division of the string and that the pressure of bowing is sufficiently large in relation to its velocity to ensure that the motion at the bowed point alternates between two and only two rigorously constant velocities, once or oftener in each period of vibration. When these conditions are satisfied, the resulting mode of vibration should in general include the complete series of harmonics. We shall now pass on to consider the cases in which owing to the position of the bow coinciding exactly with a point of rational division of the string, the series of harmonics having a node at that point is completely absent in the resulting vibration. The absence of these harmonics is evidently necessary if the mode of vibration of the string is to be fully determinate in terms of the motion at the bowed point, and in section IV it has been shown to follow from this and the known characteristics of the motion at the bowed point that the form of the velocity waves must then be that of a number of straight lines parallel to one another with intervening discontinuities. The velocity-diagram of the string at any epoch of the vibration must therefore also consist of parallel straight lines with intervening discontinuities. It will now be shown that these straight lines are all parallel to the $x$-axis, i.e. to the position of equilibrium of the string.

Let the velocity-diagram of the string at a certain epoch in the corresponding irrational type of vibration consist of parallel straight lines inclined to the $x$-axis at an angle $\alpha$, with discontinuities of magnitude $d_1$, $d_2$, $d_3$, etc. intervening at the points $x = c_1, c_2, c_3$, etc. respectively. This velocity-diagram may be readily analyzed into its Fourier components. Let the velocity at any point on the string at the epoch referred to, be represented by $\psi(x)$. Then,

$$\psi(x) = \sum_{n=1}^{n=\infty} A_n \sin\frac{n\pi x}{l},$$

where the value of $A_n$ is determined by the equation

$$A_n = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx.$$

$\psi(x)$ is equal to $x \tan\alpha$ between the limits $x = 0$ and $x = c_1$. From $x = c_1$ up to $x = c_2$, it is equal to $(x\tan\alpha - d_1)$ and then changes to $(x\tan\alpha - d_1 - d_2)$ retaining this value up to $x = c_3$, and so on. Integrating by parts, we have

$$A_n = -\frac{2}{n\pi}\left[\psi(x)\cos\frac{n\pi x}{l}\right]_0^l + \frac{2}{n\pi}\int_0^l \tan\alpha\cos\frac{n\pi x}{l}\,dx.$$

Since $\tan\alpha$ is a constant, the second term reduces to zero and the equation may be written in the form

$$A_n = -\frac{2}{n\pi}\left[\mathrm{d}_1 \cos\frac{n\pi c_1}{l} + \mathrm{d}_2 \cos\frac{n\pi c_2}{l} + \text{etc.}\right].$$

When $n = 1$, we have

$$A_1 = -\frac{2}{\pi}\left[\mathrm{d}_1 \cos\frac{\pi c_1}{l} + \mathrm{d}_2 \cos\frac{\pi c_2}{l} + \text{etc.}\right].$$

When $n = s$, $A_s$ may be written in the form

$$A_s = -\frac{2}{\pi}\left[\frac{\mathrm{d}_1}{s}\cos\frac{\pi c_1}{l/s} + \frac{\mathrm{d}_2}{s}\cos\frac{\pi c_2}{l/s} + \text{etc.}\right].$$

The summation of the series $\sum_{n=1}^{n=\infty} A_n \sin(n\pi x/l)$ of which $A_1 \sin(\pi x/l)$ is the leading term gives us the original velocity-diagram $\psi(x)$ which consists of parallel straight lines inclined to the $x$-axis at an angle $\alpha$ and has a discontinuity $\mathrm{d}_1$ at the point $x = c_1$, a discontinuity $\mathrm{d}_2$ at the point $x = c_2$, and so on. From this, it follows that the series $\sum_{n=1}^{n=\infty} A_{ns} \sin(n\pi x/l/s)$ of which $A_s \sin(s\pi x/l)$ is the leading term would similarly give us when summed, a diagram also consisting of straight lines inclined to the $x$-axis at the same angle $\alpha$, the magnitudes of the discontinuities in it being $\mathrm{d}_1/s$, $\mathrm{d}_2/s$, $\mathrm{d}_3/s$, etc. and the series being periodic for successive increments of $x$ by the length $2l/s$ instead of $2l$ as with the original series. Subtracting the ordinates of the diagram thus derived from those of the original diagram $\psi(x)$, the resulting figure in which the $s^{th}$, $2s^{th}$, $3s^{th}$ harmonics, etc. are all absent, is seen to consist of straight lines parallel to the $x$-axis with intervening discontinuities. We have already seen that with the irrational types of vibration, $\mathrm{d}_1$, $\mathrm{d}_2$, $\mathrm{d}_3$, etc. are all numerically equal to $(v_A - v_B)$, some belonging to the positive wave and some to the negative. In the particular cases in which the positions $c_1$, $c_2$, $c_3$, etc. of the discontinuities are such that the motion at the bowed point is a simple two-step zig-zag, the summation of the series of harmonics to be left out gives, when graphically represented, a diagram of the simplest possible form, that is, one straight line terminated by one discontinuity in each length of abscissa $2l/s$, the $s^{th}$ harmonic being the first of the series. This is very readily proved. Taking the first type of vibration in which there is only one discontinuity in the velocity-diagram (figure 1 in section VI), we see that at the epoch chosen as the origin of time, $c_1 = l$. From the expression for $A_s$ given above, it follows that the series $\sum_{n=1}^{n=\infty} A_{ns} \sin(n\pi x/l/s)$ has a discontinuity equal to $v_A - v_B/s$ at the same point $c_1 = l$. The next discontinuity would be at $x = (s-2)l/s$, the line of the diagram passing through the intervening node at $x = (s-1)l/s$. For the second, third, fourth types of vibration, etc. discussed in sections VII, VIII and IX, a precisely similar construction holds good when the motion at the bowed point is a simple two-step zig-zag. For, the positions of the discontinuities $c_1$, $c_2$, $c_3$, etc. in the initial velocity-diagram are such that if the bow is applied at a node of a given harmonic, the discontinuities are also situated at certain other nodes of the same harmonic. A sufficient illustration of this fact and of the method of construction is given in figure 5(*i*). This shows the initial velocity-diagram for the third type of vibration elicited by bowing at the node $5l/8$. The thin inclined lines show the velocity-diagram with the complete series of harmonics, the thick horizontal lines show the velocity-diagram obtained by dropping out the $8^{th}$, $16^{th}$ harmonics, etc. and the dotted vertical lines show the intervening discontinuities. From the subsequent movement of the discontinuities in the diagram thus drawn, the vibration-curve at any desired point on the string is readily obtained.

The types of vibration set up by bowing the string at the nodal points $l/2$, $2l/3$, $3l/4$, $4l/5$, etc. are of special interest. The initial velocity-diagram in these cases is readily derived from the first irrational type of vibration by the method of construction described above. We shall now consider these cases a little more fully.

Figure 6 illustrates the first case. The initial velocity-diagram is a straight line parallel to the string with a discontinuity at each end. These discontinuities move in towards the centre of the string, and when they meet at the expiry of a quarter-period, the velocity at every point on the string, except the centre itself, is zero. At this epoch, the form of the string consists of two straight lines meeting at the centre, being the same as at the corresponding epoch in figure 1. Incidentally, it will be noticed that the mode of vibration of a string bowed at the centre is the same as that obtained when it is plucked at the centre.[10]

Figure 7 illustrates the case of a string bowed at a point of trisection. At the two points of trisection, the vibration-curve is a simple two-step zig-zag, and at two corresponding epochs, the configuration of the string consists of two straight lines meeting at the point of trisection. The nature of the motion as observed at the centre of the string is of special interest in this case. Commencing from the position of extreme displacement in one direction, the complete period of vibration is seen to be made up of six phases of equal duration, in the first and third of which the velocity of the centre of the string is the same as that of the bow applied at the point of trisection. In the fourth and sixth phases the velocity is also the same numerically but in the opposite direction. In the second and fifth phases, the velocity is numerically double that of the bow, having the same sign in the second phase, and the opposite sign in the fifth phase. Only the first, fifth, seventh, eleventh harmonics, etc. contribute towards making up the motion at the centre of the string in this case.

[10]It is also readily seen that the velocity-diagram of a string plucked at any other point and released should consist of a straight line parallel to the $x$-axis and bounded by two discontinuities which are initially coincident at the point of plucking and start off with the velocity of wave propagation in opposite directions. The form of the vibration-curves in this and other analogous cases is much more readily found by the aid of the velocity-diagram than by tracing the configuration of the string.

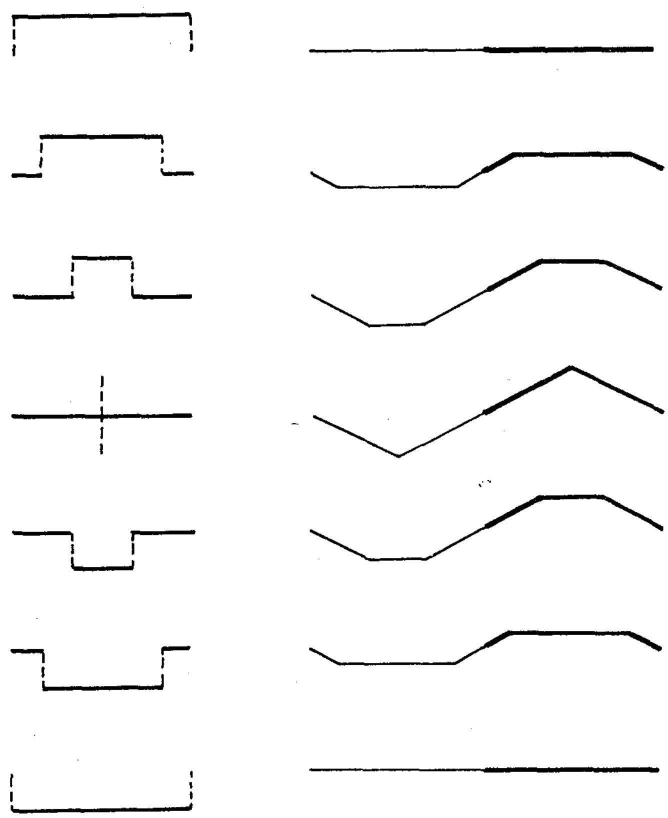



**Figure 6.** The string bowed at the centre.

Figure 8 illustrates the case in which the bow is applied at a point exactly a quarter of the length of the string from one end, the initial velocity-diagram of the string being obtained by the geometrical construction from the first type of vibration. The cases in which the bow is applied at a point one-fifth or one-sixth or one-seventh, etc., of the length from one end may be similarly dealt with.

Passing on to consider the modifications of the second, third and higher irrational types of vibration caused by the coincidence of the bow with a point of rational division of the string, we find no difficulty in constructing the initial velocity-diagram of the string in any specified case, and as we have already seen, the construction becomes further simplified when the motion at the bowed point is representable by a two-step zig-zag. Certain interesting relations then become evident. The velocity-diagrams derived from the first and second types of

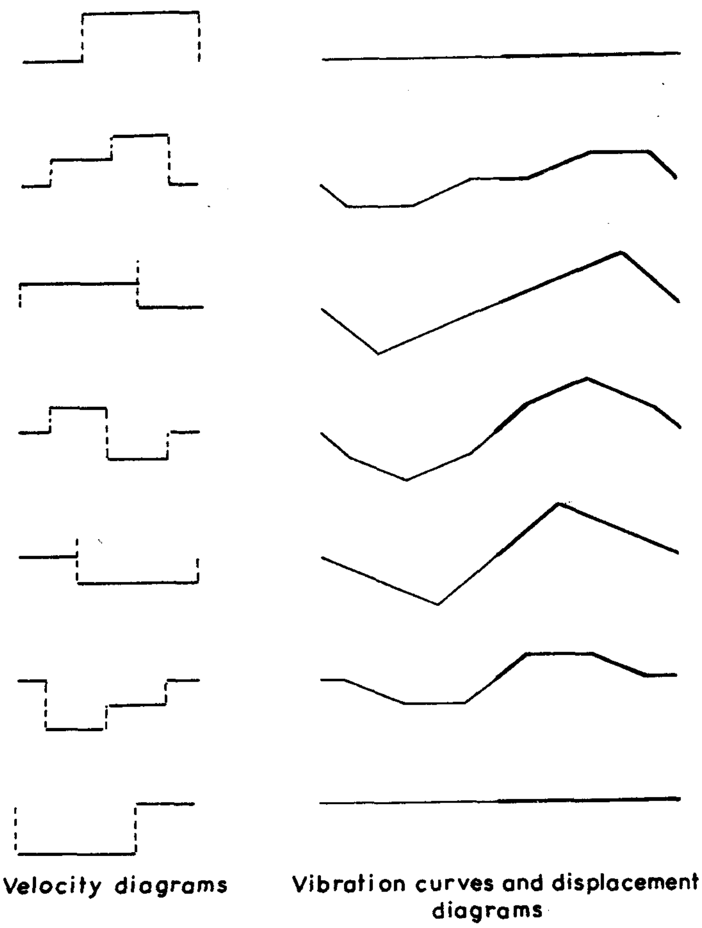



**Figure 7.** The string bowed at a point of trisection.

vibration are found to be identical when the bow is applied at a point of trisection (*cf.* section VII in which it was shown that these two types then become completely identical). Similarly, the first and third types become identical when the bow is applied at a distance $l/4$ from one end, the first and the fourth when it is applied at $l/5$ from one end and so on. Further, the second and third types give the same results when the bow is applied at a distance $2l/5$ from the end, the ratio $\omega$ being $1/5$ in both cases. The third and fourth types both give the ratio $\omega = 1/7$ when the bow is applied at $2l/7$, and so on. These relations may be stated and proved in the following general form.

*When the motion at the bowed point is given by a simple two-step zig-zag*, the $p^{th}$

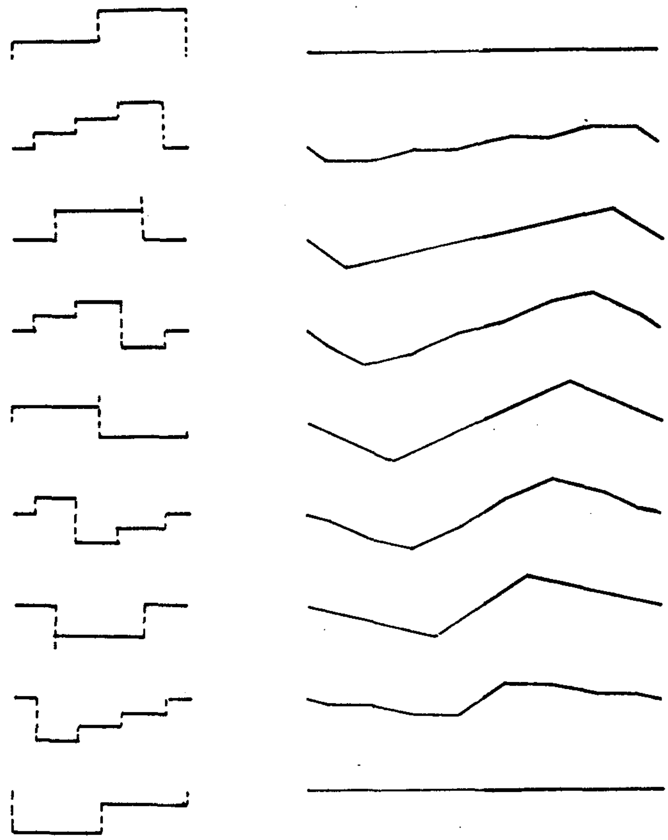



**Figure 8.** String bowed at point of quadrisection.

and $q^{th}$ types of vibration give identical vibration-curves, *provided the bowed point coincides with an intervening node of the* $(p+q)^{th}$ *harmonic.*

The proof of the relation stated above follows very simply from the general relation given in equation (17). Let $x_0$ define the position of the bowed point and let the nearest nodes of the $p^{th}$ and $q^{th}$ harmonics be the $(r+1)^{th}$ and the $(s+1)^{th}$ respectively, counting from one end. For the $p^{th}$ type, the ratio $\omega = (x_0 - rl/p) \div l/p$. For the $q^{th}$ type, the ratio $\omega = (sl/q - x_0) \div l/q$, it being assumed that $rl/p < x_0 < sl/q$. The $p^{th}$ and $q^{th}$ types give the same vibration-curves when the value of $\omega$ is the same for both at the bowed point. Equating $\omega$ for the two cases, we deduce $x_0 = (r+s)l/(p+q)$, in other words that the bowed point coincides with the

$(r+s+1)^{th}$ node of the $(p+q)^{th}$ harmonic. Similarly, if $x_0$ lies outside the limit $srl/s$ and $sl/q$, the $p^{th}$ and $q^{th}$ types become identical when the bow coincides with the $(r-s+1)^{th}$ node of the $(p-q)^{th}$ harmonic.

As an illustration of the results, we may consider the case in which the bow is applied at the point $3l/11$. The fourth and seventh types of vibration both give the ratio $\omega = 1/11$ and are thus identical in this case, but the third type gives the ratio $\omega = 2/11$, the first type gives the ratio $\omega = 3/11$ and the second type gives the largest ratio of all, viz. $\omega = 5/11$.

## Section XI—Some examples of the theoretical determination of the vibration-curves

We shall now proceed to figure a number of vibration-curves whose forms can be calculated by the aid of the theoretical principles set out in the preceding sections. To trace the vibration-curve at any given point on the string, we require data regarding the position of the bowed point and also as to the particular type of vibration (according to the method of classification explained in section V) which may be assumed to be actually elicited. In the next section we shall consider at some length the general relation connecting the position of the bowed point and the pressure, velocity and other features of the bowing adopted, with the mode of vibration elicited. Meanwhile, as a working rule, it will be assumed that if the bow is applied at a point not far from a node of the $n^{th}$ harmonic, $n$ being one of the first seven or eight natural numbers, the $n^{th}$ type of vibration is elicited. It will be further assumed that the motion at the bowed point is a simple two-step zig-zag except in cases in which, as shown in section IX, it is necessarily of a more complicated type. These assumptions are justifiable, provided the pressure of bowing is sufficient and other circumstances are favourable.

