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TIME-REPARAMETRIZATION-INVARIANT DYNAMICS OF RELATIVISTIC SYSTEMS

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The review is devoted to the description of the reparametrization-invariant dynamics of relativistic systems (special relativity, string, and general relativity) obtained by resolving constraints and constructing equivalent unconstrained systems. The constraint-shell actions allow us to give clear mathematical definitions of measurable quantities in both classical and quantum theories of the type of the geometric time interval, or the dynamic evolution parameter in the world space of dynamic variables, the energy density and the holomorphic (particle-like) variables in general relativity.

Обзор посвящен репараметризационно-инвариантному описанию динамики релятивистских систем (релятивистская частица, струна, вселенная в общей теории относительности). Такое описание достигается переходом на поверхность связей в фазовом пространстве.

Вычисление функции действия на поверхности связей (путем их решения) ведет к эквивалентным системам без связей, совместимым с простейшими вариационными принципами и с физически ясными, математически строгими определениями инвариантных измеряемых величин как в классической, так и в квантовой теориях, в частности, динамического эволюционного параметра в мировом пространстве переменных, плотности измеряемой энергии и частицеподобных (голоморфных) переменных в общей теории относительности.

1. INTRODUCTION

The modern physics grew from two different roots: the Newton mechanics and the Faraday–Maxwell electrodynamics. The first gave variational principles; the second, relativistic and gauge symmetries (see Fig. 1). Relativistic particles, strings, n branes, and general relativity are systems with constraints and are not compatible with the simplest variational principles of the Newton mechanics. One of the main difficulties is the invariance of relativistic system with respect to reparametrization of the coordinate time.

The problem of the self-consistent Hamiltonian description of relativistic systems (particles, strings, n branes, general relativity) has a long history [1–7]. There are two opposite approaches to solution of this problem in the generalized Hamiltonian formulation [1, 8–12].

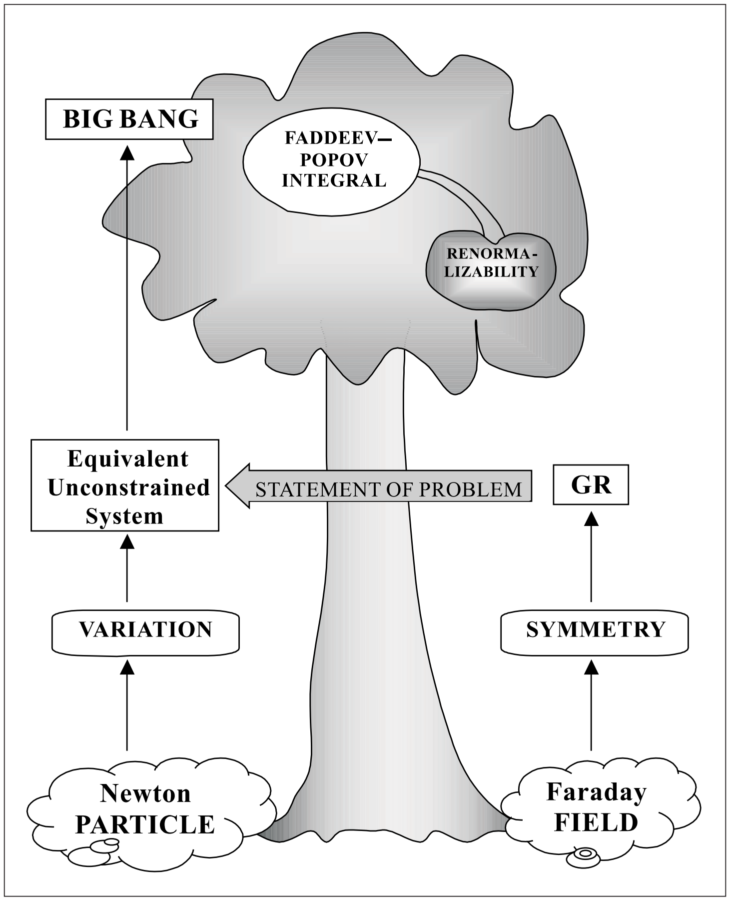

Fig. 1. The tree of modern theoretical physics grew from two different roots («particle» and «field») which gave the VARIATIONAL method and SYMMETRY principles for formulating modern physical theories as constrained systems. To obtain unambiguous physical results, one should construct Equivalent Unconstrained Systems compatible with the simplest variational method. It is just the problem discussed in the present paper

The first approach is the reduction of the extended phase space by fixing the gauge that breaks reparametrization invariance from the very beginning [2,6]. The defect of this approach is unclear correspondence between the reparametrization-noninvariant mathematical quantities and the invariant physical observables of the type of measurable time and energy; the quantum version of the first approach looks as attractable mathematical games with unnormalizable wave functions that are free from clear physical predictions.

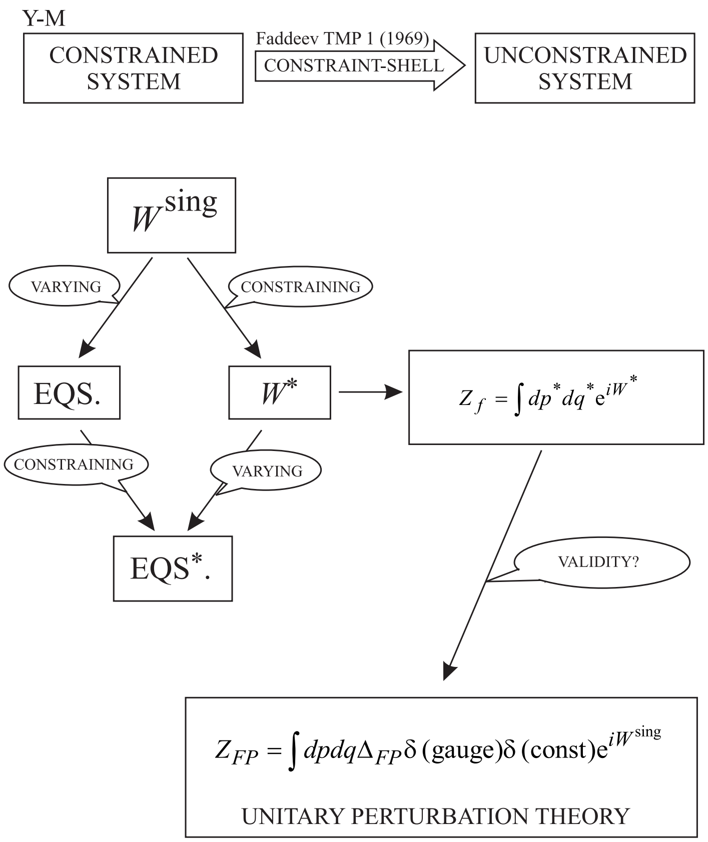

Fig. 2. An equivalent unconstrained system $W^*(p^*, q^*)$ can be obtained in the case when the operations of varying and constraining commute with each other to obtain equations (EQS.) of motion in terms of independent variables p^*, q^*. The next problem is to establish the range of validity of the standard Faddeev–Popov (FP) integral

The second approach is the reparametrization-invariant reduction of an action by the explicit resolving of the first class constraints to get an equivalent unconstrained system (see Fig. 2), so that one of the variables of the extended phase space (with a negative contribution to the energy constraint) converts into the *dynamic evolution parameter*, and its conjugate momentum becomes the nonzero Hamiltonian of evolution [3, 4, 7, 13–16].

An example of the application of such an invariant reduction of the action is the Dirac formulation of QED [17] directly in terms of the gauge-invariant (dressed) fields as the proof of the adequateness of the Coulomb gauge with the invariant content of classical equations. As was shown by Faddeev [18], the invariant reduction of the action is the way to obtain the unconstrained Feynman integral for the foundation of the intuitive Faddeev–Popov functional integral in the non-Abelian gauge theories [19–21] (see Fig. 2).

In reparametrization-invariant relativistic theories, the constraining of actions does not kill superfluous variables. They are kept in the constrained action as the evolution parameter and the corresponding Hamiltonian. This fact is the main difference of reparametrization-invariant systems from the gauge-invariant ones where the operation of constraining removes all longitudinal components from the action. This difference explains why the gauge fixing is not compatible with reparametrization invariance.

To emphasize the importance of the superfluous variables in relativistic systems, we introduce the notion of the sector of «measurable quantities» including in it (together with the sector of the Dirac observables) the superfluous variables which cannot be removed by the gauge fixing, and which play important physical roles of the dynamic evolution parameter and the corresponding evolution Hamiltonian.

In special relativity (SR), the sector of «measurable quantities» coincides with the world space. The causal structure of this world space (given in the form of the light cones of future and past) determines the causal Green functions and the arrow of the *geometric time*. The latter is defined as the reparametrization-invariant geometric interval that is always greater than zero in accordance with equations of motion.

The application of the invariant reduction of extended actions in cosmology and general relativity [4, 7] allows one to formulate the dynamics of relativistic systems directly in terms of the invariant geometric time with the nonzero Hamiltonian of evolution, instead of the non-invariant coordinate time with the generalized zero Hamiltonian of evolution in the gauge-fixing method. The formulation in terms of the geometric time is based on the Levi-Civita canonical transformation [22–24] that converts the energy constraint into a new momentum, so that the new dynamic evolution parameter coincides with the geometric time, as one of the consequences of new equations of motion.

In the present paper, we apply the method of the invariant Hamiltonian reduction (with resolving the first class constraints and the Levi-Civita canonical transformations) to express reparametrization-invariant dynamics of relativistic systems in terms of the geometric time and to construct the causal Green functions in the form of the path integrals in the world space of dynamic variables.

The content of the paper is the following. In Section 2, we consider the extended version of classical mechanics. A relativistic particle is considered in Section 3. Section 4 is devoted to the generalized Hamiltonian formulation of a relativistic string and its invariant reduction. Section 5 is devoted to the reparametrization-invariant Hamiltonian reduction of general relativity. In Section 6, we discuss the reparametrization-invariant dynamics of the Early Universe. Section 7 is devoted to conformal relativity.

2. INVARIANT HAMILTONIAN REDUCTION: MECHANICS

To illustrate the time-reparametrization-invariant Hamiltonian reduction [4] and its difference from the gauge-fixing method, let us consider an extended form of a classical-mechanical system

$$W = \int\limits_{\tau^1}^{\tau_2} d\tau \left(p\dot{q} - \Pi_0 \dot{Q}_0 - \lambda[-\Pi_0 + H(p,q)] \right), \tag{1}$$

that is invariant under reparametrizations of the coordinate evolution parameter τ and «lapse» function λ

$$\tau \to \tau' = \tau'(\tau), \qquad \lambda \ \to \ \lambda' = \lambda \frac{d\tau}{d\tau'}. \tag{2}$$

The problem of the classical description is to obtain the evolution of the physical variables of the *world space* q, Q_0 in terms of the *geometric time* T defined as

$$dT := \lambda d\tau, \qquad T = \int\limits_0^{\tau} d\tau' \lambda(\tau'), \tag{3}$$

that is also invariant under reparametrizations (2).

The second problem (connected with quantization) is to present the effective action of the equivalent unconstrained theory directly in terms of T, the equations of which reproduce this evolution. The solution of the second problem will be called the *invariant Hamiltonian reduction*.

The resolving of the first problem for the considered system is trivial, as the equations of motion of this system

$$\dot{q} = \lambda \partial_p H, \qquad \dot{p} = -\lambda \partial_q H, \qquad \dot{Q}_0 = \lambda, \qquad \dot{\Pi}_0 = 0 \tag{4}$$

in terms of the geomeric time (3)

$$\frac{dq}{dT} = \partial_p H, \qquad \frac{dp}{dT} = -\partial_q H, \qquad \frac{dQ_0}{dT} = 1, \qquad \frac{d\Pi_0}{dT} = 0 \tag{5}$$

are completely equivalent to the equations of the conventional unconstrained mechanics in the *reduced phase space* (p, q)

$$W^M = \int\limits_{T(\tau_1)=T_1}^{T(\tau_2)=T_2} dT \left(p\frac{dq}{dT} - H(p,q) \right). \tag{6}$$

The problem is how to derive this system from the extended one (1) to apply the symplest Hamiltonian quantization with a clear physical interpretation of the invariant quantities.

The solution of the problem of the *invariant Hamiltonian reduction* considered in the present review is the explicit resolving of three equations of the extended system (1):
i) for the variable λ (treated as constraint)

$$\frac{\delta W}{\delta \lambda} = -\Pi_0 + H(p,q) = 0, \tag{7}$$

ii) for the momentum Π_0 with a negative contribution to the constraint (7)

$$\frac{\delta W}{\delta \Pi_0} = 0 \ \Rightarrow \ \frac{dQ_0}{d\tau} = \lambda, \tag{8}$$

iii) for its conjugate variable Q_0

$$\frac{\delta W}{\delta Q_0} = \frac{d\Pi_0}{d\tau} = 0. \tag{9}$$

(We call these three equations (7)–(9) the *geometric sector*.)

The resolving of the constraint (7) expresses the «ignorable» momentum Π_0 through $H(p,q)$ with a positive value $\Pi_0 = H(p,q) > 0$. The second equation (8) identifies the *dynamic evolution parameter* Q_0 with the proper time (3) $Q_0 = T$. It is not the gauge but the invariant solution of the equation of motion (8). The third equation (9) is the conservation law.

As a result of the invariant Hamiltonian reduction (i.e., a result of the substitution of $\Pi_0 = H$ and $Q_0 = T$ into the initial action (1)) this action is reduced to the one of the conventional mechanics (6) in terms of the proper time T, where the role of the nonzero Hamiltonian of evolution in the proper time T is played by the constraint-shell value of the «ignorable» momentum $\Pi_0 = H(p,q)$. In other words, this constraint-shell action $W(\text{constraint}) = W^M$ determines the nonzero Hamiltonian $H(p,q)$ in the proper time T, instead of the zero generalized Hamiltonian in the coordinate time τ in (1) $\lambda(-\Pi_0 + H)$.

Thus, the equivalent unconstrained system was constructed without any additional constraint of the type:

$$\lambda = 1, \qquad \tau = T \tag{10}$$

which confuse quantities of the measurable sector with noninvariant ones. This confusion is contradictable. The «gauge-fixing» identification of the coordinate evolution parameter τ and the geometric time $T = \lambda\tau$ in the form of the gauges (10) contradicts the difference of their Hamiltonians $\lambda(-\Pi_0 + H) \neq H(p,q)$.

The second difference of the «gauge-fixing» from the invariant Hamiltonian reduction is more essential, namely, the formulation of the theory in terms of the invariant geometric time (3) is achieved by the explicit resolving of the constraint (7) and equation of motion (8), as a result of which «ignorable» variables Π_0, Q_0 are excluded from the phase space.

In the present paper, we apply the invariant Hamiltonian reduction to relativistic particle, string, and general relativity.

3. SPECIAL RELATIVITY

3.1. Statement of the Problem. To answer the question: Why is the reparametrization-invariant reduction needed?, let us consider relativistic mechanics in the Hamiltonian form

$$W[P, X|N|\tau_1, \tau_2] = \int\limits_{\tau_1}^{\tau_2} d\tau[-P_\mu \dot{X}^\mu - \frac{N}{2m}(-P_\mu^2 + m^2)], \tag{11}$$

which is classically equivalent to the conventional square root form

$$W[X|\tau_1, \tau_2] = -m\int\limits_{\tau_1}^{\tau_2} d\tau\sqrt{\dot{X}^\mu \dot{X}_\mu}. \tag{12}$$

Both these actions are invariant with respect to reparametrizations of the *coordinate evolution parameter*

$$\tau \to \tau' = \tau'(\tau), \qquad N'd\tau' = Nd\tau \tag{13}$$

given in the one-dimensional space with the invariant interval

$$dT := Nd\tau, \qquad T = \int\limits_0^\tau d\bar{\tau}N(\bar{\tau}). \tag{14}$$

We called this invariant interval the *geometric time* [4] whereas the dynamic variable X_0 (with a negative contribution in the constraint) we called *dynamic evolution parameter*.

In terms of the geometric time (14) the classical equations of the generalized Hamiltonian system (11) take the form

$$\frac{dX_\mu}{dT} = \frac{P_\mu}{m}, \qquad \frac{dP_\mu}{dT} = 0, \qquad P_\mu^2 - m^2 = 0. \tag{15}$$

The classical problem is to find the evolution of the world space variables with respect to the geometric time T.

The quantum problem is to obtain the equivalent unconstrained theories directly in terms of the invariant times X_0 or T with the invariant Hamiltonians of evolution. The solution of the second problem is called the dynamic (for X_0), or geometric (for T) *reparametrization-invariant Hamiltonian reductions*.

3.2. Dynamic Unconstrained System. The dynamic reduction of the action (11) means the substitution of the explicit resolving of the energy constraint $(-P_\mu^2 + m^2) = 0$ with respect to the momentum P_0 into this action

$$\frac{\delta W}{\delta N} = 0 \quad \Rightarrow \quad P_0 = \pm\sqrt{m^2 + P_i^2}. \tag{16}$$

In accordance with two signs of the solution (16), after the substitution of (16) into (11), we have two branches of the dynamic unconstrained system

$$W(\text{constraint}) = W_\pm^D = \int\limits_{X_0(\tau_1)=X_0(1)}^{X_0(\tau_2)=X_0(2)} dX_0 \left[P_i \frac{dX_i}{dX_0} \mp \sqrt{m^2 + P_i^2}\right]. \tag{17}$$

The role of the time of evolution, in this action, is played by the variable X_0 that abandons the Dirac sector of «observables» P_i, X_i, but not the sector of «measurable» quantities. At the same time, its conjugate momentum P_0 converts into the corresponding Hamiltonian of evolution, values of which are energies of a particle.

This invariant reduction of the action gives an «equivalent» unconstrained system together with definition of the dynamic evolution parameter (X_0) corresponding to a nonzero Hamiltonian P_0.

Thus, we need the reparametrization-invariant Hamiltonian reduction to determine the dynamic evolution parameter and its invariant Hamiltonian for a reparametrization-invariant system and to apply the symplest canonical quantization to it.

