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# Crocheting the Lorenz manifold

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You have probably seen a picture of the famous butterfly-shaped Lorenz attractor — on a book cover, a conference poster, a coffee mug or a friend’s T-shirt. The Lorenz attractor is the best known image of a chaotic or strange attractor. We are concerned here with its close cousin, the two-dimensional stable manifold of the origin of the Lorenz system, which we call the *Lorenz manifold* for short. This surface organizes the dynamics in the three-dimensional phase space of the Lorenz system. It is invariant under the flow (meaning that trajectories cannot cross it) and essentially determines how trajectories visit the two wings of the Lorenz attractor.

We have been working for quite a while on the development of algorithms to compute global manifolds in vector fields and have computed the Lorenz manifold up to considerable size. Its geometry is very intriguing and we explored different ways of visualizing it on the computer [6, 9]. However, a real model of this surface was still lacking.

During the Christmas break 2002/2003 Hinke was relaxing by crocheting hexagonal lace motifs when Bernd suggested: “Why don’t you crochet something useful?” The algorithm we developed ‘grows’ a manifold in steps. We start from a small disc in the stable eigenspace of the origin and add at each step a band of a fixed width. In other words, at any time of the calculation the computed part of the Lorenz manifold is a topological disc whose outer rim is (approximately) a level set of the geodesic distance from the origin. What we realized then and there is that the mesh generated by our algorithm can directly be interpreted as crochet instructions!

After some initial experimentation, the first model of the Lorenz manifold was

crocheted by Hinke and then mounted by Bernd with garden wire. It was shown for the first time at the 6th SIAM Conference on Applications of Dynamical Systems in Snowbird, Utah in May 2003 and made a second public appearance at the Equadiff 2003 conference in Hasselt, Belgium, in July 2003 [7]. The model is quite large, about 0.9 m in diameter, and has to be ‘flattened’ and folded for transportation.

In this article we explain the mathematics behind the crocheted Lorenz manifold and provide complete instructions that allow you to crochet your own. The images shown here are of a second model that was crocheted in the Summer of 2003. We took photos at different stages, and it was finally mounted with great care and then photographed professionally. This second model stays mounted permanently, while we use the first model for ‘touring’.

We would be thrilled to hear from anybody who produces another crocheted model of the Lorenz manifold. As an incentive we offer a bottle of champagne to the person who produces model number three. So do get in touch when you are done with the needle work!

## The Lorenz system

The Lorenz attractor illustrates the chaotic nature of the equations that were derived and studied by the meteorologist E.N. Lorenz in 1963 as a much simplified model for the dynamics of the weather [8]. Now generally referred to as the Lorenz system, it is given as the three ordinary differential equations:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - \beta z. \end{cases} \quad (1)$$

We consider here only the classic choice of parameters, namely  $\sigma = 10$ ,  $\rho = 28$ , and  $\beta = 2\frac{2}{3}$ . The Lorenz system has the symmetry

$$(x, y, z) \mapsto (-x, -y, z), \quad (2)$$

that is, rotation by  $\pi$  about the  $z$ -axis, which is invariant under the flow of (1).

A simple numerical simulation of the Lorenz system (1) on your computer, starting from almost any initial condition, will quickly produce an image of the Lorenz attractor. However, if you pick two points arbitrarily close to each other, they will move apart after only a short period of time, resulting in two very different time series. This was accidentally discovered by Lorenz when he restarted a computation from printed data rounded to three decimal digits of accuracy, while his computer internally used six decimal digits; see, for example, the book by Gleick [1].

While the Lorenz system has been widely accepted as a classic example of a chaotic system, it was proven by Tucker only in 1998 [12] that the Lorenz attractor is actually a chaotic attractor. For an account of the mathematics involved see the *Intelligencer* article by Viana [13].

## Stable and unstable manifolds

The origin is always an equilibrium of (1). The eigenvalues of the linearization at the origin are

$$-\beta \text{ and } -\frac{\sigma+1}{2} \pm \frac{1}{2}\sqrt{(\sigma+1)^2 + 4\sigma(\rho-1)}.$$

For the standard parameter values they are numerically

$$-22.828, -2.667, \text{ and } 11.828$$

in increasing order. This means that the origin is a saddle with two attracting and one repelling direction. According to the Stable Manifold Theorem [2, 11], there exists a one-dimensional unstable manifold  $W^u(0)$  and a two-dimensional stable manifold  $W^s(0)$ , defined as

$$\begin{aligned} W^u(0) &= \{x \in \mathbb{R}^3 \mid \lim_{t \rightarrow -\infty} \phi^t(x) = 0\}, \\ W^s(0) &= \{x \in \mathbb{R}^3 \mid \lim_{t \rightarrow \infty} \phi^t(x) = 0\}, \end{aligned}$$

where  $\phi$  is the flow of (1). The manifolds  $W^u(0)$  and  $W^s(0)$  are tangent at the origin to the unstable and stable eigenspace, respectively. We call  $W^s(0)$  simply the

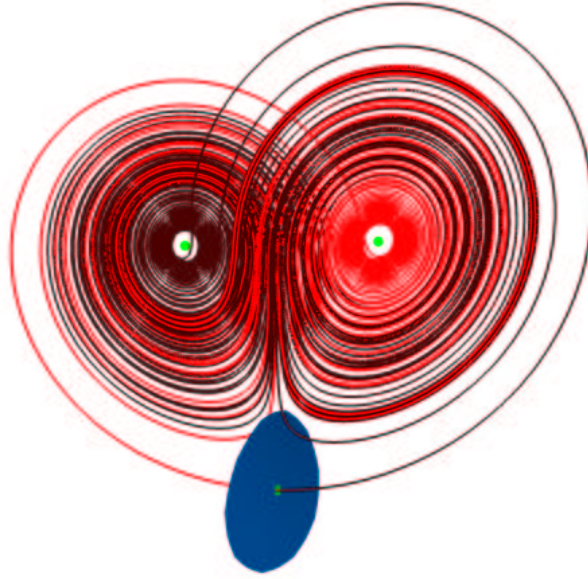


Figure 1: The two branches of the unstable manifold, one red and one brown, accumulate on the Lorenz attractor. The little blue disc is in the stable eigenspace and separates the two branches.

*Lorenz manifold.* While most trajectories end up at the Lorenz attractor, those on  $W^s(0)$  converge to the origin instead.

The  $z$ -axis, the axis of symmetry, is part of the Lorenz manifold  $W^s(0)$ , which is itself invariant under rotation by  $\pi$  around this axis. Furthermore, there are two special trajectories that are tangent to the eigenvector of  $-22.828$ , which is perpendicular to the  $z$ -axis. They form the two branches of the one-dimensional *strong stable manifold*  $W^{ss}(0)$ . All other trajectories on  $W^s(0)$  are tangent to the  $z$ -axis.

Apart from the origin, the Lorenz system (1) has two other equilibria, namely

$$(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1) \approx (\pm 8.485, \pm 8.485, 27),$$

which are also saddles. They sit in the centres of the ‘wings’ of the Lorenz attractor and are each other’s image under the symmetry (2).

Figure 1 shows an image of the Lorenz attractor that was not obtained by

simply integrating from an arbitrary starting condition, but by computing the one-dimensional unstable manifold  $W^u(0)$ . Since the origin is in the Lorenz attractor, plotting  $W^u(0)$  gives a good picture of the attractor. We computed both branches, one in red and one in brown, of the unstable manifold of the origin by integration from two points on either side of  $W^s(0)$  at distance  $10^{-7}$  away from the origin along the unstable eigendirection.

It is clear from Figure 1 that each of the branches of the unstable manifold visits both wings of the attractor, as is to be expected. In fact, due to the symmetry of equations (1), the red branch is the symmetric image of the brown branch. Locally near the origin, each branch starts on a different side of the two-dimensional stable manifold  $W^s(0)$ . In Figure 1 we show a small local piece of  $W^s(0)$  as a small blue disc.

