My Fifty Years with Finite Elements

Robert L. Taylor

Department of Civil & Environmental Engineering

University of California, Berkeley

WCCM8/ECCOMAS 2008 Congress: 20 June - 4 July 2008

My Fifty Years with Finite Elements

Outline: Presentation summarizes:

- Historical overview
  - Early FEM developments by R.W. Clough
  - Berkeley-Swansea connection.

- Some of the work of people who have influenced FEM and me!
  - Near incompressibility treatment
  - Time integration algorithms
  - Computational mechanics at UC
  - Finite deformation

- Some challenges I see today.

My Fifty Years with Finite Elements

Early History

My Fifty Years with Finite Elements

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Berkeley Campus 1940

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UC Engineering Buildings ~1960

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Karl Pister's 1962-63 Graduates!

1962

1995

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- FEM: The engineeering begining

- Clough spent summers of 1952-53 and 1953-54 at Boeing.

- Worked with M.J. Turner, L.J. Topp and H.C. Martin (U. Wash).

- Credits Turner with idea of *elements* to determine frequencies of Delta wing aircraft.

- Names FEM in 1960

- My first class in 1957!

R.W. Clough (1956)

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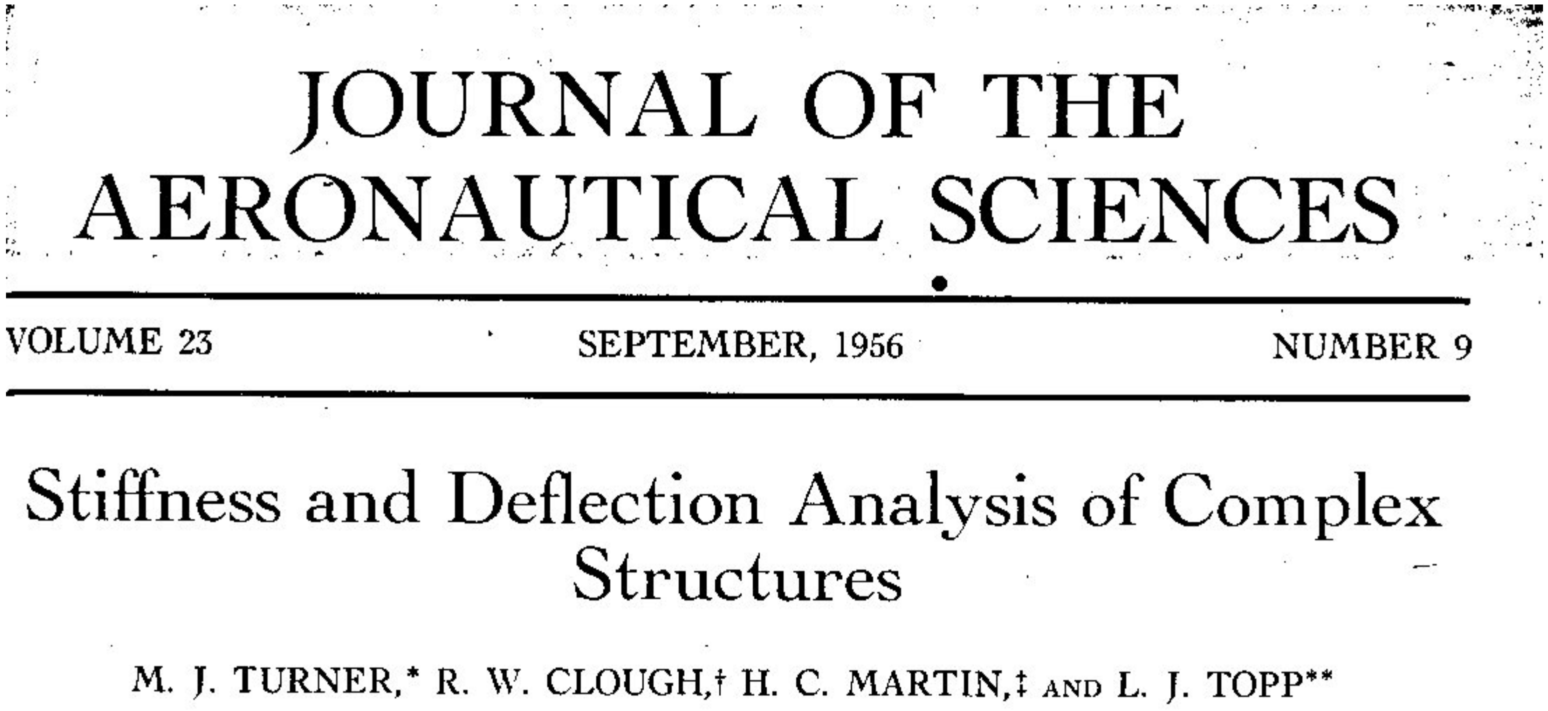
1956: First *Berkeley* and *Engineering* FEM paper.