In quantum relativistic theory, we get two Schrödinger equations

$$i\frac{d}{dX_0}\Psi_{(\pm)}(X|P) = \pm\sqrt{m^2 + P_i^2}\,\Psi_{(\pm)}(X|P), \tag{18}$$

with positive and negative values of P_0 and normalized wave functions

$$\Psi_\pm(X|P) = \frac{A_P^\pm \theta(\pm P_0)}{(2\pi)^{3/2}\sqrt{2P_0}} \exp\left(-iP_\mu X^\mu\right) \qquad \left([A_P^-, A_{P'}^+] = \delta^3(P_i - P_i')\right). \tag{19}$$

The coefficient A_P^+, in the secondary quantization, is treated as the operator of creation of a particle with positive energy; and the coefficient A_P^-, as the operator of annihilation of a particle also with positive energy. The physical states are formed by action of these operators on the vacuum $\langle 0|, |0\rangle$ in the form of out-state ($|P\rangle = A^+_P|0\rangle$) with positive frequencies and in-state ($\langle P| = \langle 0|A^-_P$) with negative frequencies. This treatment means that positive frequencies propagate forward ($X_{02} > X_{01}$); and negative frequencies, backward ($X_{01} > X_{02}$), so that the negative values of energy are excluded from the spectrum to provide the stability of the quantum system in QFT [25]. For this causal convention the geometric time (14) is always positive in accordance with the equations of motion (15)

$$\left(\frac{dT}{dX_0}\right)_{\pm} = \pm\frac{m}{\sqrt{P_i^2+m^2}} \quad\Rightarrow\quad T(X_{02},X_{01}) = \pm\frac{m}{\sqrt{P_i^2+m^2}}(X_{02}-X_{01}) \geqslant 0. \tag{20}$$

In other words, instead of changing the sign of energy, we change that of the dynamic evolution parameter, which leads to the arrow of the geometric time (20) and to the causal Green function

$$\begin{aligned} G^c(X) &= G_+(X)\theta(X_0) + G_-(X)\theta(-X_0) = \\ &= i\int\frac{d^4P}{(2\pi)^4}\exp{(-iPX)}\frac{1}{P^2-m^2-i\epsilon}, \end{aligned} \tag{21}$$

where $G_+(X) = G_-(-X)$ is the «commutative» Green function [25]

$$G_+(X) = \int\frac{d^4P}{(2\pi)^3}\exp{(-iPX)}\delta(P^2-m^2)\theta(P_0) = \tag{22}$$

$$= \frac{1}{2\pi}\int d^3Pd^3P'\langle 0|\Psi_-(X|P)\Psi_+(0|P')|0\rangle.$$

3.3. Path Integral for the Causal Green Functions. The question appears: How to construct the path integral without gauges?

To obtain the reparametrization-invariant form of the functional integral adequate to the considered gaugeless reduction (17) and the causal Green function (21), we use the version of composition law for the commutative Green function with the integration over the whole measurable sector $X_{1\mu}$

$$G_+(X-X_0) = \int G_+(X-X_1)\bar{G}_+(X_1-X_0)dX_1 \qquad \left(\bar{G}_+ = \frac{G_+}{2\pi\delta(0)}\right), \tag{23}$$

where $\delta(0) = \int dN$ is the infinite volume of the group of reparametrizations of the coordinate τ. Using the composition law n times, we got the multiple integral

$$G_+(X - X_0) = \int G_+(X - X_1) \prod_{k=1}^{n} \bar{G}_+(X_k - X_{k+1}) dX_k, \qquad (X_{n+1} = X_0). \tag{24}$$

The continual limit of the multiple integral with the integral representation for δ function

$$\delta(P^2 - m^2) = \frac{1}{2\pi} \int dN \exp\left[iN(P^2 - m^2)\right]$$

can be defined as the path integral in the form of the average over the group of reparametrizations

$$G_+(X) = \int\limits_{X(\tau_1)=0}^{X(\tau_2)=X} \frac{dN(\tau_2) d^4P(\tau_2)}{(2\pi)^3} \prod_{\tau_1 \leqslant \tau < \tau_2} \left\{ d\bar{N}(\tau) \prod_{\mu} \left(\frac{dP_\mu(\tau) dX_\mu(\tau)}{2\pi} \right) \right\} \times$$

$$\times \exp\left(iW[P, X|N|\tau_1, \tau_2]\right), \tag{25}$$

where $\bar{N} = N/2\pi\delta(0)$, and W is the initial extended action (11).

3.4. Geometric Unconstrained System. The Hamiltonian of the unconstrained system in terms of the geometric time T can be obtained by the canonical Levi-Civita-type transformation [12, 22, 23]

$$(P_\mu, X_\mu) \Rightarrow (\Pi_\mu, Q_\mu) \tag{26}$$

to the variables (Π_μ, Q_μ) for which one of equations identifies Q_0 with the geometric time T. This transformation [22] converts the constraint into a new momentum

$$\Pi_0 = \frac{1}{2m}[P_0^2 - P_i^2], \qquad \Pi_i = P_i, \quad Q_0 = X_0 \frac{m}{P_0}, \qquad Q_i = X_i - X_0 \frac{P_i}{P_0} \tag{27}$$

and has the inverted form

$$P_0 = \pm\sqrt{2m\Pi_0 + \Pi_i^2},\ P_i = \Pi_i,\ X_0 = \pm Q_0 \frac{\sqrt{2m\Pi_0 + \Pi_i^2}}{m},\quad X_i = Q_i + Q_0 \frac{\Pi_i}{m}. \tag{28}$$

After transformation (27) the action (11) takes the form

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \left[-\Pi_\mu \dot{Q}^\mu - N(-\Pi_0 + \frac{m}{2}) - \frac{d}{d\tau} S^{lc} \right], \qquad S^{lc} = (Q_0 \Pi_0). \tag{29}$$

The invariant reduction is the resolving of the constraint $\Pi_0 = m/2$ which determines a new Hamiltonian of evolution with respect to the new dynamic evolution parameter Q_0, whereas the equation of motion for this momentum Π_0 identifies the dynamic evolution parameter Q_0 with the geometric time T ($dQ_0 = dT$). The substitution of these solutions into the action (29) leads to the reduced action of a geometric unconstrained system

$$W(\text{constraint}) = W^G = \int\limits_{T_1}^{T_2} dT \left(\Pi_i \frac{dQ_i}{dT} - \frac{m}{2} - \frac{d}{dT}(S^{lc})\right) \qquad (S^{lc} = Q_0 \frac{m}{2}), \tag{30}$$

where variables Π_i, Q_i are cyclic ones and have the meaning of initial conditions in the comoving frame

$$\frac{\delta W}{\delta \Pi_i} = \frac{dQ_i}{d\tau} = 0 \Rightarrow \ Q_i = Q_i^{(0)}, \qquad \frac{\delta W}{\delta Q_i} = \frac{d\Pi_i}{d\tau} = 0 \Rightarrow \ \Pi_i = \Pi_i^{(0)}. \tag{31}$$

The substitution of all geometric solutions

$$Q_0 = T, \quad \Pi_0 = \frac{m}{2}, \quad \Pi_i = \Pi_i^{(0)} = P_i, \quad Q_i = Q_i^{(0)} \tag{32}$$

into the inverted Levi-Civita transformation (28) leads to the conventional relativistic solution for the dynamical system

$$P_0 = \pm\sqrt{m^2 + P_i^2}, \qquad P_i = \Pi_i^{(0)}, \qquad X_0(T) = T\frac{P_0}{m}, \qquad X_i(T) = X_i^{(0)} + T\frac{P_i}{m}. \tag{33}$$

The Schrödinger equation for the wave function

$$\frac{d}{idT}\Psi^{lc}(T, Q_i|\Pi_i) = \frac{m}{2}\Psi^{lc}(T, Q_i|\Pi_i), \tag{34}$$

$$\Psi^{lc}(T, Q_i|\Pi_i) = \exp\left(-iT\frac{m}{2}\right)\exp\left(i\Pi_i^{(0)} Q_i\right)$$

contains only one eigenvalue $m/2$ degenerated with respect to the cyclic momentum Π_i. We see that there are differences between the dynamic and geometric descriptions. The dynamic evolution parameter is given in the whole region $-\infty < X_0 < +\infty$, whereas the geometric one is only positive $0 < T < +\infty$, as it follows from the properties of the causal Green function (21) after the Levi-Civita transformation (27)

$$G^c(Q_\mu) = \int\limits_{-\infty}^{+\infty} d^4\Pi_\mu \frac{\exp\left(iQ^\mu \Pi_\mu\right)}{2m(\Pi_0 - m/2 - i\epsilon/2m)} = \frac{\delta^3(Q)}{2m}\theta(T), \qquad T = Q_0.$$

Two solutions of the constraint (a particle and antiparticle) in the dynamic system correspond to a single solution in the geometric system.

Thus, the reparametrization-invariant content of the equations of motion of a relativistic particle in terms of the geometric time is covered by two «equivalent» unconstrained systems: the dynamic and geometric. In both the systems, the invariant times are not *the coordinate evolution parameter*, but variables with the negative contribution into the energy constraint. The Hamiltonian description of a relativistic particle in terms of the geometric time can be achieved by the Levi-Civita-type canonical transformation, so that the energy constraint converts into a new momentum. Whereas, the dynamic unconstrained system suits for the secondary quantization and the derivation of the causal Green function that determine the arrow of the geometric time.

4. RELATIVISTIC STRING

4.1. The Generalized Hamiltonian Formulation. We begin with the action for a relativistic string in the geometrical form [26–28]

$$W = -\frac{\gamma}{2}\int d^2u\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu, \qquad u_\alpha = (u_0, u_1), \tag{35}$$

where the variables x_μ are string coordinates given in a space-time with a dimension D and the metric $(x_\mu x^\mu := x_0^2 - x_i^2)$; $g_{\alpha\beta}$ is a second-rank metric tensor given in the two-dimensional Riemannian space $u_\alpha = (u_0, u_1)$.

To formulate the Hamiltonian approach, one needs to separate the two-dimensional Riemannian space $u_\alpha = (u_0, u_1)$ on the set of space-like lines $\tau = \mathrm{constant}$ in the form of the Dirac–Arnovitt–Deser–Misner parametrization of the two-dimensional metric

$$g_{\alpha,\beta} = \Omega^2 \begin{pmatrix} \lambda_1^2 - \lambda_2^2 & \lambda_2 \\ \lambda_2 & -1 \end{pmatrix}, \qquad \sqrt{-g} = \Omega^2 \lambda_1 \tag{36}$$

with the invariant interval [2]

$$ds^2 = g_{\alpha\beta}du^\alpha du^\beta = \Omega^2[\lambda_1^2 d\tau^2 - (d\sigma + \lambda_2 d\tau)^2], \qquad u_\alpha = (u_0 = \tau, u_1 = \sigma), \tag{37}$$

where λ_1 and λ_2 are known in general relativity (GR) as the lapse function and shift «vector», respectively [29, 30]. The action (35) after the substitution (37) does not depend on the conformal factor Ω and takes the form

$$W = -\frac{\gamma}{2}\int\limits_{\tau_1}^{\tau_2} d\tau \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left[\frac{(D_\tau x)^2}{\lambda_1} - \lambda_1 x'^2\right], \tag{38}$$

where

$$D_\tau x_\mu = \dot{x}_\mu - \lambda_2 x'_\mu \qquad (\dot{x} = \partial_\tau x,\ x' = \partial_\sigma x) \tag{39}$$

is the covariant derivative with respect to the two-dimensional metric (37). The metric (37), the action (38), and the covariant derivative (39) are invariant under the transformations

$$\tau \Rightarrow \widetilde{\tau} = f_1(\tau), \qquad \sigma \Rightarrow \widetilde{\sigma} = f_2(\tau, \sigma). \tag{40}$$

A similar group of transformations in GR is well-known as the «kinemetric» group of diffeomorphisms of the Hamiltonian description [31].

The variation of action (38) with respect to λ_1 and λ_2 leads to the equations

$$\frac{\delta W}{\delta \lambda_2} = \frac{x' D_\tau x}{\lambda_1} = 0 \ \Rightarrow \ \lambda_2 = \frac{\dot{x}x'}{x'^2}, \tag{41}$$

$$\frac{\delta W}{\delta \lambda_1} = \frac{(D_\tau x)^2}{\lambda_1^2} + x'^2 = 0 \ \Rightarrow \ \lambda_1^2 = \frac{(\dot{x}x')^2 - \dot{x}^2 x'^2}{(x'^2)^2}.$$

The solutions of these equations convert the action (38) into the standard Nambu–Gotto action of a relativistic string [28, 32]

$$W = -\gamma \int\limits_{\tau_1}^{\tau_2} d\tau \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}.$$

The generalized Hamiltonian form [8] is obtained by the Legendre transformation [10] of the action (38)

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left(-p_\mu D_\tau x^\mu + \lambda_1 \phi_1\right) = \int\limits_{\tau_1}^{\tau_2} d\tau \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left(-p_\mu \dot{x}^\mu + \lambda_1 \phi_1 + \lambda_2 \phi_2\right), \tag{42}$$

where

$$\phi_1 = \frac{1}{2\gamma}[p_\mu^2 + (\gamma x'_\mu)^2], \qquad \phi_2 = x'^\mu p_\mu, \tag{43}$$

and the generalized Hamiltonian

$$\mathcal{H} = \lambda_1 \phi_1 + \lambda_2 \phi_2 \tag{44}$$

is treated as the generator of evolution with respect to the coordinate time τ, and λ_1, λ_2 play the role of variables with the zero momenta

$$P_{\lambda_1} = 0, \qquad P_{\lambda_2} = 0 \tag{45}$$

considered as the first class primary constraints [8, 10]. The equations for λ_1, λ_2

$$\frac{\delta W}{\delta \lambda_1} = \phi_1 = 0, \qquad \frac{\delta W}{\delta \lambda_2} = \phi_2 = 0 \tag{46}$$

are known as the first class secondary constraints [8, 10]. The Hamiltonian equations of motion take the form

$$\frac{\delta W}{\delta x^\mu} = \dot{p}_\mu - \partial_\sigma[\gamma \lambda_1 x'_\mu + \lambda_2 p_\mu] = 0, \qquad \frac{\delta W}{\delta p^\mu} = p_\mu - \gamma \frac{D_\tau x_\mu}{\lambda_1} = 0. \tag{47}$$

The problem is to find solutions of the Hamiltonian equations of motion (47) and constraints (46) which are invariant with respect to the kinemetric transformations (40).

There is the problem of the solution of the linearized «gauge-fixing» equation in terms of the evolution parameter τ (as the object reparametrizations in the initial theory) being adequate to the initial kinemetric invariant and relativistic invariant system. In particular, the constraints mix the global motion of the «centre-of-mass» coordinates with local excitations of a string ξ_μ, which contradicts the relativistic invariance of internal degrees of freedom of a string. In this context, it is worth to clear up a set of questions: Is it possible to introduce the reparametrization-invariant evolution parameter for the string dynamics, instead of the noninvariant coordinate time (τ) used as the evolution parameter in the gauge-fixing method? Is it possible to construct the observable nonzero Hamiltonian of evolution of the «centre-of-mass» coordinates? What is relation of the «centre-of-mass» evolution to the unitary representations of the Poincare group?

4.2. The Separation of the «Centre-of-Mass» Coordinates. To apply the reparametrization-invariant Hamiltonian reduction discussed before to a relativistic string, one should define the *proper (geometric) time* in the form of the reparametrization-invariant functional of the lapse function (of type (14)), and to point out, among the variables, a *dynamic evolution parameter*, the equation of which identifies it with the proper time of type (8). We identify this *dynamic evolution parameter* with the time-like variable of the centre of mass of a string defined as the total coordinate

$$X_\mu(\tau) = \frac{1}{l(\tau)} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma x_\mu(\tau, \sigma), \qquad l(\tau) = \sigma_2(\tau) - \sigma_1(\tau). \tag{48}$$

Therefore, the invariant reduction requires to separate the «centre-of-mass» variables before variation of the action. We consider this separation on the level of the action (38) which after the substitution

$$x_\mu(\tau,\ \sigma) = X_\mu(\tau) + \xi_\mu(\tau,\ \sigma), \qquad x'_\mu(\tau,\ \sigma) = \xi'_\mu(\tau,\ \sigma) \tag{49}$$

takes the form

$$W = -\frac{\gamma}{2}\int\limits_{\tau_1}^{\tau_2} d\tau \left\{ \frac{\dot{X}^2 l(\tau)}{N_0(\tau)} + 2\dot{X}_\mu \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{D_\tau \xi^\mu}{\lambda_1} + \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left(\frac{(D_\tau \xi)^2}{\lambda_1} - \lambda_1 \xi'^2 \right) \right\}, \tag{50}$$

where the global lapse function $N_0(\tau)$ is defined as the functional of $\lambda_1(\tau, \sigma)$

$$\frac{1}{N_0[\lambda_1]} = \frac{1}{l(\tau)} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{1}{\lambda_1(\tau, \sigma)}. \tag{51}$$

To exclude the superfluous coordinates and momenta, the local variables ξ_μ are given (according to (48) and (49)) in the class of functions (with the nonzero Fourier harmonics) which satisfy the conditions

$$\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \xi_\mu = 0. \tag{52}$$

A definition of the conjugate momenta is consistent with (48) and the equation for the momentum p_μ (47) of the local momentum is given in the same class (52)

$$\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{D_\tau \xi^\mu}{\lambda_1} = 0\ . \tag{53}$$

Then we get

$$P_\mu = \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma p_\mu(\tau, \sigma) = \frac{\delta W}{\delta \dot{X}^\mu} = -\gamma \frac{\dot{X}_\mu l}{N_0}, \quad \pi_\mu = \frac{\delta W}{\delta \dot{\xi}^\mu} = \gamma \frac{D_\tau \xi^\mu}{\lambda_1}. \tag{54}$$

This separation conserves the group of diffeomorphisms of the Hamiltonian [4] and leads to the Bergmann–Dirac generalized action

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \left[\left(\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma [-\pi_\mu D_\tau \xi^\mu - \lambda_1 \mathcal{H}] \right) - P_\mu \dot{X}^\mu + N_0 \frac{P_\mu^2}{2\bar{\gamma}} \right] \quad (\bar{\gamma} = \gamma l(\tau)), \tag{55}$$

where $\mathcal{H}$ is the Hamiltonian of local excitations

$$\mathcal{H} = -\frac{1}{2\gamma}[\pi_\mu^2 + (\gamma \xi'_\mu)^2]\ . \tag{56}$$

The variation of the action (55) with respect to λ_1 results in the equation

$$\frac{\delta W}{\delta \lambda_1} = \mathcal{H} - \left(\frac{1}{l\bar{\lambda}_1^2}\right)\frac{P^2}{2\bar{\gamma}} = 0, \tag{57}$$

where

$$\bar{\lambda}_1(\tau,\sigma) = \frac{\lambda_1(\tau,\sigma)}{N_0(\tau)} \tag{58}$$

is the reparametrization-invariant component of the local lapse function. Here we have used the variation of the functional $N_0[\lambda_1]$ (51)

$$\frac{\delta N_0[\lambda_1]}{\delta \lambda_1} = \frac{1}{l(\tau)\bar{\lambda}_1^2}.$$

In accordance with our separation of dynamic variables onto the global and local sectors, the first class constraint (57) has two projections onto the global sector (zero Fourier harmonic) and the local one. The global part of the constraint (57) can be obtained by variation of the action (55) with respect to N_0 (after the substitution of (58) into (55))

$$\frac{\delta W}{\delta N_0} = \frac{P^2}{2\bar{\gamma}} - H = 0, \qquad H = \int\limits_{\sigma_1}^{\sigma_2} d\sigma \bar{\lambda}_1 \mathcal{H}, \tag{59}$$

or, in another way, by the integration of (57) multiplied by λ_1. Then, the local part of the constraint (57) can be obtained by the substitution of (59) into (57)

$$\bar{\lambda}_1\mathcal{H} - \frac{1}{l\bar{\lambda}_1}\int\limits_{\sigma_1}^{\sigma_2} d\sigma \bar{\lambda}_1 \mathcal{H} = 0. \tag{60}$$

The integration of the local part over σ is equal to zero if we take into account the normalization of the local lapse function

$$\frac{1}{l(\tau)}\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{1}{\bar{\lambda}_1} = 1. \tag{61}$$

This follows from the definition of the global lapse function (51). We see that the local part (60) takes the form of an integral operator, orthogonal to the operator of integration over σ.