The main question is: what does the global Lorenz manifold  $W^s(0)$  look like, as it ‘wiggles’ between the red and brown curves of Figure 1? Remember that  $W^u(0)$  cannot cross  $W^s(0)$ ...

## Geodesic level sets

The Lorenz manifold, like any global two-dimensional invariant manifold of a vector field, cannot be found analytically but must be computed numerically. The knowledge of global stable and unstable manifolds of equilibria and periodic orbits is important for understanding the overall dynamics of a dynamical system, which we take here to be given by a finite number of ordinary differential equations. In fact, there has been quite some work since the early 1990’s on the development of algorithms for the computation of global manifolds. We do not give details here but refer to [5] for a recent overview over the literature. The key idea of several of these methods is to start with a uniform mesh on a small circle around the origin in the stable eigenspace and then use the dynamics to ‘grow’ this circle further away. The main problem one needs to deal with is that the flow does not evolve the initial

circle uniformly, so that the mesh quality generally deteriorates very quickly.

The goal of our algorithm is to compute ‘nice circles’ on the Lorenz manifold to obtain a uniform mesh. Nice circles on the manifold are those that consist of points that lie at (approximately) the same distance away from the origin. In other words, we want to evolve or grow the initial circle radially outward (away from the origin) and with the same ‘speed’ everywhere. To formalize this, we consider the geodesic distance between two points on the manifold, which is defined as the length of the shortest path on the manifold connecting the two points. The geometrically ‘nicest’ circle is then a geodesic level set, which is a smooth closed curve whose points all lie at the same geodesic distance from the origin.

The algorithm that we developed computes a manifold as a sequence of approximate geodesic level sets; see [4, 5] for the details. We start from a small disc in the stable eigenspace of the origin which we represent by a circular list of equidistant points around its boundary. This is our first approximate geodesic level set. The algorithm then adds at each step a new approximate geodesic level set, again given as a circular list of points. To this end we compute for every known point on the present geodesic level the closest point that lies on the new geodesic level set. (This can be achieved by solving a boundary value problem.) When the distance between neighbouring points on the new level set becomes too large, we add a new point between them by starting from a point on the present level set. Similarly, we remove a point when two neighbouring points become too close. In this way, we ensure that the distribution of mesh points along the new level set is close to uniform. At the end of a step we add an entire band of a particular fixed width to the manifold. The width of the band that is added depends on the (local) curvature of geodesics on  $W^s(0)$ .

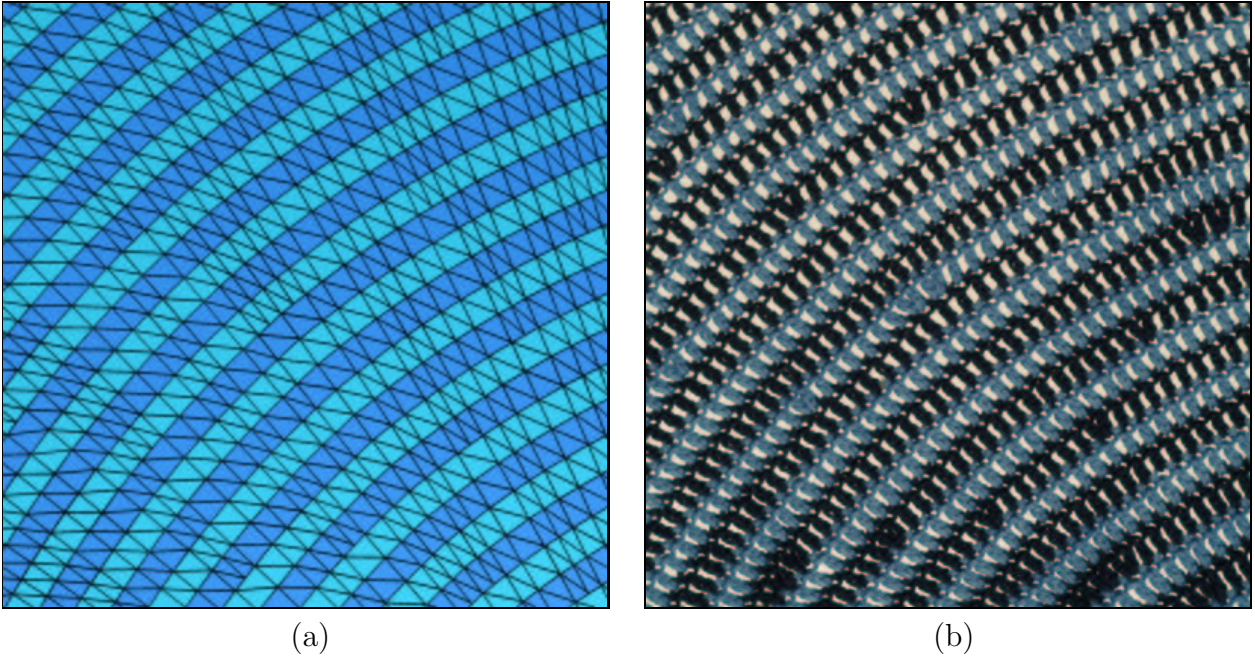


Figure 2: A close-up of the mesh generated by our algorithm, showing bands of alternating colour and the edges of the triangulation (a), and (practically) the same close up of the crocheted manifold (b). New crochet stitches are added exactly where new mesh points are added.

### Global information encoded locally

We used our algorithm to compute the Lorenz manifold up to considerable size, where we made use of the parameterization in terms of geodesic distance; illustrations and accompanying movies of how the Lorenz manifold is grown were published in [6, 9] and are not repeated here. Instead we show a crocheted model of the Lorenz manifold; see already Figures 4–6. The key observation is that, while the algorithm computes each new mesh point as a point in  $\mathbb{R}^3$ , the essential information on the shape of the manifold is actually encoded locally!

This is illustrated in Figure 2 (a), which shows an enlargement of a part of the Lorenz manifold with the triangular mesh that was computed. Consecutive bands



are shown alternatingly in light and darker blue; the mesh points in the bottom right corner are closest to the origin. The mesh is formed from the mesh points on the geodesic level sets. The diagonal mesh lines from bottom right to top left are approximations of geodesics; they are perpendicular to the level sets. Whenever two such geodesics move too far apart, a new one is started between them where a new point is added.

The image is from a part of the manifold that is almost flat. Because the circumference of a planar disc is linearly related to its diameter, the number of new points being added to the level sets depends linearly on the geodesic distance covered. If the level sets are all at the same distance from each other, as in Figure 2, then a fixed number of new points is added at each step. If the manifold is curved, on the other hand, then the number of points added during the steps varies with the geodesic distance covered. For positive local curvature fewer points are being added, while for negative local curvature more points are being added.

The crucial point is that the curvature of the manifold is given locally on the level of the computed mesh simply by the information where we added or removed points during the computation.

## **Interpretation as crochet instructions**

This observation allows us to interpret the result of a computation by our algorithm directly as a crochet pattern. Starting from a small crocheted circle, each new band is created by making one or more crochet stitches of a fixed length (translated from the width of the respective band) in each stitch of the previous round. Extra stitches are added or removed where points were added or removed during the computation; this information was written to a file.

Figure 2 (b) shows practically the same part of the crocheted Lorenz manifold that is shown in Figure 2 (a). You are encouraged to look closely for the points in the crocheted manifold where an extra crochet stitch was added and identify them

Abbreviation	British name	American name
ch	chain stitch	chain stitch
dc	double crochet	single crochet
tr	treble crochet	double crochet
dtr	double treble crochet	treble crochet

Table 1: Abbreviations of the crochet stitches used for the Lorenz manifold.

in the original mesh.

