Caspar Wessel

On the Analytical Representation of Direction

An Attempt Applied Chiefly to Solving Plane
and Spherical Polygons
1797

translated by Flemming Damhus

Introductory chapters by
Bodil Branner, Nils Voje Johansen and Kirsti Andersen

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Abstract

This book contains the first complete English translation of Caspar Wessel's essay from 1797 »On the Analytical Representation of Direction«. The essay has become famous because it gives the first geometric interpretation of the complex numbers. As the complete translation shows, it is also remarkable for the analytical representation of directions (vectors) in space and the elegant analytic solution of plane and spherical polygons. The translation of Wessel's essay is prefaced by two papers on Caspar Wessel the man and his work. They shed much new light on this first important Norwegian mathematician and set his work on »directions« in its proper historical context. Among other things it is shown that the idea of using complex numbers to represent directions in a plane occurred to Wessel as early as 1787 in connection with his work as a surveyor. Moreover the papers address the question as to why Wessel's essay has remained without influence.

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Preface

The Norwegian surveyor Caspar Wessel is generally recognized as the first to have given the now familiar geometric interpretation of the complex numbers and of their rules of composition. His paper in Danish on this subject was presented to the Royal Danish Academy of Sciences and Letters on March 10th 1797 and published two years later in the Academy's journal. It had no immediate effect and was only rediscovered a century later, when it was mentioned in 1894 by Sophus A. Christensen in his doctoral thesis on the development of mathematics in Denmark and Norway in the 18th century. Upon reading this Christian Juel called attention to the importance of Wessel's achievements, after which Sophus Lie had Wessel's paper reprinted in the *Archiv for Mathematik og Naturvidenskab* of 1895. Two years later the Academy published a French translation of it. Finally, in 1929 David E. Smith included an English translation of the first 16 sections dealing with plane geometry and the complex numbers into his widely read *Source Book in Mathematics.*

On the occasion of the bicentenary of the publication of Wessel's paper the Academy is happy to present Flemming Damhus's translation into English of the whole paper, including the last less well known parts in which Wessel gave an analytic representation of directions (vectors) in space, and applied his analytic machinery to solve plane and spherical polygons.

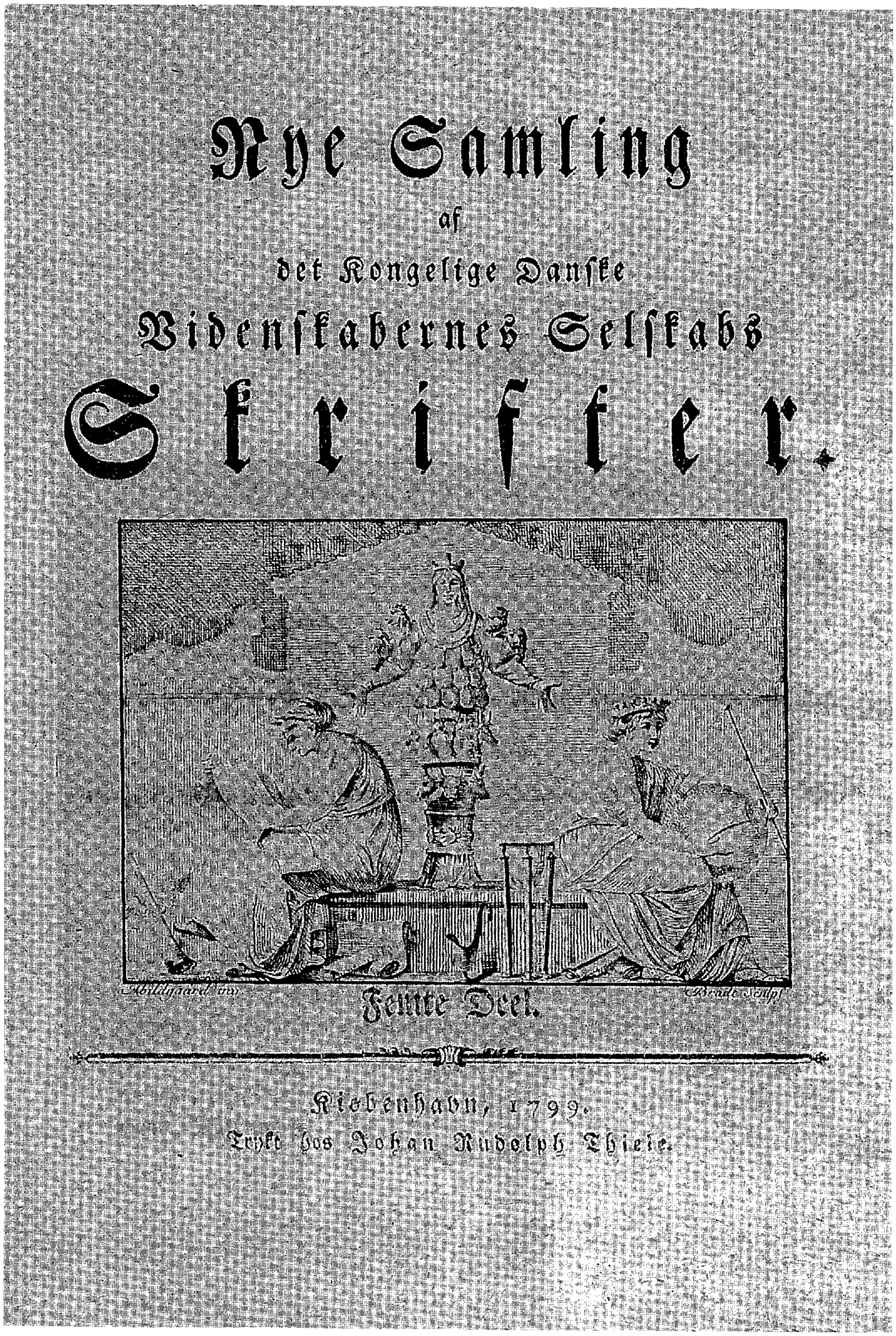
The translation of Wessel's paper is introduced by two papers on Wessel and the geometric representation of complex numbers. In the first one, Bodil Branner and Nils Voje Johansen deal with the person Caspar Wessel and his work as a surveyor. Some papers, in particular by Viggo Brun, Asger Lomholt and Otto Harms, have previously dealt with Wessel, but he has remained a somewhat shadowy figure until Bodil Branner and Nils Voje Johansen began their intensive search for Wesseliana. Their paper presents the result of this research. It throws brighter light on Wessel as well as on his work and shows in particular that the idea of using complex numbers to represent directions in a plane occurred to Wessel at least as early as 1787 in connection with his theoretical investigations of geodesy.

In the second paper Kirsti Andersen analyses Wessel's mathematical work and shows how it fits into a general history of complex numbers and their geometric representation. Again, many of the facts of this history have been recounted in various books and articles, but Kirsti Andersen's paper is the first to analyze the story in its full complexity. In particular she addresses the question why Wessel's paper remained without any influence.

Wessel's paper was the first important mathematical work published in the Collected Papers of the Royal Danish Academy of Sciences and Letters. With the present book the Academy wishes to make this contribution to the history of mathematics more accessible, to situate it into its historical and geodesic context, and to diffuse knowledge about its author.

We are indebted to professor Poul Lindegård Hjorth. In the early stages of preparing the book we discussed the project with him and we were fortunate to be able to draw on his long experience with publications from the Academy. Unfortunately he passed away in May 1998 after a short period of illness.

Bodil Branner and Jesper Lützen



Nye Samling
af
det Kongelige Danske
Videnskabernes Selskabs
Skrifter.

Abildgaard inv. Bradt Sculp.

Femte Deel.

Kiøbenhavn, 1799.
Trykt hos Johan Rudolph Thiele.

Front page of Det Kongelige Danske Videnskabernes Selskabs Skrifter, Nye Samling, V, *1799 in which Wessel's paper appeared on pages 469 -518.*

Caspar Wessel (1745-1818) Surveyor and Mathematician

by

Bodil Branner and Nils Voje Johansen

Introduction

This biography is written in honour of Caspar Wessel on the occasion of the bicentenary of the publication of his mathematical paper *Om Directionens analytiske Betegning, et Forsøg anvendt fornemmelig til plane og sphæriske Polygoners Opløsning*, published as a special print in 1798 and in the Collected Papers of The Royal Danish Academy of Sciences and Letters in 1799.

Our starting point for the investigation was Viggo Brun's chapter about Caspar Wessel in his book *Regnekunsten i det gamle Norge*, 1962, Asger Lomholt's fourth volume from 1961 in the series *Det Kongelige Danske Videnskabernes Selskab 1742-1942, Samlinger til Selskabets Historie*, the volume dealing with surveying under the auspices of the Academy, and Otto Harms' paper *Die amtliche Topographie in Oldenburg und ihre kartographischen Ergebnisse*, in Oldenburger Jahrbuch, 1961.

We have searched through a lot of different material, including letters, Caspar Wessel's surveying diaries, trigonometrical reports and trigonometrical calculations, and other primary sources, and we have found many new pieces of information. Here we only wish to emphasise that our thorough study of Wessel's surveying reports unveiled that he had the idea of using the complex numbers to represent directions in a plane at least as early as 1787. This use did not really simplify the calculations involved in the surveying. To us the idea stands out as a mathematical abstraction that was deeply satisfying to himself, and an idea that ripened over the following years until he was ready to present it in a more complete form in the paper he submitted to the Academy.

All citations in this paper are transcribed from the original hand-written material or occasionally from a contemporary hand-written copy. The transcribed citations are given in their original language (Danish or German) as footnotes. We have chosen to keep the English translation close to the original wording. The footnotes also include references to the archives where the original text can be found, as well as references to other sources utilised in the paper. The explanation of the abbreviations for the main archives we have used can be found under References. We have also included at the end an index of names of persons who are mentioned in the paper.

We would have liked to begin this paper by showing a portrait of Caspar Wessel. Perhaps there never was one, at least none seems to have survived. Nevertheless, we hope a picture of Wessel will emerge from the text. We do not offer many speculations, but mainly let the story unfold through the facts we have chosen to present.

1 Wessel's mathematical treatise presented to the Academy in 1797

On the 10th of March the Academy was assembled, present were Mr. Tetens, Councillor of State, Councillor of Conferences Colbiörnsen, Professor Krebs, Judge Advocate General Lövenörn, Captain Höyer, Surveyor Morville.

Mr. Surveyor Morville read a treatise containing calculations on the considerable loss the shareholders suffer after a division of land undertaken in such a way that the shares are given an unprofitable geometry for the enclosure, accompanied by three tables.

Councillor of State Tetens, as leader of the mathematical section, informed the Academy of a treatise sent to the Academy by Trigonometrical Surveyor Wessel concerning Calculus Situs, which by the Academy was found absolutely worthy to be included in its publications. On the occasion of the treatise Mr. Tetens, Councillor of State, made some remarks regarding the nature of this Calculus, which as well as Mr. Wessel's treatise, will be published in the Collected Papers of the Academy.

in the absence of the President

N Morville[1]

This short report (see figure 1), found in the records of the Academy, is all we know about the presentation of Caspar Wessel's mathematical treatise. It is worth noticing that Wessel was not himself present when his treatise was presented. This may simply be due to the fact that he was not a member of the Academy. In the previous year (1796) the Academy had however agreed on a change in its statutes, which allowed its members not only to invite (after acceptance by the president) a non-member to attend a presentation of a treatise at a meeting, but even more important, to encourage a non-member to submit a treatise for publication in its collection of papers.[2] Perhaps Wessel was not invited to participate on this special occasion, although he participated in several other kinds of meetings that were all related to the surveying under the auspices of the Academy. Here he played an active role when results and plans for surveying were presented and discussed. Another possible explanation of why Wessel did not attend the meeting on the 10th of March may be that he did not wish to be present; he was a very modest man, and did not at all like to stand out. Thomas Bugge, the leader of the Surveying under the auspices of the Academy, wrote about him:

I know the profound and more than usual insights and diligence of this man, [I] also particularly appreciate him due to his great humility.[3]

1 "Den 10de Martij var Videnskabernes Selskab forsamlet, nærværende var – Hr. EtatzRaad Tetens, ConfR Colbiörnsen, Prof. Krebs, G.A. Lövenörn, Capt. Höyer, Conducteur Morville.
Af Hr. Conducteur Morville blev oplæst en Afhandling Indeholdende Beregninger over det Betydelige Tab der foraarsages Lodsejeren naar Lodderne ved Jordskiftningen gives de til Indhegning ufordeelagtige Skikkelser, oplyst ved trende Tabeller.
Herr EtatzRaad Tetens som Formand for den Mathematiske Klasse, gav Selskabet Beretning om et til Selskabet indsendt Skrift fra Trigonometrisk Operateur Wessel angaaende Calculus Situs, som Selskabet fandt fuldkommen værdigt at indrykke i dets Samlinger. I Anledning af dette Skrift forelæste Herr Etatz Tetens nogle Anmærkninger angaaende Naturen af denne Cacul, som tillige med Herr Wessels Skrift blive trykte i Selskabets Samlinger.
i Præsidentens Fraværelse
N Morville."
Mødeprotokol, 1797, KDVS.

2 Asger Lomholt: *Det Kongelige Danske Videnskabernes Selskab 1742-1942. Samlinger til Selskabets Historie.* Vol. I. Munksgaard, København, 1942, p. 144.