The detailed method of tracing the vibration-curves is as follows: the initial velocity-diagram of the string is first set down according to the principles already explained. The ordinate at the point whose vibration-curves is to be drawn, gives its initial velocity. The positions and magnitudes of the discontinuities and the direction of their motion, positive or negative, being known, the successive changes of velocity at the point of observation and the intervals of time at which they take place are obvious to inspection. The successive intervals and the resulting displacements may then be pricked off on a time-displacement diagram and when joined up, give us the desired vibration-curve.

In figures 1 to 8 we have already had a number of illustrations of the graphical process worked out in detail. The succeeding figures 9, 10, 11 and 12 present additional examples of vibration-curves determined by the entirely *a priori* method set forth above. They were specially drawn for comparison with the first

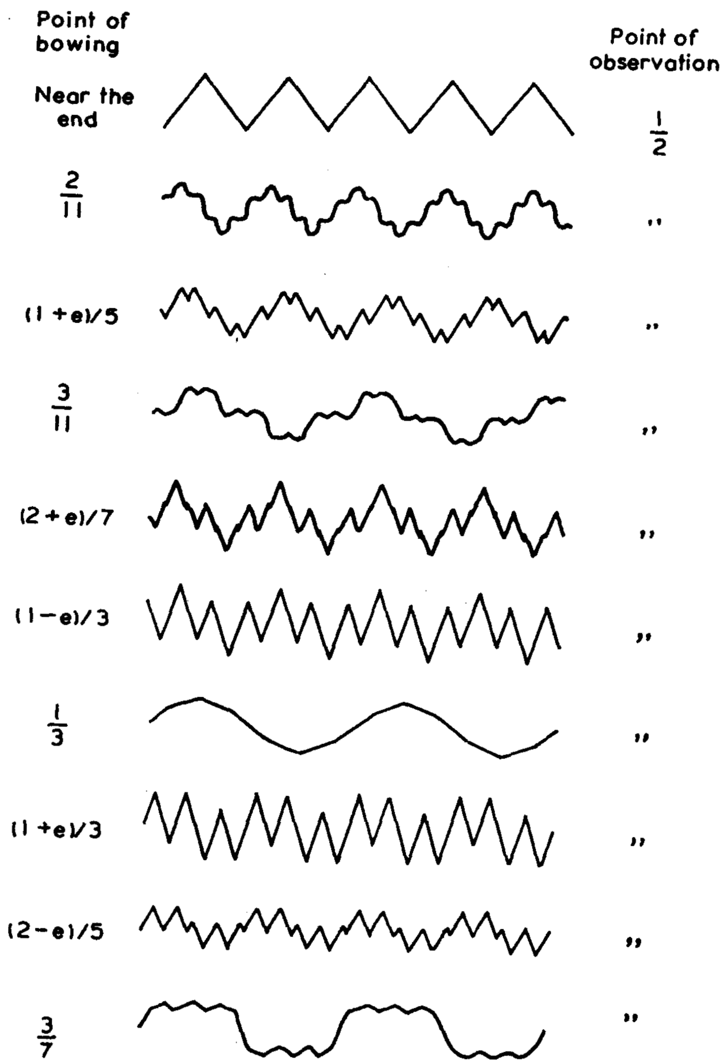



**Figure 9.** Examples of vibration-curves of a bowed string, found *a priori*.

forty of the curves found *experimentally* by Krigar-Menzel and Raps and published with their paper.[11]

*N.B.*—In these and the succeeding figures, the symbol $e$ represents a small fraction whose value was variously assumed.

The first curve in figure 9 belongs to the first type of vibration. The second and third curves belong to the fifth type of vibration, the bow being applied on opposite sides of the node of the fifth harmonic in the two cases.[12] The former curve is modified owing to the absence of the eleventh harmonic. The fifth and sixth types of vibration both give the ratio $\omega = 1/11$ when the bow is applied at the point $2l/11$. (At the same point, the first type of vibration would give the ratio $\omega = 2/11$). The fourth curve in figure 9 is a combination of the fourth and seventh types which become identical when the bow is applied at the point $3l/11$, the ratio $\omega$ at the bowed point being 1/11 for both types. The fifth curve belongs to the seventh irrational type of vibration and the sixth curve to the third irrational type. It will be noticed that these two forms are closely analogous and it can be readily shown that they merge into one another when the bow is applied exactly at the point $3l/10$, where they both give the ratio $\omega = 1/10$. The close approximation of the form of the seventh curve to that of a pure sine-wave will be noticed. The eighth curve belongs to the third irrational type of vibration which is pictured in greater detail in figure 3. The ninth curve belongs to the fifth irrational type of vibration. The tenth is a combination of the fifth and second types of vibration which become identical when the bow is applied at the point $3l/7$, both types giving the ratio $\omega = 1/7$.

The first curve in figure 10 is a combination of the second and seventh types which give a common ratio $\omega = 1/9$ when the bow is applied at the point $4l/9$. The second curve represents the second irrational type of vibration (see also figure 2 in which this is worked out in greater detail). The third curve is a combination of the fifth and sixth types which give a common ratio $\omega = 1/11$ at the bowed point $2l/11$. The fourth curve is similarly a combination of the fourth and fifth types ($\omega = 1/9$ at the bowed point $2l/9$). The fifth curve represents the third irrational type of vibration (see figure 4 for a treatment in greater detail). The sixth curve is a combination of the third and eighth types, $\omega$ being equal to 1/11 at the bowed point $4l/11$. The seventh curve in figure 10 is representative of the fourth irrational type of vibration. The eighth curve is a combination of the fourth and seventh types ($\omega = 1/11$ at the bowed point $3l/11$ for both). The ninth curve is a combination of the third and eighth types and the tenth is a combination of the third and fifth types ($\omega = 1/11$ and 1/8 respectively).

---

[11] The other experimental curves (twentyfour in all) that appear with the paper of Krigar–Menzel and Raps are mostly simple two-step zig-zags. See also Barton's *Text-Book of Sound*, page 432.

[12] The position of the bowed points for the two corresponding experimental curves have been erroneously shown interchanged in the paper of Krigar–Menzel and Raps.

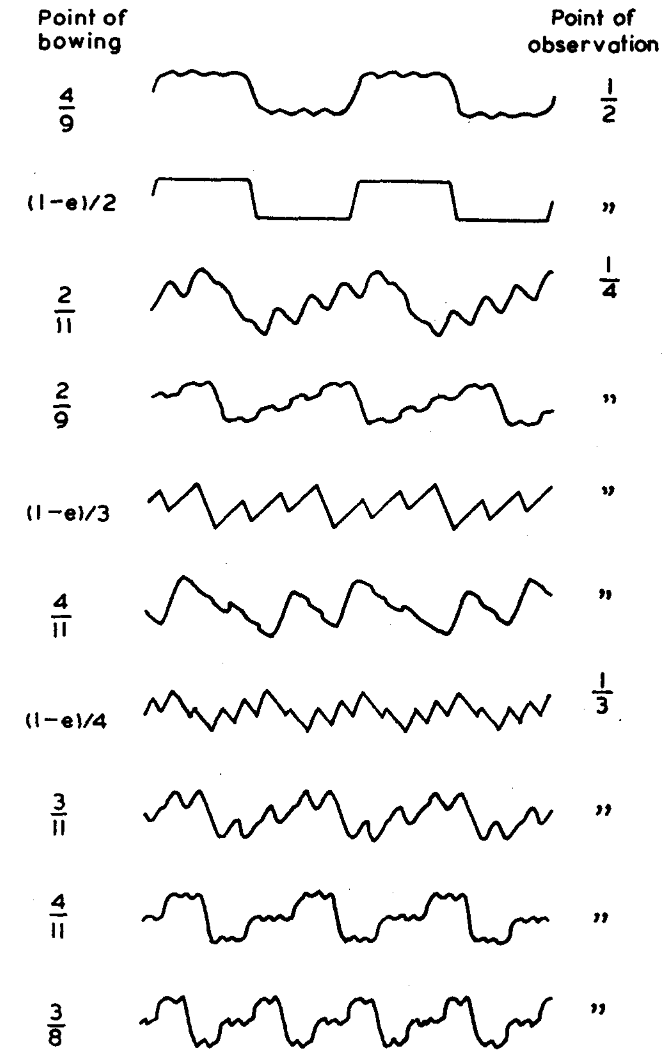



**Figure 10.** Examples of vibration-curves of a bowed string, found *a priori*.

The first curve in figure 11 belongs to the second irrational type of vibration (compare with the graphs in figure 2). The second and third curves in figure 11 need no further remarks. The fourth curve is a combination of the sixth and seventh types of vibration, $\omega$ being equal to 1/13 at the bowed point $2l/13$ for both types of vibration. The fifth and seventh curves are examples of the well-known

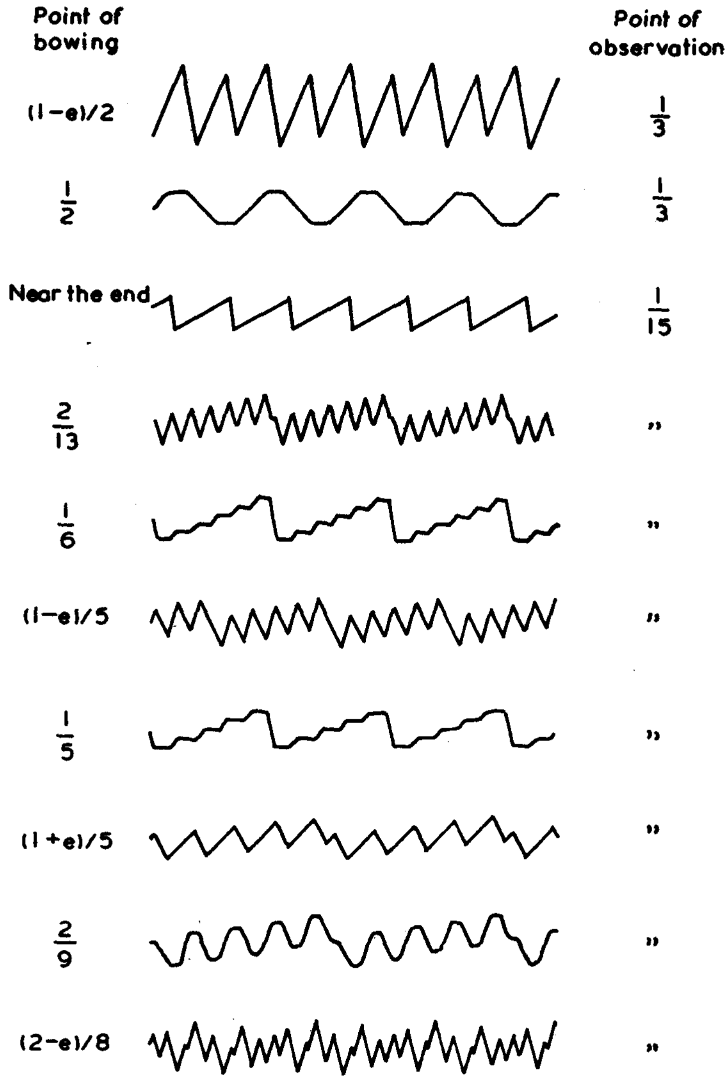



**Figure 11.** Examples of vibration-curves of a bowed string found *a priori*.

"staircase" figures discovered by Helmholtz. The sixth and eighth curves represent two cases of the fifth irrational type of vibration. The ninth curve is a combination of the fourth and fifth type of vibration, $\omega$ being equal to 1/9 at the bowed point $2l/9$ for both types. The last curve in figure 11 is of special interest, as it is a case of the eighth type of vibration maintained with a complicated motion

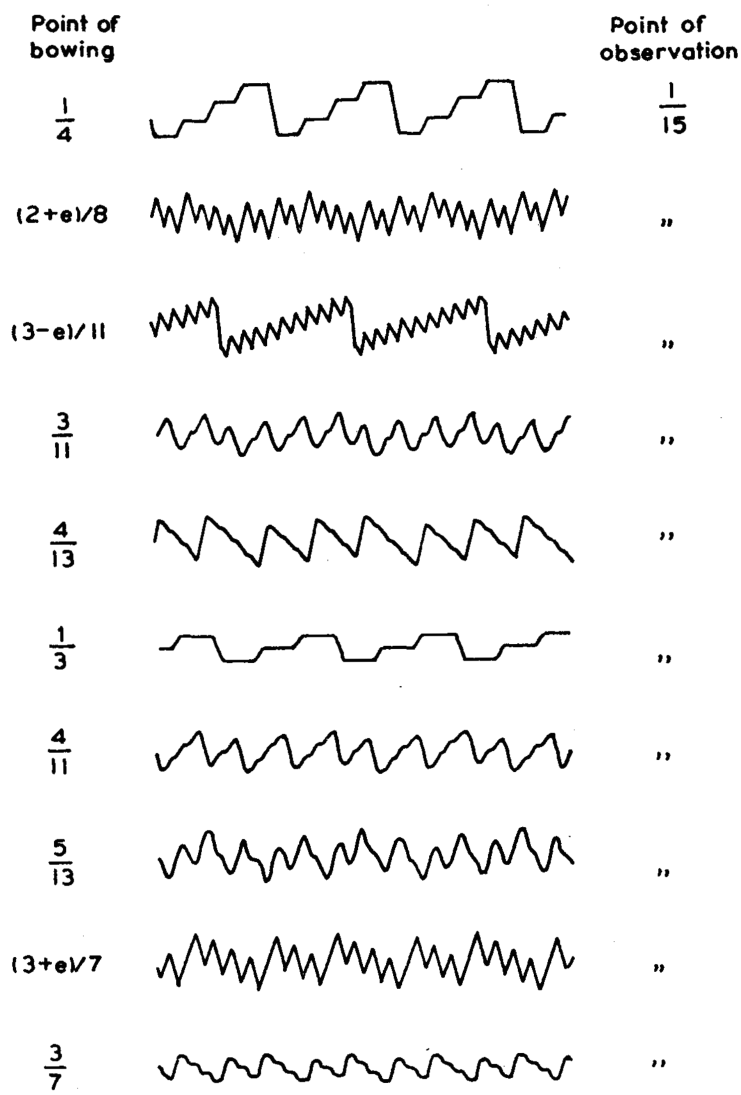



**Figure 12.** Examples of vibration-curves of a bowed string, found *a priori*.

at the bowed point. In section IX, it was shown that the motion at the bowed point cannot be a simple two-step zig-zag, if the eighth type of vibration is elicited by applying the bow near the point $l/4$. In the initial velocity-diagram from which this vibration-curve was drawn, two of the discontinuities were situated at the two ends of the string respectively, and the other six were taken to be situated in pairs at the points $(2 + 3e)/8$, $(4 - 2e)/8$ and $(6 + e)/8$ respectively, corresponding to the position of the bowed point $(2 - e)/8$.

The first vibration-curve shown in figure 12 is of the type dealt with in greater detail in figure 8. The second curve is of special interest, as the eighth irrational type of vibration which it represents, is elicited by applying the bow near a node of the fourth harmonic and the motion at the bowed point for this case is thus necessarily of a "complicated" type, viz. a four-step zig-zag instead of a two-step. The position of the bowed point being $(2 + e)/8$, the discontinuities in the initial velocity-diagram were taken to be situated in pairs at the positions $(1 - e)/8$, $(3 - 5e)/8$, $(5 + e)/8$ and $(7 - 3e)/8$ respectively. The third curve in figure 12 represents the eleventh irrational type of vibration. The velocity-diagram of the eleventh type from which this was drawn gives a simple two-step zig-zag as the vibration-curve at the point $(1 + e)/11$. An entirely different curve (in which the 4th and 11th harmonics are dominant) is however obtained from the velocity-diagram which gives a two-step zig-zag at the point $(3 - e)/11$, and actual experimental trial, I find, gives a result in agreement with this, and not the curve shown in figure 12. Krigar–Menzel and Raps however describe one of their experimental curves which is very similar to the third in figure 12 as that produced by bowing at the point $(3 - e)/11$. If this description were correct, the motion at the bowed point could not have been a simple two-step zig-zag.

The other vibration-curves obtained theoretically and shown in figure 12 do not require any special remarks. The fourth curve is a combination of the fourth and seventh types which give identical motions ($\omega = 1/11$) at the bowed point $3l/11$. The fifth curve is the third irrational type of vibration modified by the absence of the 13th harmonic with the node of which the bowed point coincides. The sixth curve is of the well-known staircase type. The seventh curve is a combination of the third and the eighth types given by the ratio $\omega = 1/11$. The eighth curve is similarly a combination of the fifth and the eighth irrational types which become identical when the bow is applied at point $5l/13$. The ninth curve illustrates the seventh irrational type of vibration, and the tenth curve is a combination of the second and fifth types of vibration which give a common ratio ($\omega = 1/7$) at the bowed point in the position $3l/7$.

## Section XII—The effect of the variation of the pressure and velocity of bowing

### Preliminary discussion

In the previous sections of the paper, we have considered the kinematics of all the possible modes of vibration of a bowed string in which the motion at the bowed point consists of ascents and descents with strictly constant velocities. It now remains to consider various subsidiary questions that arise. What are the modifications in the kinematical theory necessary, when, as foreshadowed in the discussion on the *modus operandi* of the bow, the motion at the bowed point is not rigorously of the kind postulated? In other words, what is the effect produced if the velocity with which the bowed point slips past the hairs of the bow is not exactly constant in each period of vibration? Then again, what is the effect produced by the finiteness of the region with which the bow is in contact, a region which for the purpose of discussion we have so far taken as equivalent to a mathematical point? Does any slipping occur when the string is being carried forward by the bow? Is it possible in practice that by simply removing the bow from a nodal point to another closely contiguous to it, the missing harmonics in any given type of vibration are suddenly restored to their full strength as our kinematical discussion tacitly assumed? Finally we have the all-important question, what are the conditions of excitation, e.g. pressure and velocity of bowing and so on, required for any given type of vibration to be elicited? What part does the instrument on which the string is mounted and the handling and properties of the hairs of the bow play in determining these conditions? What, for instance, is the effect on the motion of the string produced by loading the bridge over which it passes with a mute or otherwise? This is a most formidable array of questions, but it does not seem entirely hopeless to find answers to them all, provided we proceed step by step, seeking answers to the questions one at a time.