Finally, we can represent the action (55) in the equivalent form

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \left[\left(\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma[-\pi_\mu D_\tau \xi^\mu]\right) - P_\mu \dot{X}^\mu - N_0\left(-\frac{P_\mu^2}{2\bar{\gamma}} + \mathcal{N}H\right)\right], \tag{62}$$

where the global lapse function N_0 and the local one $\bar{\lambda}_1$ are treated as independent variables, with taking the normalization (61) into account after the variation.

The invariant proper time T measured by the watch of an observer in the «centre-of-mass» frame of a string is given by the expression (according to (40) and (51))

$$\sqrt{\gamma} dT := N_0 d\tau = d\bar{\tau}, \quad \bar{\tau} = \int\limits_0^{\tau} d\tau' \left[\frac{1}{l(\tau')} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{1}{\lambda_1(\tau', \sigma)} \right]^{-1}. \tag{63}$$

We include the constant $\sqrt{\gamma}$ to provide the dimension of the time measured by the watch of an observer.

Now we can see from (62) that the dynamics of the local degrees of freedom π, ξ, in the class of functions of nonzero harmonics (52), is described by the same kinemetric invariant and relativistic covariant equations (47) where x, p are changed by ξ, π, with the set of the first class (primary and secondary) constraints

$$P_{\lambda_1} = 0, \qquad P_{\lambda_2} = 0, \qquad \pi_\mu \xi'^\mu = 0, \qquad \bar{\lambda}_1 \mathcal{H} - \frac{1}{l\bar{\lambda}_1} \int\limits_{\sigma_1}^{\sigma_2} d\sigma \bar{\lambda}_1 \mathcal{H} = 0. \tag{64}$$

We see that the separation of the «centre-of-mass» (CM) variables on the level of the action removes the interference terms which mix the CM variables with the local degrees of freedom; as a result, the new local constraints (64) do not depend on the total momentum P_μ, in contrast to the standard ones (92). In other words, there is the problem: when can one separate the CM coordinates of a relativistic string — before the variation of the action or after the variation of the action? The relativistic invariance dictates the first one, because an observer in the CM frame (which is the preferred frame for a string) cannot measure the total momentum of the string.

The first class local constraints (64) can be supplemented by the second class constraints

$$\bar{\lambda}_1 - 1 = 0, \qquad \lambda_2 = 0, \qquad n^\mu \xi_\mu = 0, \qquad n^\mu \pi_\mu = 0 \qquad (\, n_\mu = (1, 0, 0, 0)\,) \tag{65}$$

so that the equations of the local constraint-shell action

$$W(\text{loc.constrs.}) = \int\limits_{\tau_1}^{\tau_2} d\tau \left[\left(\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \pi_i \dot{\xi}_i \right) - P_\mu \dot{X}^\mu - N_0 \left(-\frac{P_\mu^2}{2\bar{\gamma}} + H \right) \right] \tag{66}$$

coincide with the complete set of equations and the same constraints (64), (65) of the extended action, i.e., the operations of constraining and variation commute.

The substitution of the global constraint (59) with $\bar\lambda_1 = 1$ into the action (66) leads to the constraint-shell action

$$W^D_\pm = \int\limits_{X_0(\tau_1)}^{X_0(\tau_2)} dX_0 \left[\left(\int\limits_{\sigma_1(X_0)}^{\sigma_2(X_0)} d\sigma \pi_i \frac{d\xi_i}{dX_0}\right) + P_i \frac{dX_i}{dX_0} \mp \sqrt{P_i^2 + 2\bar\gamma H}\right]. \quad (67)$$

This action describes the dynamics of a relativistic string with respect to the time measured by an observer in the rest frame with the physical nonzero Hamiltonian of evolution. However, in this system, equations become nonlinear. To overcome this difficulty, we pass to the «centre-of-mass» frame.

4.3. Levi-Civita Geometrical Reduction. To express the dynamics of a relativistic string in terms of the proper time (63) measured by an observer in the comoving (i.e., «centre-of-mass») frame, we use the Levi-Civita–type canonical transformations [22, 24]

$$(P_\mu, X_\mu) \Rightarrow (\Pi_\mu, Q_\mu);$$

they convert the global part of the constraint (59) into a new momentum Π_0

$$\Pi_0 = \frac{1}{2\bar\gamma}[P_0^2 - P_i^2], \qquad \Pi_i = P_i, \qquad Q_0 = X_0 \frac{\bar\gamma}{P_0}, \quad Q_i = X_i - X_0 \frac{P_i}{P_0}. \quad (68)$$

The inverted form of these transformations is

$$P_0 = \pm\sqrt{2\bar\gamma\Pi_0 + \Pi_i^2}, \;\; P_i = \Pi_i, \;\; X_0 = \pm Q_0 \frac{\sqrt{2\bar\gamma\Pi_0 + \Pi_i^2}}{\bar\gamma}, \;\; X_i = Q_i + Q_0 \frac{\Pi_i}{\bar\gamma}. \quad (69)$$

As a result of transformations (68), the extended action (62) in terms of the Levi-Civita geometrical variables takes the form (compare with (1))

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \left[\left(\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma[-\pi_\mu D_\tau \xi^\mu]\right) - \Pi_\mu \dot Q^\mu - N_0(-\Pi_0 + H) - \frac{d}{d\tau}(Q_0 \Pi_0)\right]. \quad (70)$$

The Hamiltonian reduction means to resolve constraint (59) with respect to the momentum Π_0

$$\frac{\delta W}{\delta N_0} = 0 \Rightarrow \Pi_0 = H. \quad (71)$$

The equation of motion for the momentum Π_0

$$\frac{\delta W}{\delta \Pi_0} = 0 \Rightarrow \frac{dQ_0}{d\tau} = N_0 \quad (\text{i.e.,} \quad dQ_0 = N_0 d\tau) \quad (72)$$

identifies (according to our definition (63)) the new variable Q_0 with the proper time $\bar{\tau} = \sqrt{\gamma}T$, whereas the equation for Q_0

$$\frac{\delta W}{\delta Q_0} = 0 \Rightarrow \frac{d\Pi_0}{d\tau} = 0, \qquad \text{i.e.,}\ \frac{dH}{dT} = 0, \tag{73}$$

in view of (71), gives us the conservation law.

Thus, resolving the global energy constraint $\Pi_0 = H^R$, we obtain, from (70), the reduced action for a relativistic string in terms of the proper time $Q_0 = \bar{\tau}$

$$W^G = \int\limits_{T_1}^{T_2} dT \left[\left(\int\limits_{\sigma_1}^{\sigma_2} d\sigma [-\pi_\mu D_{\bar{\tau}} \xi^\mu] \right) + \Pi_i \frac{dQ_i}{dT} - H - \frac{d}{dT}(TH) \right], \tag{74}$$

where in analogy with (58) we introduced the factorized «shift-vector» $\lambda_2 = N_0 \bar{\lambda}_2$; in this case, the covariant derivative (39) takes the form

$$D_T \xi_\mu = \partial_T \xi_\mu - \bar{\lambda}_2 \xi'_\mu = \frac{D_\tau \xi_\mu}{N_0} \sqrt{\gamma}. \tag{75}$$

The reduced system (74) has trivial solutions for the global variables Π_i, Q_i

$$\frac{\delta W^G}{\delta \Pi_i} = 0 \Rightarrow \frac{dQ_i}{dT} = 0;\ Q_i = \text{const}, \tag{76}$$

$$\frac{\delta W^G}{\delta Q_i} = 0 \Rightarrow \frac{d\Pi_i}{dT} = 0,\ \Pi_i = \text{const}$$

which have the meaning of initial data.

If the solutions of equations (71), (72), and (76) for the system (74)

$$\Pi_0 = H := \frac{M^2}{2\bar{\gamma}}, \quad \Pi_i = P_i, \qquad Q_0 = T\sqrt{\gamma}, \quad Q_i = X_i(0), \tag{77}$$

are substituted into the inverted Levi-Civita canonical transformations (69)

$$P_0 = \pm\sqrt{M^2 + P_i^2}, \qquad X_0(\bar{\tau}) = T\frac{P_0}{\sqrt{\gamma}l}\ , \qquad X_i(\bar{\tau}) = Q_i + T\frac{P_i}{\sqrt{\gamma}l}\ , \tag{78}$$

the initial extended action (62) can be described in the rest frame of an observer who measures the energy P_0 and the time X_0 and sees the rest frame evolution of the «centre-of-mass» coordinates

$$X_i(X_0) = Q_i + X_0 \frac{P_i}{P_0}\ . \tag{79}$$

The Lorentz scheme of describing a relativistic system in terms of the time and energy (X_0, P_0) in the phase space $P_i, X_i, \pi_\mu, \xi_\mu$ is equivalent to the above-considered Levi-Civita scheme in terms of the proper time and the evolution Hamiltonian $(\bar{\tau}, H^R)$ in the phase space $\Pi_i, Q_i, \pi_\mu, \xi_\mu$, where the variables Π_i, Q_i are cyclic.

We identify the Levi-Civita scheme with the comoving frame with the energy

$$E_0 = -\frac{dW^G}{dT} = \frac{M^2}{2\sqrt{\gamma}l} + \frac{dS^{lc}}{dT} = \frac{M^2}{\sqrt{\gamma}l} \qquad (S^{lc} = T\frac{M^2}{2\sqrt{\gamma}l}). \tag{80}$$

This energy includes the time-surface S^{lc} term in the action (74). Then, the inverted Levi-Civita canonical transformations (69) (obtained on the level of the extended theory) play the role of the Lorentz transformation from the comoving frame to the rest frame

$$T\frac{M^2}{\sqrt{\gamma}l} - X_i^{(0)}P_i^{(0)} = \pm X_0|P_0| - X_iP_i. \tag{81}$$

4.4. Dynamics of the Local Variables. We restrict ourselves to an open string with the boundary conditions

$$\sigma_1(T) = 0, \qquad \sigma_2(T) = \pi, \qquad l(T) = \pi. \tag{82}$$

In the gauge-fixing method, by using the kinemetric transformation, we can put

$$\bar{\lambda}_1 = 1, \qquad \bar{\lambda}_2 = 0. \tag{83}$$

This requirement does not contradict the normalization of $\bar{\lambda}_1$ (61).

In view of (64), it means that the reduced Hamiltonian H (59) coincides with its density (56)

$$\bar{\phi}_1 = \mathcal{H} - \frac{1}{\pi}\int\limits_0^\pi d\sigma\mathcal{H} = 0, \qquad \bar{\phi}_2 = \pi_\mu\xi'^\mu = 0. \tag{84}$$

In this case, the reparametrization-invariant equations for the local variables obtained by varying the action (74)

$$\frac{\delta W_s^G}{\delta\xi^\mu} = 0 \Rightarrow \partial_T\pi_\mu - \partial_\sigma(\bar{\lambda}_2\pi_\mu) = \gamma\partial_\sigma(\bar{\lambda}_1\xi'_\mu), \qquad \frac{\delta W_s^G}{\delta\pi^\mu} = 0 \Rightarrow \gamma D_T\xi_\mu = \bar{\lambda}_1\pi_\mu \tag{85}$$

lead to the D'Alambert equations

$$\partial_T^2\xi_\mu - \partial_\sigma^2\xi_\mu = 0. \tag{86}$$

The general solution of these equations of motion in the class of functions (52) with the boundary conditions (82) is given by the Fourier series

$$\xi_\mu(\bar{\tau},\sigma) = \frac{1}{2\sqrt{\pi\gamma}}[\psi_\mu(z_+) + \psi_\mu(z_-)],\ \psi_\mu(z) = i\sum_{n\neq 0} \mathrm{e}^{-inz}\frac{\alpha_{n\mu}}{n},\ z_\pm = \sqrt{\gamma}T \pm \sigma,$$
$$\xi'_\mu(\bar{\tau},\sigma) = \frac{1}{2\sqrt{\pi\gamma}}[\psi'_\mu(z_+) - \psi'_\mu(z_-)], \qquad \pi_\mu(\bar{\tau},\sigma) = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}[\psi'_\mu(z_+) + \psi'_\mu(z_-)]. \tag{87}$$

The total coordinates $Q^{(0)}_\mu$ and momenta P_μ are determined by the reduced dynamics of the «centre-of-mass» (76), (77), (78), and the string mass M obtained from (59)

$$P^2_\mu = M^2 = 2\pi\gamma H = 2\pi\gamma \int\limits_0^\pi d\sigma \mathcal{H}. \tag{88}$$

The substitution of ξ_μ and π_μ from (87) into (56) leads to the density

$$\mathcal{H} = -\frac{1}{4\pi}\left[\psi'^2_\mu(z_+) + \psi'^2_\mu(z_-)\right],$$

and from (88) we obtain, for the mass, the expression

$$M^2 = -2\pi\gamma \bar{L}_0 \ = -\frac{\gamma}{2}\int\limits_0^\pi d\sigma \left[(\psi'_\mu(z_+))^2 + (\psi'_\mu(z_-))^2\right]. \tag{89}$$

The second constraint (84) in terms of the vector ψ'_μ in (87) takes the form

$$\xi'_\mu \pi^\mu = \frac{1}{4\pi}\left[\psi'^2_\mu(z_+) - \psi'^2_\mu(z_-)\right] = 0 \ \Rightarrow \ \ \psi'^2_\mu(z_+) = \psi'^2_\mu(z_-) = \text{const.}, \tag{90}$$

and the first constraint (84) $\bar{\phi}_1 = 0$ is satisfied identically. After the substitution of the constant value (90) into (89) we obtain that const. $= -M^2/\pi\gamma$; thus, finally the reparamerization-invariant constraint takes the form

$$P^2_\mu + \pi\gamma\psi'^2_\mu(z_\pm) = 0 \qquad (\ P^2_\mu = M^2\ ). \tag{91}$$

Unlike this constraint, the gauge-fixing reparametrization-noninvariant constraint [27, 28]

$$\left(P_\mu + \sqrt{\pi\gamma}\psi'_\mu\right)^2 = 0 \tag{92}$$

contains the interference of the local and global degrees of freedom $\psi'_\mu P^\mu$. The latter violates the relativistic invariance of the local excitations which form the mass and spin of a string.

Equation (91) means that ψ'_μ is the modulo-constant space-like vector. The constraint (91) in terms of the Fourier components (87) takes the form

$$\psi'^2_\mu(z_\pm) = 2 \sum_{n=-\infty}^{\infty} \bar{L}_n e^{-inz_\pm} = -\frac{M^2}{\pi\gamma}, \tag{93}$$

where $\bar{L}_n$ are the contributions of the nonzero harmonics

$$\bar{L}_0 = -\frac{1}{2}\sum_{k\neq 0} \alpha_{k\mu}\alpha^\mu_{-k}, \quad \bar{L}_{n\neq 0} = -\frac{1}{2}\sum_{k\neq 0,n} \alpha_{k\mu}\alpha^\mu_{n-k}. \tag{94}$$

From (93) we can see that the zero harmonic of this constraint determines the mass of a string

$$M^2 = -2\pi\gamma\bar{L}_0 = -\pi\gamma\sum_{k\neq 0}\alpha_{k\mu}\alpha_{-k\mu} \tag{95}$$

and coincides with the gauge-fixing value. However, the nonzero harmonics of constraint (93)

$$\bar{L}_{n\neq 0} = -\frac{1}{2}\sum_{k\neq 0,n}\alpha_{k\mu}\alpha_{n-k\mu} = 0, \qquad \bar{L}_{-n} = \bar{L}^*_n \tag{96}$$

(as we dicussed above) strongly differ from the nonzero harmonics of the gauge-fixing constraints (92) The latter (in the contrast to (91)) contains the mixing of the global motion of the centre of mass P_μ with the local excitations ψ_μ. It is clear that this mixing of the global and local motions violates the Poincare invariance of the local degrees of freedom.

The algebra of the local constraints (96) of the reparametrization-invariant dynamics of a relativistic string is not closed, as it does not contain the zero Fourier harmonic of the energy constraint (which has been resolved to express the dynamic equations in terms of the proper time).

The ideology of the invariant reduction (with the explicit resolving of constraints to exclude the superfluous variables of the type of the time-like component of the CM coordinates) can be extended onto the local constraints (84). These constraints in the form (91) can be also used to exclude the time component of the local excitations ξ_0, π_0 (with the negative contribution into energy) from the phase space, to proceed the stability of the system and the positive norm of quantum states

$$\xi'_0 = \frac{1}{2\sqrt{\pi\gamma}}[\psi'_0(z_+) - \psi'_0(z_-)], \qquad \pi_0 = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}[\psi'_0(z_+) + \psi'_0(z_-)], \tag{97}$$

where

$$\psi'_0(z_\pm) = \pm\left[(\psi'_i(z_\pm))^2 - \frac{M^2}{\pi\gamma}\right]^{1/2}. \tag{98}$$

The constraining (97) means that only the spatial components ξ_i, π_i are independent variables.

The choice of gauge (65) leads to $\xi_0 = \pi_0 = 0$ and fixes a contribution of the time-like component into the string mass. In this case, as was mentioned above, the equations for the reduced action coincide with the set of equations and the same constraints of the initial extended action. Finally, the explicit resolving of the local constraints takes the form

$$(\psi_i'(z_\pm))^2 = \frac{M^2}{\pi\gamma}. \tag{99}$$

The reparametrization-invariant dynamics of a relativistic string in the form of the first and second class constraints (64), (65) coincides with the Röhrlich approach to the string theory [33]. This approach is based on two points: i) the choice of the gauge condition

$$P_\mu \xi^\mu = 0, \quad P_\mu \pi^\mu = 0 \Rightarrow G_n = P_\mu \alpha_n^\mu = 0, \quad n \neq 0$$

and ii) the use of that condition for eliminating the states with negative norm, the physical state vectors being constructed in the «centre-of-mass» (CM) frame (in our scheme, the CM frame appears as a result of the geometric Levi–Civita reduction). This reference frame is the only preferred frame for quantizing such a composite relativistic object as the string, as only in this frame one can quantize the initial data. This is a strong version of the principle of correspondence with classical theory: the classical initial data become the quantum numbers of quantum theory. All previous attempts for quantization of the string fully ignored this meaning of the CM frame.

4.5. Quantum Theory. The Röhrlich approach distinguishes two cases: $M^2 = 0$ and $M^2 \neq 0$.

The first case, in our scheme, the equality $M^2 = 0$ together with the local constraints (96) forms the Virasoro algebra. The reparametrization-invariant version of the Virasoro algebra (with all its difficulties, including the $D = 26$ problem and the negative norm states) appears only in the case of the massless string $-2\pi\gamma\bar{L}_0 = M^2 = 0$.

