V.N.Pervushin, M.K.Volkov

LOW ENERGY SCATTERING
OF MASSIVE PIONS. II

A_{21}

1974

RANGE OF JINR PUBLICATIONS

The preprints and communications of the Joint Institute for Nuclear Research (JINR) are considered to be original publications. They are published in accordance with Article 4 of the JINR Statut. Difference between the preprints and communications consists in that text of the preprint will be published in future in one of scientific journals or in aperiodic collections.

Indexing

The preprints, communications and deposited publications of the JINR have a single numbering (four last figures of the index).

The first sign of the index - a letter, denotes the language the paper is published in:

"P" - published in Russian;

"E" - published in English;

"D" - published both in Russian and English.

The preprints and communications, which are distributed only to the Member States of the JINR, have no letter-index.

The figure following the letter denotes the subject category of the given publication. List of subject categories of the JINR publications is sent periodically to the recipients.

Above-described index is placed in the right upper corner of the cover and on the title-page of each publication.

References

In bibliographical references to the JINR preprints and communications we recommend to indicate: author's initials and name, an abbreviation of the name of the institute, index, place and year of publication.

An example of bibliographical reference:

1. I. Ivanov. JINR, P2-4985, Dubna, 1971.

V.N.Pervushin, M.K.Volkov

LOW ENERGY SCATTERING OF MASSIVE PIONS. II

1. Introduction

In papers of Lehmann and Honerkamp et al. 1-2 the elastic scattering amplitude has been found for massless pions in the low-energy range ($\sqrt{S}$ < 1 GeV) within the chiral quantum field theory. Since at the threshold energies it is rather important to take account of the finite pion mass, the problem has naturally emerged of extension of the previous results to the case of massive pions. This problem has been solved in our paper 3.

The expression we have obtained for the pion-pion scattering amplitude has turned out to be rather suitable for calculating scattering lengths and effective-range parameters for partial waves. In the energy region $0 \leq S' \leq 4m_{\pi}^2$ these partial waves satisfy all rigorous inequalities obtained from crossing symmetry and positivity properties due to unitarity 4. That these general requirements on the $\pi\pi$-scattering amplitude are fulfilled in 3 indirectly proves the fact that at low energies ($\sqrt{S'} \ll 4\pi F_{\pi}$, $F_{\pi} \approx 92$ MeV) the first orders of our perturbation theory

expansion in powers of the parameter $1/F_\pi$ do contain the basic information on the $\pi\pi$ -scattering process.

To start our consideration we briefly recall the most important principles of chiral quantum field theory 1-3 : 1) The base of the theory is the chiral-invariant Lagrangian, therefore the theory in the Born approximation satisfies the low-energy theorems of current algebra. 2) In this theory the use is made of the perturbation theory expansion in the dimensional parameter $1/F_\pi$ at energies considerably smaller than $4\pi F_\pi \simeq 1.2 GeV$, where F_π is the pion-decay constant. This energy scale arises naturally in the chiral theories 1-3,5 . 3) In calculating higher orders of the perturbation series expansion the superpropagator (SP) method is used for regularization of the divergent diagrams. By this method, for removing the ultraviolet divergences of the diagrams it suffices to exploit the nonlinearity of chiral Lagrangian. The SP regularization is achieved through considering the whole set of two-particle diagrams 6 . 4) In constructing the superpropagator a chiral-invariant method is employed for summing all the two-vertex diagrams. The method is founded on allowing for all possible reductions of matrix elements corresponding to contractions of some lines on diagrams and decrease of the number of vertices. This occurs due to the presence of derivatives in the interaction Lagrangian. Taking into account of all possible reductions allows one to obtain the results independent of choice of a form of the chiral Lagrangian (in the limit $m_\pi=0$)[*).

[*) A detailed analysis of this technique can be found in papers by D.V.Volkov 7 , J.Honerkamp 8 and V.N.Pervushin 9 .