JOURNAL OF THE
AERONAUTICAL SCIENCES

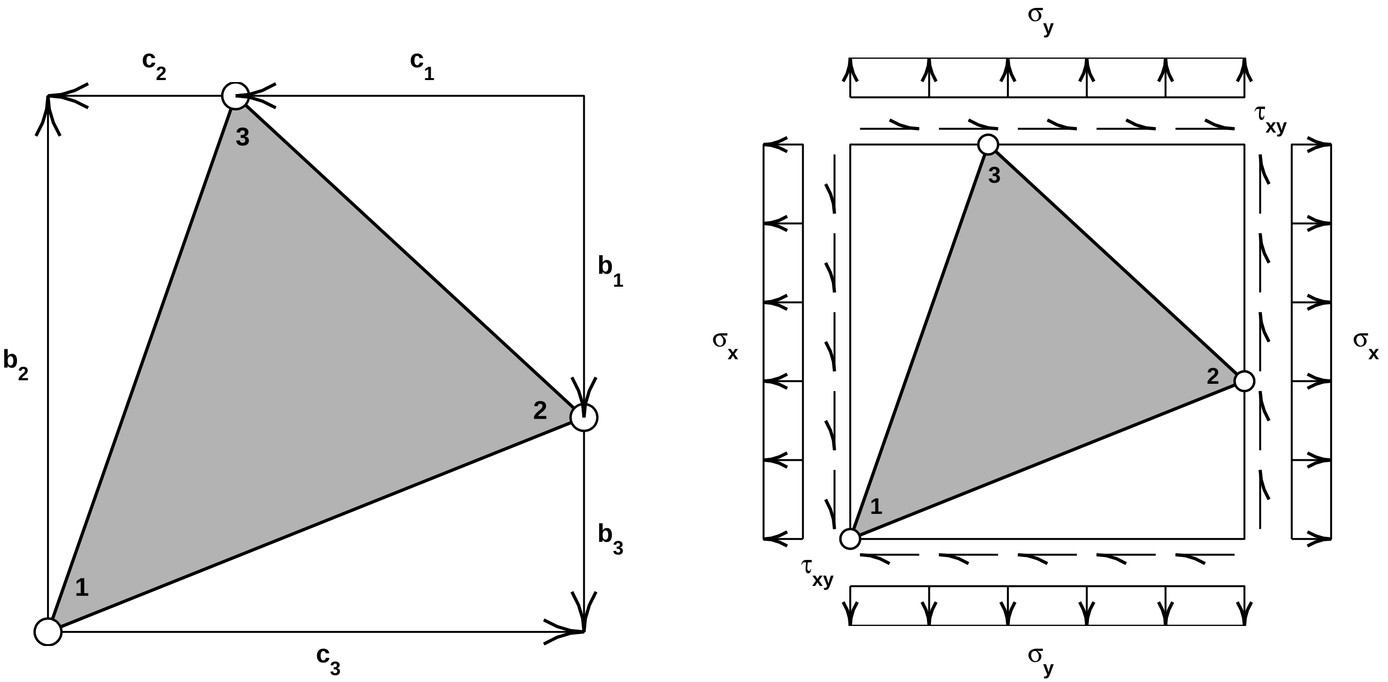
VOLUME 23 SEPTEMBER, 1956 NUMBER 9

Stiffness and Deflection Analysis of Complex Structures

M. J. TURNER,[*] R. W. CLOUGH,[†] H. C. MARTIN,[‡] AND L. J. TOPP[**]

My Fifty Years with Finite Elements

- FEM: The begining
  - Direct physical construction: Plane stress elasticity



(a) Triangle and geometry (b) Uniform stress state

  - Linear displacements: Constant Strain
    Nodal forces by equilibrium.

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- **FEM: Results from first paper**

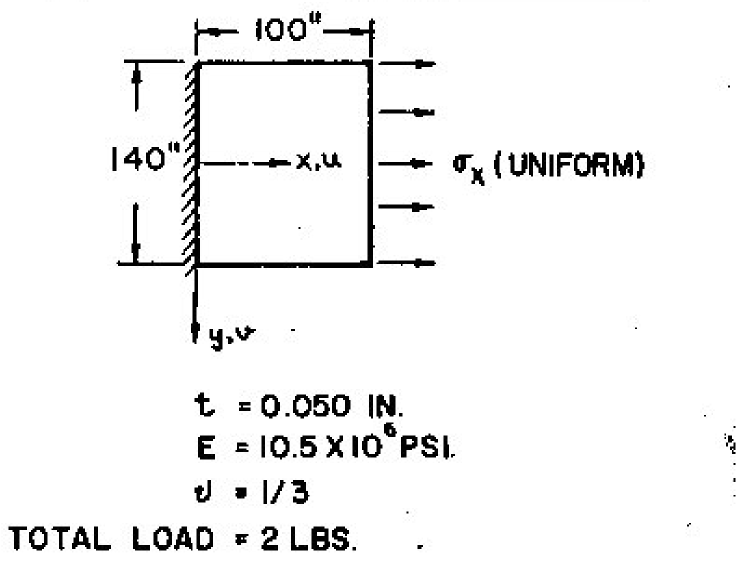
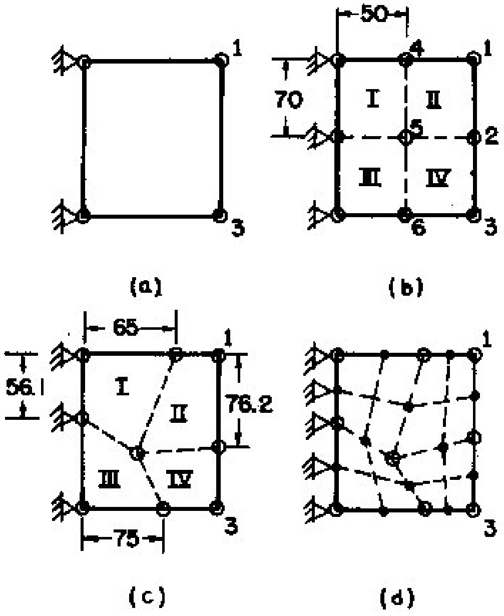


FIG. 12. Clamped rectangular plate subjected to uniform tensile loading.



(a) (b) (c) (d)

FIG. 13. Nodes and supports for clamped rectangular plate.

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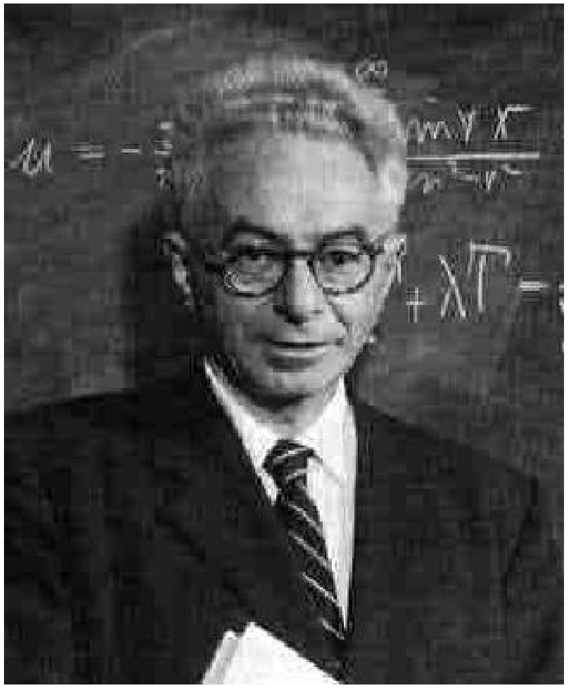
- Earlier contribution from mathematics.

- 1941 presentation to American Math Society.

- Reference: "Variational methods for the solution of problems of equilibrium and vibration", Bulletin of the American Math Society, **49**, 1943, pp. 1–61.

- Solved Laplace equation problem (torsion).

- Not known to engineers in mid-1950's.



R. Courant (1941)

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FEM Software Development:

- In addition to theoretical studies, FEM programs written.
- 1956: First campus machine IBM 701 – in Cory Hall.
- Early program in assembly code (before FORmula TRANslator developed by Backus, *et al.* at IBM – 1954-57).
- Ed Wilson (Clough's student) prepared first UC program in FORTRAN II (1958) using IBM 704.
- Wilson's programs formed basis of all our early efforts.
- All early programs used 3-node triangle as basic element.

Early Computing Environment

Keypunch room

IBM Key Punch

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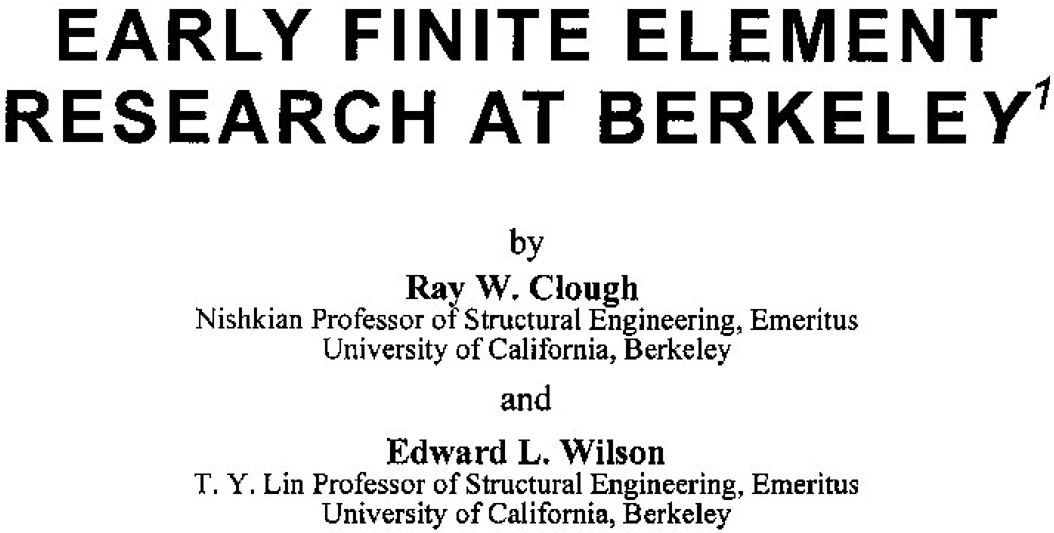
- See Wilson's web page for more on early FEM research by Clough:

  http://www.edwilson.org



EARLY FINITE ELEMENT RESEARCH AT BERKELEY[1]

by

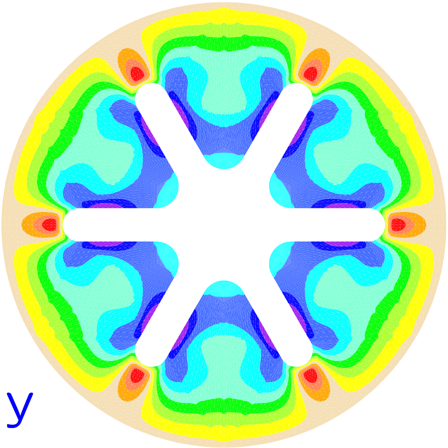
Ray W. Clough
Nishkian Professor of Structural Engineering, Emeritus
University of California, Berkeley

and

Edward L. Wilson
T. Y. Lin Professor of Structural Engineering, Emeritus
University of California, Berkeley

Nearly Incompressible Analyses

My Fifty Years with Finite Elements



- 1960-68: Solid propellant rocket analyses
- Propellant materials had properties making nearly incompressible ($\nu \approx 0.5$) and time dependent
- Displacement method gave poor results for $\nu > 0.4$.
- Improved using mixed methods – displacement/pressure
- Based on paper by: Herrmann & Toms: J. Appl. Mech, 1964
- Used 4-triangle quadrilateral with constant pressure.
- Developed 2-d programs for elastic and thermoviscoelastic materials.

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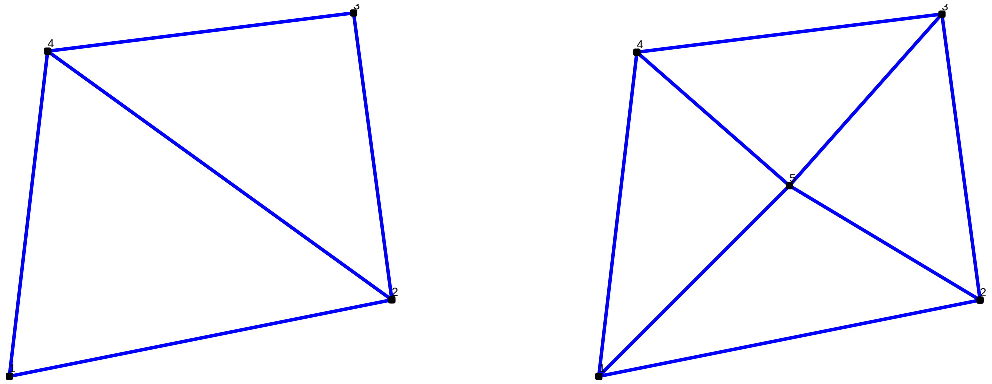
- Constitutive form: (L.R. Herrmann)

$$\sigma_{ij} = 2\mu\left[\epsilon_{ij} + (\nu H - e_T)\,\delta_{ij}\right]$$
$$H = \frac{3\sigma_{kk}}{2\mu(1+\nu)}$$
$$\epsilon_{kk} - e_T = (1-2\nu)H$$

- Elements (composite)



- Interpolations: u_i linear, H constant. (Before BB papers!)

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INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 2, 45-59 (1970)

THERMOMECHANICAL ANALYSIS OF VISCOELASTIC SOLIDS

ROBERT L. TAYLOR
Associate Professor of Civil Engineering

KARL S. PISTER
Professor of Engineering Science

and

GERALD L. GOUDREAU
Graduate Student in Civil Engineering
University of California, Berkeley

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Thermoviscoelasticity:

- Constitution: Spherical/Deviatoric split

$$\sigma_{ij} = p\,\delta_{ij} + s_{ij} \quad ; \quad \epsilon_{kl} = \theta\,\delta_{kl} + e_{kl}$$

$$p = 3\,K\,(\theta - \alpha\Delta T) \quad ; \quad s_{ij} = 2\,G \int_{-\infty}^{t} G(\xi(t) - \xi(\tau), T_0) \frac{\partial e_{ij}}{\partial \tau}\,\mathrm{d}\tau$$

- Thermorheologically simple

$$\xi(t) = \int_0^t \phi(T(t'))\,\mathrm{d}t'$$

- Relaxation function: Prony series representation

$$G(\xi) = G\left[\mu_0 + \sum_{m=1}^{M} \mu_m \exp(\xi/\lambda_m)\right] \quad ; \quad \mu_0 + \sum_{m=1}^{M} \mu_m = 1$$

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- Integration of Prony series form of constitution

$$\mathbf{s} = 2\,G\left[\mu_0\,\mathbf{e} + \sum_{m=1}^{M} \mu_m\,\mathbf{q}^m\right]$$

- Recursion for increment t_n to t_{n+1}

$$\begin{aligned}\mathbf{q}^m &= \int_{-\infty}^{t} \exp[-(\xi(t) - \xi(\tau))/\lambda_m]\,\frac{\partial \mathbf{e}}{\partial \tau}\,\mathrm{d}\tau \\ &\approx \exp[-\delta\xi_{n+1}/\lambda_m]\left[\mathbf{e}_{n+1} - \mathbf{e}_n\right] + \Delta q^m_{n+1}\,\left[\mathbf{e}_{n+1} - \mathbf{e}_n\right]\end{aligned}$$

where $[\Delta\xi_{n+1} = \xi(t_{n+1}) - \xi(t_n) \approx \frac{1}{2}\Big(\phi(T_n) + \phi(T_{n+1})\Big)\,\Delta t]$:

$$\begin{aligned}\Delta q^m_{n+1} &= \frac{\lambda_m}{\Delta\xi_{n+1}}\left[1 - \exp(-\Delta\xi_{n+1}/\lambda_m)\right] \\ &= 1 - \tfrac{1}{2}\left(\frac{\Delta\xi_{n+1}}{\lambda_m}\right) + \tfrac{1}{3!}\left(\frac{\Delta\xi_{n+1}}{\lambda_m}\right)^2 - \tfrac{1}{4!}\left(\frac{\Delta\xi_{n+1}}{\lambda_m}\right)^3 + \cdots\end{aligned}$$

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Example: Thin-walled cylinder

- Normalized temperature

$$\bar{\theta}(x,\rho) = (1-x)[1-\exp(-2\rho)]$$

where ρ is normalized time, is x normalized thickness distance

- Properties

$$K = 2.5 \times 10^{10}$$
$$G = 8.3 \times 10^{9}$$
$$\mu_0 = 0.001$$
$$\mu_1 = 0.999$$