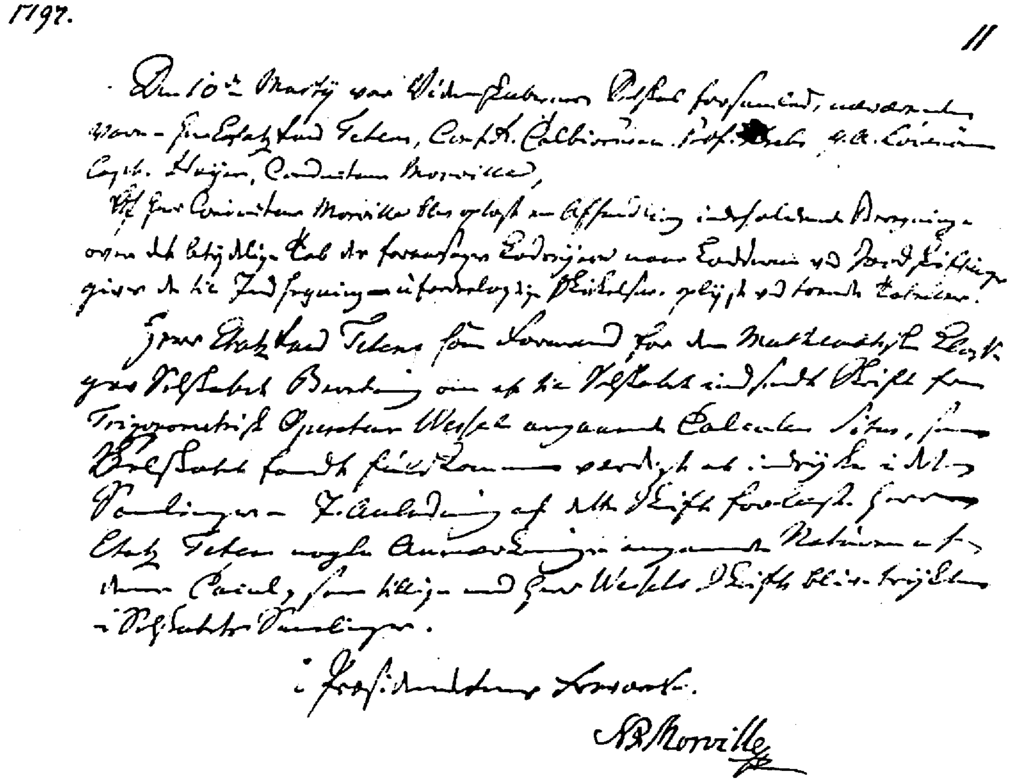

1797.

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Figure 1. The minutes of the meeting in the Academy on the 10th of March 1797, where Caspar Wessel's mathematical treatise was presented. From the records (Mødeprotokol) of the Academy. A transcript is found in footnote 1.

Wessel was indeed well known to the few that attended the meeting. We would have expected Bugge to be present as well, and we do not know what prevented him from being there. Usually he took part in the meetings of the Academy, and he was throughout the years an advocate for Wessel.

Since Wessel himself did not present the treatise, it was obvious that the choice fell on Johan Nicolai Tetens. Not only was he the leader of the mathematical section, but he had also encouraged Wessel to write the paper and to submit it to the Academy. Tetens' support was of crucial importance to Wessel. In the treatise he stated

Mr. Tetens, Councillor of State, had the patience to read these first investigations, and I owe to the encouragement, advice and guidance of this distinguished scholar that the present paper now appears less incomplete and has been deemed worthy of publication in the collection of papers of The Royal Danish Academy of Sciences and Letters.[4]

Wessel's treatise became the first contribution by a non-member to be accepted for publication. It is a pity that Tetens did not publish his remarks regarding *"the nature of this Cal-*

3 "Ich kenne dieses Mannes gründliche und mehr als gemeine Einsichten und großen Fleis, shaze Ihn auch besonders wegen seiner großen Besheidenheit." Letter from Thomas Bugge to Georg Christian Oeder (German transcript), 19 October 1782. NS, 31-4-36-1.

4 "Disse første Undersøgelser havde Hr. Etatsraad Tetens den Taalmodighed at giennemlæse, og denne navnkundige Lærdes Opmuntringer, Raad og Veiledning skylderjeg, saavel at dette Skrivt nu fremkommer mindre ufuldkomment, som og at det er værdiget, at optages i Samlingen af det Kongelige Videnskabernes Selskabs Skrivter." Caspar Wessel: *Om Directionens analytiske Betegning* ...., Kiøbenhavn, 1799.

culus" as he planned to. His well-known name could have drawn some attention to Wessel's work. Without this recommendation it was not noticed, but forgotten for a long time – as well as its author.

Let us go back to 1745, the year Caspar Wessel was born, and follow him through his life until he died in 1818.

2 The early years

2.1 Living in Norway

The father of Caspar Wessel was Jonas Wessel. Since 1735 he had been the curate in the parish of which his uncle Ole Wessel was the parson, and he had settled on the cotter allotment called Jonsrud, near the house of the parson, on the east-side of the fjord of Christiania (now Oslo), see figure 2. Ole Wessel was a brother of the famous naval hero Peter Wessel Tordenskiold and of vice admiral Caspar von Wessel. In 1738 Jonas Wessel married Helene Marie Schumacher, and the year after their first child Gjertrud was born. Caspar was born six years later on the 8th of June 1745 as child number six. After Ole Wessel's death in 1748 the father Jonas succeeded him as the parson to Vestby and the family moved into the house of the parson.

In November 1757 Caspar and two of his elder brothers Johan Herman and Ole Christopher attended the Cathedral School in Christiania.[5] Caspar was only 12 years old. The school had six grades, to which the pupils were assigned according to their knowledge. Caspar entered the fourth grade. Pupils were moved to the next grade when they had acquired the necessary skills.

The change from the village to life in Christiania was most certainly exciting and also a bit frightening for the young boy. However, he had his elder brothers along, and the number of pupils was not very large in those days. A total of 40-50 boys, so they all knew each other in the same manner that pupils today know their classmates. They attended classes in the morning and in the afternoon. In the short autobiography that Caspar Wessel wrote more than 55 years later he recalled

I entered the 4th grade, where then Master of Arts Bartholin was the teacher, he later became Secretary in the Missionary Convocation. I owe him a lot; on his own initiative he gave me free private lessons, and also taught me how to draw.[6]

The private lessons and especially the lessons in drawing must have been dear and outstanding memories to Caspar since he found them important enough to mention so many years later. It might be the fruits of these lessons that we can enjoy in much of his work as a surveyor.

5 School register (Mandtalprotokoll) of the Cathedral School in Christiania.

6 "Jeg kom i 4de Lektie, som den gang havde til Lærer Magister Bartholin, der siden blev Sekretær i Missionskollegiet. Ham skylder jeg meget; uanmodet gav han mig fri privat Undervisning, og hjalp mig dessuden til at Lære Tegning." Caspar Wessel: *Autobiography* (transcript), KB, NKS 4° 1977 b.

Figure 2. A rough sketch of a map marking Vestby (Caspar Wessel's birthplace), Christiania (now Oslo), Kiøbenhavn (Copenhagen), Siælland (Zealand), Fyen (Funen), Jylland (Jutland), the duchies of Schleswig, Holstein and Oldenburg, Oldenburg town, and the rivers Elbe and Weser.

2.2 *At the university in Copenhagen*

His two brothers Johan Herman and Ole Christopher finished school in the spring of 1761 and moved to Copenhagen to enter the university. Caspar followed his brothers two years later, most likely in June 1763. At that time the university in Copenhagen was the only one in the dual monarchy of Denmark-Norway, hence the natural place to study. Previously, both Ole and Jonas Wessel had studied theology there.

When Caspar arrived in Copenhagen, Johan Herman lived on Ulfeldt's Square in the house which is now Gråbrødretorv 3. It is not known where Ole Chrisopher lived, but it is quite possible that Caspar moved in with him. If we compare the kind of work and studies chosen by Ole Chrisopher and Caspar, it is clear that Ole Christopher had a great influence on his younger brother. From the tax census of 1766[7] we know that the two brothers lived together in Gammeltorv 6, and when the Copenhagen Guide (Københavns Vejviser) was revised and enlarged in 1772, we find that the two brothers lived in Kompagnistræde, so they may have stayed together since 1763.

Caspar entered the University of Copenhagen on the 25th of July 1763, after having passed the entrance exam. We do not know what intentions he had for further studies when he enrolled. At the time there were only few possible degrees one could complete, either in theology, in law or in medicine. Many years later Caspar – like his brother Ole Christopher – finished his studies with an exam in law. When Johan Herman and Ole Christopher entered the university, they both chose Professor Kofod Ancher to be their tutor. Kofod Ancher was a professor of law, a choice that fitted well with Ole Christopher's later studies. It is, however, interesting to notice that Caspar chose the mathematician and astronomer Professor Horrebow to be his tutor.[8] This may indicate that Caspar already at this early stage took a special interest in mathematics. He passed the second exam at the university, Examen Philosophicum, on the 12th of March 1764 with excellent marks.[9]

Latina:	optime
Græca:	bene
Ebræa:	nm
Logica:	optime
Metaphys:	optime
Physica:	optime
Ethica:	optime
Historia:	bene
Geograph:	bene-bene
Geometr:	optime
Arithmet:	optime
Sphærica:	optime

Note in particular that he obtained the best mark possible, optime, in the three mathematical disciplines: geometrica, arithmetica and sphærica.

7 Tax census (Ekstraskatten) for 1766, R.
8 Universitetspedellen, examen artium 1700-1780, KU 14.01.01, R.
9 Universitetspedellen, 2. examen 1738-1822, KU 14.01.03, R.

Caspar and his brothers had some financial support from their father, each of them about 200 Rd,[10] but this was most likely spent during the first couple of years in Copenhagen. It was common for students to have a job along with their studies, often as private teachers for children in wealthy families. But both Ole Christopher and Caspar became attached to the surveying under the auspices of the Academy. Ole Christopher earned his living this way only while he was a student, for Caspar it became an engagement for life.

3 From assistant to cartographer and geographical surveyor

3.1 The beginning

From 1761 the Academy undertook a huge project: the surveying of Denmark and the duchies of Schleswig and Holstein. A few years earlier a student by the name of Peder Koefoed had made a proposal to the King where he offered to draw accurate maps of parts of Denmark. The proposal was approved, and Koefoed had started his work in the summer of 1757, from 1759 with Thomas Bugge as his assistant. Unfortunately, Koefoed died suddenly in the summer of 1760.

But the surveying project survived. New revised and extended plans were worked out by Bugge and professor of mathematics Christen Hee. The surveying was to be tested against trigonometrical and astronomical operations, with methods similar to those behind the famous French Cassini maps. In 1761 the Academy formed a Surveying Commission to supervise the project. Its first leader became Henrik Hielmstierne, the other members were Christen Hee, Christian Horrebow, Jørgen Nicolai Holm and Bolle W. Luxdorph. The members were not replaced when they died, and from 1780 Bugge[11] became the indisputable leader of the surveying under the direction of the Academy, a position he kept to his death in 1815.

One of the members of the commission, the civil servant and man of letters Luxdorph, was a friend of Albert Peter Bartholin, the former teacher of the Wessel brothers in Christiania Cathedral School.[12] From the register of 1762 of taxpayers we know that Bartholin and Johan Herman Wessel lived in the same house on Ulfeldt's Square.[13] The Wessel brothers very likely met Luxdorph in this house, and he may have told them about the surveying that was about to begin in Denmark under the auspices of the Academy.

In any case Ole Christopher Wessel took part in the surveying as an assistant from the very beginning in 1762,[14] one year after he had arrived in Copenhagen. In 1763 he was an assistant of Bugge. In the surveying-diary of Bugge from 1763 we learn that he set his assistants to work and then took upon himself the trigonometrical operations:

10 Øvre Borgesyssels geistelige skifteprotokoll, No. 2 (1780-1824), p. 69, SO.

11 In 1759 Bugge had finished a degree in theology. In parallel to this he had studied mathematics with Hee, and he had also become an assistant of Horrebow at the Round Tower Observatory in Copenhagen. In 1775 he became a member of the Academy, and in 1777 he was appointed professor of mathematics at the university and director of the Round Tower Obervatory. The surveying of Norway began in 1779 under his guidance. Einar Andersen: *Thomas Bugge.* København, 1968 and Asger Lomholt: *Det Kongelige Danske Videnskabernes Selskab 1742-1942.* Vol. IV. Munksgaard, København, 1961.

12 *Unpublished letters from C.B.Tullin,* Illustreret Nyhedsblad No.29 and No.30, 1861, Christiania, and Bolle W. Luxdorph: *Dagbøger.* Vol. I+II, G.E.C.Gad, København, 1915-30.

13 Tax census (Skattemandtal) for 1762, R.

14 *Instrux for Landmaalerne 1762,* Christopher Hammer's private archive, Gunnerus bibliotek in Trondheim, fMS HA 24a.

When Mr. Wessel was so well trained that I could trust him with the [geographical] surveying, I informed Professor Hee about this, and asked at the same time the Academy to be good enough to let me for the remaining part of the summer, perform some trigonometrical operations as a test, while Mr. Wessel and Mr. Morville continued the [geographical] surveying. On the 11th of August Professor Hee was kind enough to inform me that both parts were approved by the Academy. As a result I left Mr. Wessel with the [geographical] surveying, and I myself started the trigonometrical surveying. [15]

Ole Christopher became a geographical surveyor the following year. Caspar had then just passed the Examen Philosophicum and was looking for a job. As surveyor Ole Christopher was in a position where he could choose his own assistants. He then picked Caspar as his assistant, a choice he knew to be wise. The very first day of Caspar Wessel's career as a surveyor is described in the surveying-diary of Ole Christopher from 1764.

30 April *I travelled from Copenhagen to Borup where the operations had ended last year.*
1 and 2 May *were used for verification of instruments and training of the assistants.*[16]

After having been an assistant under his brother in 1764 and 1765 Caspar Wessel became an "interim" geographical surveyor in 1766. This position he held until the 17th of February 1769 when he became an ordinary geographical surveyor. The previous year his brother had been promoted to trigonometrical surveyor and succeeded Bugge as such.

3.2 Geographical surveying in Denmark

To get an impression of Caspar Wessel's work as a surveyor it is necessary to look at the methods that were used. We will first give a short general description of how the geographical surveying was performed – from a theoretical point of view, and then in some detail, follow Caspar Wessel one week in April 1769. The general description is based upon two books by Bugge.[17]

In geographical surveying the aim was to show the location of towns, churches, castles, mills, woods, etc., the course of roads and streams, and the position of coastlines and islands. The objects in the terrain were marked on paper that was placed across a so-called plane table, the most important tool in geographical surveying. Afterwards several of these sheets of paper were glued together to form geographical maps (conceptkort). Before the paper was attached to the plane table it was damped. When it was cut off later on it shrank in a non-uniform way. The aim of trigonometrical surveying was to determine the position of a triangular net of points through more exact methods. The triangular

15 "Efter at Monsr. Wessel var bragt saa vit at ieg med sikkerhed kunde betroe ham Landmaalings Operationerne, berettede ieg saavel dette til Professor Hee, som udbad mig den Bevaagenhed af Societetet, at mig maatte tillades for den övrige Deel af Sommeren at foretage nogle Trigonometriske Operationer til en Pröve, medens Monsr. Wessel med Monsr. Morville continuerede Landmaalings Operationerne. Den 11. August havde Her Professor Hee den Godhed at svare mig, at begge Deele af Societetet vare accorderede. Ieg overlod altsaa Landmaalings Operationerne til Her Wessel, og selv begynte de trigonometriske." Thomas Bugge: *Journal over Landmaalings Operationerne 1763,* K&M.