To preserve the geometry of the manifold one needs to ensure that the horizontal width of the used stitch and its length are in the same ratio as the average distance between mesh points on a level set and the width of the respective band. The crocheting reader will be relieved to hear that these considerations were translated into the crochet instructions below — simply following them slavishly will give a good result.

We assume that the reader is familiar with the basic crochet stitches, as they can be found in any book on chrocheting. Throughout we use the British naming convention of stitches, which differs from the American one; the definitions and abbreviations used are summarized in Table 1. The Lorenz manifold is crocheted in rounds. Stitches in each round are counted with respect to the previous round, starting from the number 0.

The first stitch of a round is 1 ch, 3 ch, or 4 ch, depending on whether the round is done in dc, tr, or dtr, respectively. Each round is closed with a slip stich in the last ch of the first stitch. From one round to the next the colour alternates between light and dark blue, which helps identify the different bands in the finished model. We found that the end result is much better if the threads are cut after each round, rather than carrying strands up the rounds.

## Getting started

To help with the interpretation of the instructions, we explain in more detail how to get started. We used a 2.50 mm crochet hook with 4-ply mercerized cotton yarn. Note that the crochet hook is slightly smaller than recommended for the weight of the yarn; this is done to obtain a tighter gauge. The finished model Lorenz manifold up to geodesic distance 110.75 is then about 0.9 m in diameter, and required four 100 g balls of yarn. Obviously, using a thicker crochet hook and yarn will lead to an even bigger manifold. The complete crochet instructions are below; here we explain briefly how to read the compact crochet notation.

Begin (**rnd1**) with a foundation chain in light blue of 5 ch stitches that are closed with a slip stitch to form a ring. The first round consists of 10 dc. This means that one starts with 1 ch followed by 9 dc in the loop, after which the ring is closed with a slip stitch. The small disc obtained so far represents the Lorenz manifold up to geodesic distance (*gd*) 2.75. The next round (**rnd2**) is done in dark blue using a treble crochet stitch. The total number of stitches doubles to 20 in this round, which means that 2 tr stitches are made in each dc. (Recall that the 10 dc are numbered from 0 to 9.) The crocheted disc has now grown to represent the Lorenz manifold up to *gd* 4.75. In the next round (**rnd3**) the geodesic distance grows to *gd* 8.75 with dtr crochet stitches. As in **rnd2**, there are 2 dtr in each tr. Starting from **rnd4**, crochet stitches are no longer doubled at each previous stitch.

Notice that 20 new crochet stitches are added in each round from **rnd2** to **rnd7**; then the number of stitches starts to vary from round to round, but essentially remains constant when counted over two consecutive rounds. This means that roughly up to **rnd10** of *gd* 36.75 the Lorenz manifold is a flat disc, allowing the algorithm to take large steps, which is translated to dtr crochet stitches. From *gd* 36.75 onward, all rounds are worked in tr crochet stitches.

In **rnd37**, that is, at *gd* 90.75 stitches are deleted for the first time. The notation -515 means that the treble crochet stitch at position 515 merges with the one at

position 514. This is done as follows: treble crochet stitch 514 is not finished completely, namely one does not bring the yarn around the hook and pull it through the last two loops on the hook. Similarly, treble crochet stitch 515 is then crocheted except for this last step. The two stitches are crocheted together by pulling a loop of yarn through all three loops at once. Note that -515 is followed by 515 so that a second treble crochet stitch is made in position 515, which effectively undoes the deletion of the stitch. This corresponds to an *adjustment* of the mesh points by the algorithm, and we kept the instructions to be faithful to the computed mesh. In later rounds, for example, in **rnd39**, crochet stitches are deleted without being recreated again.

## Comparison with crocheting the hyperbolic plane

The idea to crochet a model of the Lorenz manifold was born quite suddenly in December 2002, but was indirectly influenced by our knowledge of the Intelligencer article “Crocheting the hyperbolic plane” by Henderson and Taimina [3]. Indeed when their article came out in 2001 we already had developed our algorithm for manifold computations, but somehow the idea of crocheting did not click. As soon as we decided to crochet a mathematical object ourselves, we had of course another look at this paper.

Their idea is to crochet a model of hyperbolic space by starting from a row (or a round) of a fixed number of chain stitches and then adding rows (rounds), all of the same basic crochet stitch. The trick is to add one extra crochet stitch every  $N$  stitches. In other words, the number of stitches increases per row (round) and this leads to negative local curvature as was explained earlier. The smaller  $N$ , the more extra crochet stitches are added and the larger the negative curvature of the resulting object. This curvature is constant as the procedure is repeated the same everywhere.

From a crocheting point of view, crocheting a model of hyperbolic space is quite

simple as it involves the same crochet stitch and counting to  $N$ . An expert needle worker will be able to do this ‘on the side’ while having a nice conversation or watching TV. Crocheting the Lorenz manifold, on the other hand, requires continuous attention to the instructions in order not to miss when to add or indeed remove an extra crochet stitch. This involves much counting and checking of each round. In fact, Hinke crocheted the Lorenz manifold in the course of two months in an estimated 85 hours, which corresponds to about 300 stitches per hour for the total of 25,511 stitches. (To translate this time estimate to your own crocheting skills, be warned that Hinke is an expert at crochet and counting alike!)

### **A shapeless crocheted topological disc**

Initially, up to a geodesic distance of about 36.75, the Lorenz manifold is virtually flat as a pancake. It then starts picking up a lot of negative curvature near the positive  $z$ -axis, around which it spirals. The lower part of the manifold with  $z < 0$  has almost zero curvature. It is impossible to flatten out the crocheted manifold on a table, as the region of strong negative curvature forms more and more folds.

Figure 3 shows the crocheted manifold at three different stages of progress and flattened out as much as possible. The images on the left show the manifold as a simply disc; the  $z$ -axis corresponds to the vertical line through the center of each panel. In the images on the right the crocheted manifold has been folded double along the  $z$ -axis. To ‘absorb’ some of the curvature the  $z$ -axis is then no longer a straight line in the upper part of the images, but even this is not enough to avoid the increasing (with diameter) rippling of the object.