The second case $M^2 \neq 0$ allows us to exclude the time Fourier components α_{n0}, and it is just these components that after quantization $[\alpha_{n,\mu}, \alpha^+_{m,\nu}] = -n\eta_{\mu\nu}\delta_{nm}$ $(n, m \neq 0)$ lead to the states with negative norm because of the system being unstable. This means that the state vectors in the CM frame are constructed only by the action on vacuum of the spatial components of the operators $a^+_{ni} = \alpha_{-ni}/\sqrt{n}, n > 0$

$$|\mathbf{\Phi}_\nu\rangle_{\rm CM} = \prod_{n=1}^{\infty} \frac{(a^+_{nx})^{\nu_{nx}}}{\sqrt{\nu_{nx}!}} \frac{(a^+_{ny})^{\nu_{ny}}}{\sqrt{\nu_{ny}!}} \frac{(a^+_{nz})^{\nu_{nz}}}{\sqrt{\nu_{nz}!}} |0\rangle, \tag{100}$$

where the three-dimensional vectors $\nu_n = (\nu_{nx}, \nu_{ny}, \nu_{nz})$ have only nonnegative integers as components. These state vectors automatically satisfy the constraint

$$\alpha_{n0}|\mathbf{\Phi}_\nu\rangle_{\mathrm{CM}} = 0, \qquad n > 0. \tag{101}$$

The physical states (100) are subjected to further constraints (96) with $n \geqslant 0$

$$\bar{L}_n|\mathbf{\Phi}_\nu\rangle_{\mathrm{CM}} = 0, \qquad n > 0, \qquad P^2 = M_\nu^2 = \pi\gamma\langle\mathbf{\Phi}_\nu \sum_{m\neq 0} \alpha_{-m,i}\alpha_{m,i}|\mathbf{\Phi}_\nu\rangle, \tag{102}$$

where $\bar{L}_n$ can be represented in the normal ordering form

$$\bar{L}_{n>0} = \sum_{k=1}^{\infty} \alpha_{k,i}^{+}\alpha_{n+k,i} + \frac{1}{2}\sum_{k>1}^{n-1} \alpha_{k,i}\alpha_{n-k,i}. \tag{103}$$

Constraints $G_n = \alpha_{n0} = 0, n > 0$ (101) and $\bar{L}_m, m \geqslant 0$ (102), taken together, represent the first class constraints, in accordance with the Dirac classification [8] as they form a closed algebra for $(n, m > 0)$

$$[G_n, G_m] = 0, \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}, \quad [G_n, \bar{L}_m] = nG_{m+n}. \tag{104}$$

Therefore the conditions (101) eliminating the ghosts and the conditions (102) defining the physical vector states are consistent. Note that the commutator $[\bar{L}_n, \bar{L}_m]$ does not contain a c-number since $n \geqslant 0$ and $m \geqslant 0$.

On the operator level, equations determining the resolution of the constraints are fulfilled in a weak sense, as only the «annihilation» part of the constraints is imposed on the state vectors.

4.6. The Causal Green Functions. Now we can construct the causal Green function for a relativistic string as the analogy of the causal Green function for a relativistic particle (23)–(25) discussed in Section 3.3.

The Veneziano-type causal Green function is the spectral series with the Hermite polynomials $\langle\xi|\nu\rangle$ over the physical state vectors $|\mathbf{\Phi}_\nu\rangle = |\nu\rangle$

$$G_c(X|\xi_1, \xi_2) = G_+(X|\xi_1, \xi_2)\theta(X_0) + G_-(X|\xi_1, \xi_2)\theta(-X_0) = \tag{105}$$

$$= i\int \frac{d^4P}{(2\pi)^4} \exp(-iPX) \sum_\nu \frac{\langle\xi_1|\nu\rangle\langle\nu|\xi_2\rangle}{P^2 - M_\nu^2 - i\epsilon}.$$

The commutative Green function for a relativistic string $G_+(X|\xi_1, \xi_2)$ can be represented in the form of the Faddeev–Popov functional integral [19, 21] in the local gauge (65)

$$G_+(X|\xi_2,\xi_1) =$$
$$= \int\limits_{X(\tau_1)=0}^{X(\tau_2)=X} \frac{dN_0(\tau_2)d^4P(\tau_2)}{(2\pi)^3} \prod_{\tau_1\leqslant\tau<\tau_2} \left\{ d\bar{N}_0(\tau) \prod_{\mu} \left(\frac{dP_\mu(\tau)dX_\mu(\tau)}{2\pi} \right) \right\}, \quad (106)$$
$$F_+(\xi_2,\xi_1),$$

where we use the representation of the spectral series in the form of the functional integral

$$F_+(\xi_2,\xi_1) = \sum_\nu \langle \xi_2|\nu\rangle \exp\left\{ iW[P,X,N_0,M_\nu]\right\} \langle \nu|\xi_1\rangle =$$
$$= \int\limits_{\xi_1}^{\xi_2} D(\xi,\pi)\Delta_{fp} \exp\left\{ iW_{fp}\right\}, \quad (107)$$

$W[P,X,N_0,M_\nu]$ is the action (11) with the mass M_ν,

$$W_{fp} = \int\limits_0^{\tau(X_0)} d\tau \left[- \left(\int\limits_0^{\pi} d\sigma \pi_\mu \dot{\xi}^\mu \right) - P_\mu \dot{X}^\mu - N_0 \left(-\frac{P^2}{2\pi\gamma} + H \right) \right] \quad (108)$$

is the constraint-shell action (66), and

$$D(\xi,\pi) = \prod_{\tau,\sigma} \prod_{\mu} \frac{d\xi_\mu d\pi_\mu}{2\pi}, \quad (109)$$

$$\Delta_{fp} = \prod_{\tau,\sigma} \delta(\phi_1))\delta(\pi_0)\delta(\phi_2))\delta(\xi_0) \det B^{-1}, \ \det B = \det\left\{\phi_1,\phi_2,\pi_0,\xi_0\right\} \quad (110)$$

is the FP determinant given in the monograph [9].

5. HAMILTONIAN DYNAMICS OF GENERAL RELATIVITY

5.1. Action and Geometry. General relativity (GR) is given by the singular Einstein–Hilbert action with the matter fields

$$W(g|\mu) = \int d^4x\sqrt{-g}\left[-\frac{\mu^2}{6}R(g) + \mathcal{L}_{\text{matter}} \right] \qquad \left(\mu^2 = M^2_{\text{Planck}}\frac{3}{8\pi} \right) \quad (111)$$

and by a measurable interval in the Riemannian geometry

$$(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta. \quad (112)$$

They are invariant with respect to general coordinate transformations

$$x_\mu \to x'_\mu = x'_\mu(x_0,x_1,x_2,x_3). \quad (113)$$

5.2. Variables and Hamiltonian. The generalized Hamiltonian approach to GR was formulated by Dirac and Arnovit, Deser and Misner [2] as a theory of a system with constraints in $3+1$ foliated space-time

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2 dt^2 - {}^{(3)}g_{ij}\breve{dx}^i\breve{dx}^j \qquad (\breve{dx}^i = dx^i + N^i dt) \quad (114)$$

with the lapse function $N(t,\vec{x})$, three shift vectors $N^i(t,\vec{x})$, and six space components ${}^{(3)}g_{ij}(t,\vec{x})$ depending on the coordinate time t and the space coordinates $\vec{x}$. The Dirac-ADM parametrization of metric (114) characterizes a family of hypersurfaces $t = \mathrm{const}$. with the unit normal vector $\nu^\alpha = (1/N, -N^k/N)$ to a hypersurface and with the second (external) form

$$\frac{1}{N}({}^{(3)}\dot{g}_{ij}) - \Delta_i N_j - \Delta_j N_i \quad (115)$$

that shows how this hypersurface is embedded into the four-dimensional space-time.

Coordinate transformations conserving the family of hypersurfaces $t = \mathrm{const}$

$$t \to \tilde{t} = \tilde{t}(t), \qquad x_i \to \tilde{x}_i = \tilde{x}_i(t, x_1, x_2, x_3), \quad (116)$$

$$\tilde{N} = N\frac{dt}{d\tilde{t}}, \qquad \tilde{N}^k = N^i\frac{\partial\tilde{x}^k}{\partial x_i}\frac{dt}{d\tilde{t}} - \frac{\partial\tilde{x}^k}{\partial x_i}\frac{\partial x^i}{\partial\tilde{t}} \quad (117)$$

are called a kinemetric subgroup of the group of general coordinate transformations (113) [4, 5, 7, 31]. The group of kinemetric transformations is the group of diffeomorphisms of the generalized Hamiltonian dynamics. It includes reparametrizations of the nonobservable time coordinate $\tilde{t}(t))$ (116) that play the principal role in the procedure of the reparametrization-invariant reduction discussed in the previous Sections. The main assertion of the invariant reduction is the following: the dynamic evolution parameter is not the coordinate but the variable with a negative contribution to the energy constraint. (Recall that this reduction is based on the explicit resolving of the global energy constraint with respect to the conjugate momentum of the dynamic evolution parameter to convert this momentum into the Hamiltonian of evolution of the reduced system.)

A negative contribution to the energy constraint is given by the space-metric-determinant logarithm. Therefore, following papers [3, 4, 13, 14, 29, 34] we introduce an invariant evolution parameter $\varphi_0(t)$ as the zero Fourier harmonic component of this logarithm (treated, in cosmology, as the cosmic scale factor). This variable is distinguished in general relativity by the Lichnerowicz conformal-type transformation of field variables f with the conformal weight (n) [35]

$${}^{(n)}\bar{f} = {}^{(n)}f\left(\frac{\varphi_0(t)}{\mu}\right)^{-n}, \quad (118)$$

where $n = 2,\ 0,\ -3/2,\ -1$ for the tensor, vector, spinor, and scalar fields, respectively, $\bar{f}$ is the so-called conformal-invariant variable used in GR for the analysis of initial data [29, 35]. In particular, for metric we get

$$g_{\mu\nu}(t,\vec{x}) = \left(\frac{\varphi_0(t)}{\mu}\right)^2 \bar{g}_{\mu\nu}(t,\vec{x}) \;\Rightarrow\; (ds)^2 = \left(\frac{\varphi_0(t)}{\mu}\right)^2 [\bar{N}^2 dt^2 - {}^{(3)}\bar{g}_{ij}\check{d}x^i\check{d}x^j]. \tag{119}$$

As the zero Fourier harmonic is extracted from the space metric determinant logarithm, the space metric $\bar{g}_{ij}(t,\vec{x})$ should be defined in a class of nonzero harmonics

$$\int d^3x \log||\bar{g}_{ij}(t,\vec{x})|| = 0. \tag{120}$$

The transformational properties of the curvature $R(g)$ with respect to the transformations (119) lead to the action (111) in the form [4]

$$W(g|\mu) = W(\bar{g}|\varphi_0) - \int\limits_{t_1}^{t_2} dt \int\limits_{V_0} d^3x \varphi_0 \frac{d}{dt}\left(\frac{\dot{\varphi}_0\sqrt{\bar{g}}}{\bar{N}}\right). \tag{121}$$

This form defines the global lapse function N_0 as the average of the lapse function $\bar{N}$ in the metric $\bar{g}$ over the kinemetric invariant space volume

$$N_0(t) = \frac{V_0}{\int\limits_{V_0} d^3x \frac{\sqrt{\bar{g}(t,\vec{x}}}{\bar{N}(t,\vec{x})}}, \qquad \bar{g} = \det({}^{(3)}\bar{g}), \qquad V_0 = \int\limits_{V_0} d^3x, \tag{122}$$

where V_0 is a free parameter which in the perturbation theory has the meaning of a finite volume of the free coordinate space. The lapse function $\bar{N}(t,\vec{x})$ can be factorized into the global component $N_0(t)$ and the local one $\mathcal{N}(t,\vec{x})$

$$\bar{N}(t,\vec{x})\bar{g}^{-1/2} := N_0(t)\mathcal{N}(t,\vec{x}) := N_q, \tag{123}$$

where $\mathcal{N}$ fulfills normalization condition:

$$I[\mathcal{N}] := \frac{1}{V_0}\int \frac{d^3x}{\mathcal{N}} = 1 \tag{124}$$

that is imposed after the procedure of variation of action, to reproduce equations of motion of the initial theory. In the Dirac harmonical variables [1] chosen as

$$q^{ik} = \bar{g}\bar{g}^{ik}, \tag{125}$$

the metric (114) takes the form

$$(ds)^2 = \frac{\varphi_0(t)^2}{\mu^2} q^{1/2} \left(N_q^2 dt^2 - q_{ij}\check{d}x^i\check{d}x^j \right) \qquad (q = \det(q^{ij})). \tag{126}$$

The Dirac–Bergmann version of action (121) in terms of the introduced above variables reads [4, 5]

$$W = \int\limits_{t_1}^{t_2} dt \left\{ L + \frac{1}{2}\partial_t(P_0\varphi_0) \right\}, \tag{127}$$

$$L = \left[\int\limits_{V_0} d^3x \left(\sum_F P_F \dot{F} - N^i \mathcal{P}_i \right) \right] - P_0\dot{\varphi}_0 - N_0 \left[-\frac{P_0^2}{4V_0} + I^{-1} H(\varphi_0) \right], \tag{128}$$

where

$$\sum_F P_F \dot{F} = \sum_f p_f \dot{f} - \pi_{ij}\dot{q}^{ij}, \tag{129}$$

$$H(\varphi_0) = \int d^3x \mathcal{N}\mathcal{H}(\varphi_0) \tag{130}$$

is the total Hamiltonian of the local degrees of freedom,

$$\mathcal{H}(\varphi_0) = \frac{6}{\varphi_0^2} q^{ij} q^{kl} [\pi_{ik}\pi_{jl} - \pi_{ij}\pi_{kl}] + \frac{\varphi_0^2 q^{1/2}}{6} {}^{(3)}R(\bar{g}) + \mathcal{H}_f, \tag{131}$$

and

$$\mathcal{P}_i = 2[\nabla_k(q^{kl}\pi_{il}) - \nabla_i(q^{kl}\pi_{kl})] + \mathcal{P}_{if} \tag{132}$$

are the densities of energy and momentum and $\mathcal{H}_f, \mathcal{P}_f$ are contributions of the matter fields. In the following, we call the set of the field variables F (129) with the dynamic evolution parameter φ_0 the field world space. The local part of the momentum of the space metric determinant

$$\pi(t, x) := q^{ij}\pi_{ij} \tag{133}$$

is given in the class of functions with the nonzero Fourier harmonics, so that

$$\int d^3x \pi(t, x) = 0, \tag{134}$$

(compare with equations (53), (54), in the previous Section).

The geometric foundation of introducting the global variable (119) in GR was given in [34] as the assertion about the nonzero value of the second form in the whole space. This assertion (which contradicts the Dirac gauge $\pi = 0$) follows from the global energy constraint, as, in the lowest order of the Dirac perturbation theory, positive contributions of particle-like excitations to the zero Fourier harmonic of the energy constraint can be compensated only by the nonzero value of the second form.

The aim of this Section is to obtain the dynamic «equivalent» unconstrained system in the field world space (F, φ_0) by explicit resolving the global energy constraint and to find the geometric unconstrained system by the Levi–Civita-type canonical transformation considered for a particle and a string in the previous Sections.

5.3. Local Constraints and Equations of Motion. Following Dirac [1] we formulate generalized Hamiltonian dynamics for the considered system (127). It means the inclusion of momenta for $\mathcal{N}$ and N_i and appropriate terms with Lagrange multipliers

$$W^D = \int\limits_{t_1}^{t_2} dt \left\{ L^D + \frac{1}{2}\partial_t (P_0 \varphi_0) \right\},$$

$$L^D = L + \int d^3x (P_{\mathcal{N}} \dot{\mathcal{N}} + P_{N^i} \dot{N}^i - \lambda^0 P_{\mathcal{N}} - \lambda^i P_{N^i}). \tag{135}$$

We can define extended Dirac Hamiltonian as

$$H^D = N_0 \left[-\frac{P_0^2}{4V_0} + I^{-1} H(\varphi_0) \right] + \int d^3x (\lambda^0 P_{\mathcal{N}} + \lambda^i P_{N^i}). \tag{136}$$

The equations obtained from variation of W^D with respect to Lagrange multipliers are called first class primary constraints

$$P_{\mathcal{N}} = 0, \qquad\qquad P_{N^i} = 0. \tag{137}$$

The condition of conservation of these constraints in time leads to the first class secondary constraints

$$\left\{ H^D, P_{\mathcal{N}} \right\} = \mathcal{H} - \frac{\int d^3x \mathcal{N} \mathcal{H}}{V_0 \mathcal{N}^2} = 0, \qquad\qquad \left\{ H^D, P_{N^i} \right\} = \mathcal{P}_i = 0 \tag{138}$$

(compare with equations (59) and (64) in Section 4). For completeness of the system we have to include a set of secondary constraints. According Dirac we choose them in the form

$$\mathcal{N}(t, \vec{x}) = 1, \qquad\qquad N^i(t, \vec{x}) = 0, \tag{139}$$

$$\pi(t, \vec{x}) = 0, \qquad\qquad \chi^j := \partial_i (q^{-1/3} q^{ij}) = 0. \tag{140}$$

The equations of motion obtained for the considered system are

$$\frac{dF}{dT} = \frac{\partial H(\varphi_0)}{\partial P_F}, \quad -\frac{dP_F}{dT} = \frac{\partial H(\varphi_0)}{\partial F}, \tag{141}$$

where $H(\varphi_0)$ is given by the equation (130), and we introduced the invariant geometric time T

$$N_0 dt := dT. \tag{142}$$

5.4. Global Constraints and Equations of Motion. The physical meaning of the geometric time T, the dynamic variable φ_0 and its momentum is given by the explicit resolving of the zero-Fourier harmonic of the energy constraint

$$\frac{\delta W^E}{\delta N_0(t)} = -\frac{P_0^2}{4V_0} + H(\varphi_0) = 0. \tag{143}$$

This constraint has two solutions for the global momentum P_0:

$$(P_0)_\pm = \pm 2\sqrt{V_0 H(\varphi_0)} \equiv H^*_\pm. \tag{144}$$

The equation of motion for this global momentum P_0 in gauge (139) takes the form

$$\frac{\delta W^E}{\delta P_0} = 0 \Rightarrow \left(\frac{d\varphi}{dT}\right)_\pm = \frac{(P_0)_\pm}{2V} = \pm\sqrt{\rho(\varphi_0)}, \quad \rho(\varphi_0) = \frac{\int d^3x\mathcal{H}}{V_0} = \frac{H(\varphi_0)}{V_0}. \tag{145}$$

The integral form of the last equation is

$$T_\pm(\varphi_1, \varphi_0) = \pm\int\limits_{\varphi_1}^{\varphi_0} d\varphi\rho^{-1/2}(\varphi), \tag{146}$$

where $\varphi_1 = \varphi_0(t_1)$ is the initial data. Equation obtained by varying the action with respect to φ_0 follows independently from the set of all other constraints and equations of motion.