16 "30. Aprilis reyste jeg fra Kiöbenhavn til Borup hvor forrige Aars operationer ophörte.
1. og 2. Maji bleve andvendte paa at verificere Instrumenter og öve Assistenterne." Ole Christopher Wessel: *Landmaaler Journal for Aar 1764,* K&M.

17 Thomas Bugge: *Beskrivelse over den Opmaalings Maade som er brugt ved de Danske geographiske Karter.* Kiøbenhavn, 1779.
Thomas Bugge: *De første Grunde til Regning, Plan-Trigonometrie og Landmaaling.* Kiøbenhavn, 1795.

net was drawn on independent triangular maps. Through comparison it was then possible to compensate for the less accurate geographical measurements and obtain more reliable maps.

We shall first focus on geographical surveying; we come back to trigonometrical surveying when we reach the time when Caspar Wessel was promoted to trigonometrical surveyor.

In Denmark a rather special method for surveying was chosen, the so-called "Method of parallel lines". The idea was to lay out parallel lines that – in principle – were in a north-south direction, and then walk these *principal-lines* along with the plane table and determine the positions of the different objects. The principal-lines were laid out with a mutual distance of 10 000 cubits (6277 m), and a surveyor had to map the objects lying within 5000 cubits (3138,5 m) from the principal-line he followed.

Three questions stand out

1) How did they lay out the lines ?
2) How did they determine the positions of the objects ?
3) How did they place the objects on the map ?

Before we answer these questions we have to get familiar with the tools the surveyors had at their disposal.

a) Poles for laying out lines
b) A measuring chain
c) A plane table
d) A diopter
e) A level
f) A compass
g) A pair of compasses

The poles for laying out lines were 1 – 2 inches thick and about 5 – 7 feet high. To lay out a line one started by setting out the two poles that defined the line. The next poles then had to be laid out behind the first two in such a way that if you went a couple of steps away from the first one it should cover all the others, whether you aimed along the top, the bottom or along a diagonal line (see figure 3a).

When moving the head slowly to the right, perpendicular to the line of poles, the pace by which the poles appeared should be the same as when moving the head to the left. The distance between the poles had to be 20 – 45 m, depending on weather conditions and the terrain. It is easy to understand that it was a time-consuming process to lay out a line, and that weather conditions had to be fair. For this reason we often find in the diaries that "no surveying was done due to rain and wind".

The measuring chain (see figure 3b) was made of steel and measured 25 cubits (15,7 m). It was used by the surveyor and his assistant to measure the distances of lines laid out as described above. The lines could either be principal-lines or lines from the principal-line to an object in the terrain.

The plane table had a tripod and the table top measured $15 \cdot 11$ inches and rested on a sphere (see figure 3c). For this reason the table top could easily be put in a horizontal po-

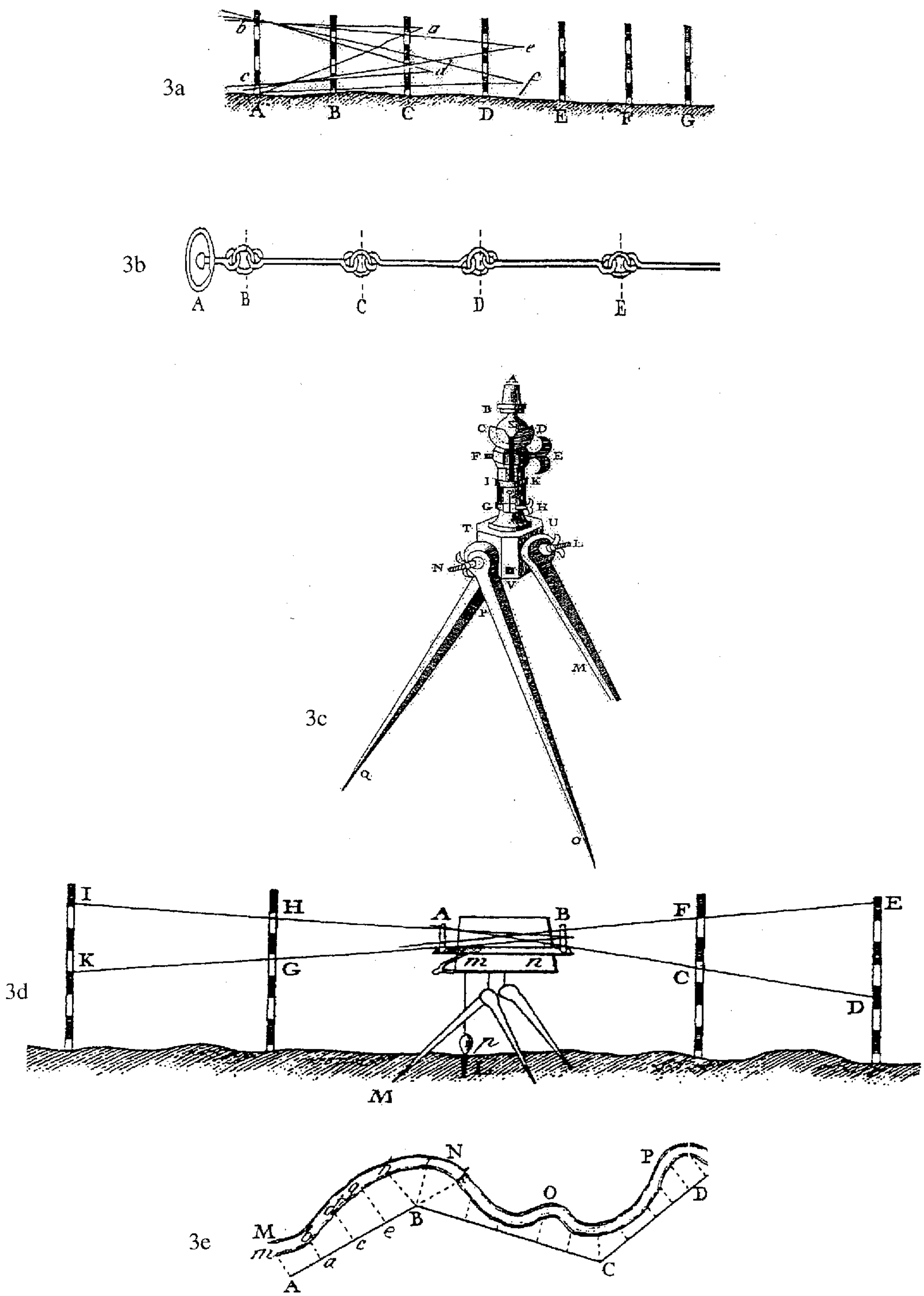

Figure 3. Surveying instruments and their use. In 3a: a line of poles determining a principal-line; in 3b: a measuring chain; in 3c: a tripod; in 3d: a plane table with a diopter, fitting the principal-line; in 3e: the principle of determining the course of a stream. From Thomas Bugge: Første Grunde til Regning, Geometrie, Plan-Trigonometrie og Landmaaling. *Kiøbenhavn, 1795.*

sition. Everything that was measured was marked on a paper placed on the table top on a scale of 1 : 20 000.

The plane table was then placed on the first station at a principal-line at the very south of the area that was to be surveyed. A straight line corresponding to the principal-line was drawn in the middle of the paper dividing it into two equal parts, and the table was oriented in such a way that the line pointed north-south. To lay out the principal-line a diopter was then placed along the line drawn and the positions of the first two poles were decided by aiming along it (see figure 3d). After this was done, the first station was marked on the paper. The diopter was placed on the table, sights were taken at the objects the surveyor wanted to determine, and the sight lines were drawn. When all sights were taken, the surveyor moved on to the next station, the distance between the stations was measured and the new station marked on the map. From this station new sight lines were drawn to the objects seen from the previous station(s). The position of an object was found at the crossing of the sight lines. Sights were then in turn taken at new objects before moving on to the next station.

When an object was not far away from the principal-line, it was often better to determine the position by laying out a line to the object and measuring the distance. If an object could not be seen from the principal-line, it was necessary to lay out a so-called *side-line* from the principal-line. Along such a side-line one would progress until the desired objects were in sight. The same principle was used to determine the borders of a forest, a stream, a coastline etc. (see figure 3e).

3.3 Wessel's geographical surveying in 1769

So far we have taken a purely theoretical approach. We will now have a look at the way Caspar Wessel did his surveying in April 1769. Below we have put in the left column Wessel's own descriptions taken from his surveying diary (see also figure 4b) and in the right column some remarks. The extract of the diary presents the dates from the 10th to the 16th of April. The remarks are based upon studies of the map (conceptkort) made from this surveying; this map is reproduced in figure 4a. Please notice that we only deal with objects mentioned by Wessel in his diary.

The cartouche on the map represents people at work at the limestone quarry close to Faxøe, it is one of the many illustrations made by Wessel. Moreover, it served as inspiration for the cartouche on the final map of the south-eastern quarter of Zealand (Siælland).

April
10 I travelled from Copenhagen to Kiöge

11 I travelled from Kiöge to Store Elmue where last year I marked my last principal-line in the direction of Ravnemöllen and the church steeple in Faxöe.

The previous year Wessel had been working as an "interim" surveyor in the same area.

12 The instruments were verified, in the afternoon I placed the plane table upon the marked station, I drew a line according to the compass, and laid out a principal-line to the north and a side-line to the west, the latter I continued to 748 cubits and from there aimed at Roeholt.

On the map we find the south-north line and the principal-line laid out to the north and the side-line to the west. We also find that in station (1) he aimed at Roeholt and then continued the side-line to 748 cubits where station (2) was set up. From here he aimed at Roeholt and Tystrup. This means that the position of Roeholt was determined.

13 I started out at 748 and advanced from there until Hyllede, zig-zagging due to marsh and brushwood, the same day a milestone on the road from Copenhagen to Vordingborg Bregnemosen, Hyllede village was determined.

Just before Hyllede he established station (3). Here the position for Roeholt was checked by aiming once more at it. In addition he aimed at Jyderup and Tystrup. This means that Tystrup was also determined. He then continued the side-line to Hyllede.

14 The line that was laid out to the north the first day was continued to 275, from where a new basis was laid out in north-west and simultaneously measured to 1125 cubits, from here I laid out another new line that was continued to 2020 cubits into a clearing in the wood at Rosenlund, along these lines Helledes Huus, Krageborg Huus, Tystrup and Jyderup as well as the border of the wood from Store Elmue to Rosenlund were determined.

15 A cross-line was laid out from 2020 through one of the avenues at Rosenlund, the same line was continued 160 cubits, from where I laid out a line to the north, right through the wood and continued it to 525 cubits, along these lines some of the hunting roads in the wood of Rosenlund were determined, the remaining roads of the same wood were drawn according to another map Chamberlain Rosenkrants has had made.

He now returned to (1) and continued 275 cubits along the principal-line before bending to the north-west and continuing for another 1125 cubits where he established station (4). Along the road the distance from the line to Helledes house was measured, which means that Helledes house was determined. From (4) he aimed at Krageborg house and Jyderup. Jyderup was then determined. A new principal-line was then continued another 2020 cubits. At 1400 cubits he established station (5) and Krageborg house was determined. The aiming at Tystrup and the determination of the border of the wood is not possible to trace on the map.

He determined some of the hunting roads and then went right through the wood until the principal-line was continued to 525 cubits.

16 Sunday

No surveying was done on Sundays, but Wessel may have done some work on the map.

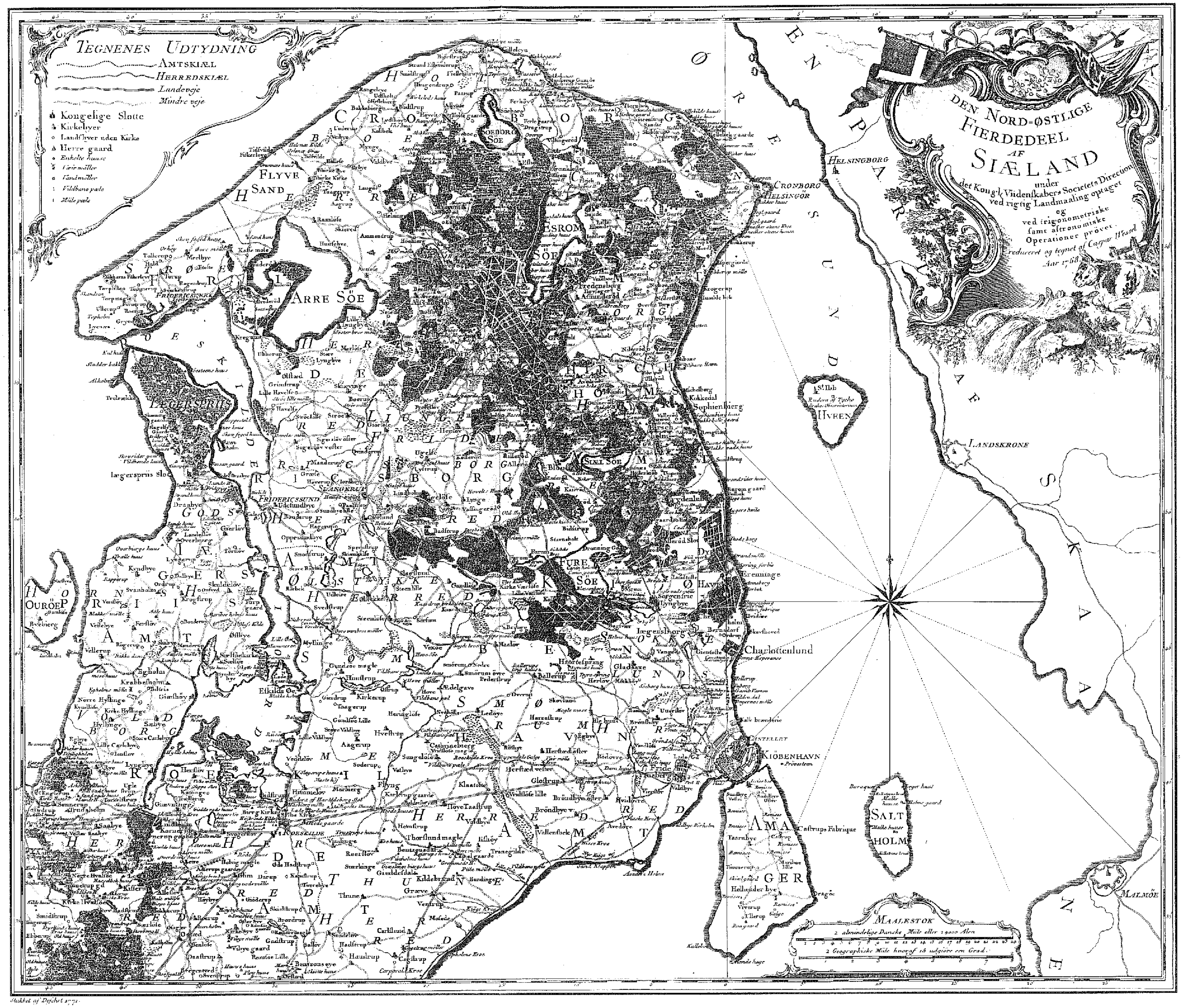

Figure 5. The North-eastern Fourth of Zealand, recorded under the auspices of the Royal Academy through real surveying and also tested by trigonometrical and astronomical Operations. Reduced and drawn by Caspar Wessel in the year 1768. Kort- & Matrikelstyrelsen. *Note the island of Hveen with the ruins of Tycho Brahe's observatory marked.*

