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The chromatic number of infinite graphs — A survey

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ABSTRACT

We survey some old and new results on the chromatic number of infinite graphs.

1. Coloring number

The coloring number of a graph X, $\mathrm{Col}(X)$, is the smallest cardinal μ such that V, the set of vertices of X, has a well-ordering $<$ such that each vertex is joined into less than μ smaller (by $<$) vertices. This notion was introduced by Erdős and Hajnal in [2]. If $\mathrm{Col}(X) = \kappa$ and $<$ is a well-ordering witnessing it, then a straightforward transfinite recursion via $<$ gives a good coloring of X with κ colors; at each step we can continue the coloring. This establishes $\mathrm{Chr}(X) \leq \mathrm{Col}(X)$ for any graph X. Equality may not hold even for very simple graphs as $\mathrm{Col}(K_{\kappa,\kappa}) = \kappa$ and $K_{\kappa,\lambda} = \kappa^+$ for $\kappa < \lambda$. Here $K_{\kappa,\lambda}$ denotes the complete bipartite graph with bipartition classes of size κ, λ, respectively.

What makes the notion of coloring number particularly useful is Shelah's Singular Cardinals Compactness Theorem: if $\lambda > \mu$, λ is singular, X is a graph such that $\mathrm{Col}(Y) \leq \mu$ holds for every subgraph Y of X with $|Y| < \lambda$, then $\mathrm{Col}(X) \leq \mu$ holds, as well [22]. This makes it possible to calculate the coloring number using stationary sets. (For the definition and basic properties of stationary sets see any textbook in set theory, e.g. [9].)

Erdős and Hajnal proved that if $\mathrm{Col}(X) > \aleph_0$ then X contains the bipartite subgraphs $K_{n,\aleph_1}$ for $n < \omega$ [2]. Hajnal later showed that the following 'half graph' also occurs in every graph with uncountable coloring number: the bipartite graph is on the countable classes $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$, and a_i is joined with b_j iff $i < j$. This is very close to the largest graph with the given property. In [12] we determined the largest countable graph Γ and the largest graph Δ (which is of cardinality $\aleph_1$) such that Γ, Δ both occur as subgraphs in every graph X with $\mathrm{Col}(X) > \omega$.

Rado asked if the de Bruijn–Erdős phenomenon (see below) holds for the coloring number as well. Answering this in the negative, Erdős and Hajnal proved the following in [2]: if $1 \leq k < \omega$, every finite subgraph Y of X has $\mathrm{Col}(Y) \leq k$, then $\mathrm{Col}(X) \leq 2k - 2$, and this is sharp.

In connection with this, E.C. Milner asked if the following holds. If k is finite, $\mathrm{Col}(X) = k + 1$ then there is a subgraph Y of X with $\mathrm{Col}(Y) = k$. This was proved in [14].

2. Chromatic number

The chromatic number of infinite graphs is defined exactly as in the finite case: the chromatic number of X, $\mathrm{Chr}(X)$, is the least number of colors required in a good coloring of the graph X. Notice that this definition uses that cardinals are well ordered, which is equivalent to the axiom of choice. Galvin and Komjáth proved in [7], that AC is actually equivalent to the fact that every graph has a chromatic number. They also showed that AC is equivalent to the existence of an *irreducible* good coloring of any graph, that is, a good coloring, where any two color classes are joined by an edge.

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An old result on the chromatic number of infinite graphs is the de Bruijn–Erdős theorem: if n is a natural number and every finite subgraph of some graph has chromatic number at most n, then so has the full graph [1]. Here the use of the axiom of choice is again essential. As Shelah observed, there are symmetric models with graphs of chromatic number n, all of whose finite subgraphs have chromatic number at most k, for any $2 \le k < n < \aleph_0$.

It is easy to see that if every connected component of a graph X can be colored with countably many colors then so can X. In other words, if $\mathrm{Chr}(X) > \omega$ then X contains a connected subgraph Y with $\mathrm{Chr}(Y) > \omega$. This was extended in [11] where it was shown that if $\mathrm{Chr}(X) > \omega$, then X has an n-connected subgraph Y with $\mathrm{Chr}(Y) > \omega$, for any finite n. With a considerably more complicated proof one can even find an n-connected subgraph in which all vertices have infinite degree. The conjecture of Erdős and Hajnal that every uncountably chromatic graph has an ω-connected subgraph remains open. It is consistent that every graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ has an ω-connected subgraph Y of X with $\mathrm{Chr}(Y) = \aleph_1$ [13]. On the other hand, it is consistent that there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ such that no subgraph Y with $|Y| = \aleph_1$ is ω-connected [16].