In quantum theory of GR (like in quantum theories of a particle and string considered in Sections 3, 4), we get two Schrödinger equations

$$i\frac{d}{d\varphi_0}\Psi^\pm(F|\varphi_0, \varphi_1) = H^*_\pm(\varphi_0)\Psi^\pm(F|\varphi_0, \varphi_1) \tag{147}$$

with positive and negative eigenvalues of P_0 and normalizable wave functions with the spectral series over quantum numbers Q

$$\Psi^+(F|\varphi_0, \varphi_1) = \sum_Q A^+_Q \langle F|Q\langle\rangle Q|\varphi_0, \varphi_1\rangle\theta(\varphi_0 - \varphi_1), \tag{148}$$

$$\Psi^-(F|\varphi_0, \varphi_1) = \sum_Q A^-_Q \langle F|Q\rangle^* \langle Q|\varphi_0, \varphi_1\rangle^*\theta(\varphi_1 - \varphi_0), \tag{149}$$

where $\langle F|Q\rangle$ is the eigenfunction of the reduced energy (144)

$$H^*_\pm(\varphi_0)\langle F|Q\rangle = \pm E(Q, \varphi_0)\langle F|Q\rangle, \tag{150}$$

$$\langle Q|\varphi_0,\varphi_1\rangle = \exp\,[-i\int\limits_{\varphi_1}^{\varphi_0} d\varphi E(Q,\varphi)],\;\; \langle Q|\varphi_0,\varphi_1\rangle^* = \exp\,[i\int\limits_{\varphi_1}^{\varphi_0} d\varphi E(Q,\varphi)]. \tag{151}$$

The coefficient A^+_Q, in «secondary» quantization, can be treated as the operator of creation of a universe with positive energy; and the coefficient A^-_Q, as the operator of annihilation of a universe also with positive energy. The «secondary» quantization means $[A^-_Q, A^+_{Q'}] = \delta_{Q,Q'}$. The physical states of a quantum universe are formed by the action of these operators on the vacuum $\langle 0|$, $|0\rangle$ in the form of out-state ($|Q\rangle = A^+_Q|0\rangle$) with positive «frequencies» and in-state ($\langle Q| = \langle 0|A^-_Q$) with negative «frequencies». This treatment means that positive frequencies propagate forward ($\varphi_0 > \varphi_1$); and negative frequencies, backward ($\varphi_1 > \varphi_0$), so that the negative values of energy are excluded from the spectrum to provide the stability of the quantum system in quantum theory of GR (similar in QFT in Section 3). In other words, instead of changing the sign of energy, we change that of the dynamic evolution parameter, which leads to the causal Green function

$$\begin{aligned} G_c(F_1,\varphi_1|F_2,\varphi_2) &= G_+(F_1,\varphi_1|F_2,\varphi_2)\theta(\varphi_2-\varphi_1) + \\ &+ G_-(F_1,\varphi_1|F_2,\varphi_2)\theta(\varphi_1-\varphi_2), \end{aligned} \tag{152}$$

where $G_+(F_1,\varphi_1|F_2,\varphi_2) = G_-(F_2,\varphi_2|F_1,\varphi_1)$ is the «commutative» Green function

$$G_+(F_2,\varphi_2|F_1,\varphi_1) = \langle 0|\Psi^-(F_2|\varphi_2,\varphi_1)\Psi^+(F_1|\varphi_1,\varphi_1)|0\rangle. \tag{153}$$

For this causal convention, the geometric time (146) is always positive in accordance with the equations of motion (145)

$$\left(\frac{dT}{d\varphi_0}\right)_\pm = \pm\sqrt{\rho} \quad\Rightarrow\quad T_\pm(\varphi_1,\varphi_0) = \pm\int\limits_{\varphi_1}^{\varphi_0} d\varphi \rho^{-1/2}(\varphi) \geqslant 0. \tag{154}$$

Thus, the causal structure of the field world space immediatly leads to the arrow of the geometric time (154) and the beginning of evolution of a universe with respect to the geometric time $T = 0$.

As we have seen in Sections 3 and 4, the way to obtain conserved integrals of motion in classical theory and quantum numbers Q in quantum theory is the Levi-Civita-type canonical transformation of the field world space (F,φ_0) to a geometric set of variables (V,Q_0) with the condition that the geometric evolution parameter Q_0 coincides with the geometric time $dT = dQ_0$ (see Fig. 3).

Equations (145), (146) in the homogeneous approximation of GR are the basis of observational cosmology where the geometric time is the conformal time

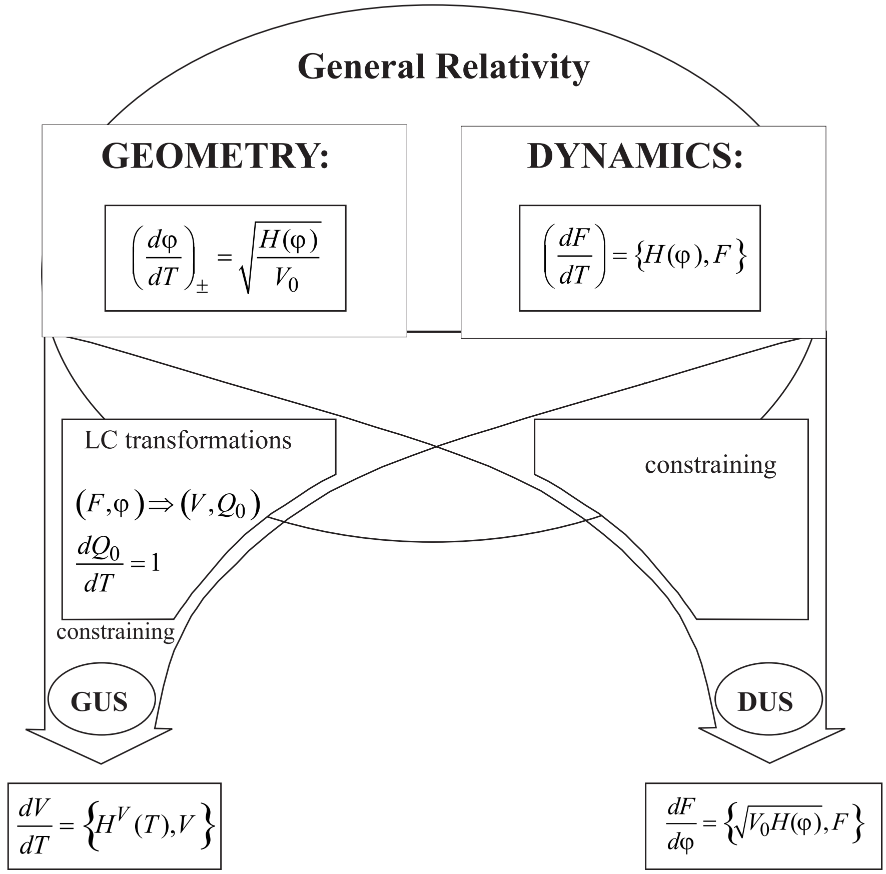

Fig. 3. Reparametrization-invariant dynamics of General Relativity is covered by the Dynamic Unconstrained Systems (DUS) and the Geometric Unconstrained Systems (GUS) connected by the Levi-Civita (LC) transformations of fields of MATTER into the vacuum fields of initial data with respect to geometric TIME

connected with the world time T_f of the Friedmann cosmology by the relation

$$dT_f = \frac{\varphi_0(T)}{\mu} dT, \tag{155}$$

and the dependence of scale factor (dynamic evolution parameter φ_0) on the geometric time T is treated as the evolution of the universe. In particular, equation (145) gives the relation between the present-day value of the dynamic evolution parameter $\varphi_0(T_0)$ and cosmological observations, i.e., the density of matter ρ and the Hubble parameter

$$\mathcal{H}^{e}_{\rm hub} = \frac{\mu\varphi_0'}{\varphi_0^2} = \frac{\mu\sqrt{\rho}}{\varphi_0^2} \qquad \Rightarrow \qquad \varphi_0(T_0) = \left(\frac{\mu\sqrt{\rho}}{\mathcal{H}_{\rm hub}}\right)^{1/2} := \mu\Omega_0^{1/4}, \qquad (156)$$

where $(0.6 < (\Omega_0^{1/4})_{\rm exp} < 1.2)$. The dynamic evolution parameter as the cosmic scale factor and a value of its conjugate momentum (i.e., a value of the dynamic Hamiltonian) as the density of matter (see equations (145), (146)) are objects of measurement in observational astrophysics and cosmology and numerous discussions about the Hubble parameter, dark matter, and hidden mass.

The general theory of reparametrization-invariant reduction described in the previous Sections can be applied also to GR. In accordance with this theory, the reparametrization-invariant dynamics of GR is covered by two unconsrained systems (dynamic and geometric) connected by the Levi-Civita canonical transformation which solves the problems of the initial data, conserved quantum numbers, and direct corrrespondence of standard classical cosmology with quantum gravity on the level of the generating functional of the unitary and causal perturbation theory [7, 15].

5.5. Equivalent Unconstrained Systems. Assume that we can solve the constraint equations and pass to the reduced space of independent variables (F^*, P_F^*). The explicit solution of the local and global constraints has two analytic branches with positive and negative values for scale factor momentum P_0 (144). Therefore, inserting solutions of all constraints into the action we get two branches of the equivalent Dynamic Unconstrained System (DUS)

$$W_{\pm}^{\rm DUS}[F|\varphi_0] = \int\limits_{\varphi_1}^{\varphi_2} d\varphi_0 \left\{ \left[\int d^3x \sum_{F^*} P_F^* \frac{\partial F^*}{\partial \varphi_0} \right] - H_{\pm}^* + \frac{1}{2}\partial_{\varphi_0}(\varphi_0 H_{\pm}^*) \right\}, \qquad (157)$$

where φ_0 plays the role of evolution parameter and $H_{\pm}^*$ defined by equation (144) plays the role of the evolution Hamiltonian, in the reduced phase space of independent physical variables (F^*, P_F^*) with equations of motion

$$\frac{dF^*}{d\varphi_0} = \frac{\partial H_{\pm}^*}{\partial P_F^*}, \qquad -\frac{dP_F^*}{d\varphi_0} = \frac{\partial H_{\pm}^*}{\partial F^*}. \qquad (158)$$

The evolution of the field world space variables (F^*, φ_0) with respect to the geometric time T is not contained in DUS (157). This geometric time evolution is described by supplementary equation (145) for nonphysical momentum P_0 (144) that follows from the initial extended system.

To get an equivalent unconstrained system in terms of the geometric time (we call it the Geometric Unconstrained System (GUS)), we need the Levi-Civita canonical transformation (LC) [12, 22, 23] of the field world phase space

$$(F^*, P_F^* | \varphi_0, P_0) \Rightarrow (F_G^*, P_G^* | Q_0, \Pi_0) \tag{159}$$

which converts the energy constraint (143) into the new momentum Π_0 (see the similar transformations for a relativistic particle and a string in Sections 3, 4).

In terms of geometrical variables the action takes the form

$$W^G = \int_{t_1}^{t_2} dt \left\{ \left[\int d^3x \sum_{F_G^*} P_G^* \dot{F}_G^* \right] - \Pi_0 \dot{Q}_0 + N_0 \Pi_0 + \frac{d}{dt} S^{\mathrm{LC}} \right\}, \tag{160}$$

where $S^{-\mathrm{LC}}$ is generating function of LC transformations. Then the energy constraint and the supplementary equation for the new momentum take trivial form

$$\Pi_0 = 0 \;; \qquad \frac{\delta W}{\delta \Pi_0} = 0 \qquad \Rightarrow \qquad \frac{dQ_0}{dt} = N_0 \qquad \Rightarrow \qquad dQ_0 = dT. \tag{161}$$

Equations of motion are also trivial

$$\frac{dP_G^*}{dT} = 0, \qquad \frac{dF_G^*}{dT} = 0, \tag{162}$$

and their solutions are given by the initial data

$$P_G^* = P_G^{*\,0}, \qquad F_G^* = F_G^{*\,0}. \tag{163}$$

Substituting solutions of (161) and (162) into the inverted Levi-Civita transformations

$$F^* = F^*(Q_0, \Pi_0 | F_G^*, P_G^*), \qquad \varphi_0 = \varphi_0(Q_0, \Pi_0 | F_G^*, P_G^*) \tag{164}$$

and similar for momenta, we get formal solutions of (158) and (146)

$$F^* = F^*(T, 0 | F_G^{*\,0}, P_G^{*\,0}),\; P_F^* = P_F^*(T, 0 | F_G^{*\,0}, P_G^{*\,0}),\; \varphi_0 = \varphi_0(T, 0 | F_G^{*\,0}, P_G^{*\,0}). \tag{165}$$

We see that evolution of the dynamic variables with respect to the geometric time (i.e., the evolution of a universe) is absent in DUS. The evolution of the dynamic variables with respect to the geometric time can be described in the form of the LC (inverted) canonical transformation of GUS into DUS (164), (165) (see Fig. 4).

There is also the weak form of Levi-Civita-type transformations to GUS $(F^*, P_F^*) \;\; \Rightarrow \;\; (\tilde{F}, \tilde{P})$ without action-angle variables and with a constraint

$$\tilde{\Pi}_0 - \tilde{H}(\tilde{Q}_0, \tilde{F}, \tilde{P}) = 0. \tag{166}$$

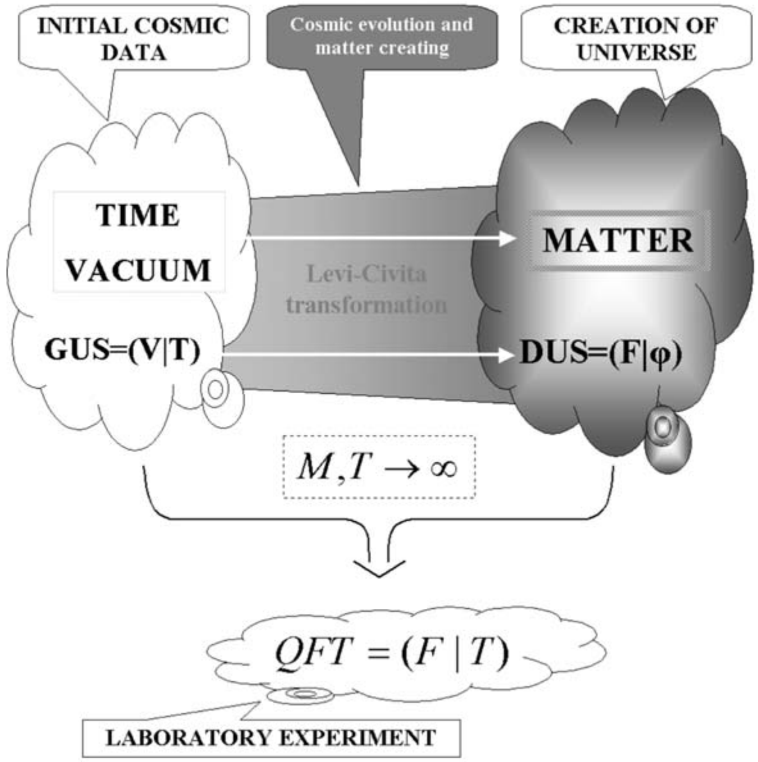

Fig. 4. Reparametrization-invariant dynamics of «Big Bang» of a quantum universe: the Dynamic Unconstrained System (DUS) describes creation of a universe in the field world space where we have only MATTER $F|\varphi$; the Geometric Unconstrained System (GUS) describes initial cosmic data (i.e., the Bogoliubov squeezed VACUUM) with respect to the geometric TIME measured by an observer; the inverse Levi-Civita canonical transformation describes the cosmic (Hubble) evolution and creation of matter from the VACUUM. The standard quantum field theory (QFT) in the form of the Faddeev–Popov generating functional for the unitary S matrix appears in the limits of tremendous mass, volume, and geometric lifetime of a universe

We get the constraint-shell action

$$\tilde{W}^{\rm GUS} = \int dT \left\{ \left[\int d^3x \sum_{\tilde{F}} \tilde{P} \frac{d\tilde{F}}{dT} \right] - \tilde{H}(T, \tilde{F}, \tilde{P}) \right\}, \qquad (167)$$

that allows us to choose the initial cosmological data with respect to the geometric time.

Recall that the considered reduction of the action reveals the difference of reparametrization-invariant theory from the gauge-invariant theory: in gauge-invariant theory the superfluous (longitudinal) variables are completely excluded from the reduced system; whereas, in reparametrization-invariant theory the superfluous (longitudinal) variables leave the sector of the Dirac observables (i.e., the phase space (F^*, P_F^*)) but not the sector of measurable quantities: superfluous (longitudinal) variables become the dynamic evolution parameter and dynamic Hamiltonian of the reduced theory.

5.6. Reparametrization-Invariant Path Integral. Following Faddeev–Popov procedure we can write down the path integral for local fields of our theory using constraints and gauge conditions (137)–(140):

$$Z_{\rm local}(F_1, F_2|P_0, \varphi_0, N_0) = \int\limits_{F_1}^{F_2} D(F, P_f)\Delta_s\bar{\Delta}_t \exp\left\{i\bar{W}\right\}, \tag{168}$$

where

$$D(F, P_f) = \prod_{t,x}\left(\prod_{i<k}\frac{dq^{ik}d\pi_{ik}}{2\pi}\prod_f\frac{df\,dp_f}{2\pi}\right) \tag{169}$$

are functional differentials for the metric fields (π, q) and the matter fields (p_f, f),

$$\Delta_s = \prod_{t,x,i}\delta(\mathcal{P}_i))\delta(\chi^j)\det\{\mathcal{P}_i, \chi^j\}, \tag{170}$$

$$\bar{\Delta}_t = \prod_{t,x}\delta(\mathcal{H}(\mu))\delta(\pi)\det\{\mathcal{H}(\varphi_0) - \rho, \pi\} \qquad \left(\rho = \frac{\int d^3x H(\varphi_0)}{V_0}\right) \tag{171}$$

are the F-P determinants, and

$$\bar{W} = \int\limits_{t_1}^{t_2} dt\left\{\int\limits_{V_0} d^3x\left(\sum_F P_F\dot{F}\right) - P_0\dot{\varphi}_0 - N_0\left[-\frac{P_0^2}{4V_0} + H(\varphi_0)\right] + \frac{1}{2}\partial_t(P_0\varphi_0)\right\} \tag{172}$$

is extended action of considered theory.