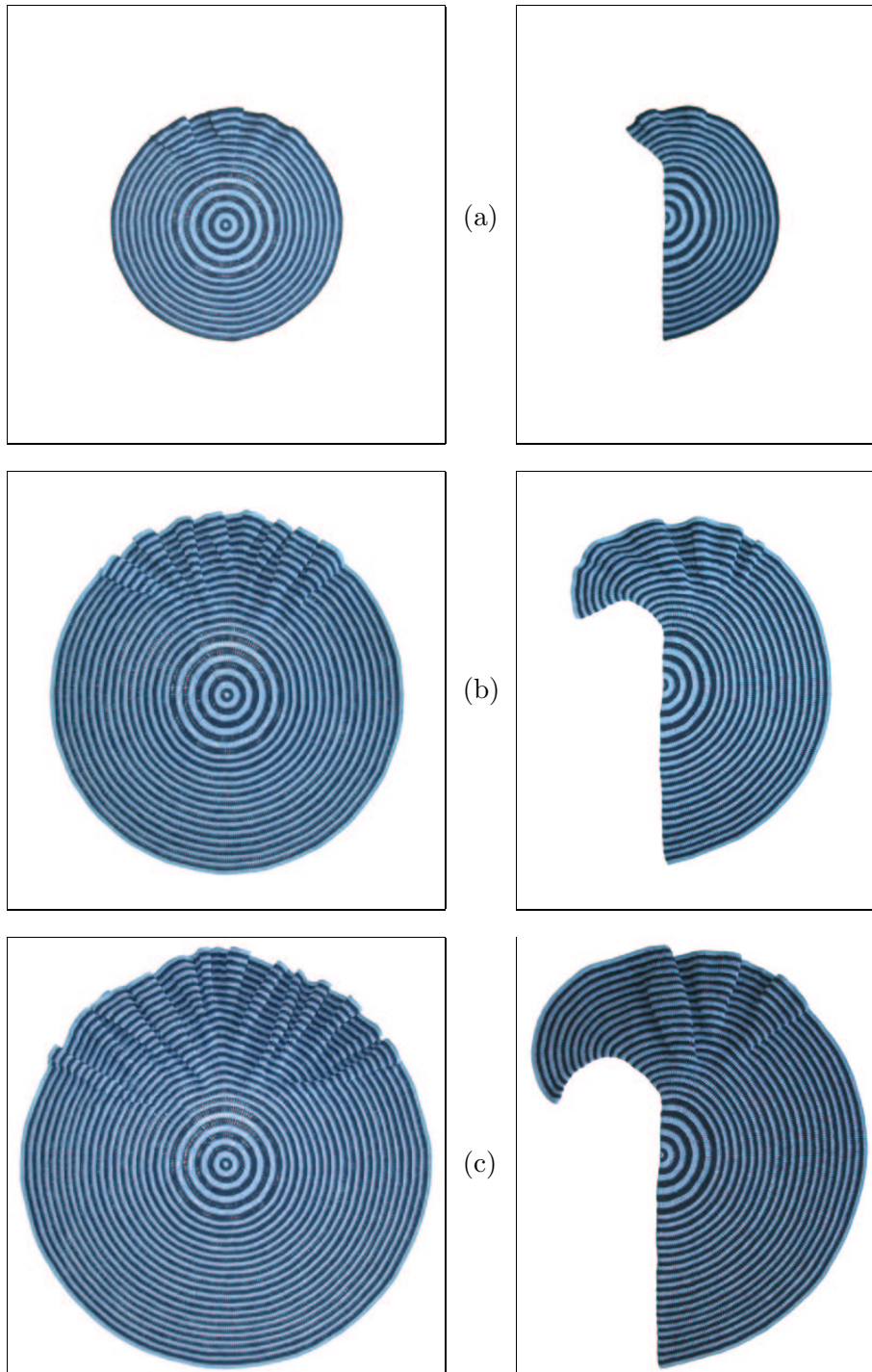


Figure 3: The Lorenz manifold in the process of being crocheted, shown ‘as flat as possible’ (left column) and doubled-up along the line of symmetry (right column); from (a) to (c) the manifold is shown up to **rnd26** (*gd 68.75*), up to **rnd39** (*gd 94.75*), and up to **rnd47** (*gd 110.75*). Where the manifold is rippled, the curvature is most negative.

## Mounting the crocheted Lorenz manifold

When we first saw the crocheted but yet unmounted Lorenz manifold shown in Figure 3 (c) we had some doubts whether we could get it into the required final shape. However, as was explained above, the crocheted manifold ‘knows’ its shape in three-space because of the locally encoded curvature information. When mounting the manifold, it (almost) automatically falls into its proper shape. To achieve this we found that only three ingredients are required:

1. fixing the  $z$ -axis with an unbendable rod;
2. supporting the outer rim with a bendable wire of the correct length;
3. Supporting the manifold in the radial direction with a single bendable wire that runs from rim to rim and through the origin.

For the third task one could choose the geodesics that are locally perpendicular to the  $z$ -axis, but we prefer to use the strong stable manifold  $W^{ss}(0)$ , which is basically the orbit of (1) that starts off at the origin in the direction perpendicular to the  $z$ -axis. Because it is an orbit, it is not a geodesic of the Lorenz manifold, but rather illustrates the difference between the geometry of the manifold and the dynamics on it. We computed  $W^{ss}(0)$  with the software from [10].

The next step was to identify the sequence of holes from one crocheted round to the next through which the positive and negative  $z$ -axis and both branches of  $W^{ss}(0)$  go. This information is collected in the weaving instructions below.

To mount the Lorenz manifold we weaved an unbendable thin kiting rod through the  $z$ -axis and bendable wires through the last round and the location of  $W^{ss}(0)$ ; details of this procedure can be found in the mounting instructions below. Modulo rotations and translations in  $\mathbb{R}^3$ , there are only two results of mounting the manifold, leading to the Lorenz manifold itself with a right-handed spiral around the  $z$ -axis, or its mirror image with a left-handed spiral around the  $z$ -axis. By giving the rim wire the right twist one can ensure that one obtains the former solution. The final step

is to bend the supporting wires so that they are nice and smooth and the crocheted model indeed resembles the Lorenz manifold.

The final result is shown in Figures 4–6. The carbon fibre rod fixing the  $z$ -axis is vertical and in the centre of the images. The image in Figures 4 shows the Lorenz manifold photographed with a white background, so that the crocheted mesh is clearly visible. Furthermore, one can see through the manifold and get an impression of the part that is hidden. This emphasizes the rotational symmetry of the Lorenz manifold. Figures 5, on the other hand, shows the Lorenz manifold photographed with a black background. One cannot see through the mesh any longer, and the manifold appears as a two-dimensional surface. Notice the wire in the position of the strong stable manifold  $W^{ss}(0)$  that supports the Lorenz manifold. Figures 6 (a) and (b) show two different views taken from different angles, again against a white and a black background to emphasize the mesh and the surface, respectively. Finally, Figure 7 is an enlargement of the Lorenz manifold that shows the strong stable manifold  $W^{ss}(0)$  running from the origin until it meets the rim. Notice that  $W^{ss}(0)$  is perpendicular to the rings only near the origin and certainly not close to the rim. In other words, it is clearly not a geodesic.

It is our experience that the crocheted model of the Lorenz manifold in Figures 4–6 is a very helpful tool for understanding and explaining the dynamics of the Lorenz system. While the model is not identical to the computer generated Lorenz manifold, all its geometrical features are truthfully represented, so that it is possible to convey the intricate structure of this surface in a ‘hands-on’ fashion. This article tried to convey this, but for the real experience you will have to get out your own yarn and crochet hook!