We may commence by retaining the assumption that the region of contact of the bow may be treated as a mathematical point and that the velocities of ascent and descent of this point are rigorously constant. We have already seen that on this assumption all the possible modes of vibration can be classified in a series of which the ordinal number $n$ is the same as the total number of discontinuities in the velocity-diagram of the string, the bowed point being taken to divide the string in an irrational ratio. If the point of bowing divides the string in a rational ratio, the mode of vibration may be derived from one of the irrational types by dropping out the series of harmonics having a node at that point (see section X). Another important point elicited by the discussion is that if $n$ be not a prime number, a two-step zig-zag motion at the bowed point is kinematically impossible for that type of vibration if the bowed point lies within a distance of $l/2n$ from any node of the $n^{th}$ harmonic which is also a node of some harmonic of lower frequency. Further, if the position of the bowed point be specified and its motion is representable by a simple two-step zig-zag, the entire mode of vibration is uniquely determined by the number $n$ of the discontinuities. But if the motion at the bowed point consists of several ascents and descents in each period, the number $n$ does not uniquely determine the motion. One or more additional constants expressing the initial position of the discontinuities in the velocity-diagram must then be specified for the motion to be completely determinate.

From the preceding, it is obvious that the first step for establishing the mode of vibration in any given case is to find the number $n$ of the discontinuities. *Prima facie*, we may exclude the consideration of all types of vibration of which the ordinal number $n$ is very large. This is evident, for the larger the number of discontinuities, the more important is the part which harmonics of high order contribute to the motion,[13] and it is known from various considerations that these harmonics of high order are difficult to elicit and maintain as part of a perfectly periodic motion with the frequency of the gravest mode. As has been well shown by Prof. E H Barton,[14] the higher the frequency of any given component in the motion, the more sensitive it is to any departure from the exact adjustment for resonance. Under ordinary circumstances, the free periods of vibration only approximate to, and do not actually form a harmonic series. We are therefore led to conclude that any type of vibration in which the number of discontinuities is very large and in which therefore harmonics of high order are dominant would not, in practice, be elicited by the bow.

Two other considerations also point to the same result as that noted above. It is very probable that the higher harmonics are subject to a greater degree of damping and the bow would therefore tend to elicit types of vibration in which they are relatively subordinate. Then again, we see that the finiteness of the region with which the bow is in contact would operate in the same direction. For, from equation (17)

$$\omega = \frac{nx_n}{l}$$

which expresses the ratio of the time during which the string moves past the hairs of the bow to the whole period of the vibration, we see that the larger $n$ is, the more rapidly does the value of $\omega$ change with any alteration in the position of the bowed point, $x_n$ being the distance of this point from the nearest node of the $n^{th}$

[13]This becomes almost self-evident on attempting to draw a velocity-diagram for the string consisting of parallel straight lines inclined at a constant angle $\alpha$ to the $x$-axis and separated by a large number of equal discontinuities situated at various intervals along the string.

[14]E H Barton, "On the Range and Sharpness of Resonance to Sustained Forcing," *Bull. Indian Assoc. Cultiv. Sci.*, No. 13.

harmonic. The disturbing effect of applying the bow over a finite region instead of at a mathematical point would therefore be much greater when the value of $n$ is large, and would thus have an unfavourable effect on the production of types of vibration involving a large number of discontinuities.

We thus see that if the motion at the bowed point admits only of discontinuous changes of velocity from one value to another and *vice versa*, the possible modes of vibration would be confined to a restricted number of types and their variants. Each type would, according to equation (17), have a characteristic value of $\omega$ if the position of the bowed point were specified, and a passage from one type to another would involve a sudden change from one value of $\omega$ at the bowed point to another.

Taking the results summarised above, together with those arrived at in the discussion on the *modus operandi* of the bow in section III, it is now possible to trace the general effects of varying the pressure or velocity with which the bow is applied. If there is a change of type, the value of $\omega$ undergoes the concomitant discontinuous change given by equation (17). If the pressure of the bow is increased or its velocity decreased, the value of $\omega$ in the concomitant change of type decreases as a rule, so that the bow and string move with a common velocity for an increased fraction of the period. The change would be in the opposite direction if the pressure of the bow were decreased or its velocity increased. The change of type may or may not, according to the circumstances, involve a change in the motion at the bowed point from a simple two-step zig-zag to one with several ascents and descents, or *vice versa*. Another variety of change that may be caused by the alteration of the pressure or velocity of the bow is a readjustment of the positions of the discontinuities with the corresponding changes in the relative amplitudes of the harmonics, the value of $\omega$ which is determined by the number of discontinuities in the motion remaining unaltered. This species of change, as already shown, would only be possible if the motion at the bowed point involved several ascents and descents in the period.

When the pressure with which the bow is applied is considerable and the other circumstances are favourable, the maintained motion would be the one which gives, as nearly as possible, the smallest value of $\omega$ at the bowed point. When the bowed point is in the neighbourhood of a node of some fairly important harmonic, say the $n^{th}$, it is readily seen from equation (17) that the $n^{th}$ type would give a very small value of $\omega$ and is therefore the type that would be maintained. This is the justification of the result which was assumed in section XI for theoretically determining the vibration-curves and comparing them with those obtained experimentally by Krigar-Menzel and Raps. From the published account of these experimenters it appears that the style of bowing actually adopted by them was the one referred to above. For smaller pressures of bowing than those adopted by them, it might be expected that the modes of vibration figured therein would be incapable of being maintained and would yield place to types giving larger values of $\omega$.

## Graphical treatment

A clearer comprehension of the foregoing results may be obtained by discussing the specific cases in which the bow is applied at some point between the extreme end of the string and the node of the fifth harmonic distant $l/5$ from the same end. This includes the "musical range" of bowing. When the bow is applied very close to the end of the string, only the first type of vibration is kinematically capable of giving a two-step zig-zag motion at the bowed point. This also gives the smallest value of $\omega$ and is accordingly the type that is elicited by firm bowing. The higher types give four, six, or eight-step zig-zags, etc. as the case may be at the bowed point, with correspondingly larger and larger values of $\omega$, and can thus be obtained only by increasing the velocity or reducing the pressure of bowing. As the position of the discontinuities in these cases is susceptible of variation, each of these types is itself capable of modification by altering the pressure or velocity of bowing, and the relative amplitudes of the harmonics may be profoundly modified without any change in the value of $\omega$.

When the bow is removed from the extreme end to a point at some distance from it, some of the higher types also are capable of giving a two-step zig-zag motion at the bowed point. Let us now assume for simplicity that the bow does not elicit vibrations involving more than nine discontinuities, so that the ninth type is the highest that need be considered. Within the range between 0 and $l/5$, the bow may elicit the first type anywhere. The table IV shows for the other types the maximum ranges within which the bow must be applied for a two-step motion to be even kinematically possible.

**Table IV**

$0 < x_b < l/5$

| | |
|---|---|
| Ninth type | $l/18 < x_b < l/6$ |
| Eighth type | $l/16 < x_b < 3l/16$ |
| Seventh type | $l/14 < x_b < 3l/14$ |
| Sixth type | $l/12 < x_b < l/4$ |
| Fifth type | $l/10 < x_b < 3l/10$ |
| Fourth type | $l/8 < x_b < 3l/8$ |
| Third type | $l/6 < x_b < l/2$ |

While this table gives the maximum ranges, the *effective* ranges should be considerably less. For, taking the ninth type for example, the value of $\omega$ is 1/9 when the bow is applied at the point $8l/81$ or at the point $10l/81$. The ninth harmonic which is the dominant of the type then entirely vanishes. The effective range for eliciting this type must thus lie within the limits noted above. Proceeding on this principle, the limits within which the effective ranges for the different types should lie are calculated and shown in table V.

**Table V**

$0 < x'_b < l/5$

| | |
|---|---|
| Ninth type | $l/10\frac{1}{8} < x_b < l/8\frac{1}{10}$ |
| Eighth type | $l/9\frac{1}{7} < x_b < l/7\frac{1}{9}$ |
| Seventh type | $l/8\frac{1}{6} < x_b < l/6\frac{1}{8}$ |
| Sixth type | $l/7\frac{1}{5} < x_b < l/5\frac{1}{7}$ |
| Fifth type | $l/6\frac{1}{4} < x_b$ |
| Fourth type | $l/5\frac{1}{3} < x_b$ |

From this table it is seen that, practically speaking, the first type is the only one that can be elicited with a two-step motion at the bowed point if this lies between 0 and $l/10$. If the bow is applied between $l/10$ and $l/5$, the higher types may also give a two-step motion at the bowed point within their respective ranges. The values of $\omega$ within these ranges for the several types as ascertained from equation (17) are shown graphically in figure 13.

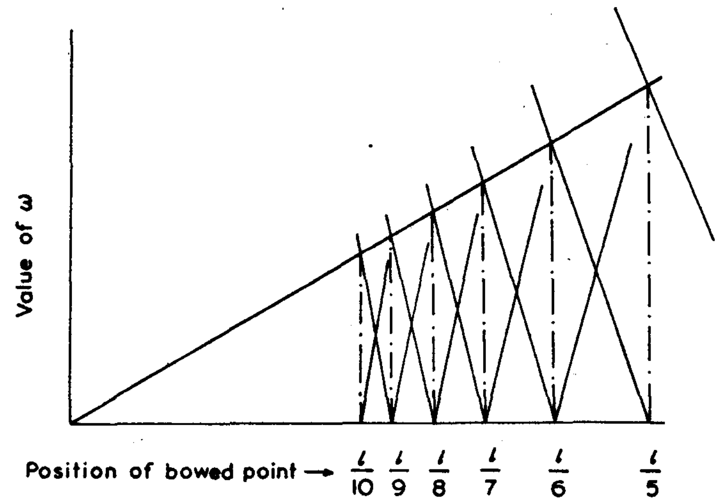



**Figure 13.** Showing the chief types of vibration, their ranges and characteristics and the effect of the pressure of bowing.

In this diagram, the heavy line is the graph of $\omega$ for the first type, and the thin lines meeting in pairs at the points $l/9$, $l/8$, etc. give the values of $\omega$ for the 9th, 8th types, etc. respectively. The most noticeable feature in the diagram is the extreme steepness of these lines compared with that for the first type. At the nodal points $l/10$, $l/9$, $l/8$, etc. the diagram shows that value of $\omega$ to be either zero, or else 1/10, 1/9, 1/8, etc. respectively as the case may be. Rejecting the zero value for $\omega$ as inadmissible, we find that at the nodal points $l/10$, $l/9$, $l/8$, etc. the value of $\omega$ is 1/10, 1/9, 1/8, etc. respectively. The lines for the higher types intersect in pairs at the points whose abscissae are $2l/19$, $2l/17$, $2l/15$, $2l/13$ and $2l/11$ respectively. At these points, the first type gives the value of $\omega$ to be 2/19, 2/17, 1/15, etc. respectively. But the higher types give the value of $\omega$ at these same points to be 1/19, 1/17, 1/15, etc. If, therefore, these types are to be elicited, considerable pressure and small velocity of bowing must be adopted, and at the same time, the region with which the bow is in contact must be very narrow. With a short, heavily-damped string such as that of a violin, these conditions are not readily obtainable and, at any rate, they differ from those actually adopted in the whole range of violin practice. We are therefore led to conclude that in the musical applications of the subject, we have to deal only with the first type of vibration and its modifications. On an ordinary monochord, however, by suitable bowing the higher types of vibration may be obtained within the ranges considered. The diagram shows that except at the nodal points already mentioned, 3 or even 4 different values of $\omega$ are possible.

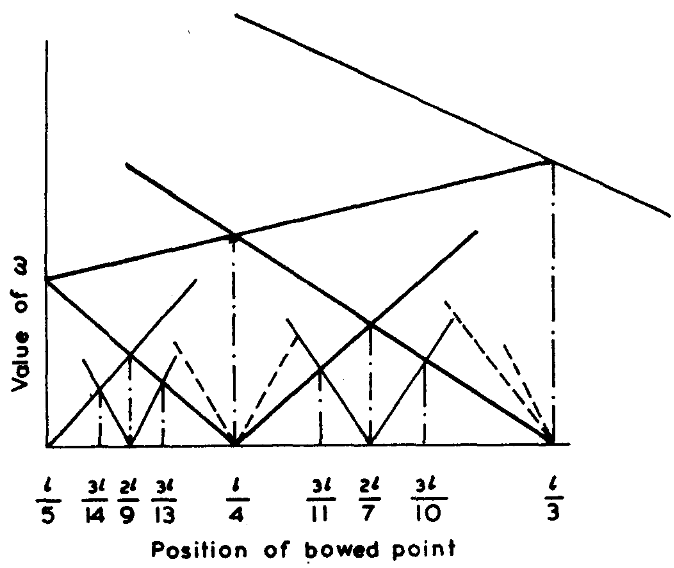



**Figure 14.** Showing the chief types of vibration, their ranges and characteristics and the effect of the variation of the pressure of bowing.

Figure 14 shows the different types of vibration with discontinuous changes of velocity that may be obtained by applying the bow anywhere within the region lying between $l/5$ and $l/3$. The heavy line, as before, is the graph for the first type. The other types are shown by lines passing through the nodes of the corresponding harmonics, thin continuous lines representing types which may give a simple two-step motion at the bowed point, and broken lines (— — — — —) representing types which necessarily involve more complicated motions at the bowed point. For instance, through the point $l/3$, we have one continuous line representing the third type and two broken lines representing the sixth and ninth types respectively. Similarly, through the point $l/4$, we have two continuous lines, one on each side, representing the fourth type and two borken lines, one on each side, representing the eighth type. The diagram enables us to find at a glance, the general effect of applying the bow at any given point within the range and of any considerable variation of the pressure or velocity with which it is applied. At the two nodal points $l/5$ and $l/3$, for a two-step zig-zag motion, the diagram shows that the value of $\omega$ can only be 1/5 and 1/3 respectively. At the point $l/4$, $\omega$ has generally the value 1/4, but another value, i.e. 1/2 belonging to the second type, is barely on the limit of possibility. Elsewhere, three, four, five or even six different values of $\omega$ are possible, the number being determined by the position of the bowed point, and the particular value out of those possible which is actually obtained being determined by the pressure, velocity and other features of the bowing adopted in any given case.

Figure 15 similarly represents the types of vibration that may be obtained by bowing at points lying between $l/3$ and $l/2$. It is noteworthy that all the three diagrams give the values of $\omega$ at the bowed point, both in the cases in which this point divides the string in a rational ratio as well as in those in which it divides the

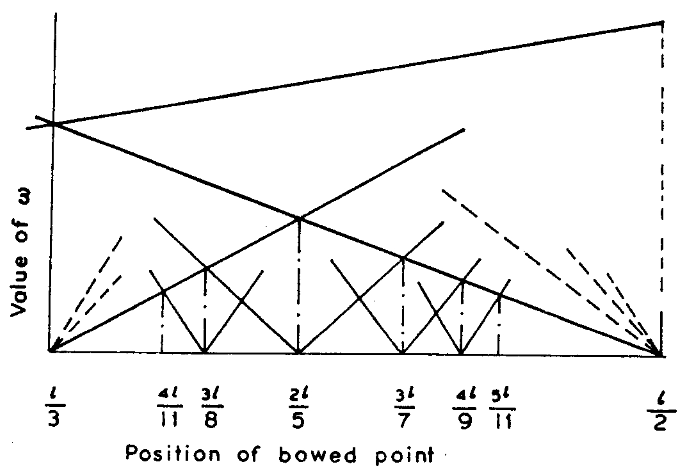



**Figure 15.** Showing the chief types of vibration, their ranges and characteristics and the effect of the variation of the pressure of bowing.

string in an irrational ratio. In cases of the latter kind, the diagrams may also be used to find the value of $\omega$ elsewhere than at the bowed point.

There is another and a very instructive point of view from which the effect of the variation of pressure may be regarded. In section II of the paper, it was shown that if the bow is applied at a point of rational division of the string, the harmonics having a node at that point are not excited. It is obvious also that if the bowed point is situated at some little distance from the node instead of actually coinciding with it, the forces required to maintain the harmonics in question with appreciable amplitudes would be of considerable magnitude relatively to those required for the maintenance of the other components in the motion. We are led therefore to conclude that the types of motion maintained would tend to be those in which harmonics having a node near the bowed point are relatively large or small in amplitude according as the pressure with which the bow is applied is much or little. We are also led to conclude that as an increase of the velocity of the bow involves an increase of the amplitude of vibration, its effect on the character of the motion would be analogous to that of a decrease in the pressure of the bow. The preceding results are in agreement with those already arrived at from somewhat different considerations.