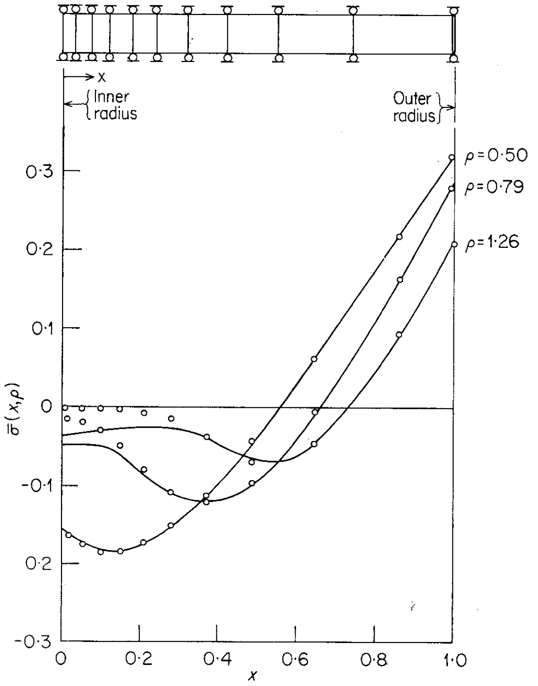

Figure 3. Hoop stress versus time, temperature-dependent properties; ○, per THVISC, Reference 16; —, per Reference 7

- Shift function for polymethylmethacrylate

$$\phi(\bar{\theta}) = 3981.1 \exp[-6.2172(1-\bar{\theta})(1.3333 + \bar{\theta} + 1.095\,\bar{\theta}^2]$$

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Connections to Swansea

My Fifty Years with Finite Elements

- 1965-70: FEM Major Advancements
  - Isoparametric elements (I. Taig – 1962: Work done 1957-58).
  - Numerical integration (Irons – 1966).
  - Thin/thick plates & shells (Zienkiewicz, *et al.* – 1965ff)
- Swansea clearly an active FEM research location!
- 1968: Introduced to Zienkiewicz by Clough on airplane to 2nd Wright-Patterson conference.
- 1969: Sabbatical leave in Swansea (returned in 1976 & 1984)

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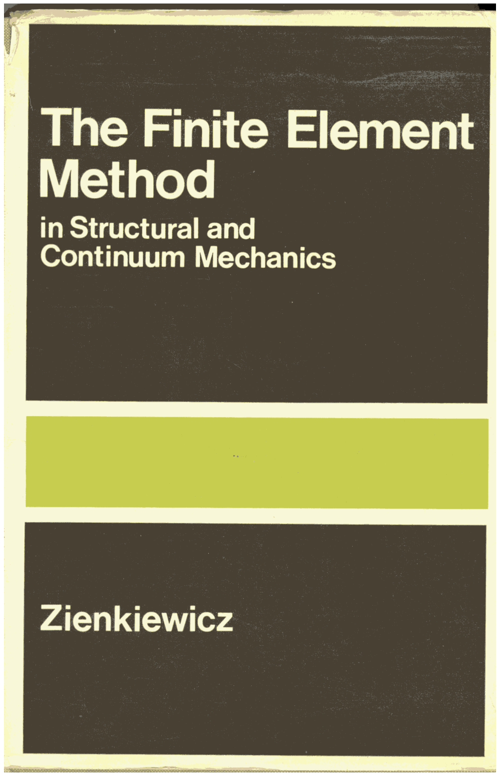
Berkeley–Swansea Connection

- 1967: First FEM book



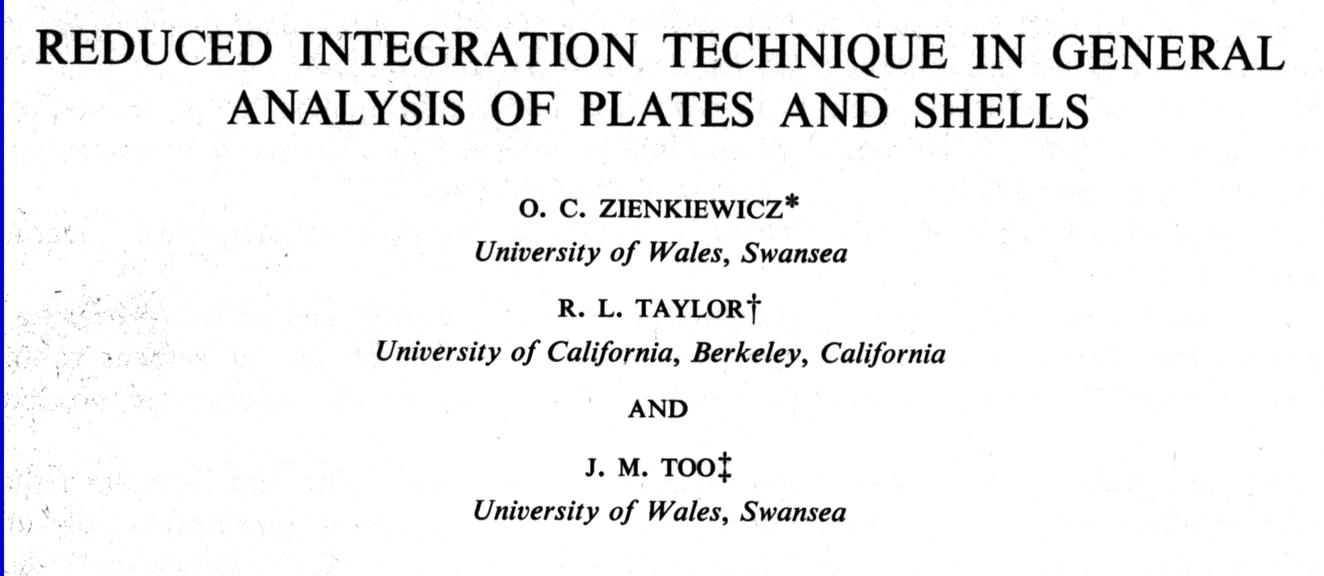
- 1977: 3rd edition – our first joint effort.

My Fifty Years with Finite Elements

Swansea in 1969

My Fifty Years with Finite Elements

Swansea – Shells work



REDUCED INTEGRATION TECHNIQUE IN GENERAL ANALYSIS OF PLATES AND SHELLS

O. C. ZIENKIEWICZ*
University of Wales, Swansea
R. L. TAYLOR†
University of California, Berkeley, California
AND
J. M. TOO‡
University of Wales, Swansea

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Swansea: Reduced Integration

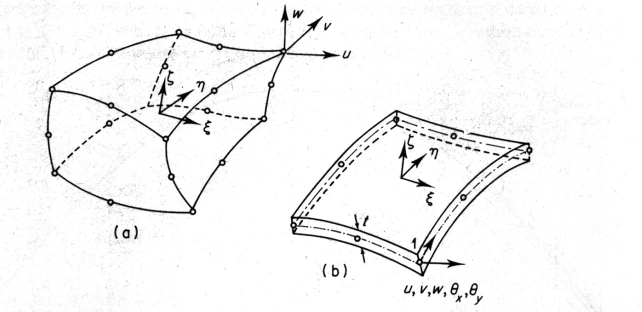


Figure 2. (a) Parabolic isoparametric hexahedron; (b) degeneration to a shell element