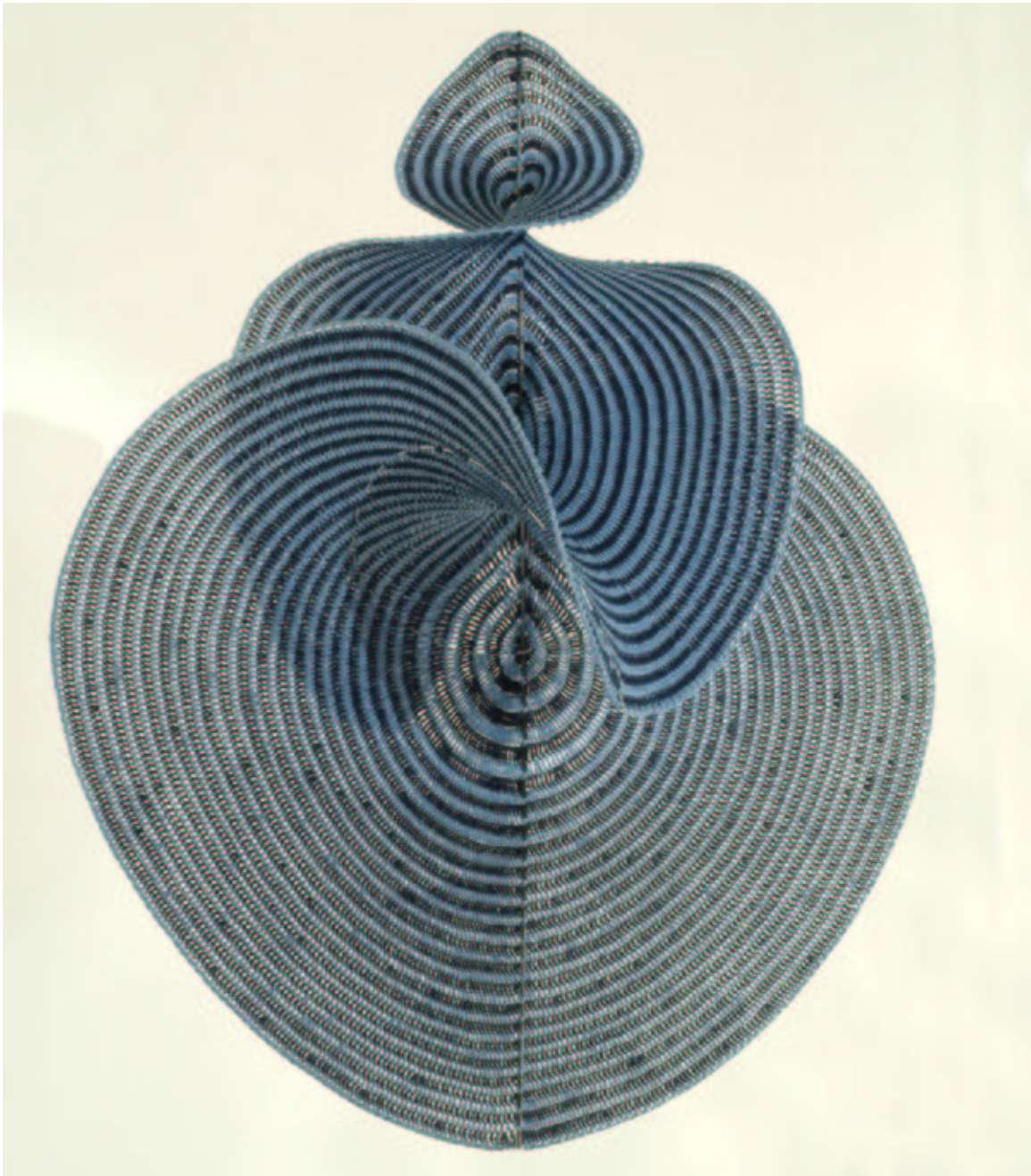


Figure 4: The crocheted Lorenz manifold in front of a white background, which brings out the mesh and the symmetry.

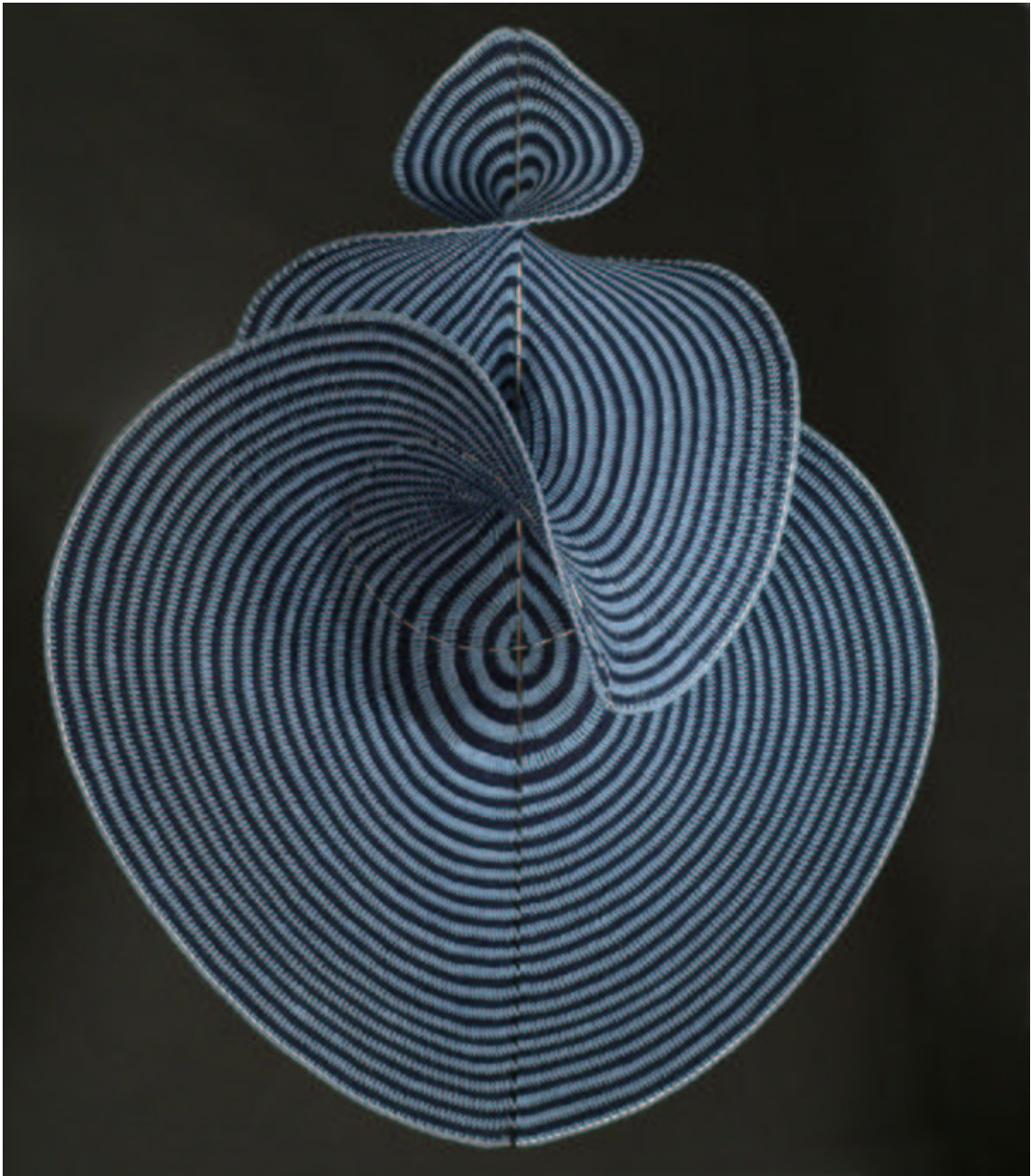


Figure 5: The crocheted Lorenz manifold in front of a black background, which gives a good impression of the manifold as a two-dimensional surface.