## Mathematical theory

We now proceed to consider the mathematical theory of the subject. For the present, we shall adhere to the simplifying assumption already made that the region of contact may be regarded as a mathematical point. Let the motion of the string in any actual case be represented by the Fourier series

$$y = \sum_{n=1}^{n=\infty} \sin\frac{n\pi x}{l}\left[a_n \sin\frac{2n\pi t}{T} + b_n \cos\frac{2n\pi t}{T}\right]$$

or more briefly

$$y = \sum_{n=1}^{n=\infty} B_n \sin\frac{n\pi x}{l} \sin\left(\frac{2n\pi t}{T} + e_n\right).$$

From the formulae found in section II of the paper it is seen that the force required to maintain this motion may be represented by

$$\sum_{n=1}^{n=\infty} k_n B_n \frac{\sin\left(\dfrac{2n\pi t}{T} + e_n + e_n'\right)}{\sin\dfrac{n\pi x_0}{l}},$$

when $k_n$ and $e_n'$ are quantities independent of the position of the bowed point and

$x_0$ is the distance of this point from the fixed end of the string. If for a particular value or values of $n$ the denominator $\sin n\pi x_0/l$ of the term in the series is very small but not actually zero, and if at the same time the amplitude $B_n$ of the corresponding harmonic or harmonics remains finite, the magnitude of the respective term or terms in the expression for the maintaining force is necessarily great. But if $\sin n\pi x_0/l$ is actually zero for certain values of $n$, the corresponding terms $B_n$ in the maintained motion must necessarily vanish as we must obviously exclude the possibility of infinitely large values for the maintaining force. The series given above represents the variable part of the frictional force at the point of contact. This force is a function of the pressure $P$ with which the bow is applied and the relative velocity $(v - v_B)$ at the point of contact. It may be written in the form $F(P, v - v_B)$, the function remaining determinate and always of the same sign so long as the relative velocity $(v - v_B)$ does not become zero or change sign. We may therefore write

$$F(P, v - v_B) = P_0 + \sum_{n=1}^{n=\infty} k_n B_n \frac{\sin\left(\dfrac{2n\pi t}{T} + e_n + e'_n\right)}{\sin\dfrac{n\pi x_0}{l}} \tag{22}$$

where $P_0$ is the non-periodic part of the frictional force.

Equation (22) may be regarded as the dynamical relation from which the motion at the bowed point and therefore also the entire mode of vibration of the string is to be determined. For, if the velocity $v$ of the bowed point be expressed as a function of the time, the amplitudes $B_n$ of the harmonics can be found therefrom by integration over the complete period. The various types of vibration whose kinematical theory was discussed in the first part of the paper may be regarded as forming a series of *limiting solutions* of this dynamical equation to one or other of which the actual results approximate more or less closely according to circumstances. How these limiting solutions are obtained has already been indicated in the discussion on the *modus operandi* of the bow. We now proceed to examine more closely the degree of approximation within which the limiting solutions represent the actual results.

Our knowledge of the form of the function $F(P, v - v_B)$ is not at present very definite for surfaces of the kind we are concerned with in this paper. It may however be accepted as demontrably correct that when the relative velocity $(v - v_B)$ is actually zero, the friction is not greater than a certain maximum statical value $P_s$ which increases continuously for increasing values of the pressure $P$ with which the bow is applied. We may also suppose that when the relative velocity $(v - v_B)$ is greater than zero, the magnitude of the friction decreases with increasing values of the relative velocity, at first somewhat rapidly and later on perhaps not so rapidly. Arguing from these premises, the relation given in (22) enables us to arrive at certain important conclusions. In the first place, since the quantities $k_n$, $e'_n$ depend on the construction of the sounding box on which the string is stretched, the summation of the series on the right-hand side of (22) cannot, in general, result in our finding a constant value for the relative velocity $(v - v_B)$ in any or all of the epochs in which it is greater than zero. The relative velocity may however approach or attain practical constancy in any or all of the epochs referred to, if the variation of the sum of the series on the right-hand side of equation (22) within those epochs is negligibly small. It is seen that such practical constancy would be attained if the pressure $P$ with which the bow is applied is sufficiently large to ensure that the necessary variation in the value of the function $F(P, v - v_B)$ can be secured by negligibly small changes in the relative velocity $(v - v_B)$. The practical constancy in the relative velocity during the epochs referred to may also be secured even for moderate values of the pressure $P$, provided that in the series on the right-hand side of (22) there are no large, rapidly-varying terms. For instance, if the string were bowed at important nodes such as $l/3$ or $l/4$ even with very moderate pressure, the absence of the corresponding series of harmonics in the resulting motion removes terms which would otherwise introduce large and rapidly varying fluctuations in the value of the series for the maintaining force. The constancy of the velocity of the bowed point in slipping past the hairs of the bow is thus remarkably perfect in such cases.

Assuming the solution of (22) to be one of the limiting types in which the relative velocity $(v - v_B)$ has a constant value in all the epochs in which it is not actually zero, we may easily find an inferior limit to the pressure with which the bow must be applied for the given type of vibration to be possible. Thus, let $(v_A - v_B)$ be the constant value of the relative velocity during any such epoch and let $P_A$ be the sum of the series on the right-hand side of (22) for that epoch. Then we must have

$$F(P, v_A - v_B) = P_0 + P_A. \tag{23}$$

Further, let $P'_A$ be the maximum value of the series during any of the epochs in which the relative velocity is zero. Then, we must have

$$P_s(\text{the statical friction}) \nless P_0 + P'_A. \tag{24}$$

Subtracting the quantities in (23) from the two sides of the inequality in (24), we find that the pressure with which the bow must be applied is such that

$$P_s - F(P, v_A - v_B) \nless P'_A - P_A.$$

If $\mu$ be the coefficient of statical friction for the pressure $P$, and $\mu_A$ be the coefficient of dynamical friction for the pressure $P$ and relative velocity $(v_A - v_B)$, the inequality may be written thus,

$$P \nless \frac{P'_A - P_A}{\mu - \mu_A}. \tag{25}$$

If the pressure of the bow is less than the critical value given by (25), the vibration would no longer be maintained and would therefore alter in character. We may now consider the effect of *increasing* the pressure above this critical value. It is obvious from equation (22) that if the relative velocity $(v - v_B)$ is assumed to have a constant value in all the epochs in which it is not actually zero, the increase in the frictional force $F(P, v - v_B)$ caused by an increase of the pressure of the bow might merely result in raising the non-periodic constant $P_0$ on the right-hand side of the equation and thus leave the periodic part of the forces acting on the string unaltered. When, therefore, any one of such types of vibration is once thoroughly established, considerable latitude is permissible in the pressure of the bow so long as it does not fall below the critical value. It must not however be understood from this that the pressure can be increased indefinitely. For, if there be some other possible type of vibration whose critical pressure be higher than the first, the motion originally maintained would be relatively less stable and would yield place to the other, when the pressure of the bow exceeds the critical value of the latter. The manner in which an increase of pressure sets up an instability of the original type of vibration is best understood by analogy with that of the equilibrium of a system with one degree of freedom. The equation of motion about the position of equilibrium of such a system under the action of the bow is

$$\ddot{x} + (k - \lambda)\dot{x} + n^2 x = 0$$

where $k$ is the damping coefficient and $\lambda$ is a positive quantity proportional to the velocity-rate of change of the frictional force which may be increased by increasing the pressure of the bow. When the velocity as given by $\dot{x}$ is initially zero, there is apparently nothing to cause the system to depart from the position of equilibrium. If, however, by increasing the pressure of the bow, $\lambda$ is increased till it exceeds $k$, the equilibrium becomes unstable and any small motion is magnified continuously till it reaches the limit set by the velocity of the bow. The mechanical disturbance caused by laying on the bow or by increasing its pressure is sufficient to give effect to the instability. Similarly in the case of the bowed string, a steady state of vibration in any given type is obviously out of the question if any slight disturbance results in a progressive change and readjustment of the amplitudes of the harmonics. Referring to equation (22), it is seen that an increase in the non-periodic part $P_0$ of the frictional force caused by an increase in the pressure $P$ of the bow would tend to set up such instability if an alternative mode of vibration with a higher critical pressure were possible.

On comparing equations (22) and (25), it is seen that, in general, the type of vibration which has a higher critical pressure than another would also contain the harmonics having nodes near the bowed point with relatively larger amplitudes.

## Analysis of the motion

$$y = \sum \sin\frac{n\pi x}{l}\left[a_n \sin\frac{2n\pi t}{T} + b_n \cos\frac{2n\pi t}{T}\right]$$

Differentiating, we have

$$\frac{dy}{dt} = \sum \frac{2n\pi}{T} \sin\frac{n\pi x}{l}\left[a_n \cos\frac{2n\pi t}{T} - b_n \sin\frac{2n\pi t}{T}\right]. \tag{26}$$

But in section X, we found that the analysis of the velocity-diagram consisting of parallel straight lines gives

$$\frac{dy}{dt} = \sum A_n \sin\frac{n\pi x}{l}$$

where

$$A_n = -\frac{2}{n\pi}\left[d_1 \cos\frac{n\pi c_1}{l} + d_2 \cos\frac{n\pi c_2}{l} + \text{etc.}\right],$$

$d_1, d_2, d_3$, etc. being the magnitudes of the discontinuities in the velocity-diagram situated at points whose abscissae are $c_1, c_2, c_3$, etc. respectively. Since some of the discontinuities move towards the origin and others away from it, it is convenient to use instead of $c_1, c_2, c_3$, etc. the quantities $C_1, C_2, C_3$, etc. where $c_r = C_r + at$ if the discontinuity $d_r$ belongs to the positive wave, and $c_r = 2l - (C_r + at)$ if it belongs to the negative. We then have, as a relation true for the whole period of vibration,

$$A_n = -\frac{2}{n\pi}\left[d_1 \cos\frac{n\pi(C_1 + at)}{l} + d_2 \cos\frac{n\pi(C_2 + at)}{l} + \text{etc.}\right]. \tag{27}$$

Comparing (26) and (27), we may write

$$a_n = -\frac{1}{n^2\pi^2 T}\left[d_1 \cos\frac{n\pi C_1}{l} + d_2 \cos\frac{n\pi C_2}{l} + \text{etc.}\right]$$

$$b_n = -\frac{1}{n^2\pi^2 T}\left[d_1 \sin\frac{n\pi C_1}{l} + d_2 \sin\frac{n\pi C_2}{l} + \text{etc.}\right]$$

$$B_n = (a_n^2 + b_n^2)^{1/2}. \tag{28}$$

This gives the amplitudes of the series of harmonics in terms of the discontinuities and their initial positions. In the case of the bowed string we have

$$d_1 = d_2 = d_3 = \ldots = (v_A - v_B). \tag{29}$$

From (28) and (29), it is easy to verify the remark already made that when the number of discontinuities is large, the higher harmonics gain considerably in their relative importance.

Substituting the values of the amplitudes $B_n$ of the harmonics found from the preceding analysis in the expression for the maintaining force on the right-hand side of equation (22), the interesting question arises whether this expression for the maintaining force is really a convergent Fourier series. If $k_n$ varies as $n^z$ where $z > 1$, the expression for the force on the right-hand side of (22) is a nonconvergent series and it follows that under such circumstances, the bow would be even theoretically incapable of eliciting an infinite series of harmonics. Even when $k_n$ varies as $n^z$ where $z$ is not greater than unity, the existence of the factor $\sin n\pi x_0/l$ in the *denominator* of each term in the series brings in a very interesting difficulty as to its convergency. If we assume $x_0/l$ to be an irrational fraction, it is obviously possible by assuming a suitable and sufficiently large value of $n$ to make the quantity $\sin n\pi x_0/l$ smaller than any specified fractional number, and, further, it is evidently possible to find an infinite number of such values of $n$ which would reduce $\sin n\pi x_0/l$ to a value below the specified limit. Under the circumstances it becomes rather a delicate mathematical question whether the expression for the force given in (22) would then be a convergent series, and, if so, for what values of $z$. This difficulty does not arise if $x_0/l$ be a rational fraction, for the values of $\sin n\pi x_0/l$ then form a recurring series and the expression for the force is obviously convergent for values of $z$ not greater than unity. It is of great interest to determine and represent the form of the expression for the force at the bowed point graphically and compare it with the form of the vibration-curve at the bowed point; if the values of $k_n$ be arbitrarily assumed and the amplitudes $B_n$ are determined from the formulae given in (28) and (29), it is to be expected that, in general, the equation for the maintaining forces given in (22) is not rigorously satisfied even if the value of the constant $P_0$ and the initial positions $C_1$, $C_2$, $C_3$, etc. of the discontinuities be suitably assumed. In other words, the motion at the bowed point cannot in general be rigorously of the type on which our kinematical discussion was based, that is, having forward motions which involve no slipping and backward motions with uniform velocity of slipping. But in a special class of cases, as we shall presently see, the motion at the bowed point may be exactly of such type consistently with the dynamical conditions expressed in equation (22).

## Frictional-force curves and motion at the bowed point

As indicated above, it is a most important matter to determine and represent graphically the nature of the variation of the frictional force at the bowed point. If the maintained motion, the position of the bowed point and the magnitudes of the series of constants $k_n$ and $e_n$ are all known, the harmonic components in the frictional force are all determinate, except those, the frequencies of which are the same as that of the missing harmonics having a node at the bowed point. If such components be assumed to be zero, or at any rate be assigned specific magnitudes, the frictional-force curve at the bowed point may be drawn by the aid of a machine for harmonic synthesis. Such curves would be very useful for comparison with the form of the vibration-curve at the bowed point, specially during these epochs at which the bowed point slips past the hairs of the bow; they could also be used for determining the critical value of the pressure of the bow for the possibility of the type of vibration considered.

The form of the frictional-force curves may be found analytically and drawn without the aid of a harmonic machine in an important class of cases which may be regarded as representative of the phenomena actually met with in experiment. We shall assume that the values of $e_n'$ can all be put equal to $\pi/2$. This would be a very close approximation to the truth, as may be seen from the formulae given in section I, if the natural frequencies of vibration of the string form a strictly harmonic series and the yielding at its ends is negligible. Further, we shall assume that the values of $k_n$ are proportional to $n$. This assumption cannot be very wide of the mark, for, if $k_n$ were proportional to a power of $n$ higher than unity, the expression for the frictional force would not be a convergent Fourier series and it would no longer be possible to consider the motion of the string as comprising an infinite series of harmonics. In practice, moreover, it seems very unlikely that the damping coefficients of the higher harmonics increase at a much lower rate than that given by the formula $k_n \propto n$, as the higher harmonics would otherwise be more conspicuous in the maintained vibration than they usually are.[15]

On these assumptions, the expression for the frictional force assumes the form

$$\sum_{n=1}^{n=\infty} \frac{nkB_n \cos\left(\dfrac{2n\pi t}{T} + e_n\right)}{\sin\dfrac{n\pi x_0}{l}}.$$

To effect the summation of this series, we have to assume that $x_0/l$ is a rational fraction and the values of the denominators therefore form a recurring series. For those values of $n$ for which the denominator is zero, $B_n$ is also equal to zero, and the series therefore contains a series of indeterminate terms. In the most general case in which the vibration-curve at the bowed point can be represented by a number of straight lines of which alternate ones are not necessarily parallel to one another, we may write the expression for the force in the form

$$\sum_{n=1}^{n=\infty} \frac{nk\left(a_n \cos\dfrac{2n\pi t}{T} - b_n \sin\dfrac{2n\pi t}{T}\right)}{\sin\dfrac{n\pi x_0}{l}},$$

the values of $a_n$ and $b_n$ being given by the formulae

$$a_n = -\frac{1}{n^2\pi^2 T}\left[\mathrm{d}_1 \cos\frac{n\pi C_1}{l} + \mathrm{d}_2 \cos\frac{n\pi C_2}{l} + \text{etc.}\right],$$

$$b_n = -\frac{1}{n^2\pi^2 T}\left[\mathrm{d}_1 \sin\frac{n\pi C_1}{l} + \mathrm{d}_2 \sin\frac{n\pi C_2}{l} + \text{etc.}\right].$$

[15] It may be noted that the assumptions $e_n' = (\pi/2)$ and $k_n \propto n$ are quite correct when the dissipation of the energy of the string is solely due to a frictional force proportional to the velocity resisting each element of the string. Cf. Andrew Stephenson, January 1911, *Philos. Mag.*

The summation of the expression for the force is thus seen to depend upon the summation of a number of separate sine and cosine series of the form

$$\sum \frac{1}{n} \frac{\cos \frac{n\pi C}{l}}{\sin \frac{n\pi x_0}{l}} \cos \frac{2n\pi t}{T} \text{ and } \sum \frac{1}{n} \frac{\sin \frac{n\pi C}{l}}{\sin \frac{n\pi x_0}{l}} \sin \frac{2n\pi t}{T}.$$

Since $x_0/l$ is, by assumption, a rational fraction, we have to exclude the indeterminate terms in which the denominator $\sin n\pi x_0/l$ is zero, in order to effect the summation of these series.

The form of the expression suggests that the graph representing the sum of each of the series given above should consist of a number of straight lines parallel to the axis of time, separated by intervening discontinuities. The summation of the series then reduces itself to finding the positions and magnitudes of these discontinuities. Taking, for instance, any one of the cosine series, we may assume that its graph has a discontinuity $\delta_1$ at either of the points $t = \pm(\alpha_1 T/2\pi)$, a discontinuity $\delta_2$ at either of the points $t = \pm(\alpha_2 T/2\pi)$, and so on. This graph is representable by the expression

$$f_0 + \sum f_n \cos \frac{2n\pi t}{T},$$

where

$$f_n = \frac{2}{n\pi}[\delta_1 \sin n\alpha_1 + \delta_2 \sin n\alpha_2 + \text{etc.}]$$

To find the values of the quantities $\delta_1, \delta_2 \ldots, \alpha_1, \alpha_2 \ldots$ etc., we have the set of equations,

$$\delta_1 \sin \alpha_1 + \delta_2 \sin \alpha_2 + \ldots = \ldots \frac{\pi \cos \pi C/l}{2 \sin \pi x_0/l}$$

$$\delta_1 \sin 2\alpha_1 + \delta_2 \sin 2\alpha_2 + \ldots = \ldots \frac{\pi \cos 2\pi C/l}{2 \sin 2\pi x_0/l}$$

............................................

$$\delta_1 \sin n\alpha_1 + \delta_2 \sin n\alpha_2 + \ldots = \ldots \frac{\pi \cos n\pi C/l}{2 \sin n\pi x_0/l}$$

............................................

............................................