By analogy with a particle and a string considered in Sections 3 and 4 we define a commutative Green function as an integral over global fields (P_0, φ_0) and the average over reparametrization group parameter N_0

$$G_+(F_1, \varphi_1|F_2, \varphi_2) = \int\limits_{\varphi_1}^{\varphi_2}\prod_t\left(\frac{d\varphi_0 dP_0 d\tilde{N}_0}{2\pi}\right) Z_{\rm local}(F_1, F_2|P_0, \varphi_0, N_0), \tag{173}$$

where

$$\tilde{N} = N/2\pi\delta(0), \qquad \delta(0) = \int dN_0. \tag{174}$$

The causal Green function in the world field space (F, φ_0) is defined as the sum

$$\begin{aligned} G_c(F_1, \varphi_1|F_2, \varphi_2) &= G_+(F_1, \varphi_1|F_2, \varphi_2)\theta(\varphi_1 - \varphi_2) + \\ &+ G_+(F_2, \varphi_1|F_2, \varphi_1)\theta(\varphi_2 - \varphi_1). \end{aligned} \tag{175}$$

This function will be considered as generating functional for the unitary S-matrix elements [25]

$$S[1,2] = \langle \text{out } (\varphi_2)|T_\varphi \exp\left\{-i\int\limits_{\varphi_1}^{\varphi_2} d\varphi (H_I^*)\right\}|(\varphi_1) \text{ in}\rangle, \tag{176}$$

where T_φ is a symbol of ordering with respect to parameter φ_0, and $\langle \text{out } (\varphi_2)|$, $|(\varphi) \text{ in}\rangle$ are states of quantum univers in the lowest order of the Dirac perturbation theory ($\mathcal{N} = 1;\ \ N^k = 0;\ q^{ij} = \delta_{ij} + h_{ij}$), H_I^* is the interaction Hamiltonian

$$H_I^* = H^* - H_0^*, \quad H^* = 2\sqrt{V_0 H(\varphi)}, \qquad H_0^* = 2\sqrt{V_0 H_0(\varphi)}, \tag{177}$$

H_0 is a sum of the Hamiltonians of «free» fields (gravitons, photons, massive vectors, and spinors) where all masses (including the Planck mass) are replaced by the dynamic evolution parameter φ_0 [7]. For example for gravitons the «free» Hamiltonian takes the form:

$$H_0(\varphi_0) = \int d^3x \left(\frac{6(\pi_{(h)})^2}{\varphi_0^2} + \frac{\varphi_0^2}{24}(\partial_i h)^2\right) \quad (h_{ii} = 0;\ \ \partial_j h_{ji} = 0). \tag{178}$$

In order to reproduce Faddeev–Popov integral for general relativity in infinite space-time [19], one should fix the dynamic evolution parameter at its present-day value $\varphi_0 = \mu$ (156), remove all the zero-mode dynamics $P_0 = \dot{\varphi}_0 = 0, N_0 = 1$, and neglect the surface Newton term in the Hamiltonian. We get

$$Z^{\text{FP}}(F_1, F_2) = Z_{\text{local}}(F_1, F_2|P_0 = 0, \varphi_{0\exp} = \mu, N_0 = 1), \tag{179}$$

or

$$Z^{\text{FP}}(F_1, F_2) = \int\limits_{F_1}^{F_2} D(F, P_f)\Delta_s\Delta_t \exp\left\{iW_{fp}\right\}, \tag{180}$$

where

$$W_{fp} = \int\limits_{-\infty}^{+\infty} dt \int d^3x \left(\sum_F P_F\dot{F} - \mathcal{H}_{fp}(\mu)\right), \qquad \mathcal{H}_{fp}(\mu) = \mathcal{H}(\mu) - \frac{\mu^2}{6}\partial_i\partial_j q^{ij}, \tag{181}$$

and

$$\Delta_t = \prod_{t,x} \delta(\mathcal{H}(\mu))\delta(\pi)\det\{\mathcal{H}(\mu), \pi\}. \tag{182}$$

The FP integral (180) is considered as the generating functional for unitary perturbation theory in terms of S-matrix elements

$$S[-\infty|+\infty] = \langle \text{out}| T \exp\left\{-i \int\limits_{-\infty}^{+\infty} dt H_I(\mu)\right\}|\text{in}\rangle. \tag{183}$$

Strictly speaking, the approximation (179) is not a correct procedure, as it breaks the reparametrization-invariance. The range of validity of FP integral (180) is discussed in next sections.

6. REPARAMETRIZATION-INVARIANT DYNAMICS OF EARLY UNIVERSE

6.1. Dynamic Unconstrained System. Possible states of a free quantum universe in S matrix (176) (see Fig. 5) are determined by the lowest order of the Dirac perturbation theory given by the well-known system of «free» conformal fields (118), (178) in a finite space-time volume [7, 36]

$$\begin{aligned} W_0^E \;\; &= \int\limits_{t_1}^{t_2} dt \left(\left[\int d^3x \sum_F P_F \dot{F} \right] - P_0 \dot{\varphi}_0 - \right. \\ &- \left. N_0 \left[-\frac{P_0^2}{4V} + H_0(\varphi_0) \right] + \frac{1}{2}\partial_0(P_0\varphi_0) \right), \end{aligned} \tag{184}$$

where H_0 is a sum of the Hamiltonians of «free» fields (gravitons (178), photons, massive vectors, and spinors) where all masses (including the Planck mass) are replaced by the dynamic evolution parameter φ_0 [7].

The classical equations for the action (184)

$$\frac{dF}{dT} = \frac{\partial H_0}{\partial P_F}, \qquad -\frac{dP_F}{dT} = \frac{\partial H_0}{\partial F}, \qquad P_0 = \pm 2\sqrt{V_0 H_0} := H_0^* \tag{185}$$

contain two invariant times: the geometric T and the dynamic $\varphi_0^\pm$ connected by the geometro-dynamic (back-reaction) equation

$$\frac{d\varphi_0^\pm}{dT} = \pm\sqrt{\rho_0(\varphi_0^\pm)}, \qquad \left(\rho_0 = \frac{H_0}{V_0}\right). \tag{186}$$

$$W^E(g,\Psi)=\int d^4x\left\{-\frac{\mu^2\sqrt{-g}}{6}R(g)+L_{\rm matter}(g,\Psi)\right\},\mu=m_{\rm Planc}\sqrt{\frac{3}{8\pi}}$$

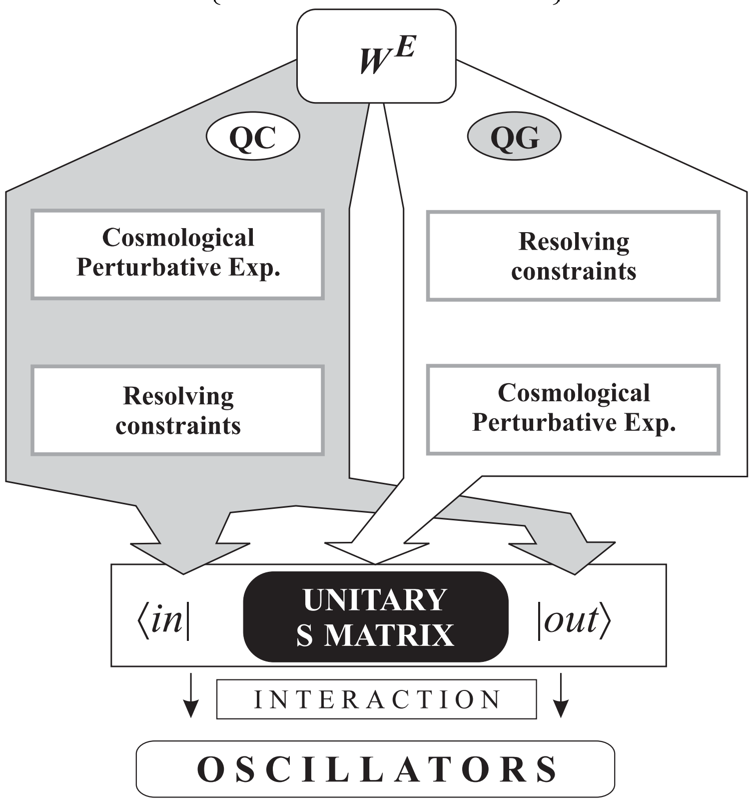

Fig. 5. To obtain the unitary S matrix in terms of invariants, one can use two ways: quantum cosmology (QC) and quantum gravity (QG). The first way (QC) is to formulate the cosmological perturbation theory and resolve constraints; this way is suitable for constructing «in» and «out» states as systems of «free oscillators». The second way (QG) is to resolve constraints and to formulate perturbation theory; this way is more suitable for constructing the unitary S matrix elements between the states of the Quantum Universe. Both the ways should be consistent

Solving the energy constraint we get the action for dynamic system

$$W_0^E(\text{constraint})=W_0^D=\int\limits_{\varphi(t_1)}^{\varphi(t_2)}d\varphi\left(\left[\int d^3x\sum_F P_F\partial_\varphi F\right]-H^*_{0\pm}+\frac{1}{2}\partial_\varphi(\varphi H^*_{0\pm})\right),\tag{187}$$

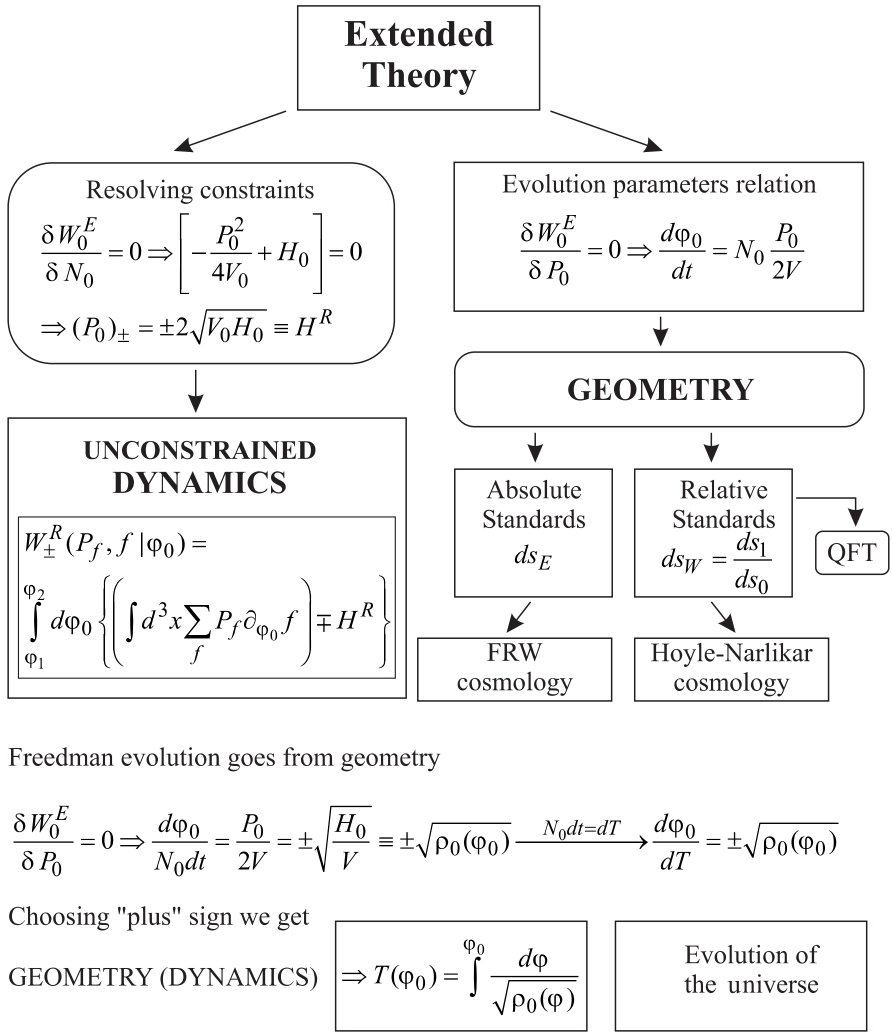

Fig. 6. The reduction means explicit resolving the energy constraint with respect to the momentum of the cosmic scale factor which gives a negative contribution to the constraint. As a result, we get an unconstrained version of «free» theory, where the cosmic scale factor φ_0 represents the dynamic evolution parameter, and its momentum converts into the reduced Hamiltonian ($H_O^R = 2\sqrt{V_0 H_0}$). However, the unconstrained dynamics is not sufficient to determine the geometrical interval of the proper time. The latter coincides with the Friedmann time for the absolute standard of measurement, in the FRW cosmology; or with the conformal time for the relative standard, in the Hoyle–Narlikar cosmology

that has two branches for a universe with a positive energy ($P_0 > 0$), and a universe with a negative energy ($P_0 < 0$). We interpret the branch with negative energy as an «antiuniverse» which propagates backward ($\varphi < 0$) with positive energy to provide the stability of a quantum system (see Fig. 6).

The content of matter in a universe is described by the number of particles $N_{F,k}$ and their energy $\omega_F(\varphi_0, k)$ (which depends on the dynamic evolution parameter φ_0 and quantum numbers k, momenta, spins, etc.). Detected particles are defined as the field variables $F = f$

$$f(x) = \sum_k \frac{C_f(\varphi_0)\exp(ik_i x_i)}{V_0^{3/2}\sqrt{2\omega_f(\varphi_0, k)}}\left(a_f^+(-k) + a_f^-(k)\right) \qquad (188)$$

which diagonalize the operator of the density of matter

$$\begin{aligned} \rho_0 &= \sum_{f,k} \frac{\omega_f(\varphi_0, k)}{V_0}\hat{N}_{f,k}, \\ \hat{N}_f(a) &= \frac{1}{2}(a_f^+ a_f^- + a_f^- a_f^+) \end{aligned} \qquad (189)$$

(see Fig. 7).

We restrict ourselves to gravitons ($f = h$) $C_h(\varphi_0) = \varphi_0\sqrt{12}$, $\omega_h(\varphi_0, k) = \sqrt{k^2}$ and massive vector particles ($f = v$) $C_v(\varphi_0) = 1$, $\omega_v(\varphi_0, k) = \sqrt{k^2 + y^2\varphi_0^2}$, where y is the mass in terms of the Planck constant.

6.2. Geometric Unconstrained System. The equations of motion (185) in terms of a^+, a^- [7] are not diagonal

$$i\frac{d}{dT}\chi := i\chi'_{a_f} = -\hat{H}_{a_f}\chi_{a_f}, \quad \chi_{a_f} = \begin{pmatrix} a_f^+ \\ a_f \end{pmatrix}, \ \hat{H}_{a_f} = \begin{vmatrix} \omega_{a_f}, & -i\Delta_f \\ -i\Delta_f, & -\omega_{a_f} \end{vmatrix}, \qquad (190)$$

where nondiagonal terms $\Delta_{f=h,v}$ are proportional to the Hubble parameter (156)

$$\Delta_{f=h} = \frac{\varphi_0'}{\varphi_0}, \qquad \Delta_{f=v} = -\frac{\omega_v'}{2\omega_v}, \qquad \varphi_0' = \sqrt{\rho_0}. \qquad (191)$$

The «geometric system» (b^+, b) is determined by the transformation to the set of variables which diagonalize equations of motion (185) and determine a set of integrals of motion of equations (185) (as conserved numbers $\{Q\}$).

To obtain integrals of motion and to choose initial conditions for a universe evolution we use the Bogoliubov transformations [37] and define «quasi-particles»

$$b^+ = \cosh(r)\,\mathrm{e}^{-i\theta}a^+ - i\sinh(r)\,\mathrm{e}^{i\theta}a, \quad b = \cosh(r)\,\mathrm{e}^{i\theta}a + i\sinh(r)\,\mathrm{e}^{-i\theta}a^+, \qquad (192)$$

"The most important aspect of any phenomenon from mathematical point of view is that of a MEASURABLE QUANTITY. I shall therefore consider electrical phenomena chiefly with a view to their measurement, describing the methods of measurement, and defining the STANDARDS on which they depend."

Maxwell J.C. 1873. *A Treatise on Electricity and Magnetism* (Oxford)

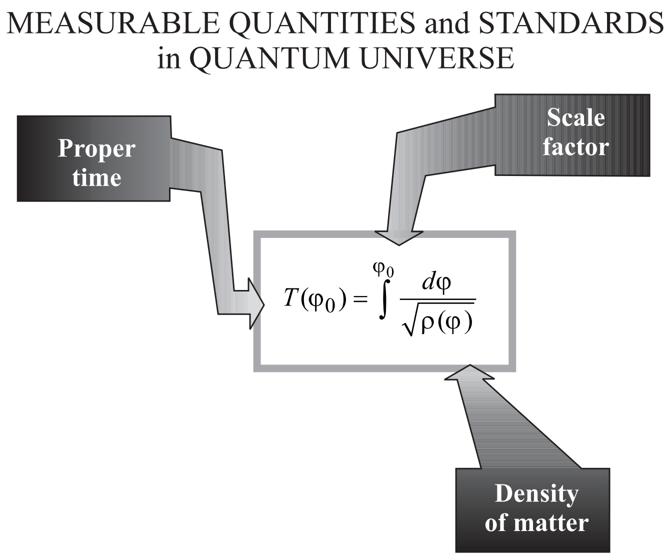

Fig. 7. The equation for dynamic evolution of the measurable time contains the energy density ρ which is treated as the measurable quantity in astrophysics and observational cosmology as the object of numerous discussions about the dark matter and hidden mass. Following the observational cosmology, we shall also treat this quantity ρ as the observable energy density and define «particles» as field variables in the holomorphic representation, which diagonalize this observable energy density

or

$$\chi_b = \begin{pmatrix} b^+ \\ b \end{pmatrix} = \hat{O}\chi_a,$$

which diagonalize the classical equations expressed in terms of particles (a^+, a), so that the number of quasiparticles is conserved

$$\frac{d(b^+b)}{dt} = 0, \qquad b = \exp\,(-i\int\limits_0^T d\bar{T}\bar{\omega}_b(\bar{T}))b_0 \qquad (193)$$

BOGOLIUBOV EQUATION
(state equations)

$$\underline{N}_0(r) = \cosh(2r) = {}_b\langle 0 \,|\, \hat{N}_a 0 \rangle_b \qquad \mathrm{H}_{\mathrm{Hubble}} = \frac{1}{\varphi}\frac{d\varphi}{dT} = \frac{\sqrt{\rho}}{\varphi}$$

$$\underline{\omega}_b^h = \omega_h \underline{N}_0 - \mathrm{H}_{\mathrm{Hubble}}\left[\sqrt{\underline{N}_0^2 - 1}\cos 2\theta + \underline{N}_0 \frac{\varphi\, d\theta}{d\varphi}\right]$$

$\underline{\omega}_b$ — quasiparticle energy

$$0 = \omega_h \sqrt{\underline{N}_0^2 - 1} - \mathrm{H}_{\mathrm{Hubble}}\left[\underline{N}_0 - \cos 2\theta + \sqrt{\underline{N}_0^2 - 1}\,\frac{\varphi\, d\theta}{d\varphi}\right]$$

$$\sqrt{\underline{N}_0^2 - 1}\,\sin 2\theta = -\frac{\varphi\, d\underline{N}_0}{2 d\varphi}$$

$$\boxed{\rho(\varphi) = \frac{{}_b\left\langle 0 \,|\, \sum_f \omega_f(\varphi) N_a^f \,|\, 0\right\rangle_b}{V_0}} = ?$$

Fig. 8. The Bogoliubov quasiparticles are defined as field variables which diagonalize the equations of motion and mark states of the universe by integrals of motion (i.e., quantum numbers in the corresponding quantum theory with a vacuum state $|0>_b$). The Bogoliubov transformation means the construction of a geometric unconstrained system (GUS), for which a new internal evolution parameter coincides with the conformal time. The Bogoliubov vacuum expectation value of the number of «particles» measured in the comoving frame, and the Hubble parameter (H_{Hubble})

(see Fig. 8). Functions r and θ in (192), and the quasiparticle energy $\bar{\omega}_b$ in (193) are determined by the equation of diagonalization

$$i\frac{d}{dT}\chi_b = [-i\hat{O}^{-1}\frac{d}{dT}\hat{O} - \hat{O}^{-1}\hat{H}_a\hat{O}]\chi_b \equiv -\begin{pmatrix} \bar{\omega}_b, & 0 \\ 0, & -\bar{\omega}_b \end{pmatrix}\chi_b \qquad (194)$$

in the form obtained in [7]

$$\bar{\omega}_{fb} = (\omega_f - \theta'_f)\cosh(2r_f) - (\Delta_f \cos 2\theta_f)\sinh(2r_f), \qquad (195)$$

$$0 = (\omega_f - \theta'_f)\sinh(2r_f) - (\Delta_f \cos 2\theta_f)\cosh(2r_f), \qquad r'_f = -\Delta_f \sin 2\theta_f.$$

Equations (191)–(195) are closed by the definition of «observable particles» in terms of quasiparticles

$$\rho(\varphi) = \frac{H_0}{V} = \frac{\sum\limits_f \omega_f(\varphi)\{a_f^+ a_f\}}{V_0}, \; \{a^+a\} = \{b_0^+ b_0\}\cosh 2r - \frac{i}{2}(b^{+2} - b^2)\sinh 2r \tag{196}$$

with

$$\bar{\omega}_{fb} = \sqrt{(\omega_f - \theta'_f)^2 + (r'_f)^2 - \Delta_f^2}, \qquad \theta'_f = -\frac{1}{2}\left(\frac{r'_f}{\Delta_f}\right)'\left[1 - \frac{(r'_f)^2}{\Delta_f^2}\right]^{-1/2},$$
$$\cosh(2r_f) = \frac{\omega_f - \theta'_f}{\bar{\omega}_{fb}}. \tag{197}$$

The constrained system in terms of geometric variables is described by the action

$$\tilde{W}^G = \int dt \left\{ \sum_f \frac{i}{2}\left(b\partial_t b^+ - b^+\partial_t b\right)_f - \tilde{\Pi}_0 \dot{Q}_0 - N_0\left[-\tilde{\Pi}_0 + \sum_f \omega_b^f(Q_0) N_f(b)\right]\right\}, \tag{198}$$

where the new dynamic evolution parameter Q_0 coincides with geometric time T on the equations of motion

$$\frac{\delta \tilde{W}^E}{\delta \tilde{\Pi}_0} = 0 \qquad \Rightarrow \qquad dQ_0 = dT. \tag{199}$$

Reduction of this system leads to the weak version of Geometric Unconstrained System (167)

$$\tilde{W}^{\rm GUS} = \int dT \left\{ \sum_f \frac{i}{2}\left(b\partial_T b^+ - b^+\partial_T b\right)_f - \sum_f \omega_b^f(T) N_f(b)\right\}. \tag{200}$$

We choose the initial data appropriate for the dynamics described by GUS (200).