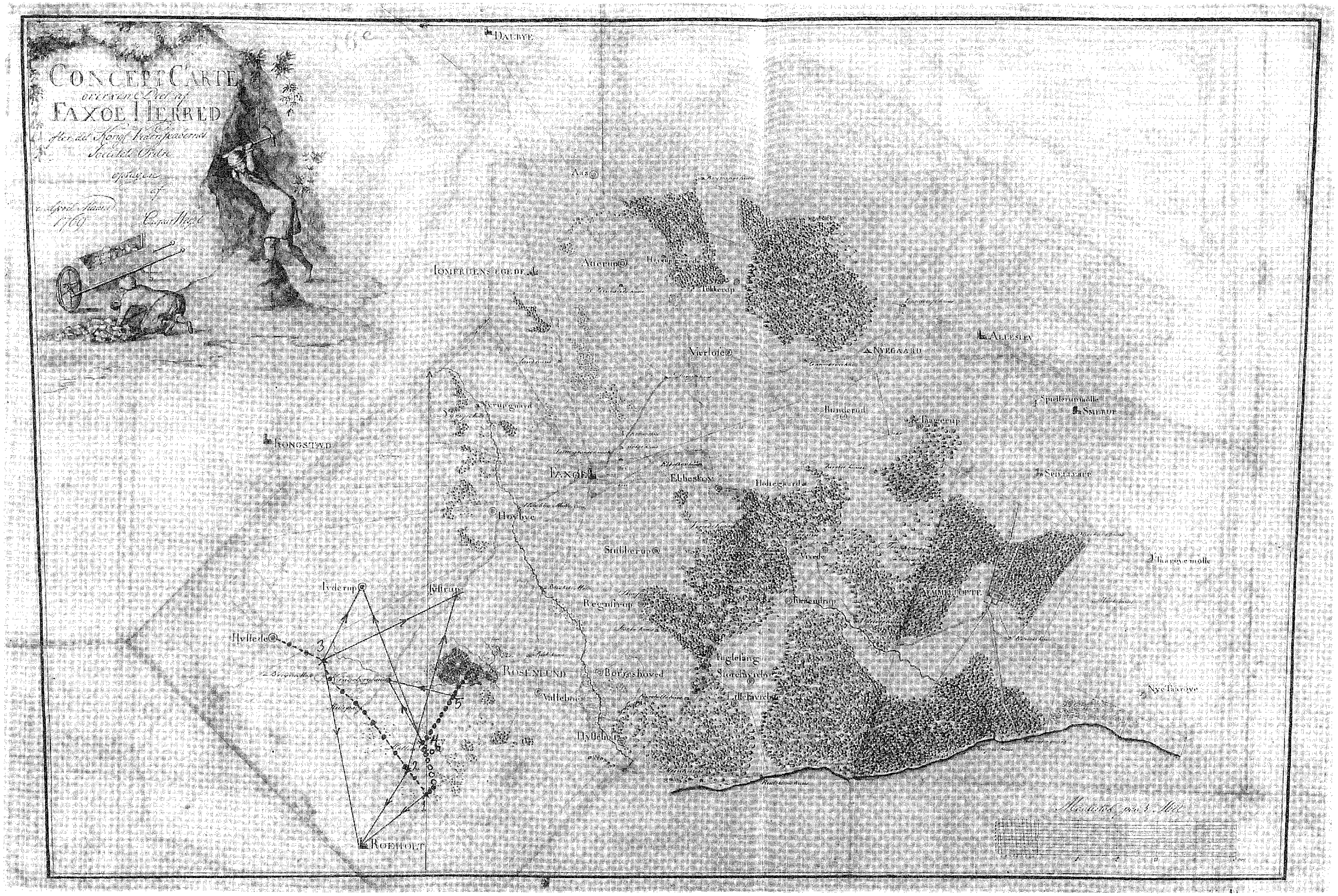

Figure 4a. Map of a part of Faxøe District recorded by Caspar Wessel after the Royal Academy's order. The month of April 1769. Kort- & Matrikelstyrelsen.

Note that the map is oriented after a compass line (indicated by a vertical arrow), not the north-south direction. In 1767/68 Ole Christopher and Caspar Wessel drew a geographical map (conceptkort) together, this one is the first drawn by Caspar Wessel all alone. To make it easier to follow the description in the diary we have emphasised the relevant lines, stations, and sight lines by adding the following symbols to the map:

••••••• : *A principal-line.*

•—•—• : *A side line.*

1 – 5 : *Stations from which sight lines were drawn.*

oooooo : *A measured line.*

____ : *A sight line, used to determine the position of an object.*

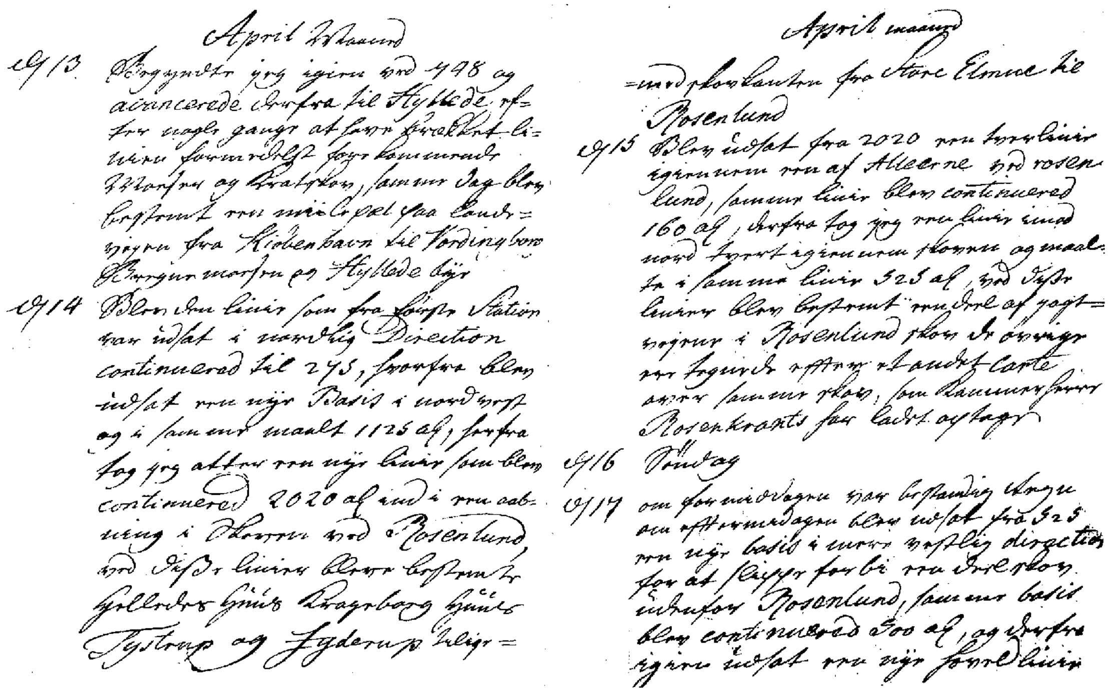

Figure 4b. Caspar Wessel's diary 13-17 April in Journal over Landmaalings Operationerne efter det Kongl. Videnskabers Societets Ordre *1769, Kort- & Matrikelstyrelsen.*

Transcripts of the entries 10-16 April:

"April Maaned
den 10 Reiste jeg fra Kiöbenhavn til Kiöge
den 11 Fra Kiöge til Store Elmue, hvor jeg forrige Aar havde udmærket min sidste Hovedlinie i direction af Ravnemöllen og Faxöe Kirketaarn.
den 12 Bleve instrumenterne verificerede, samme dags eftermiddag stillede jeg maalebordet over det udmærkede Stationspunct, hvor jeg optrak een Compaslinie, og udsatte een Basis i nordlig og en i vestlig direction, denne sidste continuerede jeg 748 alen og derfra tog intersection paa Roeholt.
den 13 Begyndte jeg igjen ved 748 og avancerede derfra til Hyllede efter nogle gange at have brækket linien formedelst forekommende Moser og Kratskov, samme dag blev bestemt een miilepæl paa landeveyen fra Kiöbenhavn til Vordingborg Bregnemosen og Hyllede bye.
den 14 Blev den linie som fra förste Station var udsat i nordlig Direction continuered til 275, hvorfra blev udsat een nye Basis i nordvest og i samme maalt 1125 alen, herfra tog jeg atter een nye linie som blev continuered 2020 alen ind i een aabning i Skoven ved Rosenlund, ved disse linier bleve bestemte Helledes Huus Krageborg Huus Tystrup og Jyderup tiligemed skovkanten fra Store Elmue til Rosenlund.
den 15 Blev udsat fra 2020 een tverlinie igiennem een af Alleerne ved Rosenlund, samme linie blev continuered 160 alen, derfra tog jeg een linie imod nord tvert igiennem skoven og maalte i samme linie 525 alen, ved disse linier blev bestemt een deel af jagtveyene i Rosenlund skov de övrige ere tegnede effter et andet Carte over samme skov, som Kammerherre Rosenkrants har ladet optage.
den 16 Söndag."

3.4 Promotion

The pay of 50 Rd (rigsdaler) per year that was offered assistants was hardly enough to live on. One of Wessel's contemporary students[18] wrote in his diary that the consumption of a year amounts to 150-200 Rd. In the spring of 1768 Wessel's financial situation became impossible, and he applied to the Academy for a rise of 100 Rd.

...since I could not make a living from my present salary as an assistant, and unless I get the rise, I have to go home to my father: in addition to assisting with the surveying I would also take upon me the drawing of maps.[19]

The reason why he had the courage to take upon himself the cartography lies in the fact that he already knew the job. In Caspar Wessel's surveying-diary of 1766 we see that he had helped Ole Christopher draw the map of the county of Copenhagen, a map that carries the name of O.C.Wessel only.

September
This month I was in Copenhagen, drawing on the map of the county of Copenhagen, my brother went instead of me to the countryside .[20]

The Academy did not want to lose an efficient surveyor, so Caspar Wessel was granted increased pay and made responsible for the construction, reduction and drawing of maps based on the results of the geographical and trigonometrical surveying. Already in 1768 he finished the first of four maps that were to cover all of Zealand on the scale of 1 : 120 000. Even though he was promoted to geographical surveyor in 1769, it seems that he did not do much surveying in the countryside during the years 1770 and 1771, but was allowed to spend almost all his working time on the drawing of maps. In a short time it was therefore possible to present the four maps of Zealand:

1768: The north-eastern quarter of Zealand;
1770: The south-eastern quarter of Zealand;
1771: The north-western quarter of Zealand;
1772: The south-western quarter of Zealand.

Caspar Wessel did a tremendous job as a cartographer (see figure 5), and at the same time he also tried to study law. No wonder his elder brother the poet Johan Herman Wessel, wrote about him[21]

18 Edvard Storm: *Regnskabsböger.* KB, NKS 396r 8°.

19 "...da jeg med min Assistents Gage kunde umuelig udkomme, og med mindre jeg kan faae den, maa jeg reise hjem til min Fader, da jeg foruden at assistere ved Landmaalingen, vilde paatage mig Carte Teigningen." KDVS's Protokol, 7 June 1768, p. 325.

20 "September Maaned
I denne Maaned var jeg i Kiöbenhavn og tegnede paa cartet over Kiöbenhavns Amt, min Broder reiste i min stæd ud paa Landet." Caspar Wessel: *Landmaalings Journal fra Aar 1766,* K&M.

21 Viggo Brun suggested in *Regnekunsten i det gamle Norge* that the verse could as well refer to Ole Christopher Wessel. We are however convinced that the verse is written about Caspar although he is not mentioned explicitly. We base this on the fact that the verse most likely was written after Ole Christopher left the surveying, and moreover, that Caspar drew many more maps than Ole Christopher.

He is drawing maps and reading the law	*Han tegner Landkaart og læser Loven*
He is as diligent as I am lazy.	*Han er saa flittig som jeg er Doven.*[22]

4 Among Norwegians in Copenhagen

4.1 Brother and poet

While the whereabouts of the two brothers Ole Christopher and Caspar Wessel are easy to track down through the sixties, Johan Herman Wessel almost vanished. But in 1772 he surfaced again with the humorous "tragedy" *Love Without Stockings* (Kierlighed uden Strømper). Johan Herman was just like his brother Caspar a very modest man, and his friends had to talk him into publishing the play. Christen H. Pram wrote some years later

It ought to be mentioned that the modesty of Mr. Vessel was so great, that he might never have published his masterpiece, if his friends had not persuaded him, or even more if his conditions had not forced him to do so.[23]

The "tragedy" had a most favourable reception when it was published on the 2nd of September 1772. Three days later Luxdorph wrote in his diaries that he had "read Love Without Stockings. A most witty piece".[24]

The play was performed at one of the two theatres of Copenhagen in March 1773. In a newspaper we find that

Friday the 26th of March between 7 and 8 pm., on the arrival of His Royal Majesty, the tragedy Love Without Stockings, a new original tragedy in 5 acts by Mr. Wessel will be performed on the Royal Danish Stage.[25]

Most likely the Norwegian group of students saw the play on the 31th of March, at the third performance, because

The receipts of this evening belong to the author, by whom tickets may be obtained, No. 127 at Ulfeldt's Square.[26]

Caspar and Ole Christopher Wessel may well have been there.