It is obvious that if $C/l$ is an irrational fraction, the quantity $(\pi/2)[(\cos n\pi C/l)/(\sin n\pi x_0/l)]$ will never recur in value, however much $n$ be increased. The number of independent equations is then infinite and the method of evaluation proposed appears to fail altogether. If however $C/l$ be a rational fraction, and the two fractions $C/l$ and $x_0/l$ be reduced to their lowest common denominator, the expression $(\pi/2)[(\cos n\pi C/l)/(2\sin n\pi x_0/l)]$ will recur when $n$ is increased by any multiple of twice this common denominator. To enable the equations to be satisfied, the quantities $\sin n\alpha_1$, $\sin n\alpha_2$, etc. should similarly recur, and the angles $\alpha_1$, $\alpha_2$, etc. must therefore all be multiples of $\pi$ divided by the lowest common denominator of the two fractions $C/l$ and $x_0/l$. The number of unknown quantities ($\delta_1$, $\delta_2$, etc.) to be evaluated is the same as the number of independent equations available, and it is thus possible to determine $\delta_1$, $\delta_2$, etc. completely.

The several series contained in the expression for the frictional force may thus be added up and their sum represented graphically. It is obvious that the frictional-force curve assumes the least complicated form, that is, has fewest discontinuities when the initial positions $C_1$, $C_2$, $C_3$, etc. of the discontinuities in the velocity-diagram of the string coincide with the nodes of the principal member of the missing series of harmonics. For, the fractions $x_0/l$ and $C_1/l$, $C_2/l$, $C_3/l$, etc. would then have the smallest possible common denominator. It may be noted that the method given above for drawing the frictional-force curve is applicable in the general case when the vibration-curve at the bowed point consists of any number of straight lines forming a continuous "curve," and is thus not restricted to the cases in which the velocities in the forward and backward movements are both constant and uniform. If these velocities are constant and uniform, the discontinuities $d_1$, $d_2$, etc. in the velocity-diagram of the string are all equal to one another (see section IV).

When the vibration-curve at some one point on the string (not necessarily at the bowed point) is a simple two-step zig-zag, the calculation of the form of the frictional-force curve becomes particularly simple. When the motion at the bowed point is itself of this type, the expression reduces to the form

$$\frac{K(v_A - v_B)}{\pi^2 T}\sum\frac{(-1)^n \sin n\pi\omega}{n\sin^2\dfrac{n\pi x_0}{l}}\cos\frac{2n\pi t}{T}.$$

When the motion at a point $x_1$ which is not the bowed point is of the two-step zig-zag form, the expression for the frictional force may be written as

$$\frac{K(v'_A - v'_B)}{\pi^2 T}\sum\frac{(-1)^n \sin n\pi\omega'}{n\sin\dfrac{\pi n x_0}{l}\sin\dfrac{n\pi x_1}{l}}\cos\frac{2n\pi t}{T},$$

the quantities $v'_A$, $v'_B$ and $\omega'$ having reference to the motion at the point $x_1$. The two formulae given above may be readily verified from (22) by analysing the motion postulated at the bowed point and substituting the values thus obtained for the quantities $B_n$. Figures 16, 17, 18, 19, 20 and 21 in the text represent no fewer than 45 curves for the frictional force calculated for various cases from the expressions given above, the vibration-curve at the bowed point which forms the basis of the calculation being shown alongside for comparison. Figures 16 to 19 represent

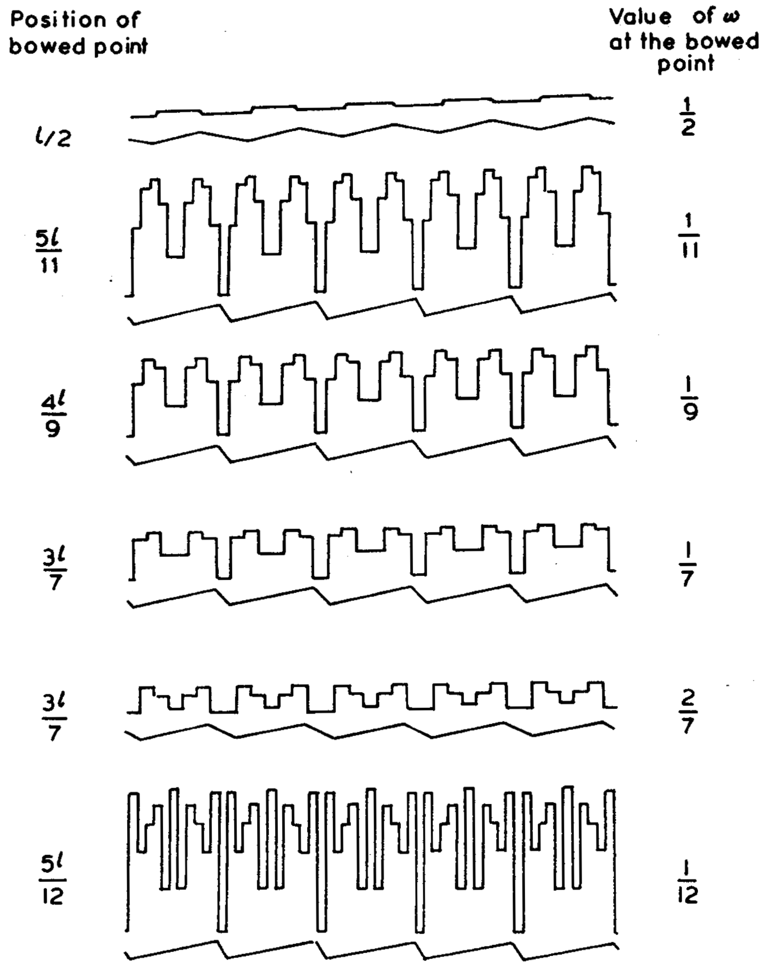



**Figure 16.** Frictional-force curves and motion at the bowed point.

cases in which the vibration-curve at the bowed point is assumed to be of the simple two-step zig-zag type, and figure 20 represents cases in which it is taken to be a four-step or a six-step zig-zag, though at some other point on the string the vibration-curve is a two-step zig-zag.

As examples of the method of calculation, we may consider a few of the cases illustrated in the figures. Let the string bowed at a distance of one-seventh of its length from one end. From figure 13 it is seen that in this case $\omega = \frac{1}{7}$. The expression for the force then reduces to

$$\frac{7kv_B}{\pi^2 T}\sum\frac{(-1)^{n-1}\cos\dfrac{2n\pi t}{T}}{n\sin\dfrac{n\pi}{7}}.$$

The positions of the discontinuities in the frictional-force curve must therefore be

$$\pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}, \pm\frac{4\pi}{7}, \pm\frac{5\pi}{7} \text{ and } \pm\frac{6\pi}{7}.$$

We then get the set of equations

$$\delta_1\sin\frac{n\pi}{7}+\delta_2\sin\frac{2n\pi}{7}+\delta_3\sin\frac{3n\pi}{7}+\delta_4\sin\frac{4n\pi}{7}+\delta_5\sin\frac{5n\pi}{7}$$
$$+\delta_6\sin\frac{6n\pi}{7}=(-1)^{n-1}\frac{7kv_B}{2\pi T\sin\dfrac{n\pi}{7}},$$

of which only the first six are independent (i.e. $n = 1, 2, 3, 4, 5$ or $6$). On writing down the equations, it is found at once that $\delta_1 = \delta_3 = \delta_5 = 0$. Multiplying both sides of the equation by $\sin n\pi/7$, simplifying and utilizing the relation

$$\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$$

we find that

$$\delta_2:\delta_4:\delta_6 \text{ as } 1:2:3, \text{ and } \delta_2=\frac{2kv_B}{\pi T}.$$

The frictional-force curve for this case is shown in figure 19. As another example, we may take the position of the bowed point to be $2l/11$ and that $\omega = 1/11$. (Referring to figure 13, it will be seen that this gives a combination of the fifth and sixth types of vibration.) On writing down the equations as before, it is found that $\delta_1 = \delta_3 = \delta_5 = \delta_7 = \delta_9 = 0$ and that the equations reduce to the form

$$\delta_2\sin\frac{2n\pi}{11}+\delta_4\sin\frac{4n\pi}{11}+\delta_6\sin\frac{6n\pi}{11}+\delta_8\sin\frac{8n\pi}{11}+\delta_{10}\sin\frac{10n\pi}{11}$$
$$=(-1)^{n-1}\frac{11kv_B\cdot\sin\dfrac{n\pi}{11}}{2\pi T\cdot\sin^2\dfrac{2n\pi}{11}}.$$

Multiplying both sides by $2\sin^2(2n\pi/11)$, i.e. by $(1-\cos(4n\pi/11))$ and simplifying, the equations may be easily solved by grouping together all the terms which have a common coefficient $\sin \pi/11$, all those which have the coefficient $\sin 2\pi/11$ and so on. The solution is found to be $\delta_2:\delta_4:\delta_6:\delta_8:\delta_{10}$ as $3:-5:9:-10:15$ and $\delta_2 = 6kv_B/\pi T$. On plotting, these values, a very interesting figure is obtained in which the fifth and sixth harmonics are very prominent and by their superposition, give an appearance similar to that of "beats." (see figure 19). All the curves shown in figures 16 to 21 were calculated by methods closely analogous to those employed in these two examples. They were all drawn to exactly the same scale, the velocity of the bow being taken to have a fixed value, and the curves are thus all strictly comparable.

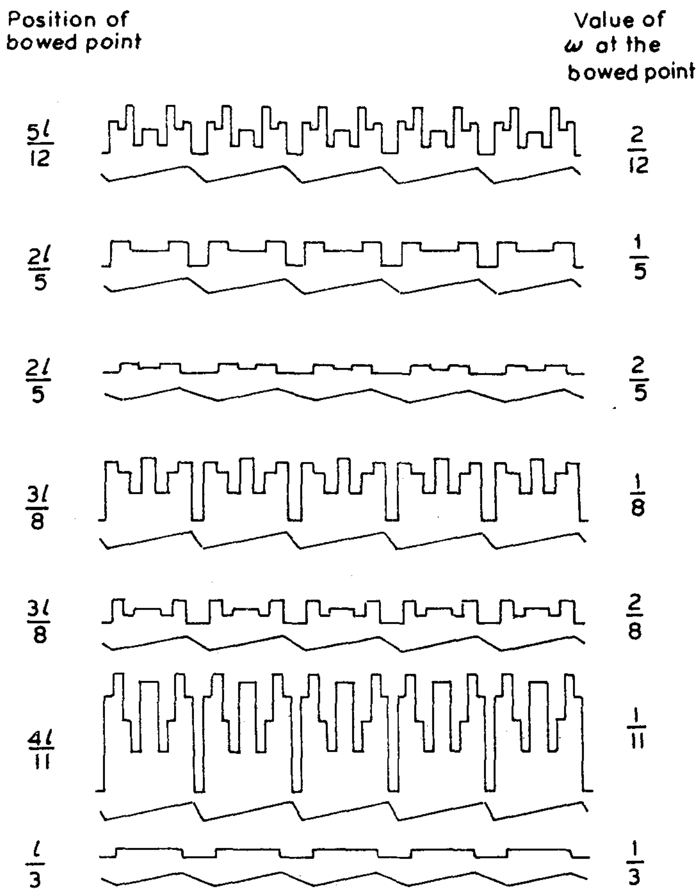
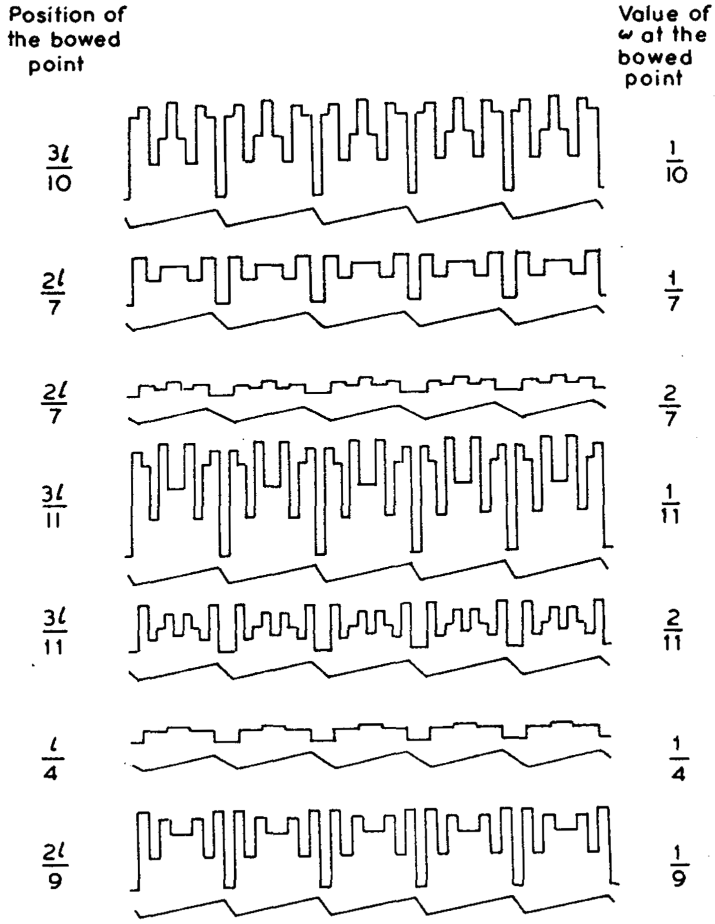



**Figure 17.** Frictional-force curves and motion at the bowed point.

From a scrutiny of the 45 frictional-force curves shown in figures 16 to 21, a number of important generalisations can be arrived at, which we now proceed to consider.

The curves shown in figures 16 to 21 are of special interest in many ways. The 29 curves shown in figures 16 to 19 all deal with cases in which the bow is applied at a point of rational division and the motion at this point is a two-step zig-zag, the value of $\omega$ for which has either 1 or 2 as its numerator. It will be seen that, in all these cases, the frictional force has a minimum constant value during the epochs at which the bowed point slips past the hairs of the bow. It is also seen that in some of these cases the frictional force has *the* maximum or *a* maximum value just before the slipping begins and just after it is over. But this is by no means a general rule, for in 12 out of the 29 cases, the frictional force before and after this epoch is not a maximum at all. Further, in all the cases in which the bow is applied at a point of aliquot division, e.g. $l/2$, $l/3$, $l/4$, $l/5$ or $l/6$, etc. the friction is a maximum at the middle of the stage during which the bowed point is carried forward by the bow. The nine curves of this kind shown in the series all bear a strong family resemblance to one another. Similarly the frictional-force curves for the bowed points $2l/5$, $2l/7$, $2l/9$, $2l/11$, etc. are worthy of careful study and inter-comparison. When the value of $\omega$ for these cases is 1/5, 1/7, 1/9, 1/11, etc. respectively, the curves have a distinctive form in which the two chief harmonics having a node on either side of the bowed point are specially prominent. But when $\omega = 2/5, 2/7, 2/9, 2/11$, etc. respectively, the form of the curve is more closely analogous to those for aliquot points of division referred to above. This is evidently because the fundamental is more prominent when $\omega$ has the larger value in such cases.

Krigar–Menzel and Raps remark in their paper as follows: "It is clear from all that has been said that the idea of the mechanical action of the bow which we form is in all the cases the same as that which has already been described by Helmholtz. The bowed point adheres to the rosined hairs of the bow and is carried forward with a constant velocity equal to that of the bow. This situation accounts for the ascending line of moderate slope in the vibration-curve at the bowed point. Finally, through the increasing tension of the string, the adhering point breaks loose and glides downwards against the bow under strong friction with a constant maximum velocity till the cycle commences to repeat itself anew." If Krigar–Menzel and Raps were quite correct in their explanation of the "breaking loose of the bowed point from the hairs of the bow" as due to the increasing tension of the string, we should have expected to find that the friction of the bow is a maximum immediately before the release takes place. As we have just seen, this is by no means generally the case, and we are therefore forced to conclude that the pure kinematics of the motion is also a factor in determining the release of the string by the bow. As we shall see presently, Krigar–Menzel and Raps were also wrong in taking the velocity of slipping against the hairs of the bow to be uniform. It is in general only approximately uniform or even largely non-uniform, and such




**Figures 18.** Frictional-force curves and motion at the bowed point.

variations of the velocity have an important significance and result.

Referring again to the curves shown in figures 16 to 19, it will be seen that in all cases, the harmonics which have a node near the bowed point are prominent in the frictional-force curve. This is what might be naturally expected from the form of the expression for the force.

Further, in all cases when $\omega$ is a small fraction, the frictional-force curves become very steep, that is, the friction becomes greater. If this friction exceeds the maximum statical value, the motion ceases to be possible. We thus see that if the value of $\omega$ is to be small, the pressure of the bow must be considerably increased. For instance, from the last five cases shown in figure 19 it is clear that the

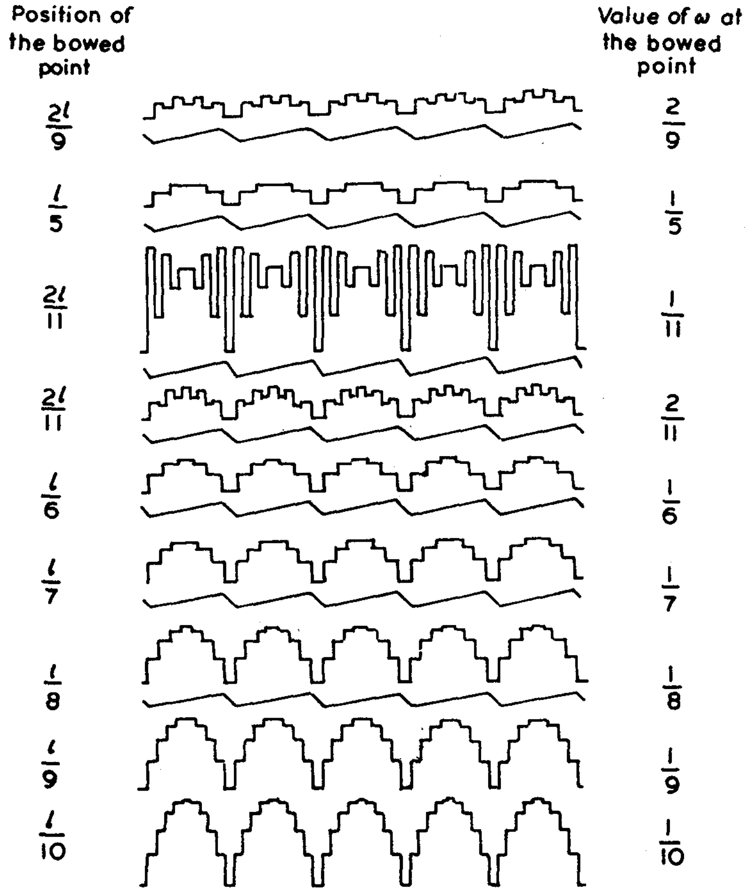



**Figure 19.** Frictional-force curves and motion at the bowed point.

minimum pressure of the bow must be increased as the bowed point approaches one end of the string. Similar effects are also noticeable in the cases in which the bow closely approaches important nodes such as $l/2$, $2l/5$, $l/3$ or $l/4$.