2. The $\overline{W}\,\overline{W}$ -scattering amplitude

Here the necessary results of paper 3 are listed briefly.
The scattering amplitude is defined in the standard way

$$\langle i_{\jmath} i_{2} | S | i_{3} i_{4} \rangle = I + i (2\pi)^{4} \delta^{(4)}(p_{1}+p_{2}-p_{3}-p_{4}) \times$$

$$\times \left[\delta_{i_{1} i_{2}} \delta_{i_{3} i_{4}} A(s,t,u) + \delta_{i_{1} i_{3}} \delta_{i_{2} i_{4}} A(t,s,u) + \delta_{i_{1} i_{4}} \delta_{i_{2} i_{3}} A(u,t,s) \right], \tag{1}$$

where $s = (p_{1}+p_{2})^{2}$, $t = (p_{1}-p_{3})^{2}$, $u = (p_{1}-p_{4})^{2}$; I the unit matrix, δ_{ij} - the Kronecker symbol.

In the one-loop approximation (i.e., all the baryon and pion loops are considered to the $1/F_{\overline{W}}^{4}$ order of perturbation series expansion) the function $A(s,t,u)$ has the form 3 :

$$\frac{A(s,t,u)}{32\pi} = \sum_{2} \alpha_{o}(3\bar{s}-1) + \sum_{2} \alpha_{o}^{2} \Pi(\bar{s},\bar{t},\bar{u}), \tag{2}$$

$$\Pi(\bar{s},\bar{t},\bar{u}) = A\bar{s}^{2} + B(\bar{t}^{2}+\bar{u}^{2}) + C\bar{s} + D - J(\bar{s})(3\bar{s}-1)^{2} - \tag{3}$$

$$- J(\bar{u})[3(\bar{u}-1)(\bar{u}-\bar{t})+3\bar{u}-1] - J(\bar{t})[3(\bar{t}-1)(\bar{t}-\bar{u})+3\bar{c}-1], $$

where

$$\bar{\xi} = \xi/4m_{\overline{W}}^{2} \qquad (\xi = s, t, u)$$

$$A = 0.63 \; ; \quad B = 20,5 \; ; \quad C = 3 \; ; \quad D = -1,5 \; , \tag{4}$$

$$J(x) = 1 - \frac{1}{2} \sum_{1}^{\infty}(4x)^{n} \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+2)} = $$

$$= \begin{cases} y \operatorname{arctg} y^{-1} & \\ \frac{\bar{y}}{2}[-i\pi + \ln\frac{1+\bar{y}}{1-\bar{y}}] & \\ \frac{\bar{y}}{2} \ln\frac{\bar{y}+1}{\bar{y}-1} & \end{cases} \begin{array}{l} 0<x<1 \; ; \; y=(1/x-1)^{1/2} \\ x>1 \quad ; \quad \bar{y}=(1-1/x)^{1/2} \\ x<0 \end{array} \tag{5}$$

(Relation (5) has been derived in Appendix of paper 10).

The first term in eq. (2) is the Born term 11 , the second one is the loop-diagram contribution. At energies much smaller than $4\pi F_{\overline{W}}$ formula (2) represents good expansion of the $\overline{W}\,\overline{W}$ elastic scattering amplitude in the small parameter α_{o}:

$$\alpha_o = \frac{4}{3}\left(\frac{m_\pi}{2\pi f_\pi}\right)^2 = \frac{2}{.03} \quad . \tag{6}$$

3. Scattering lengths

Following paper 12 let us introduce the notation

$$\alpha_\ell^{I(n)} = \lim_{\bar{s}\to 1} \frac{1}{4n}\frac{\partial^n}{\partial_{\bar{s}}n} h_\ell^I(\bar{s}), \tag{7}$$

$$\alpha_\ell^{I(0)} = a_\ell^I \quad ; \quad \alpha_\ell^{I(1)} = b_\ell^I \quad ; \quad \alpha_\ell^{I(2)} = c_\ell^I , \tag{8}$$

where

$$h_\ell^I(\bar{s}) = \frac{1}{2(\bar{s}-1)} e \int_1^1 dx \, P_\ell(x) \, T^I(s,x). \tag{9}$$

