PTOLEMY'S THEOREM AND ITS CONVERSE

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ABSTRACT. This is an expository note on Ptolemy's Theorem and its converse, giving a more algebraic proof of these results. We show that 4 points in the plane lie on a circle or straight line if and only if they satisfy Ptolemy's condition.

1. The Theorems

If A and B are points in the plane we write AB for the distance between them.

Theorem 1.1 (Ptolemy's Theorem)**.** *Let A, B, C, D be 4 points lying in order on a circle. Then*

$$AB \cdot CD + AD \cdot BC = AC \cdot BD \tag{1}$$

The same conclusion holds if the 4 points lie in order on a straight line.

We refer to (1) as Ptolemy's condition. In the usual statement the points A, B, C, D are the vertices of a quadrilateral with AC and BD being the diagonals. The theorem says that if the quadrilateral can be inscribed in a circle then Ptolemy's condition is satisfied.

There is an excellent article on Ptolemy's Theorem and its applications in [2]. The following related results are also mentioned but no proof is given (as of 2019).

Theorem 1.2 (Converse of Ptolemy's Theorem)**.** *If 4 points A, B, C, D in the plane satisfy (1), they lie on a circle or straight line.*

In other words, the quadrilateral with the given points as vertices can be inscribed in a circle or is a line segment. Note that the points may satisfy the condition in one ordering but not in a different ordering.

Theorem 1.3 (Ptolemy's inequality)**.** *Let A, B, C, D be 4 points in the plane. Then*

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD \tag{2}$$

Here the ordering of the points is irrelevant.

2. Proof of Ptolemy's Theorem

There are many well known geometric and trigonometric proofs of Theorem 1.1. See [2]. Here is a more algebraic one.

Suppose first that $a < b < c < d$ are 4 points on the line $\mathbb{R}$. For these points (1) takes the form

$$(b-a)(d-c) + (d-a)(c-b) = (c-a)(d-b) \tag{3}$$

which is easily verified.

For the case of points on a circle we identify the plane with the complex numbers $\mathbb{C}$. By translation and scaling we can assume the circle is the unit circle $\{z||z|=1\}$. Ptolemy's condition now takes the form

$$|b-a||d-c|+|d-a||c-b|=|c-a||d-b| \tag{4}$$

where a,b,c,d are 4 points in order on the unit circle.

If $z=re^{i\theta}$ with $r>0$ we choose θ to satisfy $0\le\theta<2\pi$ and choose $\arg(z)=\theta$ and $\sqrt{z}=\sqrt{r}e^{i\theta/2}$.

Lemma 2.1. *Let $w,z\in\mathbb{C}$ with $|w|=|z|=1$ and $\arg(w)\le\arg z$. Then $(z-w)=i\sqrt{w}\sqrt{z}|z-w|$*

Proof. Let $z=e^{i\theta}$ and $w=e^{i\phi}$. where $0\le\phi\le\theta<2\pi$. We have

$$(\sqrt{w}\sqrt{z})^{-1}(z-w)=\frac{\sqrt{z}}{\sqrt{w}}-\frac{\sqrt{w}}{\sqrt{z}}=e^{i\frac{\theta-\phi}{2}}-e^{i\frac{\phi-\theta}{2}}=2i\sin\frac{\theta-\phi}{2} \tag{5}$$

where $\sin\frac{\theta-\phi}{2}>0$ since $0\le\theta-\phi<2\pi$. Taking absolute values in (5) shows that $|z-w|=2\sin\frac{\theta-\phi}{2}$ so the lemma follows from (5). □

Now let a,b,c,d be 4 points in order on the unit circle. Rotate the circle so that $0\le\arg(a)\le\arg(b)\le\arg(c)\le\arg(d)<2\pi$, The lemma shows that each term of (3) is the product of the corresponding term of (4) with the factor $-\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{d}$. Since (3) is true, it follows that (4) is also true, proving Ptolemy's Theorem.

Remark 2.2. Let a, b, c, d be any 4 points of $\mathbb{C}$. In [1] Apostol observes that applying the triangle inequality to (3) gives a quick proof of Ptolemy's inequality.