For any given position of the bowed point, the larger value of $\omega$ means smaller friction and therefore a reduced minimum pressure. When the bow coincides exactly with an important node such as $l/2$ or $l/3$, the friction becomes extremely small and the pressure of bowing necessary is therefore very low indeed.

Passing on to the cases in which the motion at the bowed point is still assumed to be a two-step zig-zag, but the value of $\omega$ has 3 or some larger integer as its numerator, we see at once a remarkable difference (figure 20). *The frictional force has no longer a constant uniform value while slipping takes place. This is inconsistent*

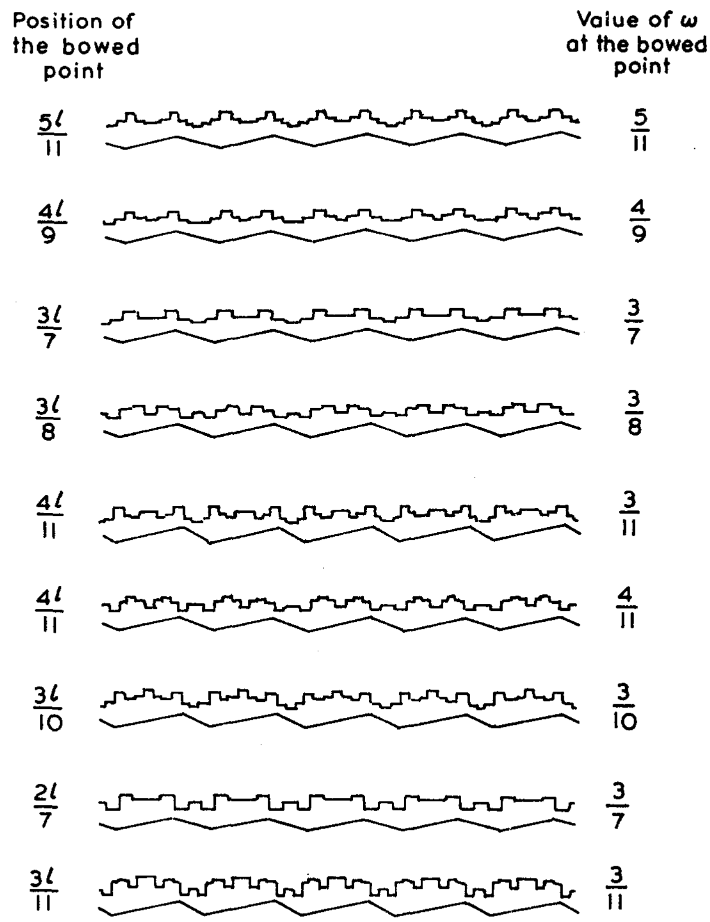



**Figure 20.** Frictional-force curves and motion at the bowed point. [Incompatible cases.]

*with the assumption that the velocity of slipping is constant, and we are thus forced to conclude that the velocity of the bowed point during the slipping stage is necessarily non-uniform in a greater or less degree, when the pressure of bowing is such that the value of $\omega$ is greater than $2/r$,* the $r^{th}$ harmonic being the principal member of the missing series.

We see, therefore, that for the particular set of damping coefficients assumed, a strictly uniform velocity of slipping is only possible when the bow is applied with such pressure that the numerator of $\omega$ is either 1 or 2, and this may not be possible at all if the denominator $r$ is too large. For other laws of damping, we would get an even more unfavourable result, that is, strict uniformity is never possible, though practical uniformity may be attained in a number of cases.

The foregoing results may, with advantage, be discussed a little more in detail. Take the case in which the bow is applied exactly at the point $l/3$. The value of $\omega$ is $\frac{1}{3}$ and the motion at the bowed point is practically a perfect two-step zig-zag. If now the bow be removed to a point a little on one side of the node, say to the point $16l/49$, we find on a reference to figure 14 that if we exclude the cases in which the motion at the bowed point is a four-step or six-step zig-zag, etc. the only possible values of $\omega$ are $\frac{1}{49}$, $\frac{16}{49}$ and $\frac{17}{49}$. The value $\frac{1}{49}$ is evidently much too small to be easily elicited. The value $\frac{16}{49}$ is evidently of importance as it corresponds to the first principal mode of vibration of a bowed string. A very small pressure of bowing is obviously sufficient to elicit this type. But from what has already been said, it is evident that for this value of $\omega$ we cannot have strictly uniform velocity of slipping of the bowed point in the backward motion, though in its forward motion, the velocity may be exactly equal to that of the bow. If the third, sixth and other harmonics of the missing series are actually restored in the motion in nearly their proper amplitudes (neither more nor less), then the velocity in the backward motion may be practically uniform, otherwise not. *Thus we see that this non-uniformity of slipping and the restoration of the missing harmonics are closely connected with one another.* If the bow is removed further still from the point of trisection and is applied at, say $5l/16$, the possible values of $\omega$ for a two-step zig-

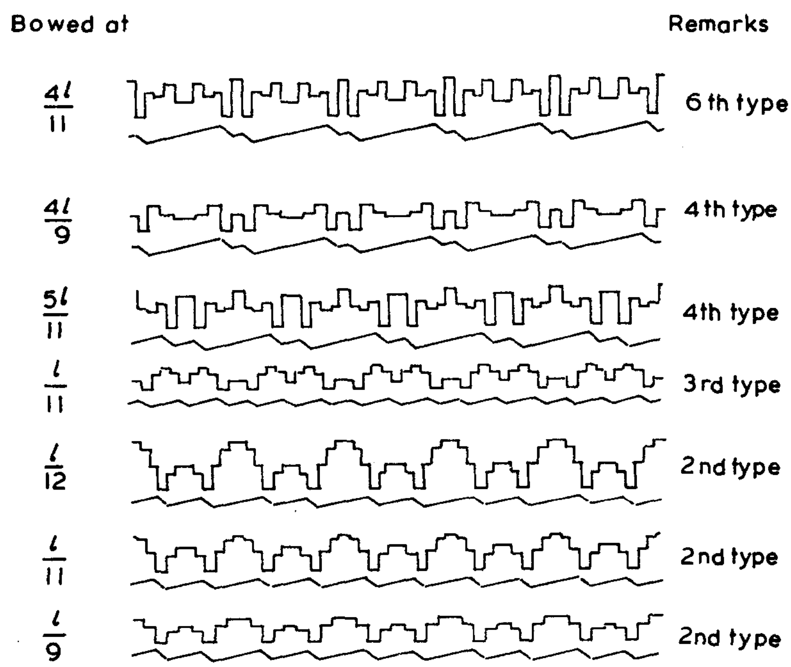



**Figure 21.** Frictional-force curves and motion at the bowed point. (Four and six-step zig-zags).

zag motion at the bowed point would be $\frac{1}{16}$, $\frac{5}{16}$ and $\frac{6}{16}$. The value $\frac{1}{16}$ may be elicited if the pressure is sufficient. If the pressure is sufficiently reduced we may get the first principal type of vibration for which $\omega = \frac{5}{16}$, but this would be modified by the non-uniform slipping which inevitably occurs in this case.

Figure 21 shows the frictional-force curves and the motion at the bowed point for six specially selected cases in which the latter is a four-step zig-zag, and one case in which it is a six-step zig-zag. In the four-step zig-zags, the velocity is assumed to be constant and the same in both stages during which slipping occurs, and the frictional-force curves are seen to be entirely compatible with this, as the friction is constant and has the same value during both the stages. It must be understood that this result holds good only in the special cases considered and in some others, but not generally. In the case of the six-step zig-zag shown, the frictional-force curve is evidently incompatible with it, as the friction in one stage is less than in the two others.

The form of the frictional-force curves for other cases in which the motion at the bowed point is representable by a six-step or an eight-step zig-zag, etc. may be investigated by the method described above. In the great majority, if not all, of such cases, it would no doubt be found that the form of the frictional force is incompatible with the perfect parallelism and straightness of the descending lines of the vibration-curve at the bowed point. In other words, the velocity of the bowed point would not be quite constant and uniform in all the stages in which it slips past the bow.

## Transition from the rational to the irrational modes of vibration and vice versa

The preceding investigation of the form of the frictional-force curves has already enabled us to form a general idea of the conditions which under the kinematical theory outlined in sections V to XII breaks down to an appreciable extent. We have seen that this departure from the comparatively simple vibration-forms so far investigated is due in the first instance to the velocity of slipping at the bowed point becoming non-uniform, and is closely connected with the progressive (as distinguished from the discontinuous) restoration of the missing harmonics which occurs when the bow is gradually moved away from some important nodal point. We are thus led to investigate the *transitional forms* of vibration, as they may be called, which are intermediate between the irrational types discussed in sections V to IX and their rational modifications worked out in section X. These transitional forms may be found, *a priori*, by a general method which will be best understood by considering some specific cases.

We may confine attention at first to the cases in which the motion at the bowed point is a two-step zig-zag or a close approximation thereto. Let the bow be applied at some point intermediate between, say, $l/5$ and $l/6$, with pressure and velocity approximately those necessary to elicit the first type of vibration (see figure 13). When the bow is applied exactly at $l/5$, the vibrational form of the string is defined by one large positive discontinuity in the velocity-diagram and four small negative discontinuities these are so situated that the motion at the bowed point is a perfect two-step zig-zag and in the resulting motion, the 5th, 10th harmonics etc. are absent. Similarly when the bow is applied at $l/6$, we have one large discontinuous change of velocity and *five* small ones: the 6th, 12th harmonics etc. are absent. When the bow is gradually moved from $l/5$ to $l/6$, the 5th, 10th harmonics etc. gradually reappear, and the 6th, 12th harmonics etc. become feebler and disappear. In these transitional stages, the forward motion of the bowed point must still take place with uniform velocity equal to that of the bow, but the velocity in the backward motion need not be strictly constant. We are thus naturally led to assume that the transitional modes should be capable of being deduced from a velocity-diagram with one large discontinuity and *five* small ones. On proceeding to find the relation that the positions and magnitudes of these discontinuities must satisfy in order that the bowed point may have a uniform velocity in the forward motion, it is found by trial that the velocity-diagram must be similar in certain respects to that for the case of bowing at $l/6$. The five small discontinuities should all be equal to one another, and their positions on the string with respect to the bowed point are perfectly determinate. If the motion of the string is assumed to be of the perfectly symmetrical type, that is involving only sine components, the position of the sixth (large) discontinuity is also uniquely determined. If, further, the inclination of the lines in the velocity-diagram and the velocity of the bow are assigned specific values, the magnitude of the discontinuities is known, and the vibration-form is completely fixed. In general, however, an asymmetrical vibration-form is possible, and it is found that when the velocity of the bow is given, the form of vibration has a possible variation determined by (*a*) the position assigned to the large discontinuity which must lie within certain limits and (*b*) the inclination of the lines of the velocity-diagram to the $x$-axis.

Except when the slope of the lines of the velocity-diagram is such that the five small discontinuities all vanish (in which case the transitional form becomes identical with the first irrational type of vibration), all the transitional forms referred to in the preceding paragraph have the common feature that the vibration-curve at the bowed point instead of being a perfect two-step zig-zag, consists of a zig-zag in which the steep descending line is not perfectly straight but consists of three (or in the extreme cases two) straight lines, all of which however are much steeper than the ascending line which is perfectly straight. Ten cases of transitional forms thus worked out, *a priori*, for bowed points lying between $l/4$ and $l/6$, are shown in figure 22, in which the corresponding vibration-curves calculated for points near the end of the string are also shown.

It will be seen that in all the ten curves, the vibration-curve at the bowed point closely approaches the simple two-step zig-zag form and in some cases (e.g. the

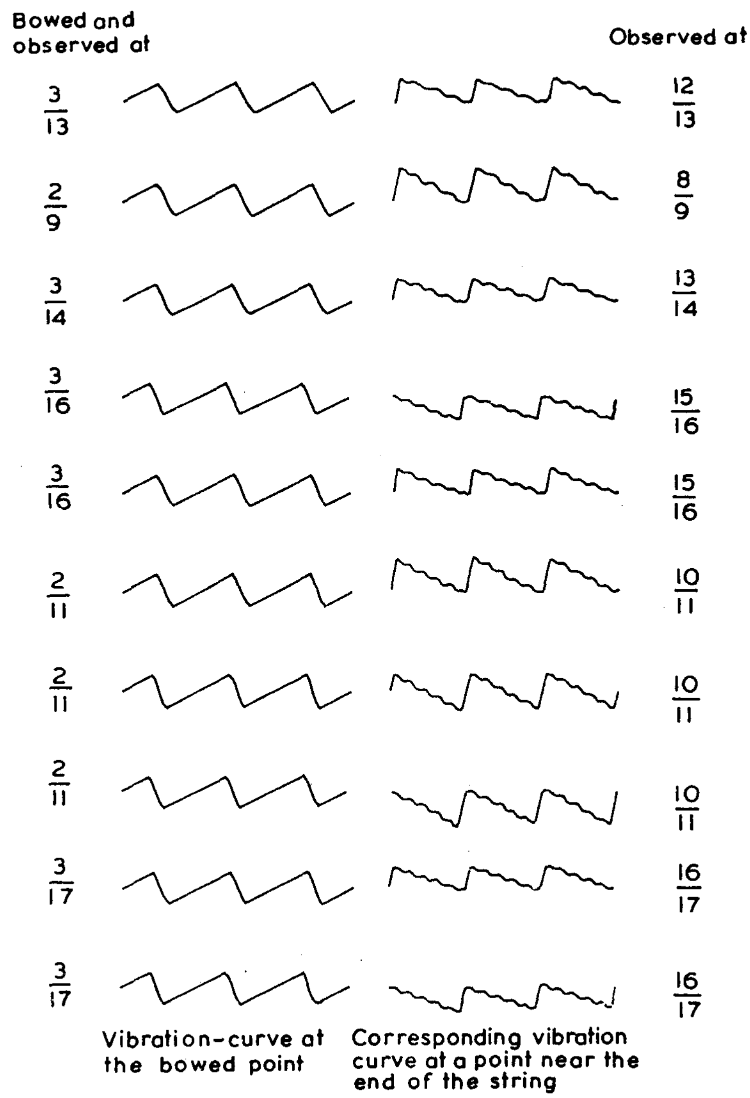



**Figure 22.** Transitional forms of vibration. (Modifications of the first irrational type).

curves for bowing at $3l/16$ and $3l/17$) is nearly indistinguishable from it. The vibration-curve for the point near the end of the string, however, shows the differences clearly enough.

Most of the curves on the right-hand side of figure 22 are seen to be asymmetrical; in other words, the appearance of the curves is not the same if the page were held upside down. The three curves for the case of bowing at $2l/11$ are particularly instructive. The first of these three curves (which is an extreme case) shows a sharp peak at the top and five horizontal steps, below. The second (which is the symmetrical case) shows six steps, of which the first and the last are shorter than the rest. The third curve (which is also an extreme case) shows five horizontal steps above, and one sharp point below.

As an example of the method of drawing these curves, we may take the case of bowing at $3l/16$. The velocity-diagram for the case is drawn in a manner similar to that for bowing at $l/6$, the positions of the discontinuities being as follows: The first small discontinuity is initially at the end of the string ($x = 0$). The point of bowing is at $3l/16$, and two other small discontinuities are therefore situate at $6l/16$. Another pair must be at $12l/16$. For, $6l/16 + 3l/16 = 12l/16 - 3l/16$. The inclination of the lines of the velocity-diagram and the position of the large discontinuity are arbitrary, though the latter must initially lie between the limits $14l/16$ and $l$. If the large discontinuity is initially at $14l/16$ and belongs to the positive wave, we have one extreme asymmetrical case in which the descending part of the vibration-curve at the bowed point consists of two straight lines only. [For, $(l - 14l/16) + (l - 3l/16) = (12l/16 + 3l/16)$]. If the large discontinuity is initially at the end $l$ of the string, we have the symmetrical case in which the descending part of the vibration-curve at the bowed point consists of three straight lines. When the large discontinuity is initially at $14l/16$ and belongs to the negative wave, we have the other extreme asymmetrical case: [$(14l/16 + 3l/16) = (l - 12l/16) + (l - 3l/16)$]. The slope of the lines of the velocity-diagram and the velocity of the bow determine the magnitudes of the discontinuities. If the lines were horizontal, the magnitudes of the discontinuities would be the same as if the string were bowed at $l/6$. Whatever the slope might be, and wherever the large discontinuity may lie initially within the limits referred to, the descending motion at the bowed point occupies exactly 3/16 of the whole period of vibration. In other words, *the transitional form also obeys the kinematical law given in equation* (17), i.e. $\omega = (nx_n/l)$, $n$ being put equal to 1, as this is a modification of the first irrational type. The same kinematical law is satisfied for all the ten cases shown in figure 22, and is, in fact, true for all the transitional modifications of the first irrational type, ($n = 1$).