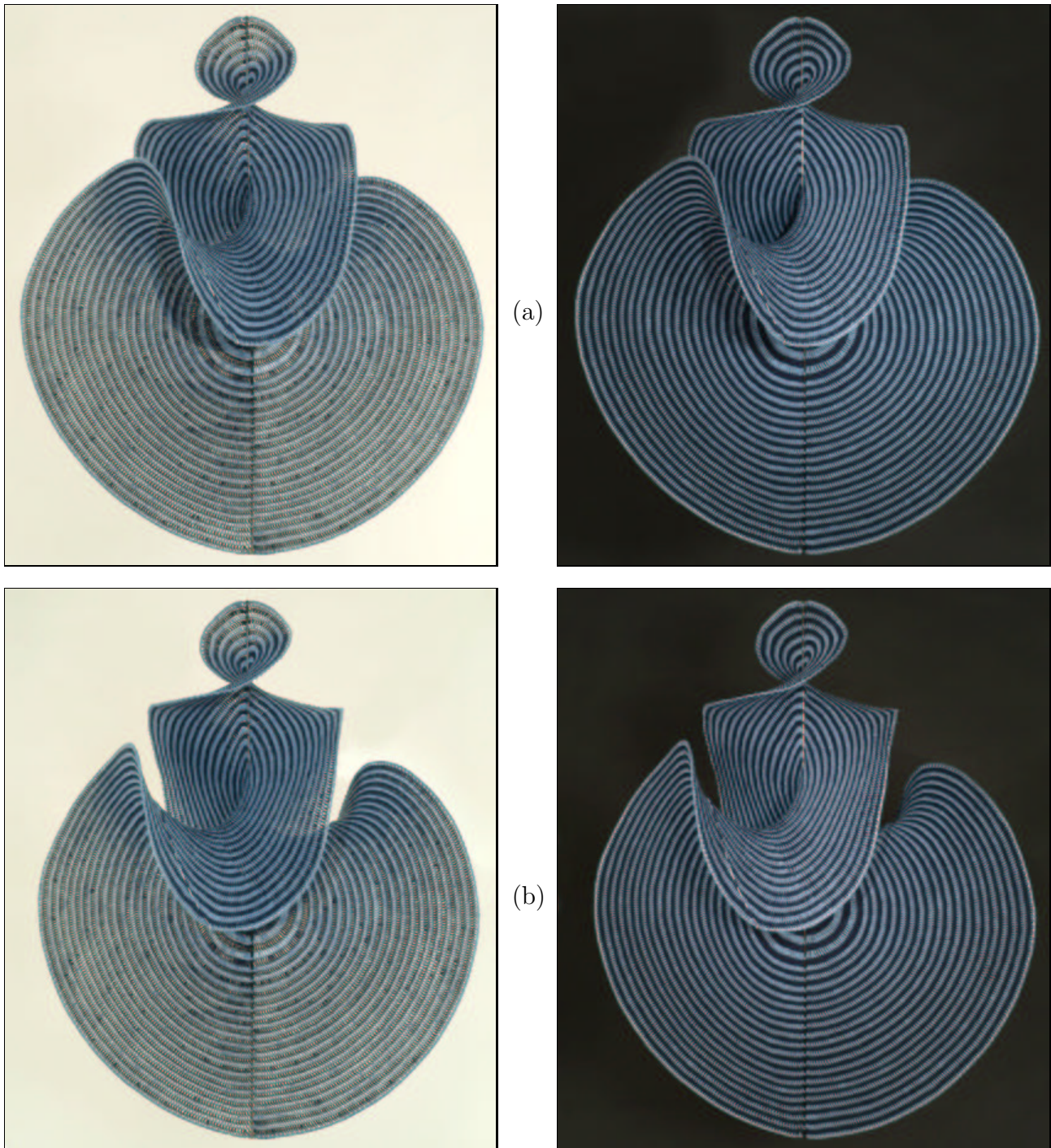


Figure 6: Two more views of the crocheted Lorenz manifold in front of a white and a black background. The difference between the views in Figures 4 and 5, panel (a) and panel (b) is a rotation of about 15 degrees.



Figure 7: A close-up view of the crocheted Lorenz manifold in front of a white background. The vertical rod is the  $z$ -axis, and the wire emerging from the origin is the strong unstable manifold  $W^{ss}(0)$ ; notice also the wire supporting the outer rim of the manifold.

## Acknowledgements

The unmounted Lorenz manifold in Figure 3 was digitally photographed by B.K. at the Design Office of the Faculty of Engineering. The mounted Lorenz manifold was carefully set up by the authors at the photo studio of the Photographic Unit of the University of Bristol, and then digitally photographed by Perry Robbins. Electronic postprocessing was expertly done by Greg Jones at the Design Office of the Faculty of Engineering. We are very grateful to both Perry and Greg for their patience and attention to detail, and for dealing so well with our enthusiasm and perfectionism.

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## Complete instructions

**Materials:** 200 g light blue and 200 g dark blue 4-ply mercerized cotton yarn; 2.50 mm crochet hook; embroidery needle; about 3 m leftover yarn of a contrasting colour; 0.9 m unbendable 4 mm or 5 mm rod; 1.45 m and  $2 \times 2.70$  m bendable 2 mm wire; 2 electrical wire connectors (come in bars; available from DIY stores); wire cutter; plyers; small screwdriver.

**Abbreviations and Notation:** see Table 1 and Figure 8.

## Crochet instructions

Work 5 ch in light blue and join with a slip stitch to form a ring. Odd rounds are worked with light blue and even ones with dark blue yarn. Work each round with the stitch as indicated; count the stitches starting with 0 for the first stitch and make two stitches in one for each stitch mentioned in the list. If the stitch appears with a minus sign, crochet it together with the previous stitch (delete the stitch). Geodesic distance ( $gd$ ) of the Lorenz manifold after each round is given for orientation and motivation.

**rnd1:** foundation round of **5 ch** then **10 dc** in loop ( $gd$  2.75); **rnd2:** **20 tr** 0 1 2 3 4 5 6 7 8 9 ( $gd$  4.75); **rnd3:** **40 dtr** 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 ( $gd$  8.75); **rnd4:** **60 dtr** 0 3 4 7 8 11 12 15 16 19 20 23 24 27 28 31 32 35 36 39 ( $gd$  12.75); **rnd5:** **80 dtr** 0 5 6 11 12 17 18 23 24 29 30 35 36 41 42 47 48 53 54 59 ( $gd$  16.75); **rnd6:** **100 dtr** 3 4 11 12 19 20 27 28 35 36 43 44 51 52 59 60 67 68 75 76 ( $gd$  20.75); **rnd7:** **120 dtr** 0 9 10 19 20 29 30 39 40 49 50 59 60 69 70 79 80 89 90 99 ( $gd$  24.75); **rnd8:** **122 dtr** 111 116 ( $gd$  28.75); **rnd9:** **148 dtr** 0 3 8 11 12 15 20 27 32 39 44 51 56 63 68 75 80 87 92 95 96 99 104 107 108 121 ( $gd$  32.75); **rnd10:** **171 dtr** 7 8 30 31 44 45 58 59 72 73 86 87 100 101 108 123 124 135 138 139 140 141 144 ( $gd$  36.75); **rnd11:** **189 tr** 3 6 25 26 41 42 57 58 73 74 89 90 105 106 121 137 142 145 ( $gd$  38.75); **rnd12:** **192 tr** 13 16 149 ( $gd$  40.75); **rnd13:** **214 tr** 0 25 28 33 36 46 51 118 123 133 136 141 144 167 169 170 172 175 176 185 189 191 ( $gd$  42.75); **rnd14:** **234 tr** 3 48 61 68 71 76 79 86 89 94 97 104 107 112