Here α_ℓ^I are scattering lengths, b_ℓ^I and c_ℓ^I effective-range parameters, T^I the amplitude in the channel with isospin I, $P_\ell(x)$ the Legendre polynomials:

$$T^I = \frac{5}{2}\alpha_o T_\beta^I + \frac{5}{2}\alpha_o^2 \Pi^I . \tag{10}$$

$$T_\beta^o = 2(3\bar{s}-1) \; ; \quad T_\beta^1 = 3(\bar{s}-1)x \; ; \quad T_\beta^2 = -(3\bar{s}-1) . \tag{11}$$

$$\Pi^0 = (2B+3A)\bar{s}^2 + \frac{(4B+A)}{2}(\bar{s}-1)^2(1+x^2) + 2C\bar{s} + C + 5D - 4(3\bar{s}-1)^2 J(\bar{s}) - 12 G^{(0)}(x),$$

$$\Pi^1 = (B-A)(\bar{s}-1)^2 x + C(\bar{s}-1)x - 3(\bar{s}-1)^2 x J(\bar{s}) + 2 G^{(1)}(x),$$

$$\Pi^2 = 2B\bar{s}^2 + \frac{(B+A)}{2}(\bar{s}-1)^2(1+x^2) - C\bar{s} + C + 2D - (3\bar{s}-1)J(\bar{s}) - 6C^{(2)}(x). \tag{12}$$

$$G^{(0)}(x) = J\left(-(\bar{s}-1)\frac{(1+x)}{2}\right)\left\{(\bar{s}-1)^2\left[5\left(\frac{1+x}{2}\right)^2 - \frac{1+x}{2}\right] + (\bar{s}-1)\left[3\left(\frac{1+x}{2}\right)^2 - 1\right]\right\},$$

$$G^{(1)}(x) = J\left(-(\bar{s}-1)\frac{(1+x)}{2}\right)\left\{(\bar{s}-1)^2\left[6\left(\frac{1+x}{2}\right)^2 - 3\frac{1+x}{2}\right] + 3(\bar{s}-1)\left[\frac{1+x}{2} - 1\right] - 1\right\},$$

$$G^{(2)}(x) = J\left(-(\bar{s}-1)\frac{(1+x)}{2}\right)\left\{(\bar{s}-1)^2\left[4\left(\frac{1+x}{2}\right)^2 + \frac{1+x}{2}\right] + (\bar{s}-1)\left[3\frac{1+x}{2} + 1\right] + 1\right\}. \tag{13}$$

Making use of the formula

$$\frac{1}{2}\int_1^1 dx \, P_\ell(x)\left(\frac{x+1}{2}\right)^\ell = \frac{(\ell!)^2}{(2\ell+1)!} \tag{14}$$

we obtain the values for scattering lengths and effective-range

parameters presented in Table I.

For $\ell \geq 3$ the above given formulae allow one to derive the following simple expressions for scattering lengths

$$\alpha_\ell^0 = (2\ell+1)(4\ell+7) Z_\ell ,$$

$$\alpha_\ell^1 = \tfrac{1}{3}(4\ell^2 - 2\ell - 1) Z_\ell , \qquad Z_\ell = 3\pi\alpha_0^2 \, \frac{2^{2(\ell-1)} \Gamma^3(\ell+1)\Gamma(\ell-2)}{\Gamma^4(2\ell+2)} \quad (15)$$

$$\alpha_\ell^2 = (4\ell^2 + 3\ell + 8) Z_\ell ,$$

The values we have calculated for scattering lengths (given in Table I) are in rather good agreement with known experimental data [12,13] and with the results of phenomenological approach of Palou and Yndurain [12]. Their model uses the Froissart-Gribov representation. Table I also lists the calculation results corresponding to the choice of parameters of their model which gives the value $\alpha_0^0 = 0.15$ (see Table II in [12]).