In the preceding treatment, we have obviously neglected to take into account the existence of certain nodal points of minor importance lying between $l/5$ and $l/6$ or between $l/4$ and $l/5$. For instance, between $l/5$ and $l/6$, we have the nodal points $3l/16$, $2l/11$ and $3l/17$. For a more complete theory, we have also to take into account the gradual dropping out and reappearance of the 16th, the 11th and the 17th harmonics and their trains of harmonics of high order, as the bow is gradually moved across from $l/5$ to $l/6$. As however these harmonics are of very small amplitudes, the correction may be neglected altogether if the bow be not applied exactly at any one of these nodal points. If it is applied exactly at such nodal point, the correction may be effected by the method described in section X for rational points of bowing, and in any case, the correction so made would have no effect if some other node of the missing harmonic is chosen as the point of observation. When, however, the intervening nodal point is of importance, a more accurate method of correction is required. For example, when the bow is moved step by step from $l/2$ to $2l/5$, the gradual reappearance of the second harmonic and the gradual dropping out of the fifth harmonic have both to be taken into account. This may be done by drawing a velocity-diagram similar to that for bowing at $l/5$, that is, with one large and four small discontinuities, and readjusting the positions of these discontinuities so as to give a uniform velocity in the forward motion at the bowed point. Figure 23 (first three graphs in the left-

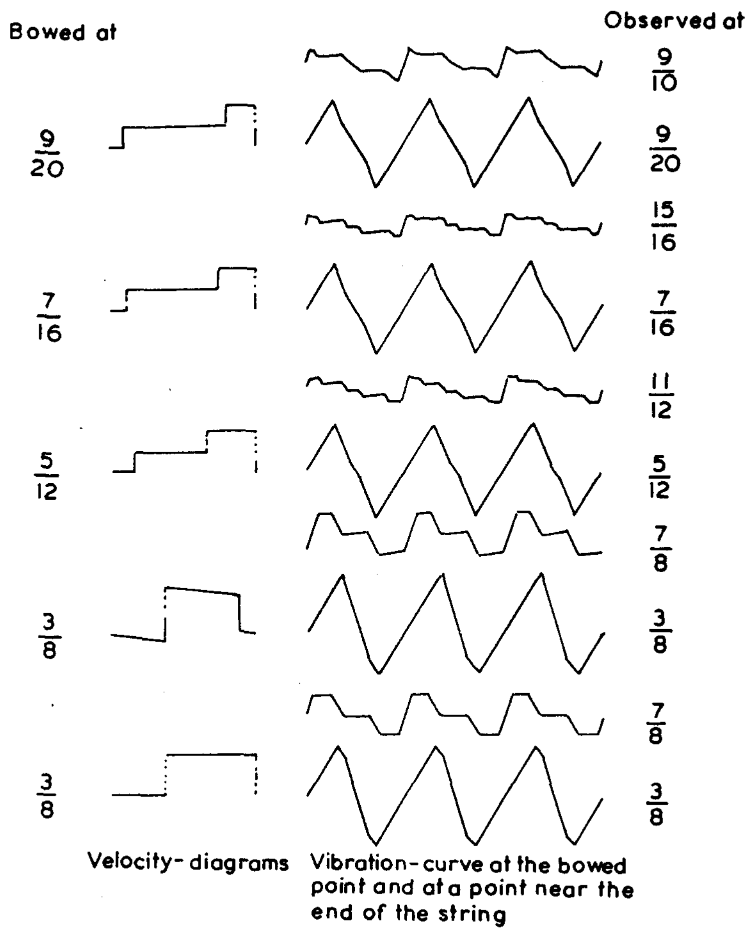



**Figure 23.** Transitional forms of vibration. (Modifications of the first irrational type).

hand column) shows three velocity-diagrams of this type, and against each of these diagrams is shown the corresponding motion at the bowed point and the vibration-curve at a point near the end of the string. The gradual transition from the type corresponding to a string bowed at $l/2$ to one bowed at $2l/5$ is clearly seen.

The graphs in figure 23 for the case of bowing at $3l/8$ show the dropping out of the 3rd harmonic. One of the cases is of the symmetrical type and the other of the asymmetrical type. In the five vibration-curves at the bowed point shown on the right-hand side of figure 23, it will be seen that the ascent with the bow is made with a uniform velocity, but that the velocity of descent is not constant.

Similarly when the bow is applied at some point intermediate between $l/3$ and $2l/7$, the velocity-diagram of the transitional mode may be drawn in a manner similar to that for bowing at $l/7$, that is, with one large and six small discontinuities, the positions of these being readjusted in the manner requisite to give a uniform velocity in the forward motion at the bowed point. These vibration-forms would show the gradual dropping out of the seventh harmonic as the bow approaches the point $2l/7$. There is however an alternative set of transition-forms in which the velocity-diagram is similar to that for bowing at $l/4$, i.e. has one large and three small discontinuities, and these forms secure predominance as the bow is removed farther and farther from $l/3$ towards $l/4$. Since both types of transition-forms give a forward motion with uniform velocity at the bowed point for identically the same fraction of the period, they may be superposed in any desired proportion so as to secure the specified velocity at this point. In this way, the various modifications of the first irrational type of vibration obtained by bowing at points between $l/3$ and $l/4$ may be accurately drawn. Results approximating to the truth may however be obtained without utilizing the principle of superposition of transitional forms here suggested, by merely choosing the form with seven discontinuities for bowed points lying between $l/3$ and say $3l/10$, and the form with four discontinuities for bowed points lying between $3l/10$ and $l/4$. The curves thus drawn are shown in figure 24.

Of the five cases shown in figure 24, the second, third and fifth are of the asymmetrical type and the others are symmetrical forms.

In all these transitional forms, the descending motion at the bowed point is not executed with a uniform velocity but consists of two or three stages in which the velocities are different. The relative duration of these stages and the velocity of the bowed point during these intervals are capable of adjustment within a considerable range, and any such alteration involves important changes in the amplitudes and, for the asymmetrical types, also in the phases of the harmonics which have a node near the bowed point. There is thus, *prima facie*, reason to consider that these transitional forms should be capable of adjusting themselves in such manner as to secure the maintenance of the motion, and also its stability, over a considerable range of bowing pressures and velocities. A further mode of adjustment is provided by the principle of superposition of transitional forms

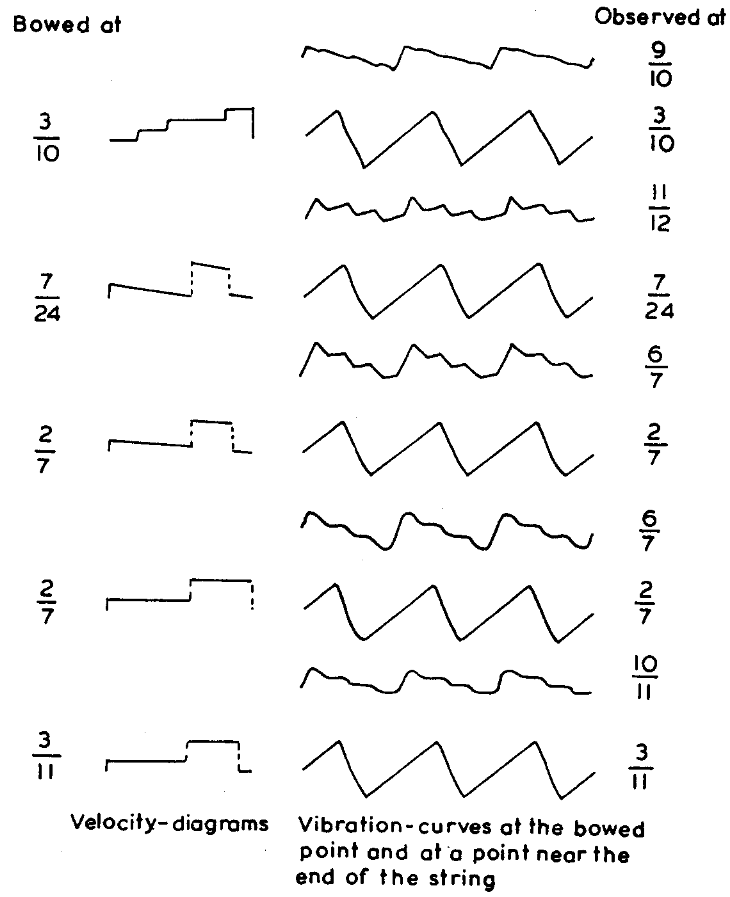



**Figure 24.** Transitional forms of vibration. (Modifications of the first irrational type).

referred to previously. The theory may be further elaborated by working out the form of the frictional-force curve in a representative set of transitional modes of vibration by the general method described in the preceding sub-section, and proving its compatibility with the motion at the bowed point corresponding to each mode. This detailed investigation must be reserved for part II of the monograph. For the present, however, it will be sufficient to point out some indications of theory. It is evident in the first place that any change either in the pressure or the velocity of bowing involves some change in the relative amplitudes of the harmonics in these transitional forms. For, the velocity of the bowed point being non-uniform when slipping past the hairs of the bow, equation (22) given on page 298 cannot, in general, continue to be satisfied at every instant by merely altering the non-periodic part $P_0$ of the frictional force when the pressure is changed, or by multiplying the periodic part on the right-hand side by a constant when the velocity of the bow is altered. Another result which is indicated by theory is that the position of the discontinuities in the velocity-diagram of the string would, in general, be compatible with the position of the discontinuities in the frictional-force curve, only if all of the former are, at some one epoch of the vibration, situated at points dividing the string into an integral number of equal parts, one of such points being that at which the bow itself is applied. This is evident from the method which was used for finding the form of the frictional-force curve.

The principle used for finding the form of the transitional modifications of the first irrational type may evidently be employed also for finding the transitional modifications of the second, third and higher irrational types. It is sufficient at first to confine attention to those cases in which the motion at the bowed point is approximately a two-step zig-zag, though the method may also be extended to cover even more complicated cases.

The first four curves in each column on the two sides of figure 25 show the vibration-curves at the bowed point and at a point near the end of the string respectively, for some transitional modifications of the second irrational type. In the first case the position of the bow is at the point $5l/13$ whose distance from the end of the string is less than $2l/5$. The curves may be drawn from a velocity-diagram with two large equal discontinuities and three smaller discontinuities also equal to one another; the magnitudes of the discontinuities being the same as for the case of bowing at $2l/5$, but their positions being different, in order that the point $5l/13$ might have a uniform velocity in the forward motion. It is seen that the curves are of the *asymmetrical* type, the two large discontinuities in the velocity-diagram not being coincident, when one of the three small discontinuities is at the end of the string and the other two are coincident. The next two pairs of curves in figure 25 show the transitional modifications, one symmetrical and the other asymmetrical, for the case of bowing at the point $5l/12$ which lies between $2l/5$ and $3l/7$. The velocity-diagram in these cases has two large and five small discontinuities. The bowed point for the fourth pair of curves in figure 25 is $7l/16$ which lies between $3l/7$ and $4l/9$. The velocity-diagram for this case has two large and seven small discontinuities and is of an asymmetrical type.

The transitional forms of the second irrational type for bowed points lying between $4l/9$ and $5l/11$ or between $5l/11$ and $6l/13$ and so on, may be similarly drawn. When the bowed point lies further away from the centre of the string than the node $5l/13$, in other words is in the neighbourhood of the node $3l/8$, the eighth harmonic and its train would partially drop out, and the velocity-diagram would have two large and six small discontinuities, and the form of the vibration-curves may be easily deduced therefrom. For example, if the bow be applied at the node $8l/21$, the two large discontinuities would be initially at the same point, and the small discontinuities would be initially situated in pairs at the points $2l/21$, $14l/21$ and $18l/21$. If the bow be applied between $3l/8$ and $l/3$, the second irrational type

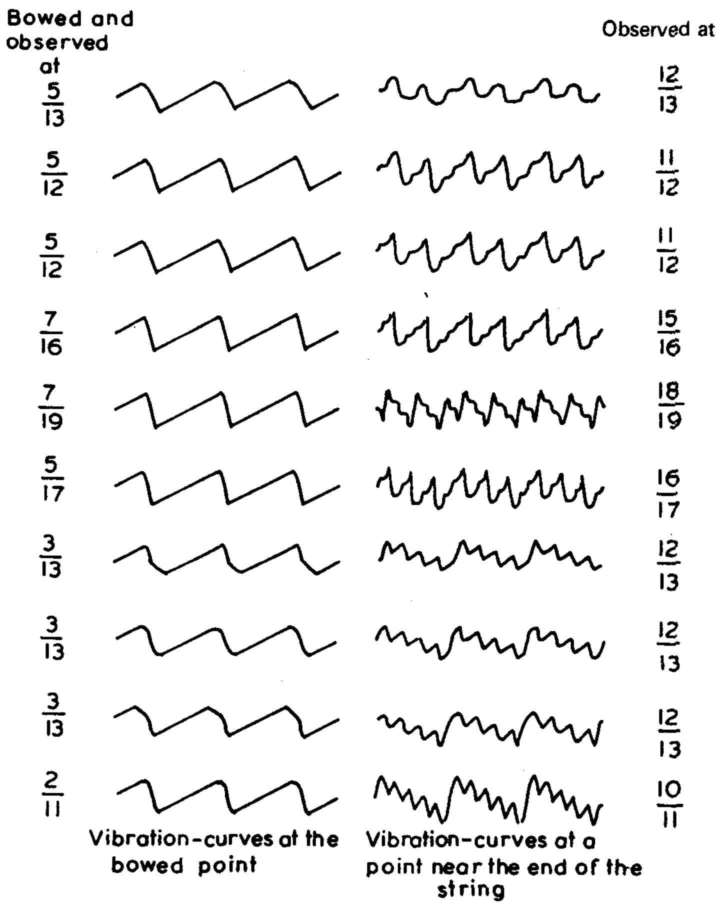



**Figure 25.** Transitional modifications of the second and third irrational types, and the transition from the first to the fourth and fifth types.

would be modified by the dropping out of the eighth, eleventh and third harmonics and their respective trains. In all these transitional modifications of the second irrational type, the motion at the bowed point is approximately but not exactly a two-step zig-zag, and the fractional part of the period of vibration taken up by the slipping motion is given rigorously by the usual kinematical law $\omega = (nx_n/l)$, $n$ being given the integral value 2.

The fifth and sixth pairs of curves in figure 25 represent transitional modifications of the third irrational type, the bowed point in the two cases being situated on opposite sides of the point of trisection of the string. The bowed point $7l/19$ is situated between the nodes $3l/8$ and $4l/11$, and the velocity-diagram for this type has therefore three large and eight small discontinuities. The bowed point $5l/17$ is situated between the nodes $2l/7$ and $3l/10$, and the velocity-diagram which has therefore three large and seven small discontinuities is of the asymmetrical type, as is shown by the vibration-curves deduced from it. Further examples of the modifications of the third, fourth and higher irrational types may be readily worked out by analogous methods. In all such cases, the fraction of the period of vibration during which slipping occurs at the bowed point is given by the kinematical law $\omega = (nx_n/l)$, $n$ being given the appropriate integral value 3, 4 or 5 etc.

## Transition from one irrational type to another

The theory given in the preceding sub-section may be extended so as to cover another important and interesting class of transitional vibration-forms, the nature of which will, as before, be best understood by considering a specific case. Let the bow be applied at some point intermediate between the nodes $l/4$ and $l/5$. We have already seen in the preceding sub-section that the transitional modifications of the first irrational type then obtained may be drawn from a velocity-diagram which has one large positive discontinuity and four small equal negative discontinuities. The magnitude of the large discontinuity bears to that of the small ones, a ratio depending on the inclination of the lines of the velocity-diagram to the $x$-axis. If this slope be at first zero and is gradually increased in the positive direction, a stage will be arrived at, when the four small discontinuities completely vanish, and the *mode of vibration becomes identical with the first irrational type*. On further increasing the slope of the lines, the four small discontinuities re-appear, this time with a positive value, and when the inclination is such that all the five discontinuities have the same magnitude, *the mode of vibration becomes identical with the fifth irrational type*. On the other hand, if the slope of the lines is negative and is numerically increased, the four small discontinuities become larger and larger, and the fifth becomes smaller and smaller, till at a particular inclination, it vanishes altogether. *The mode of vibration then becomes identical with the fourth irrational type*. These transitional changes are obviously of great interest and importance, and it is very instructive to trace the corresponding changes in the motion at the bowed point by the method described in the preceding sub-section. Till the final stage in which the type becomes identical with either the fourth or fifth irrational type, the fraction of the period of vibration during which there is slipping at the bowed point continues to be given by the kinematical law $\omega = nx_n/l$ in which $n$ is equal to unity. It then changes to the smaller value given by the same formula when $n$ is put equal to 4 or 5, as the case may be. It is to be noticed, however, that before this final stage is reached, the motion at the bowed point undergoes an important modification. For instance, in the transition from the first to the fourth irrational type, as four of the discontinuous changes of velocity increase in magnitude and the fifth decreases, slipping at first occurs at the bowed point only when its velocity is in a direction opposite to that of the bow; when the fifth discontinuity is so small that it is equal to or less than the velocity of the bow, this is no longer true, and slipping also occurs during a part of the forward motion, and this ceases only when the fifth discontinuity finally vanishes. During the latter part of the transitional stages, the fraction of the period occupied by the backward motion is given by the ratio $nx_n/l$, in which $n$ is put equal to 4, whereas the total fraction of the period during which slipping occurs is found by putting $n = 1$. Similarly in the transition from the first to the fifth irrational type, slipping is confined to the backward motion of the bowed point, only if the fifth discontinuity is greater than any one of the others plus the velocity of the bow. When it is equal to or less than this sum, slipping occurs also during part of the forward motion, and the fraction of the period occupied by the backward motion is found by putting $n = 5$ in the ratio $nx_n/l$, whereas the fraction during which slipping occurs is found by putting $n = 1$.