115 122 135 199 200 206 (*gd 44.75*); **rnd15: 243 tr** 23 24 177 178 193 204 213 215 222  
 (*gd 46.75*); **rnd16: 261 tr** 47 48 69 70 135 136 157 158 199 204 213 214 221 228 229 236  
 237 240 (*gd 48.75*); **rnd17: 269 tr** 7 8 95 96 117 118 216 217 (*gd 50.75*); **rnd18: 283 tr**  
 0 2 198 202 206 207 215 216 230 235 237 238 255 268 (*gd 52.75*); **rnd19: 307 tr** 16 17  
 21 25 32 49 56 169 176 193 216 225 226 229 234 237 246 258 259 269 271 276 277 280 (*gd*  
*54.75*); **rnd20: 325 tr** 2 7 8 80 87 104 111 128 135 152 159 268 269 272 274 279 285 290  
 (*gd 56.75*); **rnd21: 343 tr** 7 16 44 200 209 225 235 251 261 262 267 283 284 287 297 298  
 300 314 (*gd 58.75*); **rnd22: 375 tr** 0 6 30 56 73 82 99 108 125 134 151 160 177 186 209  
 234 243 246 249 260 264 265 279 280 281 310 328 329 330 337 338 342 (*gd 60.75*); **rnd23:**  
**381 tr** 54 55 265 272 314 341 (*gd 62.75*); **rnd24: 411 tr** 5 6 84 85 112 113 140 141 168  
 169 196 197 224 232 239 240 254 269 272 273 274 277 280 302 311 328 335 336 352 378  
 (*gd 64.75*); **rnd25: 432 tr** 2 23 24 31 42 43 50 300 301 308 310 315 340 345 352 355 358  
 397 400 405 406 (*gd 66.75*); **rnd26: 451 tr** 4 5 18 72 222 237 276 313 329 355 362 363  
 365 368 372 373 376 377 405 (*gd 68.75*); **rnd27: 491 tr** 0 16 24 35 83 84 91 106 113 114  
 121 136 143 144 151 166 173 174 181 196 203 204 211 234 235 271 275 288 295 327 343  
 360 363 376 380 383 419 435 436 450 (*gd 70.75*); **rnd28: 511 tr** 13 51 289 300 314 332  
 340 361 362 403 404 413 416 417 420 421 436 450 469 481 (*gd 72.75*); **rnd29: 534 tr** 2 17  
 22 25 26 48 49 60 69 78 273 282 339 348 360 375 385 427 432 433 455 475 497 (*gd 74.75*);  
**rnd30: 563 tr** 5 10 13 16 17 22 46 113 249 258 339 342 345 348 355 356 357 358 369 372  
 415 439 440 446 454 458 492 529 531 (*gd 76.75*); **rnd31: 591 tr** 0 4 10 13 29 47 48 104  
 130 155 223 232 277 284 309 332 359 367 368 381 383 414 425 466 471 472 473 488 (*gd*  
*78.75*); **rnd32: 637 tr** 9 10 18 45 86 119 148 155 174 183 190 199 208 217 224 253 260  
 325 360 362 363 371 377 405 414 439 470 487 488 492 501 504 507 508 509 512 513 514 523  
 535 576 577 580 583 586 590 (*gd 80.75*); **rnd33: 655 tr** 2 13 89 394 420 425 431 434 506  
 509 520 523 533 534 537 563 573 597 (*gd 82.75*); **rnd34: 670 tr** 5 69 115 322 383 388 408  
 423 426 436 450 513 538 539 542 (*gd 84.75*); **rnd35: 695 tr** 4 10 27 119 135 324 380 398  
 416 423 424 428 535 540 541 545 548 554 555 558 565 575 646 658 667 (*gd 86.75*); **rnd36:**  
**720 tr** 0 8 21 38 138 161 178 275 292 313 314 345 370 380 438 441 529 540 541 552 566  
 589 591 592 680 (*gd 88.75*); **rnd37: 754 tr** 12 22 50 111 168 183 206 207 222 223 244



245 260 261 284 299 351 375 376 378 435 450 451 464 -515 515 563 568 582 602 603 608  
 619 -661 661 -668 668 -671 671 701 714 719 (*gd 90.75*); **rnd38: 782 tr** 11 16 33 42 93  
 95 96 120 390 408 453 463 464 466 484 489 490 494 502 -524 524 -533 533 -536 -543  
 543 -559 559 603 605 616 617 624 627 634 642 -705 705 -715 715 742 748 (*gd 92.75*);  
**rnd39: 804 tr** 2 4 8 21 29 31 64 67 72 93 105 379 386 434 458 472 478 497 500 -564  
 607 608 613 614 627 648 652 673 -708 -711 -721 -734 (*gd 94.75*); **rnd40: 840 tr** 4 7  
 22 23 26 45 47 48 128 131 176 344 349 382 395 407 423 433 446 465 474 480 489 512 527  
 532 -559 559 -567 -572 -579 579 -584 584 -591 639 645 653 665 666 682 686 700 705  
 712 714 716 -729 729 -752 752 801 (*gd 96.75*); **rnd41: 887 tr** 0 11 21 33 40 43 69 85  
 100 145 146 167 168 192 210 229 234 271 276 313 318 336 337 380 381 399 459 481 523  
 531 537 544 554 645 653 666 681 689 690 695 696 719 734 739 740 752 -772 772 829 (*gd*  
*98.75*); **rnd42: 921 tr** 11 16 51 64 69 94 102 114 137 161 164 202 203 226 269 270 313  
 314 381 382 389 429 526 534 550 559 564 572 -611 -622 -644 -654 696 712 719 731 732  
 739 758 767 768 788 -853 886 (*gd 100.75*); **rnd43: 958 tr** 2 37 61 66 76 84 206 223 254  
 261 262 269 300 307 308 315 346 353 354 361 392 433 509 559 588 589 613 -684 684 742  
 748 750 751 757 762 778 779 794 -839 -868 903 914 915 (*gd 102.75*); **rnd44: 994 tr** 5  
 9 20 23 25 44 54 59 82 92 93 135 189 204 227 427 439 528 551 561 604 605 607 627 631  
 632 697 740 787 790 791 792 793 803 805 844 850 -872 872 -916 916 -926 (*gd 104.75*);  
**rnd45: 1025 tr** 4 7 9 19 25 33 48 113 230 254 419 431 511 536 543 578 582 584 589 622  
 645 647 659 -730 -741 779 780 783 791 803 811 814 829 830 833 863 -960 (*gd 106.75*);  
**rnd46: 1072 tr** 0 10 15 33 116 126 144 279 291 302 314 329 341 352 364 379 391 402 414  
 546 579 580 590 609 636 648 661 667 674 685 -703 -706 706 -712 712 -716 716 770 779  
 813 840 845 846 847 848 849 850 851 852 855 856 861 873 886 1022 (*gd 108.75*); **rnd47:**  
**1104 tr** 31 94 98 104 117 136 137 177 535 608 653 654 658 668 689 691 699 704 705 -740  
 740 761 777 -787 829 842 871 894 895 906 929 934 944 1008 1017 1022 (*gd 110.75*).

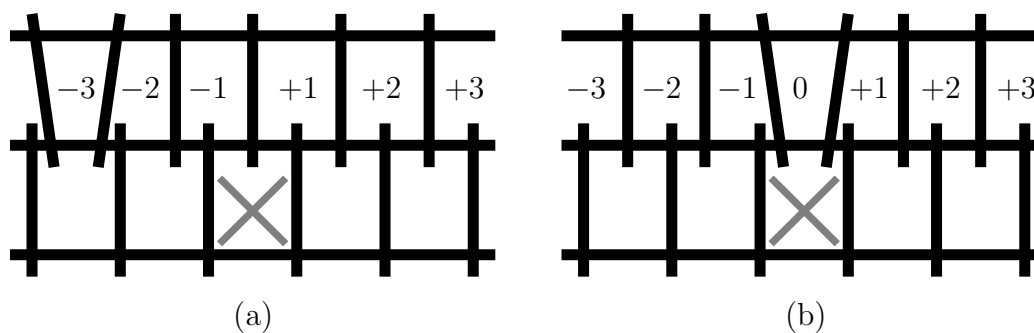


Figure 8: Numbering of holes between stitches in a round relative to a hole position (gray cross) in the previous round, as used in the weaving instructions. Since new stitches are added in front of holes, there are two cases: one stitch in front of the hole position when no extra stitch was added (a), and two stitches in front of the hole position when an extra stitch was added (b).

## Weaving instructions

To mount the Lorenz manifold it is best to first indicate the positions of the rod and the wires by weaving differently coloured yarn through the holes between stitches. Start from the centre in the hole between the two stitches indicated in **rnd1** below. Then weave the yarn through holes from one round to the next, where the position of the next hole is indicated relative to the present position as shown in Figure 8. After weaving in the  $z$ -axis and the strong stable manifold  $W^{ss}(0)$ , fold the manifold over along the  $z$ -axis weave. You should get a result as shown on the right of Figure 3 (c); the two branches of the  $W^{ss}(0)$  weave should be symmetric with respect to the  $z$ -axis weave.