All the scattering lengths for $\ell \geq 3$ obey the inequality:

$$\alpha_{\ell+2}^I \leq \alpha_\ell^I \frac{(\ell+1)(\ell+2)}{4(2\ell+3)(2\ell+5)} \qquad (16)$$

obtained from unitarity and analyticity of scattering amplitude in papers [4,14].

4. The $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude in the interval $0 \leq \bar{S} \leq 1$

For the $\pi^0\pi^0 \to \pi^0\pi^0$ S-wave

$$f_0^{00}(\bar{s}) = \frac{1}{64\pi} \int_{-s}^{4} dx \left[A(s,t,u) + A(t,s,u) + A(u,t,s) \right] \qquad (17)$$

in the interval $0 \leq \bar{S} \leq 1$ A.Martin in paper [4] has found various rigorous inequalities from unitarity and crossing symmetry. To recall, let us write down these inequalities:

1) $\quad f_0^{00}(\bar{P}) < f_0^{00}(1), \quad 0 \le \bar{P} \le 1$

2) $\quad \dfrac{d\,f_0^{00}(\bar{P})}{d\,\bar{P}} > 0, \quad 0.5 \le \bar{P} \le 1$.

3) $\quad f_0^{00}(\bar{S}) \ge 2\int_0^1 f_0^{00}(\bar{S})\,d\bar{S}$,

4) $\quad f_0^{00}(0) > f_0^{00}\left(\tfrac{1}{2}(1 + \tfrac{1}{\sqrt{3}})\right) \simeq f_0^{00}\left(\tfrac{3.15}{4}\right)$, $\qquad$ (18)

5) $\quad \dfrac{d\,f_0^{00}(\bar{S})}{d\,\bar{S}} < 0, \quad 0 \le \bar{S} \le 1.21/4$

6) $\quad \dfrac{d\,f_0^{00}(\bar{S})}{d\,\bar{S}} > 0, \quad 1.21/4 \le \bar{S} \le 1.76/4$

7) $\quad f_0^{00}(0.8) > f_0^{00}\left(\tfrac{0.21}{4}\right) > f_0^{00}\left(\tfrac{1.98}{4}\right)$.

The direct calculations show that amplitude (2) obeys all the above inequalities. Our subsequent consideration makes it easy to see this.

Our amplitude in the range $0 \le \bar{S} \le 1$ is well described by the approximate formula:

$$A(s,t,u) = \tfrac{\pi}{2}\alpha_0(3\bar{S}-1) + \tfrac{\pi}{2}\alpha_0^2\left[A_r\bar{P}^2 + B(\bar{t}^2 + \bar{u}^2) + C\bar{P} + D\right]. \qquad (19)$$

Hence,

$$f_0^{00}(\bar{S}) \cong \tfrac{\pi}{6}\alpha_0^2\left[(2B+A)(5_r\bar{S}^2 - 4\bar{S} + 2) + 3D + C\right]. \qquad (20)$$

It is quite easy to check that the parabolic function

$$f_0^{00}(\bar{S}) = \beta(5_r\bar{S}^2 - 4\bar{S} + 2) + \gamma \qquad (21)$$

obeys conditions (18) for arbitrary $\beta > 0$ and γ. Thus, to fulfil the Martin inequalities (18) it is sufficient to require that the inequality

$$2B + A > 0 \qquad (22)$$

holds.

Since the condition (22) is fulfilled (see formula (4)) the Martin inequalities (18) are satisfied.

5. Conclusion

The performed here analysis of the $\pi\pi$ scattering amplitude calculated in 3 completely proves the validity of perturbation theory even for strong interactions in the chiral quantum field theory at energies much smaller than the energy scale $4\pi F_\pi \sim 1.2\,Gev$.