These transitional modes would be symmetrical or asymmetrical according as the position of the discontinuities in the velocity-diagram is symmetrical or otherwise. The three curves for the bowed point $3l/13$ and the observed point $12l/13$ shown in figure 25 represent the transition from the first to the fourth irrational type. Two out of the three pairs of curves are asymmetrical and the other is symmetrical. The last pair of curves in figure 25 shows the transition from the first to the fifth irrational type when the bow is applied between the nodes $l/5$ and $l/6$.

The theory indicated above may be extended to cover a very large number of cases in which there is a transition from one irrational type to another, the motion at the bowed point remaining approximately a two-step zig-zag. For example, as may be seen from figures 14 and 15, we may have transitions from the second irrational type to the fifth, seventh or ninth type, or transitions from the third type to the seventh or eighth type and so on.

## Transitional forms involving complicated motions at the bowed point

It has been shown in section IX that if the number of discontinuities in an irrational type is not a prime integer, and the bowed point lies anywhere within certain ranges, its vibration-curve is necessarily of a more complicated form than a two-step zig-zag. We shall now consider some transitional forms in which the slipping velocity of the bowed point is not the same in all the stages of descent in such cases. For example, if the bow is applied near the centre of the string and the fourth type is elicited, the vibration-curve at the bowed point is a four-step zigzag; if the sixth type is elicited, it is a six-step zig-zag, and so on. [See Figures 5(*b*), (*c*), (*d*) and (*e*), page 269.] In the transitional forms, all the discontinuities cannot be equal to one another as in the irrational types. On drawing the velocity-diagrams it is found, however, that when the motion at the bowed point is a four-step zig-zag, the first, third and other odd discontinuities, if any, should be equal in magnitude, and similarly, the second, fourth and other even discontinuities, if any, should be equal in magnitude. If the motion at the bowed point is a six-step zig-zag, the discontinuities should be capable of being arranged in 3 sets of equal discontinuities, and so on. These relations are perfectly general, and the vibration-curve for a point near the end of the string calculated from a velocity-diagram of this kind shows some immediately recognisable characteristics, every alternate or every third ascent or descent being of the same steepness and length. This is clearly seen in the curves on the right-hand side column of figure 26.

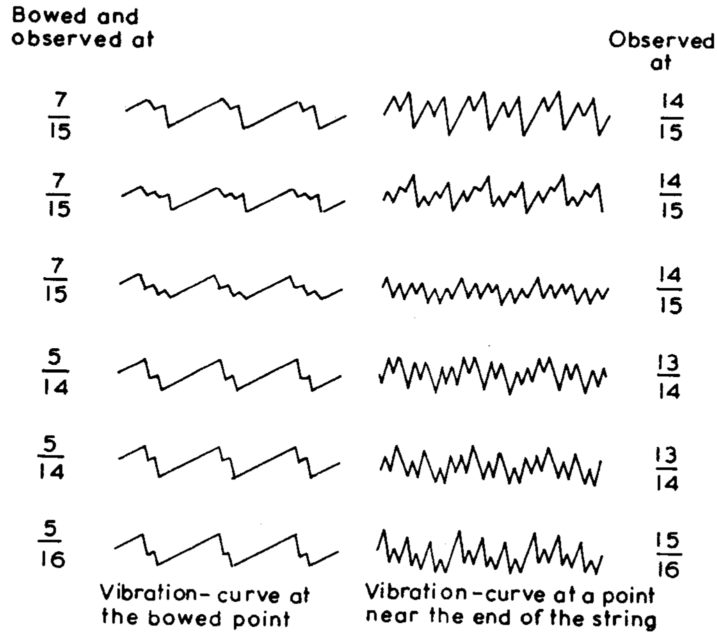



**Figure 26.** Transitional modifications of the fourth and sixth irrational types.

The first pair of curves shows the fourth type and the next two pairs the sixth type, obtained by bowing the string near the centre. The two following pairs show the sixth type obtained by bowing on one side of the point of trisection, and the last pair also show the sixth type obtained by bowing on the other side of the trisection-point of the string. The eighth type obtained by bowing near the point of quadrisection of the string may be similarly investigated. The curves are all necessarily asymmetrical. The motion at the bowed point in all these cases obeys

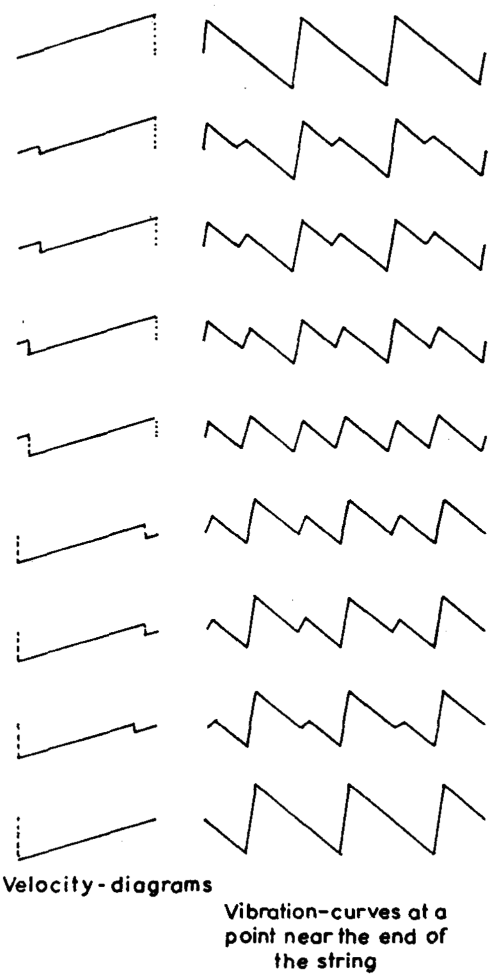



**Figure 27.** Transition from the first to the second irrational type and *vice versa* (also illustrating cyclical change of vibration-form at the "wolf-note" pitch).

the kinematical law $\omega = (nx_n/l)$, $n$ being given the appropriate integral value, and $\omega$ signifying the fractional part of the period during which slipping occurs at the bowed point.

The transition from a two-step to a four-step or a six-step motion at the bowed point, obtained by applying the bow close to the end of the string with smaller and smaller pressures, may be readily traced by drawing a velocity-diagram with unequal discontinuities, and deducing the vibration-curves in the manner explained above. The transition from a two-step to a four-step motion for bowed points close to the end of the string was observed by Helmholtz.[16]

## Limiting form of vibration for very small pressures of bowing

In the discussion on the *modus operandi* of the bow and the subsequent detailed treatment of the modes of vibration maintained under various conditions, it has so far been assumed that the pressure of the bow is sufficient to ensure that there is no slipping at the bowed point during at least a part of each period of vibration. On this assumption, it has been found that when the pressure of the bow is gradually reduced, the mode of vibration passes through various stages which are either very close approximation to certain standard types or else form transitions between those types. There is little difficulty in finding which of the standard types of vibration is capable of being elicited when the bowing pressure is the minimum admissible. The exact method of ascertaining the critical pressure for any given type of vibration has already been indicated. Generally speaking, the type which involves slipping at the bowed point for the largest fraction of the period has the smallest critical pressure. From figure 15, on page 296, it is seen that when the bow is applied at any point in the range $l/2$ to $l/3$, the fraction of the period during which there is slipping at the bowed point is largest for the first irrational type in which the fundamental is dominant. Within this range therefore, the smallest pressure of bowing for which the theory given is valid would elicit the first type of vibration, or rather a transitional modification of it in which harmonics having nodes near the bowed point are absent or relatively deficient. If the pressure of bowing be further reduced, the motion at the bowed point necessarily suffers further modification. Under the conditions postulated, the fundamental is dominant, and the higher harmonics, especially those whose nodes are nearest the bowed point, require proportionately much larger maintaining forces than the fundamental. The pressure of the bow being small, the forces exerted by it are insufficient to maintain the higher harmonics which accordingly tend to fall out, leaving the fundamental practically by itself. The limiting form of vibration of the string for small pressures of bowing within the range $l/2$ to $l/3$ is thus a simple oscillation. On reducing the pressure of bowing to the smallest possible values, even the fundamental ceases to be vigorously maintained, and the amplitude of the vibration tends to zero. The effect of an increase in the velocity of the bow on

[16] *Sensations of Tone*, English translation by Ellis, page 85. Figure 26 of Helmholtz's work representing this transition is not quite correct. For, when the two discontinuities are unequal, the two steeper lines in the vibration-curve of a point near the end of the string would not be quite parallel, except in the limit when the point observed is infinitely near the end of the string. The shorter descent would be less steep than the longer one. Further, Helmholtz shows only the symmetrical transition forms in which the break first appears in the middle of the ascending line of small slope. In general, the actual transition forms are asymmetrical. See figure 27 of this Bulletin.

the form of the maintained vibration is of course analogous to that of a diminution of its pressure.

Practically similar results would be obtained by applying the bow with a small pressure-velocity ratio at points lying between $l/3$ and $l/4$. For, within this range, the second irrational type of vibration is that which requires the smallest pressure to elicit, and when the harmonics which have nodes at $l/3$, $2l/7$ and $l/4$ drop out, it leaves little besides the fundamental and a feeble octave. The case is however very different if the bow be applied in the range between 0 and $l/4$. The first type of vibration in which the fundamental is dominant then has a fairly high critical pressure, and when the pressure of the bow is less than this value, the mode of vibration alters into one in which the motion at the bowed point is representable by a four-step or a six-step zig-zag or even a more complicated curve. In other words, the fundamental and hormonics of low order have then a tendency to fall out, leaving the higher harmonics in possession of the field. The string may even divide up into segments and vibrate with twice, thrice or four times its usual frequency.

## Instability of periodic vibration under high bowing pressures

The fundamental assumption on which the treatment given so far rests is that the bow maintains the string in a periodic vibration. If the yielding at the ends of the string be negligible, the period of this vibration is the same as that of its free oscillations, the phase of each harmonic component in the force exerted by the bow being in advance of the phase of the corresponding component in the maintained motion by 90° (see figures 16 to 19). The question arises whether a periodic motion is always possible under the action of the bow. This is best discussed by considering a specific case. Assume that the bow is applied exactly at the node $l/3$. A very small pressure is then sufficient to maintain the usual periodic motion. (See the last pair of curves in figure 17 on page 308). It is obvious that in this case, any increase in the pressure of the bow over the minimum necessary to establish the usual type of vibration can have no effect at all, so long as the motion is periodic. But it does not follow from this that the pressure of the bow may be increased indefinitely without affecting the possibility of the steady vibration. That a maximum exists beyond which the pressure may not be increased if a regular vibration is to be possible is indicated by the following considerations. The string determines its own frequency of vibration under the action of the bow. But during the fraction of the period in which the bow and bowed point adhere, the frictional force is in a sense arbitrary as it may have any value smaller than the statical friction. If, therefore, the pressure of the bow be sufficiently large, the frictional force is capable of variation in an arbitrary manner between large limits, and therefore also of setting up a motion of the string having an arbitrary period and phase. The motion of the string thus tends to become irregular when the pressure of the bow is much in excess of that required to maintain a regular vibration having the maximum critical pressure.

The maximum bowing pressure at which a steady vibration ceases to be possible evidently depends on the position of the bowed point, on the velocity of the bow, and to some extent also on the other conditions of bowing, e.g. the finite width of the bow. At important nodes such as $l/2$, $2l/5$, $l/3$ and $l/4$, the maximum admissible pressure is necessarily low, as the critical pressure necessary to maintain the usual steady vibration is small. But at nodes such as $5l/11$ or $5l/12$ or $5l/13$, a considerably larger pressure is admissible. In the general case in which the bow is applied at *any* point on the string, the factors that determine the possibility of the mode of vibration having the highest critical pressure, also determine the maximum pressure beyond which a steady vibration ceases to be possible.

## Effect of the finite width of the bow

Except by way of passing reference (see pages 247, 248, 291, 294, 295 and the last para of the previous sub-section), we have not so far considered in detail the effect of the finite width of the region of contact between the bow and the string. The necessary modifications in the equation of maintenance (22) given on page 298 may be made without difficulty. We may represent an element of the region of contact by $dx$, the pressure exerted on it by $P_x$ and $dx$ its velocity by $v_{xt}$. The frictional force acting on the element may be written as $F(P_x, v_{xt} - v_B)\,dx$. To find the expression for the maintained motion we have to analyse the force acting on each element $dx$ into its harmonic sine and cosine components, multiply each component by the corresponding factor $\sin(n\pi x/l)$ and then integrate over the region of contact. We thus obtain the two equations.

$$\int_{x_0}^{x_0'} \sin\frac{n\pi x}{l}\,dx \int_0^T F(P_x, v_{xt} - v_B)\sin\frac{2n\pi t}{T}\,dt = \frac{k_n T B_n}{2}\cos(e_n + e_n')$$

$$\int_{x_0}^{x_0'} \sin\frac{n\pi x}{l}\,dx \int_0^T F(P_x, v_{xt} - v_B)\cos\frac{2n\pi t}{T}\,dt = \frac{k_n T B_n}{2}\sin(e_n + e_n'),$$

where $k_n$, $B_n$, $e_n$ and $e_n'$ have the same significance as in equation (22), and $x_0$, $x_0'$ are the limits of the region of contact.

Comparing the expressions now obtained with those for the ideal case of a string bowed at a mathematical point, it is evident that the theory for a finite region of bowing is considerably the more complicated of the two. It is important to realise clearly the essential point of difference between the two cases. When the pressure of the bow is regarded as applied at a single point on the string, the frictional force is expressible as a function of the relative velocity of that single point with respect to the bow, and may therefore have any arbitrary value (positive or negative) smaller than the statical friction when this relative velocity is absolutely zero. On the other hand, in the actual case of the finite region of contact, the relative velocity of every element of this region with respect to the bow enters in the expression for the system of forces acting on the string, and while *it is possible for a single point on the string to have absolutely the same velocity as the bow in every part of its forward motion*, kinematical theory shows that *it is not possible for every element on a finite region to have absolutely the same velocity as the bow in every part of its forward motion.* The frictional forces do not therefore have the same freedom of adjusting themselves to the conditions of the maintained motion as in the ideal case.

The only line of attack on the present problem open to us appears to be the tentative one of assuming that the maintained motion is one of the various types we have discussed in the preceding pages, and then finding in what manner its maintenance is affected by the finiteness of the region of contact. As the simplest case possible, let us assume that the bow is applied near the end of the string and that the maintained motion is of the first irrational type (see figure 1 on page 261). The vibration-curve at every point on the region of contact is then a simple two-step zig-zag, but with different velocities of ascent and descent. The amplitude of vibration of the string then evidently depends on the position of the particular point whose velocity of ascent is the same as that of the bow; it is a maximum if the relative velocity in the ascending motion is zero at the extreme edge of the region of contact nearest the end of the string, and a minimum if it is zero at the edge farthest from the end. If the relative velocity be zero at some intermediate point, it is positive on one side of it and negative on the other. The frictional forces exerted by the bow on the region of contact on either side of this point are then in opposite directions, one set tending to increase the motion and the other to oppose it. The forces exerted by the bow thus tend mutually to cancel out their own effects, and only the difference left over, if any, is available for the maintenance of the motion. It must be remembered that, element for element, the forces acting on the string furthest from the end that tend to reduce its vibration are the more powerful, as most of the factors $\sin(n\pi x/l)$ are larger here than in the part nearer the end. The tendency of the vibrations to increase up to the maximum possible is thus strongly opposed. As the bow is applied nearer and nearer the end of the string, the point at which the relative velocity is zero tends to approach the boundary of the region of contact furthest from the end.

Conclusions similar to those stated above are also arrived at if we assume that the bow is applied with sufficient pressure, close to but not exactly at an important node such as $l/2$ or $l/3$, and that the corresponding irrational type of vibration (that is the 2nd type or the 3rd type) is elicited. It will be noticed that in every such case, the forces acting on the string furthest from the node tend to oppose the motion, while those acting nearer the node tend to support it; and the force available to maintain the motion is only the difference between the effects of the two sets. On the assumptions made, the force acting on each element of the string is perfectly determinate, and the integrals given above should therefore be capable of complete evaluation.

We may now pass on to consider the effect of the finiteness of the region of contact on the assumption that this region includes some fairly important node and that the vibration elicited is one of the rational types discussed in section X. The velocity diagram of the string then consists of horizontal lines parallel to the $x$-axis (that is, to the string) and separated by discontinuities. If the relative velocity is zero throughout the ascending motion at some one point in the region of contact, this would practically be the case also at immediately contiguous points; at points which are considerably removed from it, however, the relative velocity would be zero only during a succession of intervals making up a part of the ascending motion and would have a finite value during the other intervals, this value being positive in one part of the region of contact and negative in the other part. (For instance see figure 8 on page 282). During the intervals of the ascending motion in which the relative velocity is finite, the frictional forces would evidently act in opposite directions in the two parts of the region of contact, and would thus tend to cancel each other's effects. The more numerous the discontinuities are and the greater the width of the bow is in relation to the length of the string, the greater would be the reduction in the effective action of the bow produced by this opposition of forces. As the intervals in which the relative velocity is finite are distributed regularly over the period, the effect of the finite width may be regarded as approximately equivalent to a reduction of the pressure of the bow, though of course this is not an absolutely accurate statement. Since the number of discontinuities in the velocity-diagram is one of the factors determining the effective reduction in the pressure of bow produced by the finiteness of the region of contact, the tendency would be to elicit a mode of vibration having the fewest discontinuities. The particular point on the region of contact the motion of which approximates most closely to that for the ideal case of a bowed 'point' is, of course, a variable factor which enters into the equations determining the maintained vibration. Since this point would not in general actually coincide with the node falling within the region of contact, the mode of vibration would approximate to one of the transitional types discussed on pages 314–326 rather than to one of the standard types of vibration modified by the dropping out of a set of harmonics which we discussed in section X.

Owing to the fact that the frictional forces in the cases considered in the preceding paragraph are partly determinate and partly indeterminate, a complete evaluation of the integrals and a rigorous detailed treatment do not appear