*Positive  $z$ -axis:* **rnd1:** 9-0; **rnd2:** +2; **rnd3:** +1; **rnd4:** +1; **rnd5:** +1; **rnd6:** +1; **rnd7:** +1; **rnd8:** +1; **rnd9:** +1; **rnd10:** +2; **rnd11:** +1; **rnd12:** +1; **rnd13:** +1; **rnd14:** +1; **rnd15:** +2; **rnd16:** +1; **rnd17:** +1; **rnd18:** +1; **rnd19:** +1; **rnd20:** +2; **rnd21:** +1; **rnd22:** +1; **rnd23:** +1; **rnd24:** +2; **rnd25:** +1; **rnd26:** +1; **rnd27:** +1; **rnd28:** +2; **rnd29:** +1; **rnd30:** +1; **rnd31:** +1; **rnd32:** +1; **rnd33:** +1; **rnd34:** +1; **rnd35:** +1; **rnd36:** +1; **rnd37:** +1; **rnd38:** +1; **rnd39:** +1; **rnd40:** +1; **rnd41:** +1;

**rnd42:** +1; **rnd43:** +1; **rnd44:** +1; **rnd45:** +1; **rnd46:** +1; **rnd47:** +1;

*Negative z-axis:* **rnd1:** 4-5; **rnd2:** +2; **rnd3:** +1; **rnd4:** +1; **rnd5:** +1; **rnd6:** +1;  
**rnd7:** +1; **rnd8:** +1; **rnd9:** +1; **rnd10:** +1; **rnd11:** +2; **rnd12:** +1; **rnd13:** +1;  
**rnd14:** +1; **rnd15:** +1; **rnd16:** +1; **rnd17:** +1; **rnd18:** +1; **rnd19:** +1; **rnd20:** +1;  
**rnd21:** +1; **rnd22:** +1; **rnd23:** +1; **rnd24:** +1; **rnd25:** +1; **rnd26:** +1; **rnd27:** +1;  
**rnd28:** +1; **rnd29:** +1; **rnd30:** +1; **rnd31:** +1; **rnd32:** +1; **rnd33:** +1; **rnd34:** +1;  
**rnd35:** +1; **rnd36:** +1; **rnd37:** +1; **rnd38:** +1; **rnd39:** +1; **rnd40:** +1; **rnd41:** +2;  
**rnd42:** +1; **rnd43:** +1; **rnd44:** +1; **rnd45:** +1; **rnd46:** +1; **rnd47:** +1;

*Left branch of  $W^{ss}(0)$ :* **rnd1:** 1-2; **rnd2:** +1; **rnd3:** +1; **rnd4:** +2; **rnd5:** +2; **rnd6:**  
+2; **rnd7:** +2; **rnd8:** +1; **rnd9:** +2; **rnd10:** +3; **rnd11:** +2; **rnd12:** +1; **rnd13:** +3;  
**rnd14:** +1; **rnd15:** +2; **rnd16:** +3; **rnd17:** +2; **rnd18:** +2; **rnd19:** +2; **rnd20:** +2;  
**rnd21:** +3; **rnd22:** +2; **rnd23:** +2; **rnd24:** +3; **rnd25:** +3; **rnd26:** +3; **rnd27:** +2;  
**rnd28:** +3; **rnd29:** +3; **rnd30:** +3; **rnd31:** +3; **rnd32:** +3; **rnd33:** +5; **rnd34:** +5;  
**rnd35:** +4; **rnd36:** +4; **rnd37:** +5; **rnd38:** +6; **rnd39:** +5; **rnd40:** +6; **rnd41:** +5;  
**rnd42:** +7; **rnd43:** +7; **rnd44:** +5; **rnd45:** +5; **rnd46:** +5; **rnd47:** +5;

*Right branch of  $W^{ss}(0)$ :* **rnd1:** 7-8; **rnd2:** +2; **rnd3:** +2; **rnd4:** +2; **rnd5:** +1;  
**rnd6:** -1; **rnd7:** -1; **rnd8:** -1; **rnd9:** +1; **rnd10:** -1; **rnd11:** -1; **rnd12:** +1;  
**rnd13:** -1; **rnd14:** -1; **rnd15:** -1; **rnd16:** -1; **rnd17:** -1; **rnd18:** -1; **rnd19:** -1;  
**rnd20:** -1; **rnd21:** -1; **rnd22:** -1; **rnd23:** -1; **rnd24:** -2; **rnd25:** -1; **rnd26:** -1;  
**rnd27:** -2; **rnd28:** -2; **rnd29:** -1; **rnd30:** -3; **rnd31:** -2; **rnd32:** -3; **rnd33:** -2;  
**rnd34:** -4; **rnd35:** -3; **rnd36:** -5; **rnd37:** -4; **rnd38:** -3; **rnd39:** -5; **rnd40:** -4;  
**rnd41:** -5; **rnd42:** -5; **rnd43:** -5; **rnd44:** -5; **rnd45:** -4; **rnd46:** -4; **rnd47:** -5;

## Mounting instructions

Weave the unbendable thin rod of 0.9 m length through the manifold by following the  $z$ -axis weave; we used a 5 mm carbon fibre rod used in kiting, which is lightweight and very stiff for its diameter. Starting from the top of the  $z$ -axis, weave a 2.70 m length of the bendable wire through the outer crocheted round of the manifold until

you reach the bottom of the  $z$ -axis. Repeat the procedure with the second length of 2.70 m around the other half of the outer crocheted round of the manifold, again starting from the top of the  $z$ -axis. Try to spread the stitches evenly over the wire; you will find that this introduces twist into the rim wire. Make sure the twisting is clockwise near the  $z$ -axis in the direction of increasing  $z$ , so that you get a right-handed helix, just like a cork screw.

Cut two single electrical wire connectors from a bar and strip them of their isolating plastic cover. The stripped connectors are now unsuitable for electrical connections, but ideal for connecting the bendable wires. Connect the two 2.70 m pieces of wire at the top and bottom with the connectors by sliding in both ends and tightening the screws.

Make a mark 0.1 m from each end of the 1.45 m length of bendable wire; the middle piece of 1.25 m is the length of  $W^{ss}(0)$ . Starting from the rim, weave this wire through the manifold following the marking yarn. Using the plyers make two small loops at both ends where you made the mark and cut off the excess wire. Sow the ends in place with light blue yarn.

Finally, remove the differently coloured yarn. The Lorenz manifold should now be recognisable. With the help of the figures in this paper, tuck and bend it into its final shape, making sure that the bendable wires are nice and smooth, that is, without noticeable kinks. This may take some time depending on your level of perfectionism. Good luck!

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**Hinke Osinga** learnt crocheting, and other handcraft techniques, from her mother around the age of seven. A bit later she got a Ph.D. in Mathematics from the University of Groningen in 1996 under the direction of Henk Broer and Gert Vegter. Her research was on the computation of normally hyperbolic invariant manifolds and the ideas developed then have been generalized (and disguised) throughout her work on algorithms for other invariant manifolds. She spent 22 months at the Geometry Center, University of Minnesota, in Minneapolis, followed by 18 months in the Control and Dynamical Systems Department at the California Institute of Technology in Pasadena. She then moved to England and after 15 months in the School of Mathematical Sciences at the University of Exeter, she joined the Department of Engineering Mathematics at the University of Bristol in 2001 as a Lecturer in Mathematics. The Lorenz manifold is her first project that combines handcraft with mathematics.

**Bernd Krauskopf** got his Ph.D. in Mathematics from the University of Groningen in 1995 under the direction of Floris Takens. After a year as visiting professor at Cornell University, and a two-year Postdoctoral position at the Division of Physics of Vrije Universiteit Amsterdam, he joined the Department of Engineering Mathematics at the University of Bristol, where he is now Professor of Applied Nonlinear Mathematics. He also holds a visiting chair (bijzondere leerstoel) in Mathematical Physics at VU Amsterdam. Bernd works in the general area of dynamical systems theory, specifically on theoretical and numerical problems in bifurcation theory and their application to models arising in laser physics. The collaboration with Hinke on global manifolds started in 1997 when Bernd visited Hinke at the Geometry Center, University of Minnesota in Minneapolis — the basic idea of how to grow a global manifold emerged over a bagel with Hinke at a bagel store on Nicolette Mall.