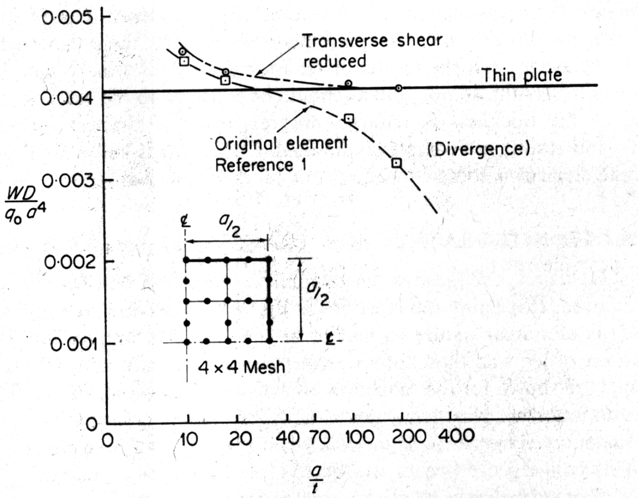
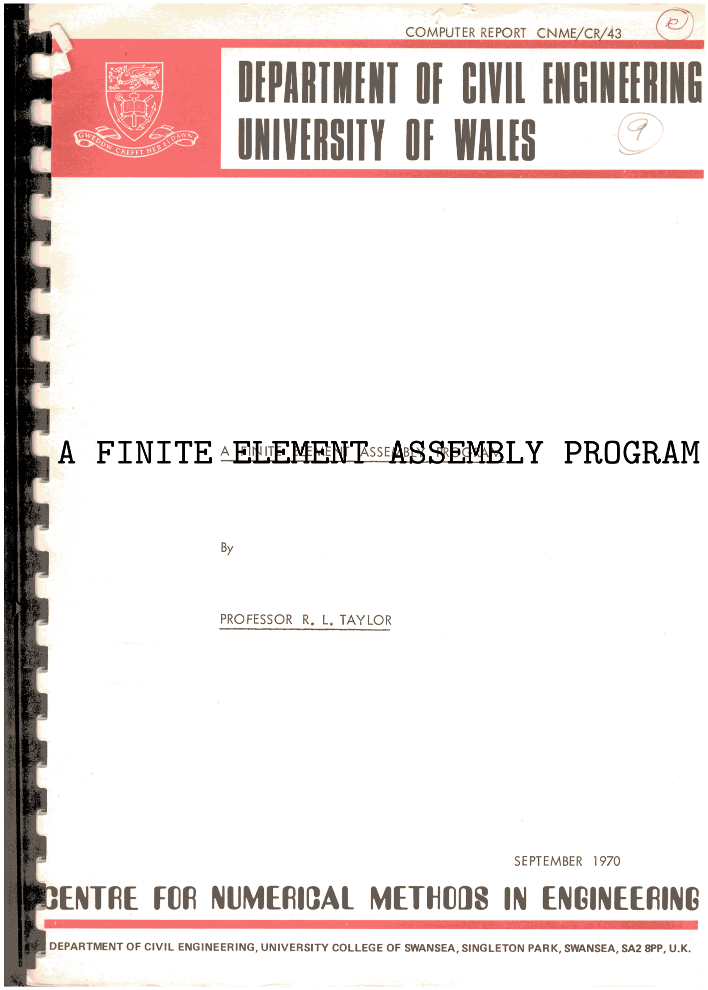


Figure 3. A simply supported square plate under uniform load q_0. (a) Plot of central deflection for element of Reference 1; (b) with reduced transverse shear integration; (c) displacement at centre versus thickness parameter

My Fifty Years with Finite Elements

The first Swansea year

- First *FEAP*



COMPUTER REPORT CNME/CR/43 (e)

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF WALES (9)

A FINITE ELEMENT ASSEMBLY PROGRAM

A FINITE ELEMENT ASSEMBLY PROGRAM

By

PROFESSOR R. L. TAYLOR

SEPTEMBER 1970

CENTRE FOR NUMERICAL METHODS IN ENGINEERING

DEPARTMENT OF CIVIL ENGINEERING, UNIVERSITY COLLEGE OF SWANSEA, SINGLETON PARK, SWANSEA, SA2 8PP, U.K.

Time Integration Developments

My Fifty Years with Finite Elements

COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING, 2, (1972) 69-97

EVALUATION OF NUMERICAL INTEGRATION METHODS IN ELASTODYNAMICS

G.L. GOUDREAU
Engineer, Lawrence Livermore Laboratory

and

R.L. TAYLOR
Associate Professor, Department of Civil Engineering
University of California, Berkeley

Methods Development Group

Saturday, March 15, 2008

Mission – MDG goals and vision

Projects – Current MDG interests

Codes – About MDG computer codes

People – MDG developers, analysts,

MDG Codes

Click on the links below to obtain information about the MDG codes. You will find information about applications, developers, and manuals for each code.

DYNA3D	NIKE3D	TOPAZ3D	ParaDyn	Diablo	FACET	Griz
DYNA2D	NIKE2D	TOPAZ2D	Maze	Orion	MONT3D	

My Fifty Years with Finite Elements

Error and stability analysis of discrete elastic problem

- Modal equation (elastic and undamped) $\ddot{d} + \omega^2 d = f$
- Newmark $$d_{n+1} = d_n + \Delta t v_n + \left(\tfrac{1}{2} - \beta\right) \Delta t^2 a_n + \beta \Delta t^2 a_{n+1}$$ $$v_{n+1} = v_n + (1-\gamma)\,\Delta t a_n + \gamma \Delta t a_{n+1}$$
- Let $\theta = \omega \Delta t$; $\delta = \gamma - \frac{1}{2}$ and $\alpha^2 = \theta^2/(1+\beta\theta^2)$
- Eliminate velocity and acceleration and assume $d_k = \lambda^k$ gives $$\lambda^2 - (2 - \alpha^2 - \delta\alpha^2)\lambda + (1 - \delta\alpha^2) = 0$$ for homogeneous equation.
- Solution: $\lambda = \left(1 - \alpha^2\delta\right)^{1/2} \exp(\pm i a)$ where $$a = \tan^{-1}\left[\frac{\alpha\sqrt{1 - \frac{1}{4}\alpha^2(1+\delta)^2}}{1 - \frac{1}{2}\alpha^2(1+\delta)}\right]$$

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Error and stability analysis (cont.)

- Results:
  - If $\gamma < \frac{1}{2}$ (or $\delta < 0$): Negative damping, unstable.
  - If $\gamma > \frac{1}{2}$ (or $\delta > 0$): Postive damping
  - For oscillatory response $1 + \frac{1}{4}\alpha^2(1+\delta)^2 \geq 0$
  - Gives stability limit on Δt: $\theta \leq \left[\frac{1}{4}(1+\delta)^2 - \beta\right]^{-1/2}$
  - Unconditional stability requires: $\beta \geq \frac{1}{4}(1+\delta)^2$

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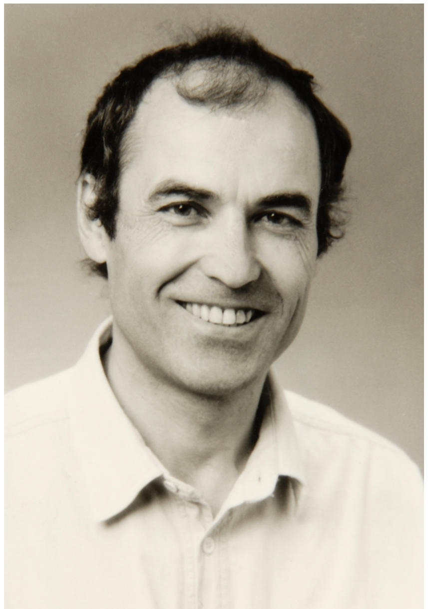
EARTHQUAKE ENGINEERING & STRUCTURAL DYNAMICS, 5, (1976) 283-292