Galvin asked if the chromatic number of subgraphs has the Intermediate Value Property in the following way. Is it true that if X is a graph with $\kappa < \lambda = \mathrm{Chr}(X)$ then there exists a subgraph Y of X with $\mathrm{Chr}(Y) = \kappa$? Galvin himself solved the problem in a restricted case, he proved the following statement in [6]. If $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2}$ then there exists a graph X such that $\mathrm{Chr}(X) \ge \aleph_2$ and X has no induced subgraph of exact chromatic number $\aleph_1$. We solved the original problem in [13] where the consistency of the following statement is proved: there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_2$ which does not have a subgraph Y (induced or not) with $\mathrm{Chr}(Y) = \aleph_1$. In the positive direction Shelah [22] proved that if the axiom of constructibility holds, X is a graph with $|X| < \kappa^{+\kappa}$ and $\mathrm{Chr}(X) > \kappa$ then there is an induced subgraph Y of X with $\mathrm{Chr}(Y) = \kappa$. Also, still under the axiom of constructibility, if X is a graph with $|X| = \mathrm{Chr}(X) = \aleph_2$, then there is a subgraph Y such that $\mathrm{Chr}(Y) = \aleph_1$.

Notice that for $\kappa \le \aleph_0$ the de Bruijn–Erdős theorem gives a positive answer to Galvin's question. We can, therefore, ignore these cardinals. In [15] we define and investigate the following two invariants of infinite graphs. If X is an uncountably chromatic graph then $S(X)$ consists of all uncountable cardinals that occur as the chromatic number of subgraphs. We define $I(X)$ similarly, but with induced subgraphs. In [15] we show that $I(X)$ is a closed set of ordinals and every singular cardinal $\lambda \in I(X)$ is a limit point of $I(X)$. Further, if a set A consisting of uncountable cardinals has these two properties, then a ccc forcing adds a graph X with $I(X) = A$. For $S(X)$ we only have the second of the above two properties. Closure does hold at singular cardinals, but not necessarily at regular ones; the existence of a measurable cardinal is equiconsistent to the existence of a graph X and a regular cardinal κ such that κ is a limit point of $S(X)$ but $\kappa \notin S(X)$.

3. Finite subgraphs of uncountably chromatic graphs

Extending a finite graph theorem Erdős and Rado showed in [4] that for any cardinal κ there is a triangle-free graph X with $\mathrm{Chr}(X) > \kappa$. As the corresponding finite result holds according to Erdős, it was natural to conjecture that there exist arbitrarily large chromatic graphs omitting $C_3, C_4, \ldots, C_s$, i.e., all circuits up to a given length. Much to their surprise, Erdős and Hajnal proved that uncountably chromatic graphs must contain all finite bipartite graphs (see above, in the section on coloring number). After this, it was easy to find the right form of the result; if $1 \le s < \omega$, κ is an infinite cardinal, then there is a graph X omitting the odd circuits $C_3, C_5, \ldots, C_{2s+1}$ and $\mathrm{Chr}(X) > \kappa$. Erdős and Hajnal even introduced two canonical examples: the shift graph, and the Specker graph. The shift graph $\mathrm{Sh}_n(\lambda)$ consists of the n-element subsets of some cardinal λ as vertex set, with $\{x_1, \ldots, x_n\}_<$ joined to $\{y_1, \ldots, y_n\}_<$ if $x_2 = y_1, x_3 = y_2, \ldots, x_n = y_{n-1}$, (or vice versa). A simple calculation gives that this graph omits $C_3, \ldots, C_{2n-1}$. Using the Erdős–Rado theorem from partition calculus one can show that $\mathrm{Chr}(\mathrm{Sh}_n(\lambda)) > \kappa$ if λ is sufficiently large. The Specker graphs are similarly defined but with inequalities rather than equalities between the elements of n-tuples of some cardinal.

These results settle the question which finite graphs can be omitted from uncountably chromatic graphs: they are exactly the non-bipartite graphs. The next step would be the investigation of countable subgraphs. Not much is known in that direction. As we mentioned above, Hajnal proved that if $\mathrm{Chr}(X) > \aleph_0$ then X contains the half graph. He also proved that uncountable chromatic graphs exist which do not contain the complete bipartite graph $K_{\aleph_0,\aleph_0}$ [8].

Erdős and Hajnal then suggested to investigate the families of finite subgraphs occurring in uncountably chromatic graphs. They conjectured, and it was proved by Erdős et al. [3] and Thomassen [24], independently, that if $\mathrm{Chr}(X) > \aleph_0$, then X contains all sufficiently large odd circuits. This implies that any two uncountably chromatic graphs have a common 3-chromatic subgraph. It is not known if this can be extended to a common 4-chromatic subgraph.