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3. Proof of the converse theorem

Given 4 points A,B,C,D in the plane satisfying Ptolemy's condition

$$AB\cdot CD+AD\cdot BC=AC\cdot BD \tag{6}$$

we want to show that the points lie on a circle or straight line. Note that the condition depends on the ordering of the points. We can avoid this nuisance by using the following easily verified identity.

$$(p+q+r)(-p+q+r)(p-q+r)(p+q-r)=-p^4-q^4-r^4+2p^2q^2+2p^2r^2+2q^2r^2 \tag{7}$$

Let F denote either side of (7) with $p=AB\cdot CD$, $q=AD\cdot BC$ and $r=AC\cdot BD$. Then $F=0$ if and only if the points in some order satisfy Ptolemy's condition.

As above we identify the plane with $\mathbb{C}$. If every set of 3 points out of A,B,C,D lies on a line, then all 4 points lie on a line and we are done. Therefore we can assume that A, B, C lie on a circle which we can assume is the unit circle. To avoid confusing $AB=|A-B|$ with the product AB I will write a,b,c for A,B,C considered as complex numbers, and write z for D. As usual we write $z=x+iy$ where x and y are real. We fix a, b, and c, and let z vary.

Ptolemy's condition now becomes

$$|b-a||z-c|+|z-a||c-b|=|c-a||z-b| \tag{8}$$

As above we let $p=|b-a||z-c|$, $q=|z-a||c-b|$, $r=|c-a||z-b|$ and let F be the expression in (7). We write $F(z)$ or $F(x,y)$ to refer to the dependence on z.

Ptolemy's theorem implies that $F(z) = 0$ if $|z| = 1$. Our aim is to show conversely that $F(z) = 0$ implies $|z| = 1$ so that z lies on the circle.

Now if $a = a_1 + ia_2$, then $|z-a|^2 = (x-a_1)^2 + (y-a_2)^2$ which is a polynomial of degree 2 in x and y. Similar arguments on the right hand terms of (7) now show that $F(x, y)$ is a polynomial of degree 4 in x and y.

Lemma 3.1. *Let $P(x, y)$ be a polynomial over $\mathbb{C}$ which vanishes when $x^2 + y^2 = 1$ with real x and y. Then $x^2 + y^2 - 1$ divides P.*

Proof. Regard $g = x^2 + y^2 - 1$ as a monic polynomial in y and divide getting $P = gh + r$ where the remainder r has degree 1 in y so $r = h(x)y + k(x)$. If $-1 < x < 1$ there are 2 values of y for which $g(x, y) = 0$. Since $r = 0$ for these 2 values of y, h and k must be 0 for each x with $-1 < x < 1$ so h and k are 0 as polynomials. □

This shows that we have $F(x, y) = (x^2 + y^2 - 1)G(x, y)$ where G is a polynomial in x and y of degree 2.

$$F(x, y) = (x^2 + y^2 - 1)G(x, y) \tag{9}$$

Lemma 3.2. *Let $h(z) = |z - a|$ with $a, z \in \mathbb{C}^*$ and $|a| = 1$. Then $h(z) = |z|h(\frac{1}{\bar{z}})$*

Proof. Using $\bar{a} = a^{-1}$ we get $|z - a| = |z||a||\frac{1}{a} - \frac{1}{z}| = |z||\bar{a} - \frac{1}{z}| = |z||a - \frac{1}{\bar{z}}|$ □

Applying this to the terms of F we see that

$$F(z) = |z|^4 F(\frac{1}{\bar{z}}) \tag{10}$$

We claim that G vanishes on the unit circle. Suppose $G(w) \neq 0$ where $|w| = 1$. Choose z very close to w with $|z| < 1$. Then $\frac{1}{\bar{z}}$ is very close to $\frac{1}{\bar{w}}$ and G is non–zero on the line joining z to $\frac{1}{\bar{z}}$ and so has the same sign at these points. The same is true of F by (10) and therefore also for $x^2 + y^2 - 1 = |z|^2 - 1$ by (9). This contradiction show that our assumption was incorrect and so G must vanish on the unit circle. By Lemma 3.1 $x^2 + y^2 - 1$ divides G so, by degrees, $G = C(x^2 + y^2 - 1) = C(|z|^2 - 1)$ where C is a constant and $F(z) = C(|z|^2 - 1)^2$ showing that $F(z) = 0$ implies $|z| = 1$.

References

[1] T. M. Apostol, Ptolemy's inequality and the chordal metric, Math. Mag 40(1967), 233–235.
[2] Wikipedia entry for Ptolemy's Theorem.

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