IMPROVED NUMERICAL DISSIPATION FOR TIME INTEGRATION ALGORITHMS IN STRUCTURAL DYNAMICS

H.M. HILBER, T.J.R. HUGHES and R.L. TAYLOR
University of California, Berkeley



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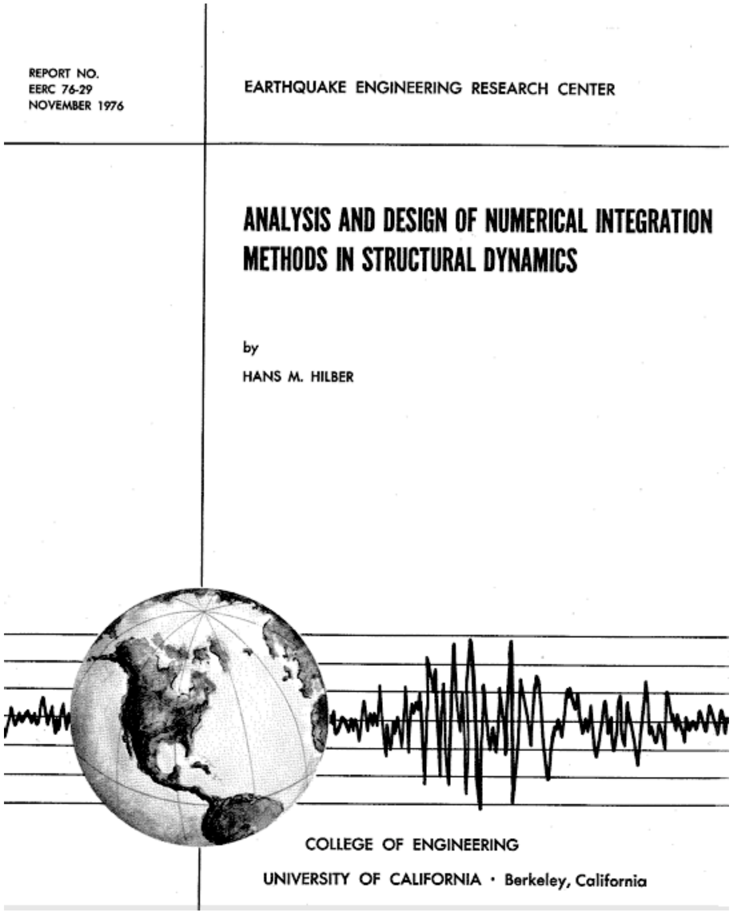
- HHT - Algorithm

$$\begin{aligned}\mathbf{M}\mathbf{a}_{n+1} + \mathbf{C}\mathbf{v}_{n+1} + (1-\alpha)\mathbf{K}\mathbf{d}_{n+1} \\ = \mathbf{F}_{n+1} + \alpha\mathbf{K}\mathbf{d}_n \\ \mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t\mathbf{v}_n + (\tfrac{1}{2} - \beta)\Delta t^2\mathbf{a}_n \\ + \beta\Delta t^2\mathbf{a}_{n+1} \\ \mathbf{v}_{n+1} = \mathbf{v}_n + (1-\gamma)\Delta t\mathbf{a}_n + \gamma\Delta t\mathbf{a}_{n+1}\end{aligned}$$

- Parameters:

$$\begin{aligned}\beta &= \tfrac{1}{4}(1-\alpha)^2 \\ \gamma &= \tfrac{1}{2} - \alpha \\ -\tfrac{1}{3} &\leq \alpha \leq 0\end{aligned}$$



REPORT NO.
EERC 76-29
NOVEMBER 1976

EARTHQUAKE ENGINEERING RESEARCH CENTER

ANALYSIS AND DESIGN OF NUMERICAL INTEGRATION METHODS IN STRUCTURAL DYNAMICS

by

HANS M. HILBER

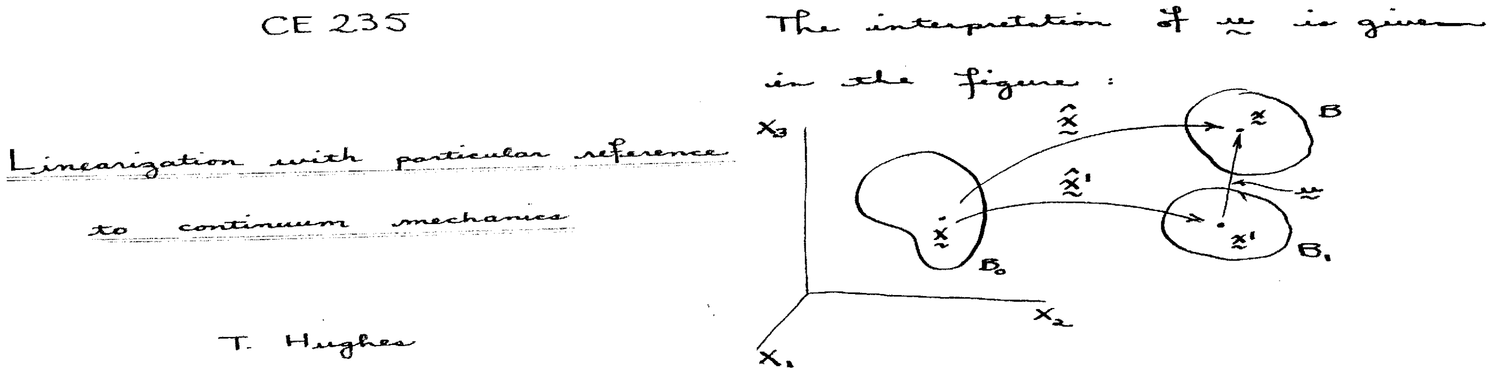
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA · Berkeley, California

Inelastic and Finite Deformation Problems

My Fifty Years with Finite Elements

- Inelastic, contact & finite deformation developments. (with T.J.R. Hughes, W. Kanok-nukulchia, & A. Curnier)

- First UC Computational Mechanics Course (Spring 1975) (Taught by: K.S. Pister & T.J.R. Hughes)



CE 235

Linearization with particular reference to continuum mechanics

T. Hughes

The interpretation of $\underset{\sim}{u}$ is given in the figure:

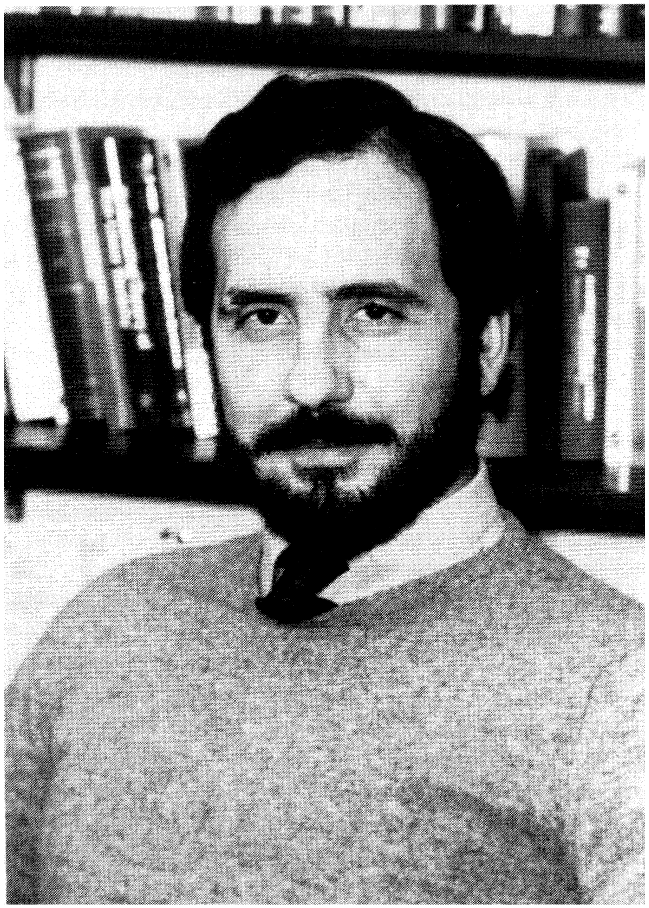
- Established much of notation and methods we use today