It seems to be a hard problem to describe those families of finite graphs that must occur in uncountably chromatic graphs. In a clever way to circumvent this problem, Erdős and Hajnal introduced the following function. If X is a graph, let $f_X(n)$ be the largest chromatic number of its n-vertex subgraphs. Clearly, f_X is weakly increasing, and by the de Bruijn–Erdős theorem, converges to infinity. Can its growth be arbitrarily slow? That is, is the following true? For every monotone function $g(n) \to \infty$, there exists a graph X with $\mathrm{Chr}(X) > \aleph_0$ and with $f_X(n) \le g(n)$, for n sufficiently large. The consistency of this statement was established by Shelah (see [20]).

Another conjecture on uncountably chromatic graphs was Taylor's conjecture [23,3] claiming that if X has $\mathrm{Chr}(X) > \aleph_0$ and κ is a cardinal then there is a graph Y with $\mathrm{Chr}(Y) > \kappa$ and containing the same finite subgraphs as X. In [3] the following much stronger conjecture was formulated. If $\mathrm{Chr}(X) > \aleph_0$ then it contains all finite subsets of $\mathrm{Sh}_n(\omega)$ for some n. This conjecture, however, was disproved in [10]. In [19] Shelah and the author proved what can be considered the simplest case of Taylor's conjecture. There are countably many classes $\mathcal{K}_0, \mathcal{K}_1, \ldots$ of finite graphs with the following properties: if λ is a cardinal with $\lambda^{\aleph_0} = \lambda$ and X is a graph with $|X| = \mathrm{Chr}(X) = \lambda^+$ then X contains all graphs in $\mathcal{K}_n$, for some n. On the other hand, if κ is a regular cardinal, then there is a cardinal preserving forcing which adds a graph X with $|X| = \mathrm{Chr}(X) = \kappa^+$, all of whose finite subgraphs are in $\mathcal{K}_n$.

Despite this positive result, Taylor's conjecture is, at least consistently, false. Namely, building upon Shelah's above result on the function f_X, I proved the consistency of the following statement [20]. There is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ such that for every graph Y which has the same finite subgraphs as X we have $\mathrm{Chr}(Y) \le \aleph_2$. On the other hand, it is consistent that if X is a graph with $\mathrm{Chr}(X) \ge \aleph_2$ and κ is an arbitrary cardinal, then there is a graph Y with $\mathrm{Chr}(Y) \ge \kappa$ all of whose finite subgraphs are subgraphs of X. These result suggest that some weak form of Taylor's conjecture may hold outright.

Erdős asked if it is true that every uncountably chromatic graph contains an uncountably chromatic triangle-free subgraph. A few minutes after he had heard the problem, Shelah disproved this. He constructed a model of set theory in which a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ exists, such that each triangle-free subgraph of X is countably chromatic. After this, we gave a model in which an X as above exists which does not even contain a K_4 [18]. Clearly, this graph has the property that if n is finite and the edges are colored with n colors, then there is a monocolored triangle. Compactness arguments (i.e., Gödel's compactness theorem) give the existence of a finite such graph, a famous result of Folkman [5], Nesetril, and Rödl [21]. Our proof is not, however, an independent proof for that theorem, as it uses it.

4. List-chromatic number

We borrow the notion of list-chromatic number from finite graph theory. The list-chromatic number, $\mathrm{List}(X)$, of a graph X, is the least cardinal κ such that if a set $F(v)$ of cardinality κ is assigned with every vertex v, then there is a choice function $f(v) \in F(v)$ which is a good coloring. That is, $f(v) \neq f(w)$ whenever v, w are joined in X.

There is also a modified form of the list-chromatic number: $\mathrm{List}^*(X)$ is the smallest cardinal κ such that whenever $F(v) \in [\kappa]^\kappa$, then there is a choice function $f(v) \in F(v)$ which is a good coloring.

Notice the following chain of inequalities:

$$\mathrm{Chr}(X) \le \mathrm{List}^*(X) \le \mathrm{List}(X) \le \mathrm{Col}(X) \le |X|.$$

In [17] we show that this is the most that can be said; if we restrict to graphs of cardinality $\aleph_1$, in different models of set theory, both statements $\mathrm{List}(X) = \aleph_1$ iff $\mathrm{Chr}(X) = \aleph_1$ and $\mathrm{List}(X) = \aleph_1$ iff $\mathrm{Col}(X) = \aleph_1$ may hold. Further, $\mathrm{List}(X)$ and $\mathrm{List}^*(X)$ may differ; it is consistent that the Generalized Continuum Hypothesis holds and there is a graph X with $|X| = \aleph_1$ and $\mathrm{List}^*(X) = \aleph_0$, $\mathrm{List}(X) = \aleph_1$.

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