The information on the scattering amplitude contained in the two first orders of perturbation theory ($1/f_\pi^2$ and $1/f_\pi^4$) quite correctly reflects all properties of the amplitude resulting from the very general assumptions 4 . The values for scattering lengths and effective-range parameters calculated in this approximation are also well consistent both with the known experimental data and with the results of phenomenological description of the $\pi\pi$ -scattering amplitude.

We would remind here that the value for the pion form factor we have obtained in the same approach 5a is in good agreement with experiment in the threshold energy range, as well. Besides, the pion radius

$$\sqrt{\langle \tau^2 \rangle_\pi} \sim 0.65\,fm \qquad (23)$$

is rather consistent with recent experimental data 15 . Correct results have been found for the pion photoproduction and their polarizabilities 5b .

In conclusion the authors thank D.I.Blokhintsev, V.A.Meshcheryakov, Yu.V.Parfenov, V.V.Serebryakov and D.V.Shirkov for fruitful discussions.

R e f e r e n c e s

1. H.Lehmann.Phys.Lett.,41B,529 (1972); Preprint DESY 72/54 (1972).

2. G.Ecker,J.Honerkamp.Nucl.Phys.,B52,211 (1973); B62, 509 (1973).

3. V.N.Pervushin,M.K.Volkov. Preprint JINR,E2-7661 (1974).

4. A.Martin.Nuovo Cimento,47,265 (1967); 58A,303 (1968).

5. M.K.Volkov,V.N.Pervushin a)Yadernaya Fizika,19,652 (1974);
b)Preprint JINR,E2-8100 (1974).

6. M.K.Volkov.Ann.Phys.,49, 202 (1968); Teor.Mat.Fizika,6,21(1971).

7. D.V.Volkov. Particles and Nucleus, 4, 3 (1973).

8. J.Honerkamp. Nucl. Phys., B36, 130 (1972).

9. V.N.Pervushin. JINR preprint, P2-7644 (1973);JINR,E2-8009 (1974).

10. M.K.Volkov. TMF,6, 21 (1971).

11. S.Weinberg, Phys.Rev. Lett.,18, 567 (1971).

12. F.P.Palou, F.J.Yndurain.Nuovo Cimento,19A, 245 (1974).

13. J.P.Baton, G.Laurens and J.Reiguier.P.L.33B, 528, 1970.

14. F.J.Yndurain. Nuovo Cimento,64A, 225 (1969).

15. G.Adylov, F.Aliev, D.Bardin et al. JINR,E2-8047 (1974).

Table I.

$\mathcal{L}_{\ell}^{I\pi o}$	Experiment	Our values	Values of paper 12
a_o^o	$[0,10;0,60]$	$0,15$	$0,15 \pm 0,02$
a_o^2	$[-0,10;-0,03]$	$-0,042$	$-0,065 \pm 0,025$
a_1^1	$[0,032;0,040]$	$0,031$	$0,0341 \pm 0,0036$
b_1^1		$1,14 \cdot 10^{-3}$	$(1,07 \pm 0,27) \cdot 10^{-3}$
a_2^o	$[1,4 \cdot 10^{-3};1,8 \cdot 10^{-3}]$	$1,85 \cdot 10^{-3}$	$(1,48 \pm 0,08) \cdot 10^{-3}$
a_2^2	$[-2 \cdot 10^{-4}; 3 \cdot 10^{-4}]$	$2,6 \cdot 10^{-4}$	$(-3 \pm 8) \cdot 10^{-5}$
b_2^o		$-1,02 \cdot 10^{-4}$	$(-3,8 \pm 1,1) \cdot 10^{-5}$
b_2^2		$-5,1 \cdot 10^{-5}$	$(-4,4 \pm 1,1) \cdot 10^{-5}$
c_2^o		$2 \cdot 10^{-5}$	$(1,13 \pm 0,36) \cdot 10^{-5}$
c_2^2		$1,04 \cdot 10^{-5}$	$(1,27 \pm 0,36) \cdot 10^{-5}$
a_3^1		$1,33 \cdot 10^{-5}$	$(3,8 \pm 0,5) \cdot 10^{-5}$
a_4^o		$5 \cdot 10^{-6}$	$(4,8 \pm 0,8) \cdot 10^{-6}$
a_4^2		$2 \cdot 10^{-6}$	$(1,7 \pm 0,8) \cdot 10^{-6}$