My Fifty Years with Finite Elements

The Juan Simo Years

My Fifty Years with Finite Elements

- J.C. Simo (1952-1994)
- 1981-94: Interactions with many!

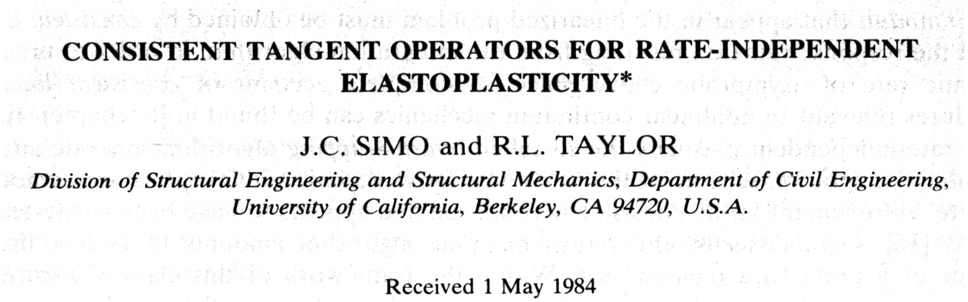


My Fifty Years with Finite Elements

- 1981-94: Interactions with J.C. Simo
  - Developed method of solution for:
    - Integration of plasticity (plane strain & plane stress);
    - Enhanced strain elements;
    - Elasticity & viscoelasticity constitution in principal stretches;
    - Flexible-rigid body solutions;
    - Energy-momentum conserving integration methods;
  - Contributed to development of *FEAP* for finite deformation

My Fifty Years with Finite Elements

Material Modeling: Elasto-Plastic



CONSISTENT TANGENT OPERATORS FOR RATE-INDEPENDENT ELASTOPLASTICITY*

J.C. SIMO and R.L. TAYLOR

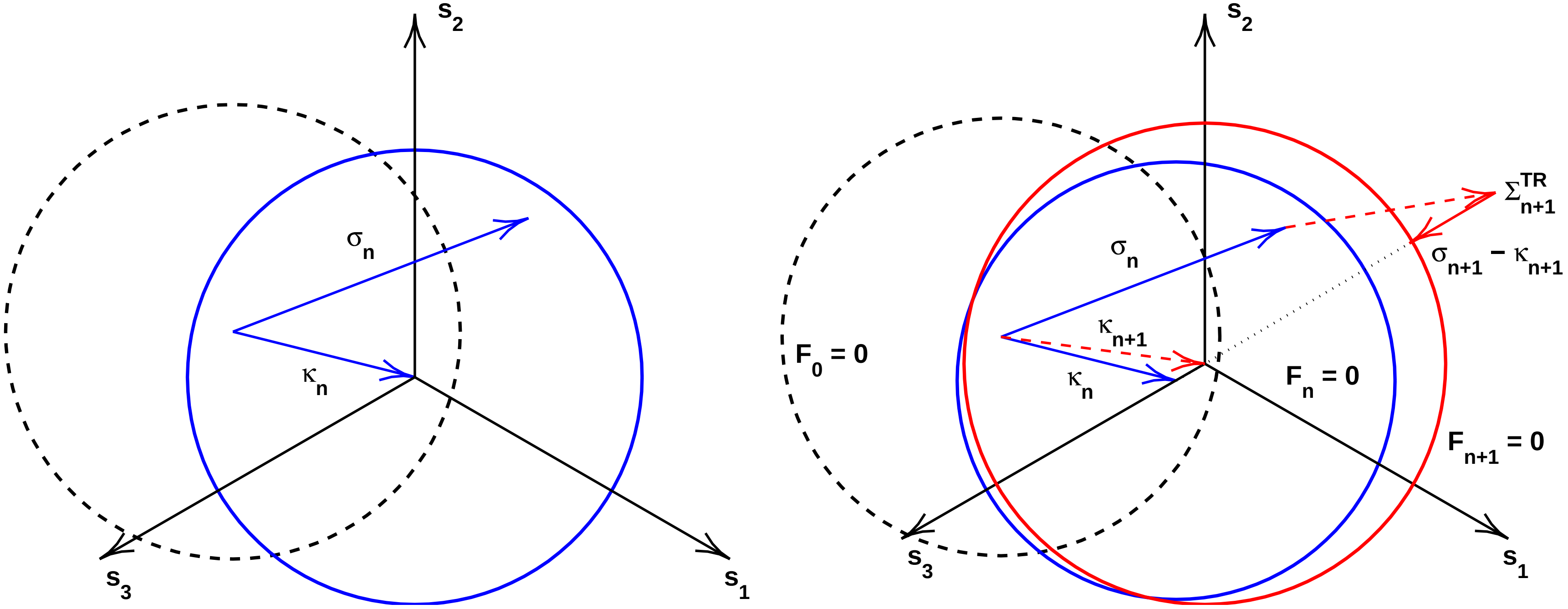
Division of Structural Engineering and Structural Mechanics, Department of Civil Engineering, University of California, Berkeley, CA 94720, U.S.A.

Received 1 May 1984

- Developed algorithm for J_2 (Mises) plasticity
- Included isotropic and kinematic hardening
- Linearized return map algorithm
- Unaware Hibbitt had done perfect plasticity in Abaqus
- Later did plane stress case also.

Material Modeling: J_2 Elasto-Plastic Model

- Graphically, return map for J_2 form is



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Material Modeling: Elasto-Plastic

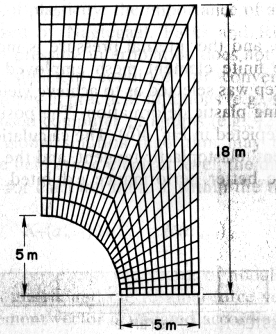

Fig. 5. Perforated strip. Finite element mesh.

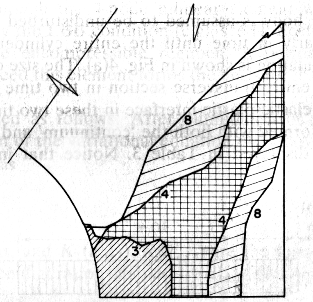

Fig. 6. Perforated strip. Elastic-plastic interface.