6.3. Quantization. The initial data b_0, b_0^+ of quasiparticle variables (193) form the set of quantum numbers in quantum theory.

Let us suppose that we manage to solve equations (193)–(197) with respect to the geometric time T in terms of conserved numbers b_0^+, b_0. This means that the wave function of a quantum universe can be represented in the form of a series over the conserved quantum numbers $Q = n_{f,k} = \langle Q|b_f^+ b_f|Q\rangle$ of the Bogoliubov states (compare with the similar series for a relativistic string in Section 4)

$$\Phi_Q(T) = \prod_{f,n_f} \exp\left\{-i\int\limits_0^T dT n_f \bar{\omega}_b(T)\right\} \frac{(b_f^+)^{n_f}}{\sqrt{n_f!}}|0\rangle_b. \tag{201}$$

In this geometric system, we have an arrow of the geometric time T for a universe

$$T_+(\varphi_2, \varphi_1) = \int\limits_{\varphi_1}^{\varphi_2} d\varphi \rho(\varphi)^{-1/2} > 0, \qquad \varphi_2 > \varphi_1, \tag{202}$$

and for an antiuniverse

$$T_-(\varphi_2, \varphi_1) = -\int\limits_{\varphi_1}^{\varphi_2} d\varphi \rho(\varphi)^{-1/2} = \int\limits_{\varphi_2}^{\varphi_1} d\varphi \rho(\varphi)^{-1/2} > 0, \qquad \varphi_1 > \varphi_2. \tag{203}$$

The dynamic system (187) of particle variables a^+, a is connected with the geometric one by the Bogoliubov transformations. Using these transformations we can find wave functions of a universe, for $\varphi_2 > \varphi_1$ and an antiuniverse, for $\varphi_1 > \varphi_2$

$$\Psi_Q(T) = A_Q^+ \Phi_Q(T_+(\varphi_2, \varphi_1))\theta(\varphi_2 - \varphi_1) + A_Q^- \Phi_Q^*(T_-(\varphi_2, \varphi_1))\theta(\varphi_1 - \varphi_2), \tag{204}$$

where the first term and the second one are positive ($P_0 > 0$) and negative ($P_0 < 0$) frequency parts of the solutions with the spectrum of quasiparticles $\bar{\omega}_b$, A_Q^+ is the operator of creation of a universe with a positive «frequency» (which propagates in the positive direction of the dynamic evolution parameter) and A_Q^- is the operator of annihilation of a universe (or creation of an antiuniverse) with a negative «frequency» (which propagates in the negative direction of the dynamic evolution parameter).

We can see that the creation of a universe in the field world space and the creation of dynamic particles by the geometric vacuum $(b^+|0\rangle = 0)$ are two different effects.

The second effect disappears if we neglect gravitons and massive fields. In this case, $d\rho/d\varphi = 0$, and one can represent a wave function of a universe in the form of the spectral series over eigenvalues ρ_Q of the density ρ

$$\Psi(f|\varphi_2, \varphi_1) = \sum_Q \frac{A_Q^+}{\sqrt{2\rho_Q}} \exp\left\{-i(\varphi_2 - \varphi_1)\sum \frac{\bar{\omega}_f n_f}{\sqrt{\rho_Q}}\right\}\langle f|Q\rangle + \tag{205}$$

$$+ \sum_Q \frac{A_Q^-}{\sqrt{2\rho_Q}} \exp\left\{i(\varphi_2 - \varphi_1)\sum \frac{\bar{\omega}_f n_f}{\sqrt{\rho_Q}}\right\}\langle f|Q\rangle^*,$$

where $\langle f|Q\rangle$ is a product of normalizable Hermite polynomials.

6.4. Evolution of Quantum Universe. The equations of diagonalization (194) for the Bogoliubov coefficients (192) and the quasiparticle energy $\bar{\omega}_b$ (195) play the role of the equations of state of the field matter in a universe. We can show that the choice of initial conditions for the «Big Bang» in the form of the Bogoliubov (squeezed) vacuum $b|0\rangle_b = 0$ reproduces all stages of the evolution of the Friedmann–Robertson–Walker universe in their conformal versions: anisotropic, inflation, radiation, and dust (see Fig. 9).

The squeezed vacuum (i.e., the vacuum of quasiparticles) is the state of «nothing». For small φ and a large Hubble parameter, at the beginning of a universe, the state of vacuum of quasiparticles leads to the density of matter [7]

$$ {}_b\langle 0|\rho(a^+,\ a)|0\rangle_b = \rho_0 \frac{1}{2}\left(\frac{\varphi^2(0)}{\varphi^2(T)} + \frac{\varphi^2(T)}{\varphi^2(0)}\right), \qquad \theta = \frac{\pi}{4}, \tag{206} $$

where $\varphi(0)$ is the initial value, and ρ_0 is the density of the Casimir energy of vacuum of «quasiparticles». The first term corresponds to the conformal version of the rigid state equation (in accordance with the classification of the standard cosmology) which describes the Kasner anisotropic stage $T_\pm(\varphi) \sim \pm\varphi^2$ (considered on the quantum level by Misner [38]). The second term of the squeezed vacuum density (206) (for an admissible positive branch) leads to the stage with inflation of the dynamic evolution parameter φ with respect to the geometric time T

$$ \varphi(T)_{(+)} \simeq \varphi(0) \exp\left[T\sqrt{2\rho_0}/\varphi(0)\right]. $$

It is the stage of intensive creation of «measurable particles». After the inflation, the Hubble parameter goes to zero, and gravitons convert into photon-like oscillator excitations with the conserved number of particles.

At the present-day stage, the Bogoliubov quasiparticles coincide with particles, so that the measurable density of energy of matter in a universe is a sum of relativistic energies of all particles

$$ \rho_0(\varphi) = \frac{E}{V_0} = \sum_{n_f} \frac{n_f}{V_0}\sqrt{k_{fi}^2 + y_f^2\varphi^2(T)}, \tag{207} $$

where y_f is the mass of a particle in units of the Planck mass. The case of massless particles ($y = 0, \rho_0(\varphi) = \text{constant}$) corresponds to the conformal version of radiation stage of the standard FRW-cosmology. And the massive particles at rest ($k = 0, \rho_0(\varphi) = \rho_{\text{baryons}}\varphi/\mu$) correspond to the conformal version of the dust universe of the standard cosmology with the Hubble law

$$ \varphi' = \pm\sqrt{\rho_0} \;\Rightarrow\; \varphi_\pm(T) = \left(\frac{\rho_{\text{baryons}}}{4\mu}\right) T^2, \qquad q = \frac{\varphi''\varphi}{\varphi'^2} = \frac{1}{2}. \tag{208} $$

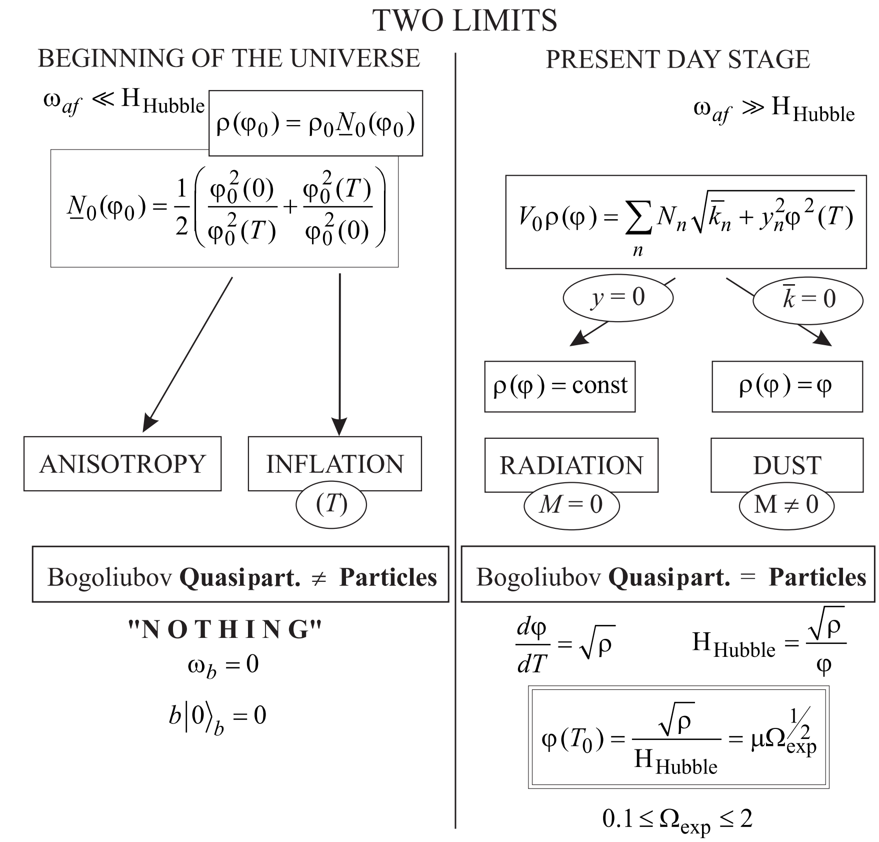

Fig. 9. These equations can be explicitly solved in two limits: at the beginning of the universe, and at the present-day stage. At the beginning of the universe in the state of the Bogoliubov vacuum, we get the density of measurable gravitons which corresponds to the well-known anisotropic stage. The anisotropic stage is changed by the stage of inflation-like increase of the cosmic scale factor with respect to the geometric (i.e., conformal) time. At the present-day stage, the Bogoliubov quasiparticles coincide with the measurable particles, so that the measurable energy of matter in the universe is a sum of relativistic energies of all particles in it. Neglecting masses, we get the conformal version of the radiation stage. Neglecting momenta, we get the conformal version of the dust stage, where an observer with the relative standard observes the Hubble law of the «accelerating» universe. According to the global equation for the cosmic scale factor φ_0 discussed before, it can be expressed in terms of astrophysical data of the observational cosmology, the density of matter, and the Hubble parameter in agreement with the value of the Newton coupling constant of gravity with $\Omega_{\mathrm{theor}} = 1$

The dynamic evolution parameter is expressed through the geometric time of a quantum asymptotic state of a universe $|\text{out}\rangle$ and conserved quantum numbers of this state: energy E_{out} and density $\rho_0 = E_{\text{out}}/V_0$.

It is well known that E_{out} is a tremendous energy (10^{79} GeV) in comparison with possible real and virtual deviations of the free Hamiltonian in the laboratory processes:

$$\bar{H}_0 = E_{\text{out}} + \delta H_0, \quad \langle \text{out}|\delta H_0|\text{in}\rangle \ll E_{\text{out}}. \tag{209}$$

We have seen that the dependence of the scale factor φ_0 on the geometric time T (or the «relation» of two classical unconstrained systems: dynamic and geometric) describes the «Big Bang» and evolution of a universe.

Therefore, from the point of view of unconstrained system «Big Bang» is the effect of evolution of the geometric interval with respect to the dynamic evolution parameter which goes beyond the scope of Hamiltonian description of a single classical unconstrained system.

Reparametrization-invariant dynamics of GR is covered by Geometric and Dynamic Unconstrained Systems connected by the Levi-Civita transformation of the matter fields into the vacuum fields of initial data with respect to geometric time (see Fig. 3).

6.5. QFT Limit of Quantum Gravity. The simplest way to determine the QFT limit of Quantum Gravity and to find the region of validity of the FP integral (180) is to use the quantum field version of the reparametrization-invariant integral (173) in the form of S-matrix elements [25] (see (176), (177)). We consider the infinite volume limit of the S-matrix element (177) in terms of the geometric time T for the present-day stage $T = T_0, \varphi(T_0) = \mu$, and $T(\varphi_1) = T_0 - \Delta T, T(\varphi_2) = T_0 + \Delta T = T_{\text{out}}$. One can express this matrix element in terms of the time measured by an observer of an out-state with a tremendous number of particles in a universe using equation (208) $d\varphi = dT_{\text{out}}\sqrt{\rho_{\text{out}}}$ and approximation (209) to neglect «back-reaction». In the infinite volume limit, we get from (177)

$$d\varphi_0[H_I^*] = 2d\varphi_0\left(\sqrt{V_0(H_0 + H_I)} - \sqrt{V_0 H_0}\right) = dT_{\text{out}}[\hat{F}\bar{H}_I + O(1/E_{\text{out}})], \tag{210}$$

where H_I is the interaction Hamiltonian in GR, and

$$\hat{F} = \sqrt{\frac{E_{\text{out}}}{H_0}} = \sqrt{\frac{E_{\text{out}}}{E_{\text{out}} + \delta H_0}} \tag{211}$$

is a multiplier which plays the role of a form factor for physical processes observed in the «laboratory» conditions when the cosmic energy E_{out} is much greater than the deviation of the free energy

$$\delta H_0 = H_0 - E_{\text{out}}; \tag{212}$$

due to creation and annihilation of real and virtual particles in the laboratory experiments.

The measurable time of the laboratory experiments $T_2 - T_1$ is much smaller than the age of the universe T_0, but it is much greater than the reverse «laboratory» energy δ, so that the limit

$$\int\limits_{T(\varphi_1)}^{T(\varphi_2)} dT_{\rm out} \Rightarrow \int\limits_{-\infty}^{+\infty} dT_{\rm out}$$

is valid. If we neglect the form factor (211) that removes a set of ultraviolet divergences, we get the matrix element (183) that corresponds to the standard FP functional integral (180) and S-matrix element (183) with the geometric (conformal) time T (instead of the coordinate time t) and with conformal-invariant fields $t \to T_{\rm out}$:

$$S[-\infty| + \infty] = \langle {\rm out}| T \exp \left\{ -i \int\limits_{-\infty}^{+\infty} dT_{\rm out} \hat{F} H_I(\mu) \right\} |{\rm in}\rangle \qquad (\hat{F} = 1). \quad (213)$$

Thus, the standard FP integral and the unitary S matrix for conventional quantum field theory (QFT) appears as the nonrelativistic approximation of tremendous mass of a universe and its very large lifetime (see Fig. 4). Now, it is evident that QFT are not valid for the description of the early universe given in the finite spatial volume and the finite positive interval of geometrical time $(0 \leqslant T \leqslant T_0)$ where T_0 is the «present-day value» for the early universe that only begins to create matter.

On the other hand, we revealed that standard QFT (that appears as the limit of quantum theory of the Einstein general relativity) speaks on the language of the conformal fields and coordinates. If we shall consider the standard QFT as the limit case of quantum gravity, we should recognize that, in QFT, we measure the conformal quantities, as QFT is expressed in terms of the conformal-invariant Lichnerowicz variables and coordinates including the conformal time ($T_{\rm out}$) as the time of evolution of these variables.

The conformal invariance of the variables can testify to the conformal invariance of the initial theory of gravity. What is this theory?

7. CONFORMAL RELATIVITY

7.1. Action and Geometry. There are observations [5,39,42] that the classical equations of Einsten's GR (111) are dynamically equivalent to the conformal-invariant theory described by the Penrose–Chernikov–Tagirov [43] action with a negative sign

$$W(g|\Phi) = \int d^4x[-\sqrt{-g}\frac{\Phi^2}{6}R(g) + \Phi\partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) + \mathcal{L}^c_{\rm matter}] \qquad (214)$$

and with the additional «dilaton» field Φ referred to as a conformal compensator and with the corresponding Lagrangian of fields of matter $\mathcal{L}^c_{\rm matter}$ [42].

The conformal-invariant version of Einstein's dynamics (214) is not compatible with the absolute standard of measurement of lengths and times given by the Einstein interval in the Riemannian geometry (112) as the latter is not conformal-invariant. As it was shown by Weyl in 1918 [44], a conformal-invariant theory corresponds to the relative standard of measurement of a conformal-invariant ratio of two intervals

$$(ds)_w = \frac{(ds_1)}{(ds_2)} \qquad (215)$$

given in the geometry of similarity as a manifold of Riemannian geometries connected by conformal transformations. The geometry of similarity is characterized by a measure of change of the length of a vector in its parallel transport. In the case (214), it is the gradient of the dilaton Φ [5, 39]. In the following, we call the theory (214) with intervals (215) the conformal relativity (CR), to differ it from the original Weyl [44] theory where the measure of change of the length of a vector in its parallel transport is a vector field.