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22 Johan Herman Wessel: *Samtlige skrifter*, 2. deel, Kiøbenhavn, 1832, p. 158.

23 "Det bør her anmærkes, at Hr. Vessels Beskedenhed var saa stor, at han maaskee aldrig havde udgivet dette sit Mestervek , dersom ikke hans Venner havde overtalt, og endnu kraftigere hans Forfatning tvunget ham dertil." Christen H. Pram's remarks to *Elegie ved Vessels grav.* Minerva, Tredie Bind, Kiøbenhavn, 1786, pp. 108-109.

24 "Læst Kiærlighed uden Strømper. Et meget vittigt Støkke." Bolle W. Luxdorph: *Dagbøger*, Bind I, 5 September 1772, G.E.C.Gad, København, 1915-30.

25 "Paa den Kongel. Danske Skueplads bliver Fredagen den 26 Martii imellem Kl. 7 og 8, ved Hans Kongelige Majestets Ankomst, opført Tragoedien: Kiærlighed uden Strømper, et nyt Original SørgeSpil i 5 Optog af Hr. Wessel." Kiøbenhavns Kongelige Adresse-Contoirs Efterretninger, 24 March 1773.

26 "Denne Aftens Indtægter tilhøre Forfatteren, hos hvem Sedlerne faaes, No.127 paa Ulfeldts Plads." Kiøbenhavns Kongelige Adresse-Contoirs Efterretninger, 29 March 1773.

4.2 *The Norwegian Society*

Johan Herman Wessel was suddenly known all over Copenhagen. However he most often wrote for himself and his Norwegian friends. In particular he was excellent at writing epigrams, verse and impromptus. A group of mainly Norwegian students often gathered around the punch bowl in a café, run by a woman named Mdm. Juul and her husband Niels Juul. The gatherings turned more and more into a literary society, and in April 1774 the Norwegian Society was founded. One of the leading characters was Johan Herman Wessel.

Caspar Wessel was also a member of the Norwegian Society, and a rather central one.[27] Several years later (in 1792) Mdm. Juul was moving to the Finmark in Norway. She had really been like a mother to many of the Norwegian students in Copenhagen. Some weeks prior to her departure, Caspar Wessel, as one of the old-timers of the Norwegian Society, wrote in her autograph book

You will abandon us, we are suffering the loss, and regret our inability to repay your often proved kindness. However should it so be; then live, wherever you go, prosperously, relish in the Finmark the tranquillity that you by us for such a long time have missed and remember in the happy days still to come, Your abandoned friends.[28]

On the other hand Ole Christopher Wessel cannot have been a regular member of the group. In 1770 he had passed the exam in law (latinsk juridisk examen), and after that he became an assessor at the aulic court (Hof- og Stadsretten) in 1771. He was the kind of person that stood out to such an extent that it would have been mentioned by the Norwegian Society if he had been a regular participant. In 1778 Hee, who had succeeded Hielmstierne as the leader of the Surveying Commission when he became the President of the Academy, described Ole Christopher as

the most brilliant genius I ever knew, he had a great assurance, was enterprising and when he was short of money, he borrowed from his good friends, went to the masquerade and sometimes won more in one evening than he earned as a surveyor in one year, and one has to say for him, Audaces fortuna juvat.[29]

Ole Christopher Wessel left the surveying when he became an assessor in 1771, but Caspar Wessel stayed on. As a surveyor he was free to choose his own assistants, and in the spring of 1773 the choice fell on one of his friends from the Norwegian Society, Johan Randulf Bull. They spent the two next summers together in Funen (Fyn) and had good opportunities to discuss their common interests in mathematics and law.
On the day before leaving (either in 1773 or 1774) for Funen Wessel had to pack all the equipment, but Bull as an assistant was free to hang out with their friends, drinking wine and punch. Before he left, two of the main characters of the Society wrote

27 Knud Lyne Rahbek: *Erindringer af mit Liv*, 4. del. Kiøbenhavn, 1827, p.5.

28 "De vil forlate os vi föle vort Tab, og beklage vor Uformuenhet at giengielde Deres beviiste Godheder. Men skal det saa være; saa lev da, hvor De kommer, lykkelig, nyd i Finmarken den Roelighed, De her hos os saa længe have savnet, og tænk endnu i mange glade Dage paa Deres eftterladte Venner." Madam Juul: *Stambok*, Norske Selskab in Oslo.

29 "det ypperligste Genie jeg haver kiendt, havde stor Hardiesse, var entreprenant og naar ham fattedes Penge, laante hand af sine gode Venner, gik paa Masceraden og vandt undertiden i en Aften, meere end hand fortiente i et Aar ved Landmaalingen, og mand maae sige om Ham, Audaces fortuna juvat." Letter from Christen Hee to Henrik Hielmstierne, 6 February 1778, KDVS, Prot. Nr.pg. 15/1778.

Give my regards,	*Formeld min Respect,*
You lecherous chap,	*Din liderlig' Knegt,*
To Caspar, if I kindly may ask	*Til Caspar, om du saa behager.*
J:H:Wessel	*J:H:Wessel*
Give mine as well,	*Formeld ogsaa min,*
You lecherous pig,	*Dit liderlig' Svin,*
If you're not snatched by the devil	*Ifald Fanden Dig ei tager.*
Joh: Wibe	*Joh: Wibe*[30]

The geographical map (conceptkort) based on the surveying in 1773 is one of the best ever made by Wessel, and it has a beautiful cartouche, see figure 6.

During the winter of 1774 - 75 Wessel appointed a new assistant. This time it was the secretary of the Norwegian Society, Johan Wibe. On the evening before their departure for Funen in 1775 the secretary told the members to guard the spirit of the Society while he was away, and in their records he wrote

I'm a surveyor of the kind	*Jeg er en Landmaaler af den profession,*
That may be called an assistant;	*Som man Assistent monne kalde;*
My station is found in wonderful Funen,	*I liflige Fyen der er min Station,*
Strange how Fate may arrange it!	*Saa underlig Lodden mon falde!*
I'm going thereover	*Jeg reyser derhen*
With Caspar my friend;	*Med Caspar min Ven;*
But you're losing two selected.	*Men I miste tvende Udvalde.*
......	
I'm putting my secretarial pen down;	*Jeg lægger nu ned Secretairiske Pen;*
I have to exchange it for the compasses.	*Jeg må den med Cirklen*[31] *ombytte.*
Pick it, noble Wessel! pick it up again.	*Tag du, ædle Wessel! tag du den igien.*[32]

Wessel is of course Johan Herman who was appointed secretary while Johan Wibe was away.

Two of Johan Wibe's brothers, Ditlev and Niels Andreas, were members of the Society and also surveyors. Ditlev Wibe and Johan Rick began in 1779 the surveying in Norway under the guidance of Bugge.[33] A couple of years later Niels Andreas Wibe joined them. Many of the members of the Norwegian Society became surveyors, the most outstanding among them was indeed Caspar Wessel.

30 *Norske Selskabs Vers-Protokoll,* Norske Selskab, Oslo 1935, p. 424.
31 *Cirkel* is short for *Stang-Cirkel,* which is *a pair of compasses.*
32 *Norske Selskabs Vers-Protokoll,* Norske Selskab, Oslo 1935, pp. 114-117.
33 Several of the old surveying instruments which were used in Norway are displayed in the museum of Statens Kartverk in Hønefoss in Norway. Among them are two geographical instruments made by Johan Ahl. One of them carries the initials DW.

Rudkøbing

Svendborg

Figure 6. Cartouche drawn by Caspar Wessel as part of his geographical map of Funen and Langeland, 1773. In the top he included the town arms of the most important towns on the (full) map: Svendborg (on the south coast of Funen) and Rudkøbing (on the west coast of Langeland). Kort- & Matrikelstyrelsen.

5 Towards becoming trigonometrical surveyor

In 1777 Caspar Wessel promised the Academy to practise trigonometrical surveying so that he could succeed the trigonometrical surveyor Niels Morville in the summer of 1778. Morville had worked as such since Ole Christopher Wessel quitted the job in 1771. Morville left the surveying under the auspices of the Academy in 1778 and was for the rest of his life occupied with economical surveying.

Caspar Wessel spent the summer of 1777 in Jutland, continuing the geographical surveying; in the winter he and Hans Skanke worked hard to finish the general map of Zealand; Wessel drew the eastern part, Skanke the western.[34]

5.1 An unusual year – 1778

Wessel had been out in the countryside surveying almost every summer since 1764. Fourteen years had passed, and he had to some extent neglected his studies of the law. His plan was to read during the whole winter of 1777, but he was disturbed in his effort when he had to finish the map of Zealand. In a desperate attempt to complete his studies he therefore applied for leave of absence with full pay.

Christen Hee wrote, as leader of the Surveying Commission, a note to the President of the Academy explaining that

None of the surveyors has been more useful to us than he has, during the summers he has been surveying and in the winter time he has been working as a designator, which in the fourteen years he has stayed with the surveying has ruined his health and been an obstacle to his studies in such a way, that if he once again has to interrupt his studies he is perdu and will never pick them up again. Last winter when he nolens volens had to draw the general map of Zealand he was once more distracted in his studies, and then I promised him never again to disturb his circles.[35]

Wessel needed a person to stand up for him. In the same letter Hee mentioned that Wessel did not dare to hand over the application to President Hielmstierne personally since he was afraid it would not be well received. Instead he delivered it on an occasion when he knew Hielmstierne was not at home. In Hee's attempt to help Wessel he had to convince Hielmstierne that it was vital for him to be granted permission, because as Hee put it, Caspar Wessel could not be compared to his brother Ole Christopher.

This one [Caspar] has a bright, but very slow head, and when he sets out to study something, he can have no peace, before he completely understands it, he does not have a memory like his brother but has to stick to the subject, because he forgets the first part in corpore juris while he is studying the second part, his financial position is also very middling, and this often makes him more melancholic and indocile than he actually is.[36]

34 Caspar Wessel: *Autobiography* (transcript), KB, NKS 4° 1977 b.

35 "Ingen af alle Landmaalerne har vaeret os et nyttigere redskab end hand, i det hand har opmaalt om sommeren og giort tieneste som desinateur om vinteren, hvilket paa de 14 aar hand har vært i landmaalingen har saa fordervet hands helbred og hindret ham i hans studering, at om hand nu end engang skal afbryde sine studeringer er hand perdu og begynder aldrig igien paa nye. Forrige vinter da han nolens volens maatte paatage sig at tægne det generale carte blev hand ligeledes troubleret i sit studio, og da lovede ieg ham, at ieg aldrig meere skulle turbere hands cirkler." Letter from Christen Hee to Henrik Hielmstierne, 6 February 1778, KDVS, Prot. Nr. pg. 15/1778.

He was granted leave of absence with full pay, and on the 7th of November he was ready for the exam. One of the examiners was Professor Jacob Edvard Colbiørnsen, an old friend from the Christiania Cathedral School, a man highly admired for his multifarious knowledge. It is also interesting to notice that Colbiørnsen was among the six that were present when Wessel's treatise was read on the 10th of March 1797. At last Wessel finished his studies, and to his great delight he passed the examination in law with first-class honours. This of course made him happy and proud.

In 1778 I passed the exam in law (latinsk juridisk Examen), with the best mark.[37]

He now had to make a decision, should he continue with the surveying, or try to create a completely different career. He only made one unsuccessful attempt to obtain a new job.[38] Being rather scared to try something new and at the same time being very conscientious, he then chose to stay on, living up to the promise he had given earlier to take over the responsibility of the trigonometrical surveying. The description given by Bugge is to the point:

If he had been in possession of more courage and assurance when it comes to trying unaccustomed work, then with his insight and talent, he could have done a lot for the benefit of the community as well as for himself.[39]

From 1779 to 1796 Wessel worked as trigonometrical surveyor, still spending the summers from May to September or October out in the countryside measuring. During the winters he was occupied by trigonometrical calculations, based on the collected data, judgement of the validity of the measurements, and by the construction and drawing of the resulting triangular maps. The work required both practical and theoretical skills, as well as accuracy, patience and a breadth of outlook. It was well-known to the Surveying Commission that Wessel had all these qualifications when they asked him to succeed Morville.

5.2 Trigonometrical surveying

In the following we shall not try to give a general account of trigonometrical surveying under the auspices of the Academy, but focus on parts which differ the most from geographical surveying and which may be seen as an inspiration and motivation for Wessel towards his later mathematical paper. This includes types of measurements and calculations as well as very selected examples of the more theoretical reflections he undertook.

36 "Denne derimod har et godt, men meget langsom hoved, og naar hand foretager sig at studere en ting, har hand ingen roe, förend hand kand approfondere ret, hand har icke en hukommelse som broderen, men maae bestandig hænge i tingen, fordj hand forglemer den förste deel in corpore juris medens hand studerer den anden, hans oeconomiske omstændigheder ere og meget maadelige, hvilket ofte giör ham meere tungsindig og tungnemmet, end hand virkelig er." Ibid.

37 "I 1778 tog jeg latinsk juridisk Examen med bedste Karakter." Caspar Wessel: *Autobiography* (transcript), KB, NKS 4° 1977 b.

38 Letter from Caspar Wessel to the King, 19 February 1805. Finanskollegiet, Sekretariatet, Journalsager 1805. No. 672.

39 "Wenn er etwas mehr Mut und Dristigkeit besäße, um ungewohnte Arbeiten anzugreifen, so könnte er mit seinen Einsichten und Talenten zum gemeinen Besten und zur Beförderung seines eigenen Glückes mehr ausrichten." Letter from Thomas Bugge til Georg Christian Oeder (German transcript), 25 December 1781. NS, 31-4-36-1.