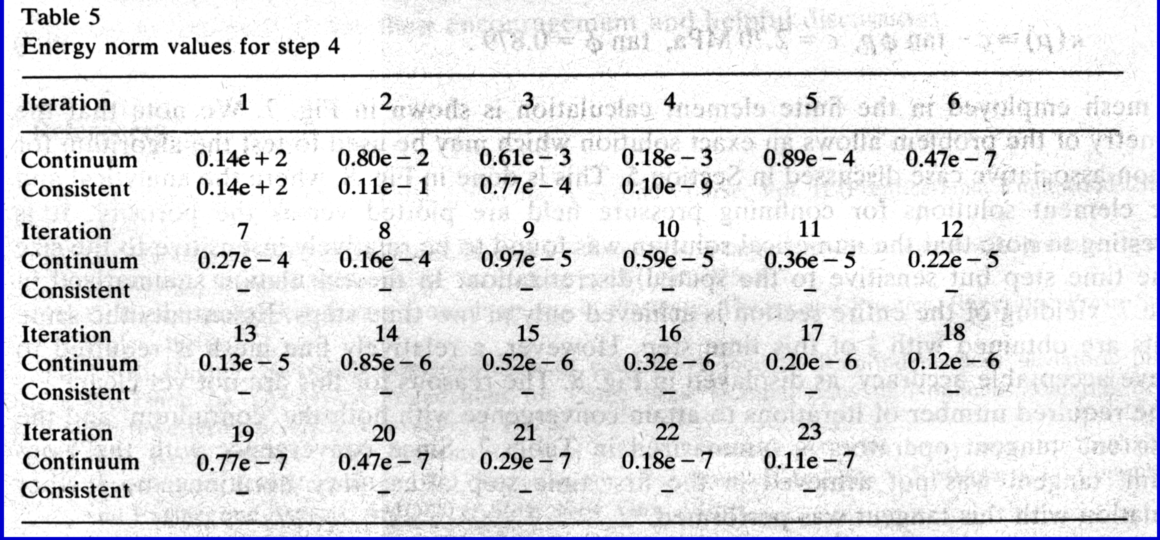

Table 5
Energy norm values for step 4

Iteration	1	2	3	4	5	6
Continuum	0.14e + 2	0.80e − 2	0.61e − 3	0.18e − 3	0.89e − 4	0.47e − 7
Consistent	0.14e + 2	0.11e − 1	0.77e − 4	0.10e − 9		
Iteration	7	8	9	10	11	12
Continuum	0.27e − 4	0.16e − 4	0.97e − 5	0.59e − 5	0.36e − 5	0.22e − 5
Consistent	–	–	–	–	–	–
Iteration	13	14	15	16	17	18
Continuum	0.13e − 5	0.85e − 6	0.52e − 6	0.32e − 6	0.20e − 6	0.12e − 6
Consistent	–	–	–	–	–	–
Iteration	19	20	21	22	23	
Continuum	0.77e − 7	0.47e − 7	0.29e − 7	0.18e − 7	0.11e − 7	
Consistent	–	–	–	–	–	

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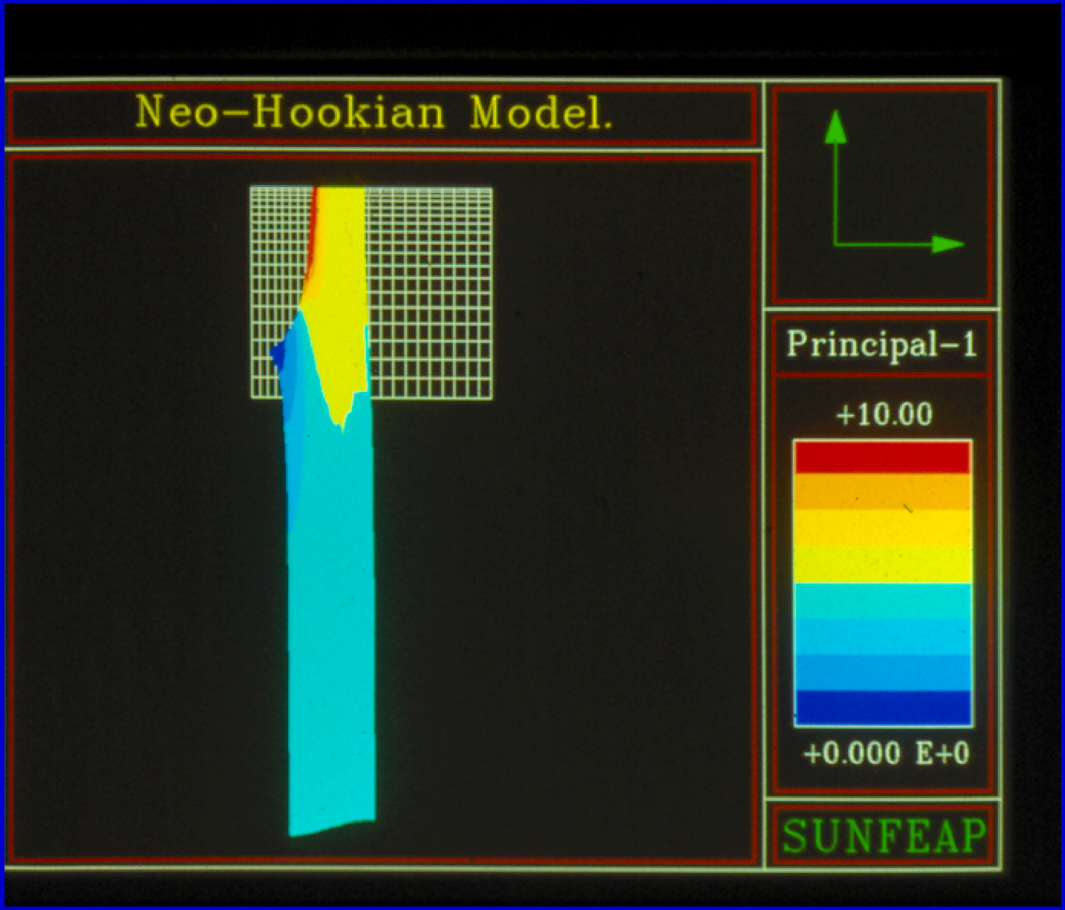
Material Modeling: Finite Elasticity



My Fifty Years with Finite Elements

Today

My Fifty Years with Finite Elements

The FEM Today:

- By mid 1990's FEM for solids was fairly well established
  - Solve finite deformation solids, rods & shells
    - Treat near incompressibility; integrate inelastic constitutive models, etc.
  - Sparse solvers and eigen-problem methods available.
  - Research and commercial software available.
  - Personal computer/workstation costs reasonable.
- Thermal, Fluids, Electro-magnetics solvers also available.

My Fifty Years with Finite Elements

Some Challenges for Today (and Tomorrow!)

- Multi-physics: Coupling of solids to multiple inputs.
  - Thermal; Electro-magnetics; Chemistry; Fluids, etc

- Multi-scale
  - Coupling between continuum-scale; meso-scale; etc.

- Automated analysis
  - From solid models to results with minimal user intervention
    - Challenges for element technology (tetrahedra)
    - Robust solvers (iterative & non-linear)

My Fifty Years with Finite Elements

Summary:

- Described some of what I have witnessed in the last 50 years.
- Key contributions always occurred in collaboration with others!
- *FEAP* remains my "hobby" – but still FEM is a lot of fun!
- I thank all (mentioned and unmentioned) I have had an opportunity to know during the last 50 years.
- Much accomplished, but much to do.
- I look forward to many more years of learning!

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Thank you for your attention!