Will you fill blank spaces in your library?

You can receive by post the books listed below. Prices - in US $, including the packing and registered postage

16-4888	Dosimetry of Radiation and Charged Particle Protection Physics. Dubna, 1969.	250 pg	4.85
D-6004	Binary Reactions of Hadrons at High Energies. Dubna, 1971.	768 pg	12.59
D10-6142	Proceedings of the International Symposium on Data Handling of Bubble and Spark Chambers. Dubna, 1971.	564 pg	9.96
D13-6210	Proceedings of the VIth International Symposium on Nuclear Electronics. Warsaw, 1971.	372 pg	6.36
D-6465	International School on Nuclear Structure, Alushta, 1972.	525 pg	9.54
D- 6840	Proceedings of the Second International Symposium on High Energy and Elementary Particle Physics (Štrbské Pleso, ČSSR, 1972).	398 pg	6.78
D2-7161	Proceedings of the III International Seminar on Non-Local Quantum Field Theory. Alushta, 1973.	280 pg	5.00
	Proceedings of the VI European Conference on Controlled Fusion and Plasma Physics. Moscow, 1973. v. I.	666 pg	21.00
D1,2-7411	Deep Inelastic and Many-Body Processes. Dubna, 1973.	507 pg	8.76
P1,2-7642	Proceedings of the International School of Young Physicists on High Energy Physics. Gomel, 1973.	623 pg	10.53

Conditions of Exchange

The preprints and communications of the Joint Institute for Nuclear Research are distributed free of charge on the mutual exchange basis to the universities, institutes, libraries, scientific groups and individual scientists of more than 50 countries.

We expect that the recipients of the JINR publications will have a possibility to display initiative in sending free of charge publications to Dubna. We receive, on the exchange basis, scientific books, journals, preprints and other types of publications on the subject categories of our Institute.

The only kind of publications that we do not require are the reprints of the articles already published in scientific journals.

In a number of cases we address to scientific institutions - the most prominent recipients of our publications - with the request to send us free of charge some books or subscribe for our library to the scientific journals published in their countries.

Requests

The Publishing Department fulfils annually about 3000 individual requests for our preprints and communications. The index of our publication must be obligatory indicated in such requests.
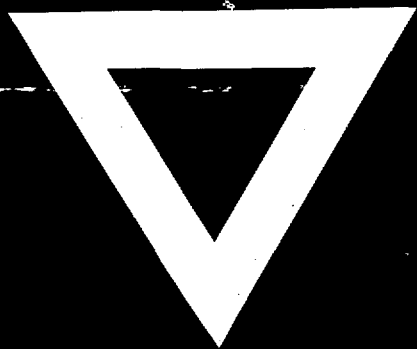
Addresses

Letters on all the questions concerning the exchange of publications as well as requests for individual publications are to be sent at the address:

> Publishing Department
> Joint Institute for
> Nuclear Research
> Head Post Office,
> P.O. Box 79
> 101000 Moscow,
> U.S.S.R.

We kindly ask to send all the publications on the exchange basis and also free of charge subscriptions to scientific journals at the address:

> Scientific-Technical Library
> Joint Institute for
> Nuclear Research
> Head Post Office
> P.O. Box 79
> 101000 Moscow,
> U.S.S.R.

75.03.07