The purpose of trigonometrical surveying was to establish a network of points determined by triangulation, and in some cases, to supplement it with an astronomical determination of the latitude and longitude of selected points as well as of the direction of the meridian through such a point. From the beginning of the surveying under the auspices of the Academy it was clear to Hee that astronomical knowledge would be essential. The University was therefore asked to make sure that the Round Tower Observatory, in the heart of Copenhagen, was modernised so that it would equal the principal observatories in Europe. The Observatory naturally became the origin of the triangulation, it was therefore important to determine the position of the Round Tower. A lot of effort by Bugge and his assistants went into determination of the latitude and longitude of the Observatory and the direction of the meridian through it. This determination was also important in order to coordinate the Danish efforts with measurements in other countries.

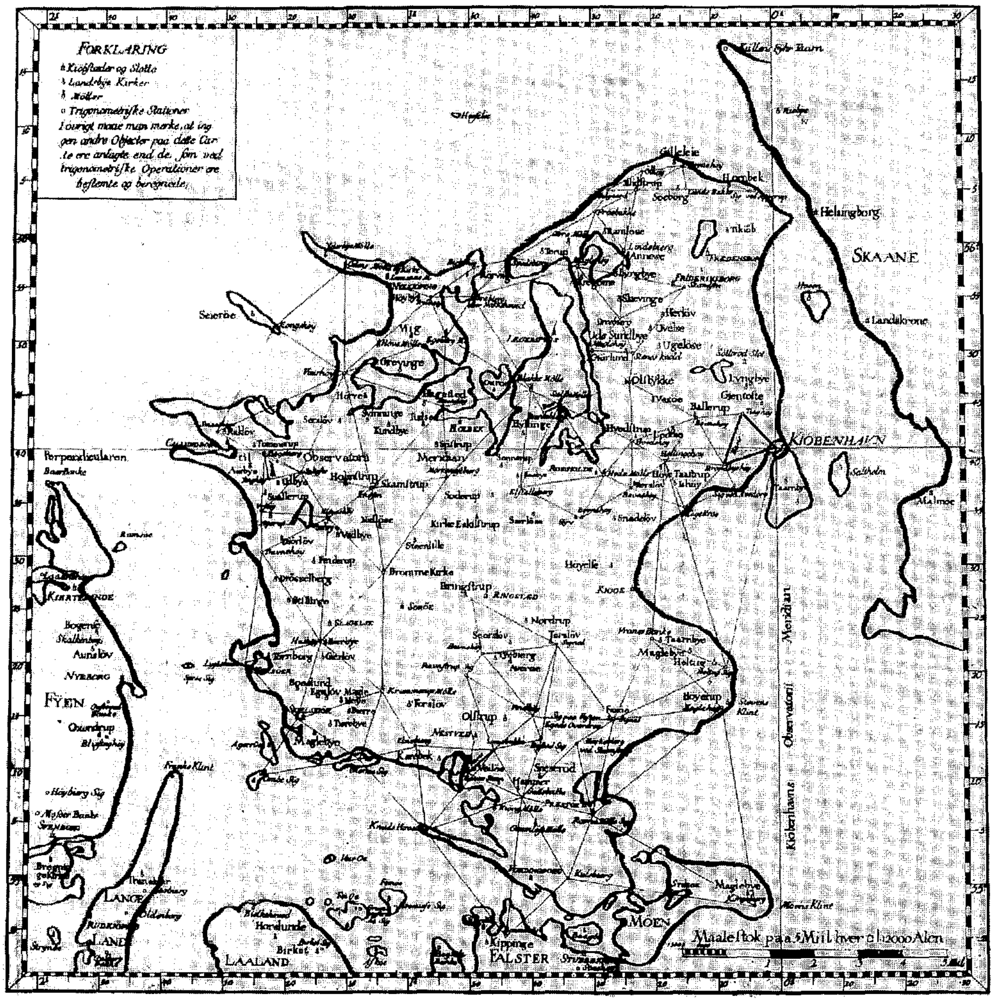


Figure 7. The primary triangular net (determined by Thomas Bugge and Ole Christopher Wessel 1765 – 1771) added to the outline of the general map of Zealand (drawn by Caspar Wessel and Hans Skanke 1777). Note the axes in the coordinate system in which the trigonometrical points were specified: the meridian of Copenhagen's Observatory and the Perpendicular. *From Thomas Bugge:* Beskrivelse over den Opmaalings Maade som er brugt ved de Danske geographiske Karter. *Kiøbenhavn, 1779. We have emphasised the triangle with the Observatory as the vertex opposite to the edge formed by the base line.*

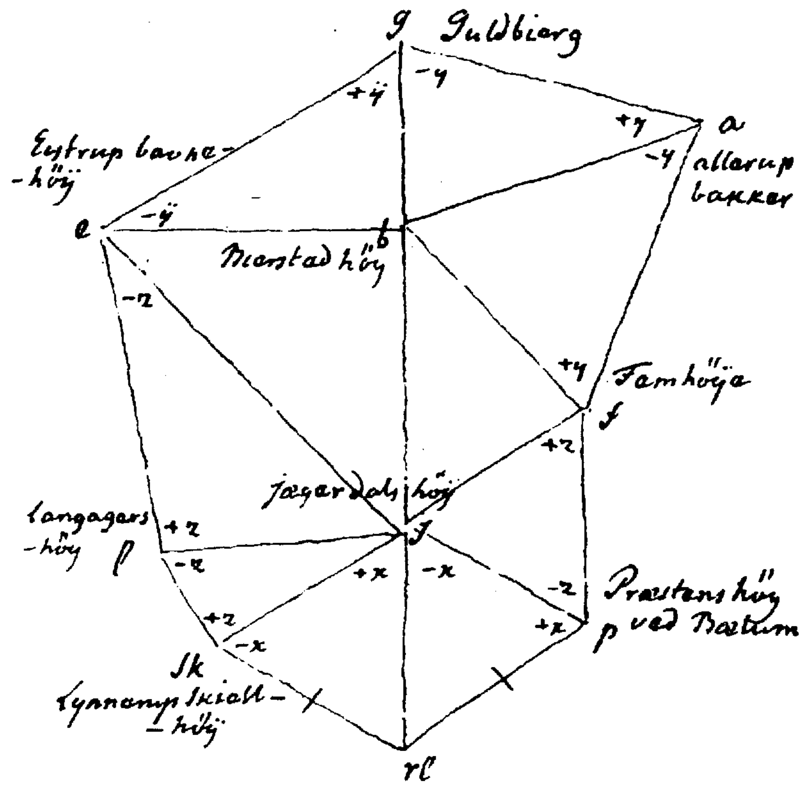


Figure 8. Part of the primary triangular net in northern Jutland determined by Wessel. From Caspar Wessel: De Trigonometriske Beregninger for 1787, *p. 42. Kort- & Matrikelstyrelsen.*

The other trigonometrical points in the triangular net were chosen by the trigonometrical surveyor, usually as the peak of a hill with a reasonable view; often an ancient burial mound. Many of these points have remained trigonometrical points and are today marked with a pillar. These trigonometrical points formed the vertices in the so-called primary (or first order) triangulation, as illustrated in figure 7, showing the triangular net of Zealand, and as illustrated in figure 8, from Wessel's surveying report of 1787, as part of the triangular net he measured that year.

The primary triangular net was determined through measurement of all angles in all triangles and calculation of the length of all edges, based on the measurement of a few edges, the so-called base lines. A base line was chosen as a line between two neighbouring trigonometrical points, for which the landscape was particularly flat and without obstacles. The length of a base line was measured a couple of times by rods (of verified length), and often also by a measuring chain, exactly as geographical surveyors determined lengths. But the chain alone was not considered to be sufficiently accurate. All angles were measured several times and tested against the fact that the sum of angles in a triangle should equal two right angles. Corrections to (the mean of all the measurements of) each angle were made in order to obtain the correct sum of angles in each triangle. Theoretically it was sufficient to measure the length of one base line in a connected triangular net in order to calculate the length of the rest of the edges, but the accuracy of the calculated edges was tested and corrected against the measurements of the other base lines.

A secondary triangulation was obtained as a refinement of the primary triangulation by adding as secondary vertices characteristic points on the most remarkable buildings, such as the steeple of a church or the tower of a windmill. All the new angles – seen from the trigonometrical points – in the secondary triangulation were measured, and the lengths of new edges were calculated. To get a feeling for the order of the number of triangles determined in one year we note that Caspar Wessel in his first year as a trigonometrical surveyor chose 12 trigonometrical points and determined 12 primary triangles and 105 secondary triangles.

It should be clear from the above that the work in the countryside consisted in a careful choice of trigonometrical points, a careful marking of these points and a careful determination of lots of angles. In the primary and secondary triangles the angles were measured in (almost) horizontal planes. The angle that a sight line from one trigonometrical point to another made with the horizontal plane in the first point was measured in the vertical plane containing the two points. Since the landscape was generally rather flat, it was in most cases unnecessary to reduce the angles in the triangles to angles in a horizontal plane.

In order to determine the latitude and longitude of selected trigonometrical points as well as the direction of the meridian through such a point the surveyor would again measure angles, but all of these angles were measured in vertical planes. The latitude of a trigonometrical point was determined by the height of the culmination of the sun in the middle of the day. The direction of the meridian through a point and the longitude were determined in different ways from measurements of the height of the sun and certain stars in different vertical planes containing an edge in the triangular net. We shall not dig further into the astronomical measurements and calculations, but only conclude that the trigonometrical surveyor along with the horizontal angles also measured a lot of vertical angles.

The instrument designed to measure angles, both in horizontal and vertical planes, was the so-called *geographical instrument*. In the trigonometrical surveying of France they had used a quadrant for such measurements. In Sweden a similar instrument, but with a complete circle, had been invented and constructed by the Swedish instrument maker Daniel Ekström, therefore the instrument was also called an Ekström Circle. An instrument with a complete circle was an improvement, since it was more stable when moved around and easier to verify. The first one of the geographical instruments, a slight variant of the Ekström Circle, was finished in Copenhagen in 1763 and constructed by the Swedish instrument maker Johan Ahl who had recently moved from Stockholm where he had worked for the Swedish Academy. Figure 9 shows the copper plate of the instrument as it occurred in Bugge's book.[40] The instrument consisted of a complete circle of brass of diameter two feet (62,8 cm) connected with four arms A, B, C, D to the centre. It could be fixed in a vertical position (as shown) for astronomical measurements and for measurements in the triangular net in a horizontal position or a position that was tilted up to 10 degrees compared with the horizontal plane. The brass circle had two scales, the inner one with the four right angles divided from 0 to 90 to 0 to 90 back to 0 (ordinary) degrees, and the outer one from 0 to 96 to 0 to 96 back to 0 parts or extraordinary degrees. Each degree was subdivided further through consecutive halvings so one could read off angles

40 Thomas Bugge: *Beskrivelse over den Opmaalings Maade som er brugt ved de Danske geographiske Karter.* Kiøbenhavn, 1779.

with an accuracy down to between 5 and 15 seconds. Bugge explained in his book that compared to the Swedish instrument

One has given it a more steady and better stand and a larger radius; one has divided it into 90 and 96 degrees; and one has added several other improvements to ease the verification.[41]

One of the advantages of adding the 96 scale was that one could obtain a more accurate division of the peripheral. An angle of 60 degrees corresponds to 64 parts on the 96 scale, a power of two, that can easily be halved six times. Another advantage of having two different scales was that each reading of a direction would give two results.[42]

Attached to the circle were two telescopes. The movable telescope HI was mounted to a so-called alhidade, which was open over the scales in both ends, allowing in fact four readings for each position. The fixed telescope OP was joined to a level. Another level was attached to one of the arms, making it possible to adjust the circle to a vertical position. Moreover, each telescope could be lifted off and turned around.

The horizontal plane of the Round Tower Observatory (the tangent plane) was thought of as identical to the plane of the maps. A point on the map was specified by its coordinates in the coordinate system that had the Observatory as its origin, one axis equal to the tangent to the meridian through the Observatory, oriented positively from north to south, and the other axis perpendicular to the first and oriented positively from west to east ("against the sun"). Longitudes were also determined relative to the Observatory, and with the same convention: positive from west to east. This may seem strange. It fits, however, well with the long-standing convention of using Ferro as a common reference point for longitudes. Ferro is the most western island of the Canary Islands. It was the most western point in the old world, that is, when Ptolemy made his map of the world.

5.3 Wessel's first year as trigonometrical surveyor – 1779

From the trigonometrical surveying diary we can as before follow Wessel's movements and work day by day. The instruments were heavy, in particular the geographical instrument, and he was completely dependent on the help of local people, and stuck when no help was available. For instance, we find

July

The 19th *The instruments were finally taken down, because on the 14th all the peasants were doing villeinage, so that I neither could have carriage nor people to help.*[43]

Through a royal decree local peasants were obliged to help; the team should include a couple of younger men who were strong enough to handle the instruments, and an older man who knew the names of the places to be reported. Trigonometrical points were cho-

41 "Man har givet det et stadigere og bedre Stativ og en større Radius; man har indeelt det i 90 og 96 Grader; og man har tillagt adskillige andre Forbedringer til Verifikationens Nemhed." Ibid., p. 22.

42 Three of the old geographical instruments are preserved in Denmark, but none of them have double scales. One is displayed in the Ole Rømer Museum. Hemming Andersen: *Historic Scientific Instruments in Denmark*, Det Kongelige Danske Videnskabernes Selskab, København, 1995, pp. 188-189, 287, 352.

43 "Julius D 19de Bleve först Instrumenterne nedtagne, thi den 14de vare alle Bönderne til Hove, saa at jeg hverken kunne faae vogn eller Folk til hielp." Caspar Wessel: *Journal over De Trigonometriske Observationer som efter Det Kongl Videnskabernes Societets Ordre ere foretagne, 1786*, K&M.