Thus, the choice between two dynamically equivalent theories — general relativity (GR) and conformal relativity — (CR) (214) is the choice between the Riemannian geometry (112) and Weyl's geometry of similarity (215). The evident fact of the correspondence of the conformal-invariant theory (214) to the geometry of similarity (215) is ignored in the current literature (see, for example, paper [42]).

7.2. Variables and Hamiltonian. The dynamic equivalence of GR and CR becomes evident in the generalized Hamiltonian approach to solution of the problems of dynamics and initial data, as in both the theories, these problems are considered in terms of the Lichnerowicz conformal-invariant variables [29, 35].

In terms of the Lichnerowicz conformal-invariant variables formed by the determinant of the spatial metric $|^{(3)}g_{ij}| = \mathbf{g}$

$$f_c^{(n)} = f^{(n)}\mathbf{g}^{-n/6} \qquad (216)$$

GR (111) locally coincides with CR. The conformal-invariant dilaton in CR φ_c corresponds to the determinant of the space metric multiplied by the Planck constant in GR: μ ($\mathbf{g}^{1/6}\mu = \varphi_c$) [5] (see the Table).

In CR (214), we obtain the same Hamiltonian equations, the same reduction, and the same Levi-Civita transformation with the only one difference: the conformal variables, coordinates, and geometric time T are considered not as a mathematical tool, but as measurable quantities in the conformal relativity (214) [5].

Table. In terms of the Lichnerowicz conformal invariant variables (g_c), the Einstein general relativity (GR) (with the scale factor $\phi_g = \mu \parallel^3 g \parallel^{1/6}$) can be treated as the scalar version of the Weyl conformal invariant theory (with the scalar conformal field ϕ_c instead of the scale factor). In the Conformal Unified Theory (CUT), the Weyl scalar field forms both the Planck mass (in agreement with the present-day astrophysical data) and masses of elementary particles (in agreement with the principle of equivalence)

<table>
<tr><th colspan="2">TWO VERSIONS</th></tr>
<tr><th>GR</th><th>CUT</th></tr>
<tr><td>$\sqrt{-g}\left[-\frac{\mu^2 R}{6} + \mathcal{L}_{\rm mat}(g,\Psi)\right]$
(Lichnerowicz)</td><td>$\sqrt{-g}\left[-\frac{\Phi^2 R}{6} + \frac{\Phi}{\sqrt{-g}}\partial(\sqrt{-g}\partial\Phi) + \mathcal{L}_{\rm c}^{\rm SM}\right]$
$\Phi \propto$ modulus of Higgs field in SM</td></tr>
<tr><td>$N_c = N \parallel^{(3)} g \parallel^{-1/6}$
$g_{ij}^c =^{(3)} g_{ij} \parallel^{(3)} g \parallel^{-1/3}$
$\Phi_g = \mu \parallel^{(3)} g \parallel^{-1/6}$
$-N_c \frac{\phi_g^2}{6} R_c + \phi_g \partial(N_c \partial \phi_g) + N_c \mathcal{L}_{\rm mat}$</td><td>$N_c = N \parallel^{(3)} g \parallel^{-1/6}$
$g_{ij}^c =^{(3)} g_{ij} \parallel^{(3)} g \parallel^{-1/3}$
$\Phi_c = \Phi \parallel^{(3)} g \parallel^{-1/6}$
$-N_c \frac{\phi_c^2}{6} R_c + \phi_c \partial(N_c \partial \phi_c) + N_c \mathcal{L}_{\rm c}^{\rm SM}$</td></tr>
<tr><td>ABSOLUTE STANDARDS
$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$</td><td>RELATIVE STANDARDS
$(ds)^2 = g_{\mu\nu}^c dx^\mu dx^\nu$</td></tr>
<tr><th colspan="2">DIFFERENCES</th></tr>
<tr><td>♦ mixing of internal evolution parameter and metric
♦ evolution of 3d-volume in FRW approximation
♦ singularity of 3d-volume</td><td>♦ separation of internal evolution parameter from metric
♦ evolution of particle masses
♦ 3d-volume conformal singularity moved to dynamics</td></tr>
</table>

7.3. Physical Consequences. In a space with the geometry of similarity, an observer can measure only the conformal-invariant ratio of lengths of two vectors (215). In particular, in the homogeneous approximation,

$$\varphi_c(t,x) = \varphi_0(t) a(t,x), \qquad a(t,x) = 1$$

a Weyl observer measures the conformal time by his watch and obtains the conformal version of the Friedmann cosmology, i.e., the Hoyle–Narlikar-type cosmology [40] with the conformal Hubble parameter $\mathcal{H}_{\rm hub}^c = \varphi'/\varphi$.

The action of conformal relativity (214) does not contain any dimensional parameter, except for a finite time interval and finite volume, as the universe has the beginning $T = 0$ and the end $T = T_0$, i.e., the present-day stage, where the value of the scalar field

$$\varphi(T = T_0) = \frac{\sqrt{\rho_{\rm baryons}}}{\mathcal{H}^c_{\rm hub}} = \mu \tag{217}$$

coincides with the coupling constant of the Newton interaction, in agreement with equations of motion and astrophysical observational data (156) [5]. Equation (217) is not the gauge $\Phi(x) = \mu$ [39, 42] but the experimental fit [5, 7].

In the conformal cosmology, the Hubble law is explained by the evolution of the masses of elementary particles [5], so that the photon on a star remembers the «size» of a star atom at the moment of emission, and this «size» increases during the time of traveling; as a result, we get the red shift of a star photon in comparison with a photon emitted by a standard atom on the Earth at the moment of observation. The conformal version at the dust stage (208) corresponds to the «accelerating universe» with

$$q_c = \frac{\varphi''\varphi}{\varphi'^2} = \frac{1}{2}, \tag{218}$$

instead of $q_F = -1/2$ for the Friedmann version (with the measurable time $dT_F = (\varphi/\mu)dT$).

7.4. Quantum Conformal Relativity: Cosmological Scenario. The universe was created with a zero reduced energy from the state of «nothing» in the world space of the conformal-invariant variable F_c, φ_0 at the moment of the geometric time $T = 0$. The stability of quantum theory explains the arrow and beginning of the geometric time.

The classical and quantum evolutions of the universe coincide and are described by the Levi-Civita-type transformation (see Fig. 4) to the set of new variables $(F_c, \varphi_0) \Rightarrow (V, Q_0)$ where the new dynamic evolution parameter is the geometric time T ($dQ_0 = dT$). This transformation is the Bogoliubov one from «particle-like» variables (which diagonalize the measurable Hamiltonian) to «quasiparticle-like» variables (which diagonalize equations of motion). In particular, the Levi-Civita transformation defines the state of «nothing», i.e., initial data, as the vacuum of the Bogoliubov «quasipaticles», or squeezing vacuum.

The Levi-Civita evolution from «nothing» has four stages: «anisotropic», the squeezing vacuum «inflation» of the dilaton with respect to the geometric time, «radiation», and «dust» with accelerating evolution (218) (considered in Section 6). In the first two stages, the intensive creation of the matter fields (including gravitons) takes place, as «quasipaticles» differ from «particles».

In the last two stages, «quasipaticles» coincide with «particles», and these stages are the conformal version of the standard FRW cosmology.

7.5. Conformal Unified Theory. In the conformal theory (214), the Higgs mechanism of the formation of particle masses becomes superfluous and, moreover, it contradicts the equivalence principle, as, in the case of the standard Higgs mechanism, the Planck mass and masses of particles are formed by different scalar fields (see Fig. 10).

To save the equivalence principle we identify the modulus of the Higgs field with the Weyl dilaton [5, 39]. As a result, the Conformal Unified Theory (CUT) is described by the action [5, 39]

$$W_{\rm CUT} = -W_{\rm PCT} + W^c_{\rm SM}, \tag{219}$$

where $-W_{\rm PCT}(\varphi, g)$ is the Penrose–Chernikov–Tagirov action (214), and

$$W^c_{\rm SM}[\varphi, V, \psi, g] = \int d^4x \left(\mathcal{L}^{\rm SM}_{(\varphi=0)} + \sqrt{-g}[-\varphi F + \varphi^2 B]\right) \tag{220}$$

is the conformally invariant part of the SM action (i.e., the conventional SM action without the «free» part for the modulus of the Higgs $SU(2)$ doublet φ and without the Higgs mass term), B and F are the mass terms of the vector V and fermion ψ fields, respectively,

$$B = V_i \hat{Y}_{ij} V_j\,; \;\; F = \bar{\psi}_\alpha \hat{X}_{\alpha\beta} \psi_\beta, \tag{221}$$

$\hat{Y}$, $\hat{X}$ are the ordinary matrices of vector meson and fermion mass couplings in the WS theory multiplied by a rescaling parameter [5, 39].

The dilaton field φ forms both the Planck mass (in agreement with the present-day astrophysical data) and masses of elementary particles [5] (in agreement with the principle of equivalence). In other words, instead of the Higgs effect, we have the cosmic formation of all masses including the Planck one.

The effective Higgs potential could not be restored by the Coleman–Weinberg perturbation theory, as the vertices with scalar field interactions are eliminated from perturbation theory by the Bogoliubov transformations. Instead of the effective Higgs potential, in the exact theory, these interactions form cosmological evolution of the universe as the pure relativistic and quantum phenomenon which reproduces the conformal version of the standard Friedmann model (developed by Hoyle and Narlikar [40]).

The Weyl geometrization of the modulus of the Higgs field removes the Higgs potential with its problems of tremendous vacuum energy, monopole creation, the domain walls [41], and the violation of causality as the monotonous dependence ($\varphi(T)$). The conformal scalar field plays the role of the dynamic time and forms the Newton potential. As a consequence, the conformal version of the Higgs field loses its particle-like excitations [39] like the time component of the electromagnetic field. In CUT (219), we obtain the σ version of the Standard Model [4, 5, 39] without Higgs particles and with the prescription (211) which removes ultraviolet divergences from the SM sector.

EQUIVALENCE PRINCIPLE

GR: $W_{\rm tot} = W_{\rm GR}(\varphi_g) + W_{\rm matter}(\varphi_g)\,;\quad \varphi_g = \mu \left\| {}^{(3)}g \right\|^{1/3}$

$$\frac{\delta}{\delta\varphi_g}\left(W_{\rm GR} + W_{\rm matter}\right) = 0 \quad \Rightarrow \quad \boxed{\text{Eq. Princ.!}}$$

GR + SM: $W_{\rm tot} = W_{\rm GR}(\varphi_g) + W^{\rm c}_{\rm SM}(\varphi_{\rm H}) + \Delta W_{\rm Higgs}(\varphi_g, \varphi_{\rm H})$

$$\frac{\delta}{\delta\varphi_g}\left(W_{\rm GR} + \Delta W_{\rm Higgs}\right) = 0$$

eq. of motion $\varphi_g \propto \varphi_{\rm H}$ $\quad \not\Rightarrow \quad \boxed{\text{Eq. Princ.!}}$

$$\frac{\delta}{\delta\varphi_{\rm H}}\left(W^{\rm c}_{\rm SM} + \Delta W_{\rm Higgs}\right) = 0$$

CUT: $W_{\rm tot} = W_{\rm GR}(\varphi_g \equiv \varphi_{\rm c}) + W^{\rm c}_{\rm SM}(\varphi_{\rm H} \equiv \varphi_c)$

$$\frac{\delta}{\delta\varphi_c}\left(W_{\rm GR} + W^{\rm c}_{\rm SM}\right) = 0 \quad \Rightarrow \quad \boxed{\text{Eq. Princ.!}}$$

Fig. 10. In the Conformal Unified Theory (CUT), the Higgs mechanism of the formation of particle masses becomes superfluous, and, moreover, it contradicts the equivalence principle, as, in the case of a naive unification of general relativity (GR) and the Standard Model (SM), the Planck mass and masses of particles are of a different nature and are formed by different fields. The Weyl geometrization of the modulus of the Higgs field removes the Higgs potential with its problems of tremendous vacuum energy, monopole creation, and the domain walls

8. CONCLUSIONS

All relativistic systems (a particle, a string, a universe in general relativity) considered in the present review are given in their world spaces of dynamic variables by their singular actions (as integrals over the coordinate space) and by the geometric interval.

The peculiarity of relativistic systems is the invariance of their actions and the geometric intervals with respect to reparametrizations of the coordinate space, i.e., the general coordinate transformations, in general relativity. These reparametrization-invariant relativistic theories are not compatible with the symplest variational principles of the Hamiltonian dynamics.

The main mystery of relativistic systems (which we tried to reveal in the review) is the following: the reparametrization symmetry means that the measurable geometric time is a time-like variable in the geometric world space (obtained by the Levi-Civita transformation to the action-angle-type variables) rather than the coordinate.

This mystery of the dynamic origin of the «time» was reliably covered by the gauge condition that the lapse-function is equal to unity.

This noninvariant gauge-fixing method of describing the Hamiltonian dynamics of relativistic systems was a real obstacle for understanding this dynamics. This noninvariant method confuses reparametrization-invariant (or measurable) quantities and noninvariant (nonobservable) ones and hides the necessity of constraining by the Levi-Civita transformation that converts ambiguous and attractive «mathematical games» with noninvariant quantities into a harmonious theory of invariant dynamics in the world space which includes an unambigouos description of quantum gravity with its relation to the standard cosmology of a classical universe.

To obtain the invariant dynamics, one should choose the dynamic evolution parameter and the homogeneous component of the lapse-function (separating the global motion of a relativistic system as a whole from the local one) to define the geometric time. This geometric time is converted into a new dynamic evolution parameter by the Levi-Civita canonical transformation.

The constraining of the initial dynamic system (to get a Dynamic Unconstrained System) loses the geometric time but determines the causal structure of a world space that follows from the stability of the quantum relativistic theory. Whereas, the constraining of the geometric system (after the Levi-Civita transformation in the strong version of the action-angle variables) loses any dynamics, as a Geometric Unconstrained System is only initial data with respect to the geometric time.

The evolution of the initial world space with respect to the geometric time (i.e., the evolution of a particle, a string, a universe) is described by the inverse Levi-Civita transformation.

The generating functionals for causal Green functions of the unitary perturbation theory in the form of path integrals are constructed by averaging over a space of the reparametrization group, instead of the gauge-fixing.

The operations of separation of the «centre-of-mass» coordinates and variation of the action do not commute. As a result, the invariant local constraints differ from the standard ones for a relativistic string. The invariant local constraints satisfy the Virasoro algebra only for the case of a string with a single value of the mass in the spectrum (in classical theory, this value is equal to zero) that corresponds to the light-like branch of the representation of the Poincare group.

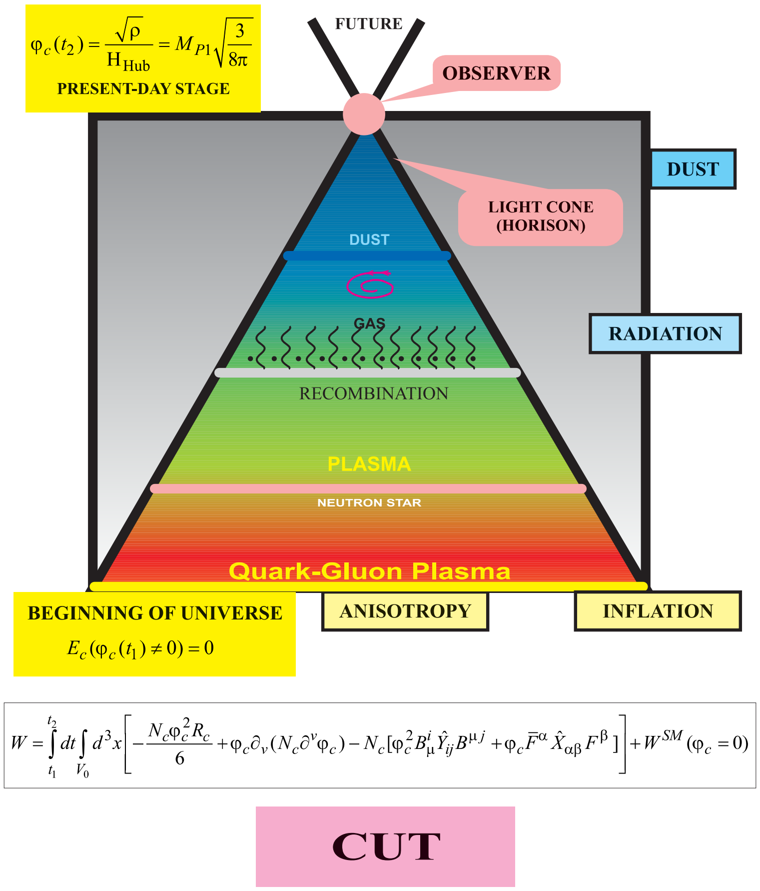

Fig. 11. Interactions of matter fields with a scalar field, in CUT, lead to the cosmic evolution of Quantum Universe with the set of predictions, including the Hoyle–Narlikar cosmology with the squeezed vacuum inflation, the accelerating evolution at the present-day dust stage. In CUT, we got the σ version of the Standard Model without Higgs particles, and with the «back-reaction» form factor to be free from the ultra-violet divergences for the precision calculations

In other words, for a string with a nontrivial spectrum of masses, the Virasoro algebra (with all its difficulties, including the $D = 26$ problem and the negative norm states) is an artefact of the reparametrization-noninvariant description.

To separate the global motion of a universe in general relativity, we used the wonderful effectivity of the Lichnerowicz conformal-invariant variables in solving the problems of the initial data and in formulating quantum field theory in the

Riemannian space. This effectivity was a signal of hidden conformal symmetry of the initial Einstein theory of gravitation. Really, the dynamics of Einstein's theory coincides with the dynamics of a conformal scalar field (dilaton) with the Penrose–Chernikov–Tagirov action with negative sign. However, the conformal-invariant theory is compatible with the Weyl geometry of similarity but not with the Riemannian one. The geometry of similarity converts the conformal-invariant Lichnerowicz variables from an effective mathematical tool to physical observables, consistent with large time and spatial volume limits of the obtained quantum gravity, where the standard Hamiltonian description of the evolution of matter fields with respect to the geometric time is possible.

The discovered conformal symmetry allows us to unify the conformal version of the Einstein theory with the Standard Model of electroweak and strong interactions on the basis of the equivalence principle that identifies the dilaton with the modulus of the Higgs field [5].

This unification of general relativity and Standard Model leads to a set of predictions, including the Hoyle–Narlikar cosmology with the «accelerating» evolution of the universe at the dust stage, the squeezed vacuum inflation from «nothing» at the beginning of the universe, and the negative result of the CERN experiment on the search of the Higgs particle [39], as the Weyl scalar field (like the determinant of the space metric in GR) has no particle-like excitations (see Fig. 11).

We would like to emphasize that we obtained the unification of a universe and an observer who appeared at the end of the evolution of the universe with respect to the geometric time; he measures the rhythm of the evolution by the rhythm of his heart and knows that any of his motions contributes to the global motion of the universe that forms its geometric time. «*Any motion*, if it makes sense, possesses also a *freedom*, and its task is to realize a good moral life, the final aim of which will be the *meaning of an everlasting existence*» (St. Maximus [45]).

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