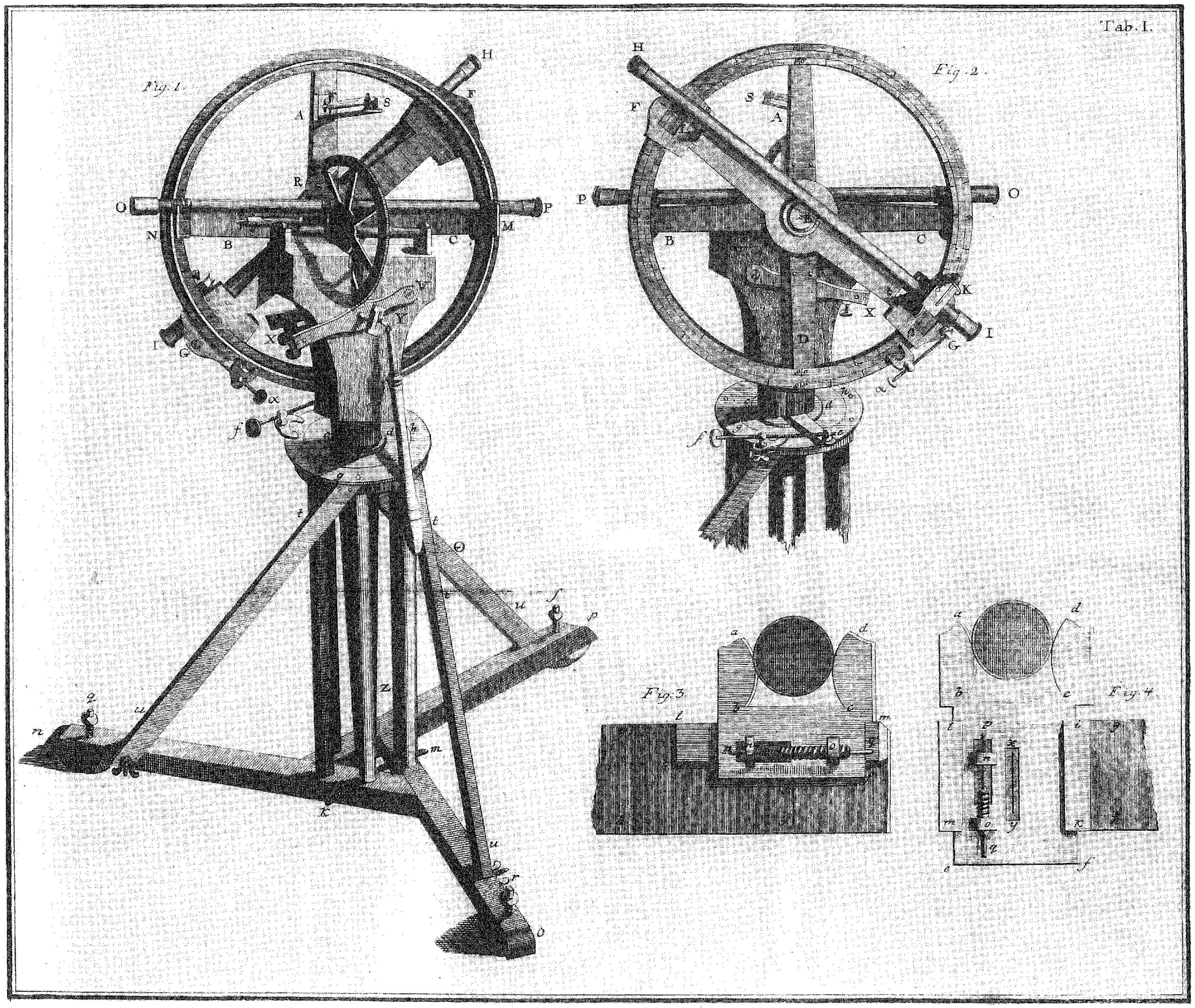


Figure 9. The geographical instrument shown in vertical position. Note the double scale on the circle, a right angle was set equal to 90 degrees in the inner scale and 96 parts in the outer scale.

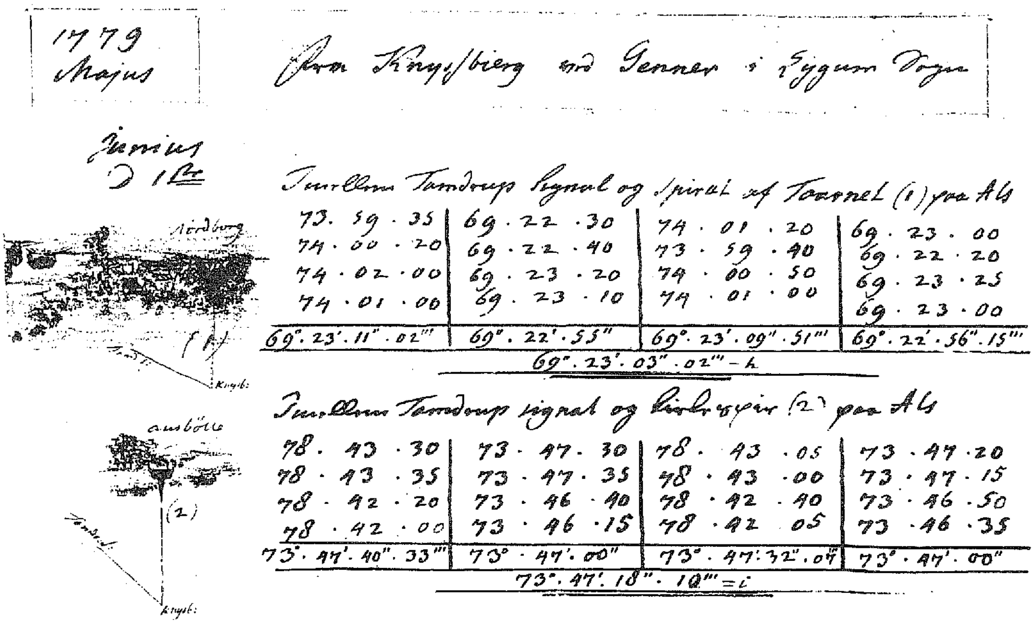
From Thomas Bugge: Beskrivelse over den Opmaalings Maade som er brugt ved de Danske geographiske Karter. Kiøbenhavn, 1779.

sen and signals placed at these indicated points before the measurements could start. The local people should allow the surveyor to pass their fields and to let signals stay undisturbed until the following year, or sometimes for several years.

Whenever the geographical instrument had been moved to a new place, it was always carefully verified before measurements could start. Nevertheless, a failure could occur so that the measurements had to be repeated, as described in the report of the trigonometrical calculations of 1779.

Notice: The angles between the principal objects which are observed from the first two Stations, and are given from pages 2 to 14, are not used in the calculations of these triangles, since the instrument when these observations took place, had the failure that the alhidade did not close up with the Limbus of the graduated circle; after this failure was corrected, the same angles are repeated.[44]

To reduce the errors due to inaccuracies in the instrument Bugge had advised the surveyor to perform four adjustments in order to determine each angle in the triangulation. We illustrate this by showing in figure 10a part of page 6 of the 1779 trigonometrical surveying report.[45] On June 1st the instrument was placed at one trigonometric point K



1779 Majus

Imellem Tamdrups Signal og Spiret af Taarnet (1) paa Als

73 . 59 . 35	69 . 22 . 30	74 . 01 . 20	69 . 23 . 00
74 . 00 . 20	69 . 22 . 40	73 . 59 . 40	69 . 22 . 20
74 . 02 . 00	69 . 23 . 20	74 . 00 . 50	69 . 23 . 25
74 . 01 . 00	69 . 23 . 10	74 . 01 . 00	69 . 23 . 00
69°. 23'. 11". 02'''	69°. 22'. 55"	69°. 23'. 09". 51'''	69°. 22'. 56". 15'''

69°. 23'. 03". 02''' – h

Imellem Tamdrups signal og Kirkespiir (2) paa Als

78 . 43 . 30	73 . 47 . 30	78 . 43 . 05	73 . 47 . 20
78 . 43 . 35	73 . 47 . 35	78 . 43 . 00	73 . 47 . 15
78 . 42 . 20	73 . 46 . 40	78 . 42 . 40	73 . 46 . 50
78 . 42 . 00	73 . 46 . 15	78 . 42 . 05	73 . 46 . 35
73°. 47'. 40". 33'''	73° . 47'. 00"	73°. 47'. 32". 04'''	73° . 47'. 00"

73°. 47'. 18". 10''' = i

Figure 10a. Table of some angles in the secondary triangular net measured by the geographical instrument. The instrument was placed in one trigonometrical point (Knyssbierg in northern Schleswig, now southern Jutland). One telescope pointed towards another trigonometrical point (Tamdrup) and the other telescope to the steeple of some churches on the island of Als. Note that the numbers in the first and third column refer to the 96-scale. The page is reproduced in reduced size (68%). From Caspar Wessel: Journal over de trigonometriske opmaalinger som aar 1779 ere foretagne i den östlige part af Hertugdömmet Slesvig, *p. 6. Kort- & Matrikelstyrelsen.*

44 "Anmærkning: De Vinkler mellem Hoved Objecterne som ere observered fra de 2 förste Stationer, og ere anförte fra pag 2 til 14 er ei brugte i disse Trekanters Udregning, saasom Instrumentet, da disse Observationer skeedte, havde den Feil, at Alhidaden ei sluttede nöie til Grad Cirklens Limbus; da denne Feil siden var rettet, ere de samme Vinkler repeterede." Caspar Wessel: *Trigonometriske Beregninger for 1779,* p. 65. K&M.

45 Caspar Wessel: *Journal over de trigonometriske opmaalinger som aar 1779 ere foretagne i den östlige part af Hertugdömmet Slesvig,* p. 6. K&M.

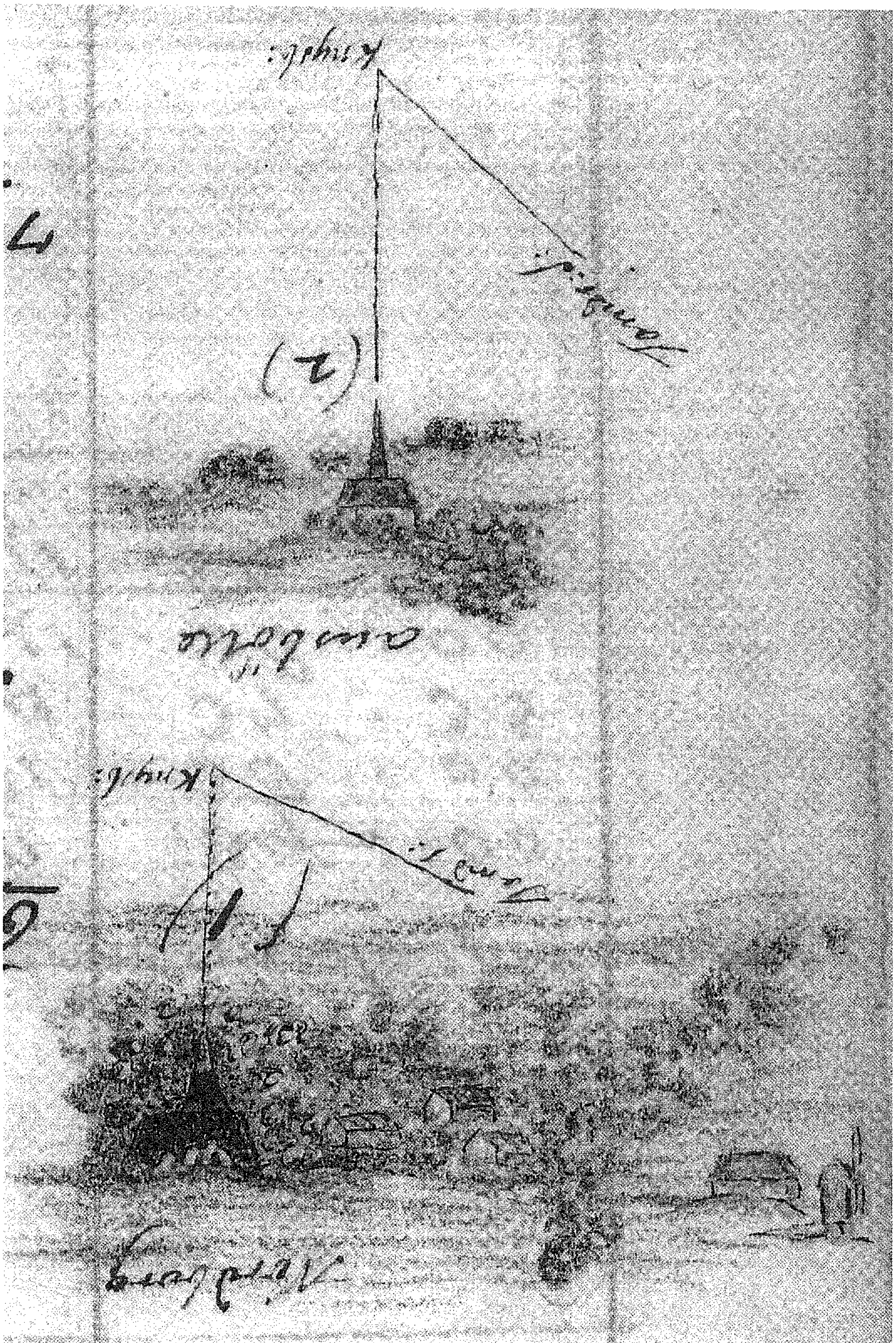
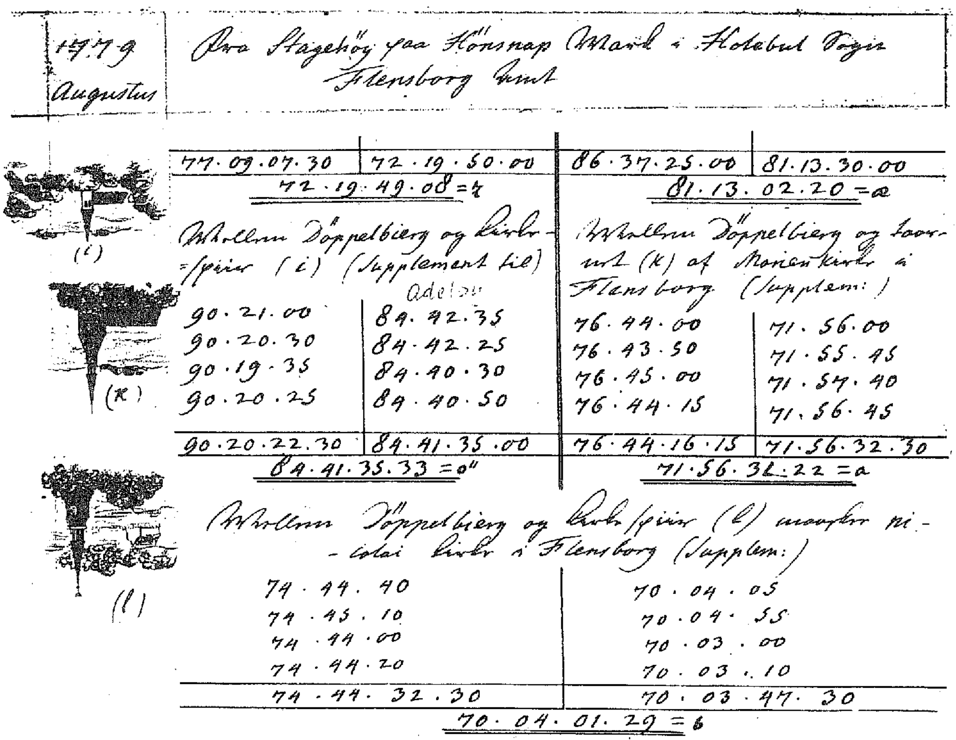


Figure 10b. Enlargement of the churches shown in figure 10a.

(Knyssbierg close to Genner in the parish of Hygum). The angle between the direction to another trigonometric point T (Tamdrup) and the direction to the steeple of church (1) respectively (2) (on Als) was determined by letting the fixed telescope be directed towards T and the removable one towards the steeple(s) and vice versa, and to repeat these measurements after both telescopes had been turned around. In this way the removable telescope pointed exactly once into each of the four quadrants with each position giving rise to four readings. For later use the angle was then set equal to the mean of these 16 numbers (after the extraordinary degrees had been converted into ordinary degrees).

As shown in figure 11a, part of page 45 of the same report, some of the determinations of angles were based on (only) 8 readings. In this first trigonometrical surveying report of Wessel there are close to 50 drawings of churches and mills with surrounding landscape; drawn as they were seen – upside down – in the telescope, and beautifully coloured. They served two purposes: to state exactly which point had been chosen as reference and to make it possible to recognise the church later on. The level of details is quite impressive; when enlarged even more so. Again we can enjoy Wessel's skill as a competent illustrator. In his later trigonometrical surveying reports such drawings are more sparse and much more sketchy. They have just enough details to fulfil their purpose. The trigonometrical surveying report of 1779 is certainly more colourful than any of the later reports. He was clearly eager to express his gratitude towards the Academy through an extra effort and – after all – he must have been satisfied to be back as a surveyor with new responsibilities after the sabbatical year.

Over the following years, up to 1796, Wessel spent more time and effort on theoretical problems, in particular in connection with the construction of maps from the trigono-


1779 Augustus

(i)

77.09.07.30	72.19.50.00	86.37.25.00	81.13.30.00
72.19.49.08 = ǂ		81.13.02.20 = a	

(k)

90.21.00	84.42.35	76.44.00	71.56.00
90.20.30	84.42.25	76.43.50	71.55.45
90.19.35	84.40.30	76.45.00	71.54.40
90.20.25	84.40.50	76.44.15	71.56.45
90.20.22.30	84.41.35.00	76.44.16.15	71.56.32.30
84.41.35.33 = o"		71.56.32.22 = a	

(l)

74.44.40	70.04.05
74.45.10	70.04.55
74.44.00	70.03.00
74.44.20	70.03.10
74.44.32.30	70.03.47.30
70.04.01.29 = b	

Figure 11a. Table of some angles in the secondary triangular net in northern Schleswig. One telescope pointed towards the steeple of some churches in Flensborg.

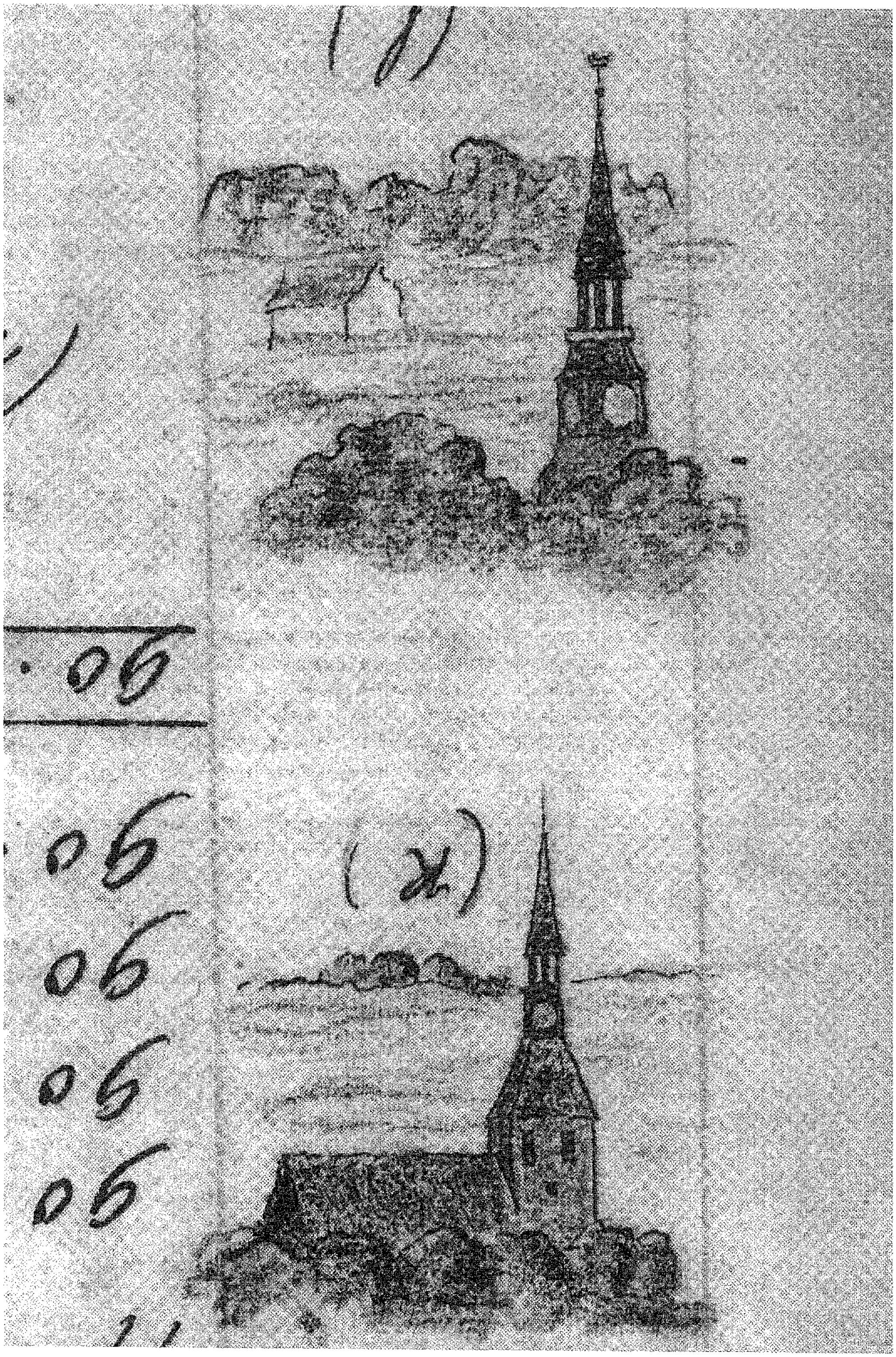


Figure 11b. Enlargement of two of the churches shown in figure 11a.

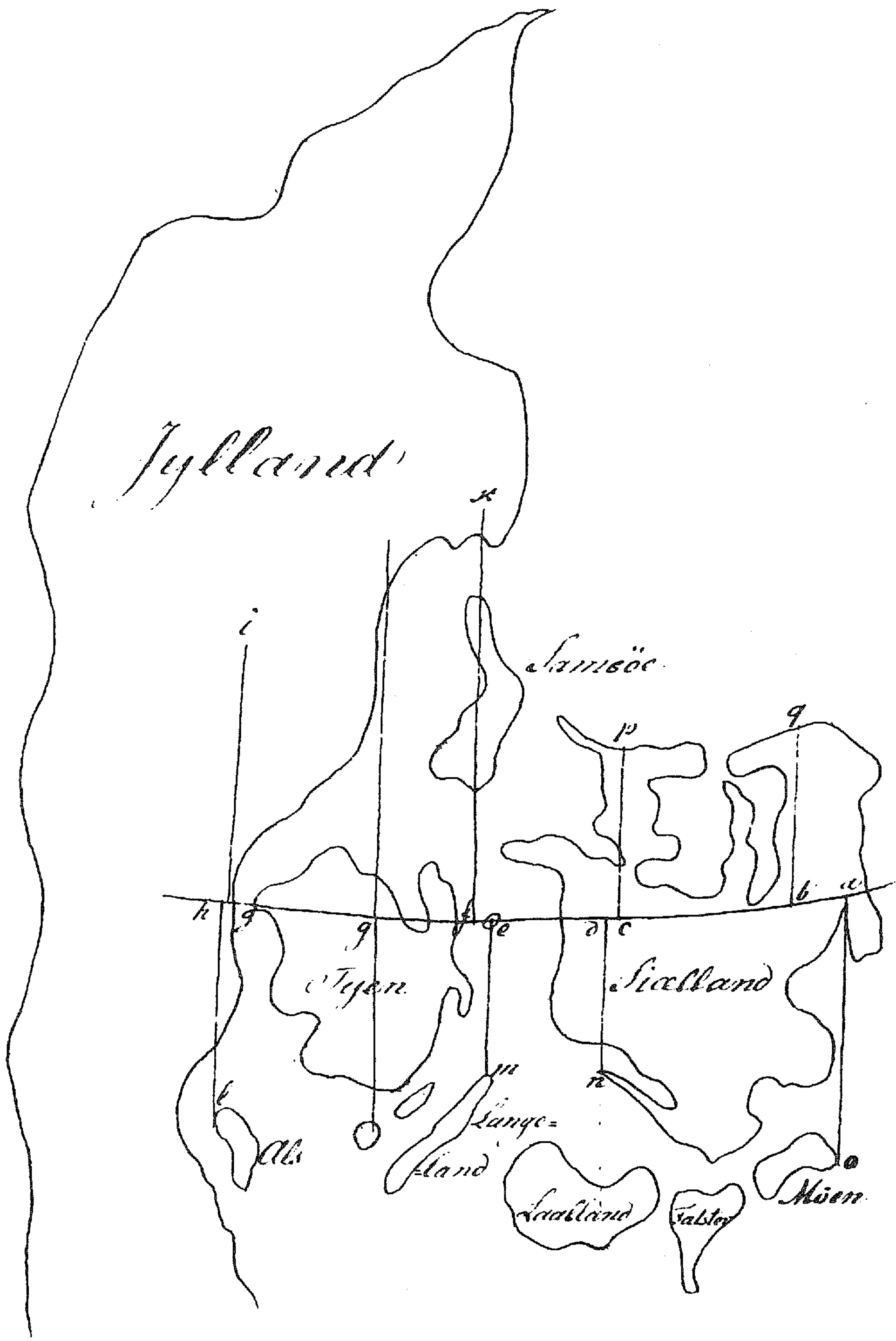


Figure 12. Sketch of the principle in the map projection which was used. Part of the parallel circle of latitude through the Round Tower Observatory is seen as an arc of a circle, and parts of some meridians are seen as line segments (in principle) perpendicular to this arc. From Caspar Wessel: Trigonometriske Beregninger for 1779 *(transcript) p. 90. Kort- & Matrikelstyrelsen.*

metrical and astronomical measurements. Each year he would describe some of the ideas and methods behind the calculations, never repeating himself. At the end of the summer period the report was handed over to the Academy. The reports constituted the scientific documents which justified the validity of the final maps. It is obvious that Wessel made it a point of honour to write comprehensive and satisfactory reports. He also did so in order to defend himself against accusations of mistakes.[46] However, we are convinced that a large part of his motivation originated from a personal scientific ambition. Over the years he also developed a style that came closer and closer to the one used in the mathematical treatise; that is to start by making a claim, then to give a proof in several paragraphs, and finally to apply the method.

In the 1779 report he commented on the astronomical determination of latitude and longitude, the assignment of coordinates to the trigonometrical points in the Observatory coordinate system, and he made a comparison between the latitude and longitude determined through astronomical observations with the one he deduced from the coordinates. In order to understand the deduction of latitude and longitude from the coordinates we have to explain the underlying assumptions.

The earth was assumed to be an ellipsoid of revolution with the ratio of the major axis to the minor axis equal to 230/229, in agreement with the shape determined by Isaac Newton in Principia Mathematica. The points on the ellipsoid were then assumed to be projected onto the cone generated by the tangents to the meridians through the points with the same latitude as the Observatory in such a way that a point on a meridian was projected onto the point of the tangent of this meridian in the correct distance from the parallel circle. Note that the tangent plane to this cone at the Observatory coincides with the tangent plane to the ellipsoid at the Observatory. One can then think about the conical map projection as obtained by unfolding the above cone in the common tangent plane, keeping the tangent to the meridian through the Observatory in place.

In figure 12 we present the illustration from Wessel's report on trigonometrical calculations[47] that he showed in order to explain the result of the conical map projection. Part of the parallel circle through the Observatory is seen as an arc of a circle, and some of the meridians perpendicular to this parallel circle are seen as straight lines, all passing through the same point if extended.

In the same report on page 69 Wessel explained how the latitude and longitude of a trigonometrical point G could be determined from its coordinates in the Observatory coordinate system.

In order to explain this Wessel drew the two figures shown in figure 13. The first of his figures, Fig I, showed a planar section through the meridian of the Observatory A. The meridian was part of the ellipse with half major axis HJ and half minor axis HE. The angle V was the complement of the latitude of A. It is easy to express the length of AV and of AF in terms of the length of HJ and HE and the angle V, and such a formula was stated. The lengths of HJ and HE were taken from Lambert.[48] From these lengths (given in toise) and the eccentricity of the ellipse one could easily calculate the length of one degree along a parallel circle of a specified latitude, and one could also estimate part of one degree on a

46 Letter from Caspar Wessel to Thomas Bugge, undated, KB, NKS 1299e 2° No. 365.

47 Caspar Wessel: *Trigonometriske Beregninger for 1779* (transcript), K&M.

48 J. H. Lambert: *Beiträge zu d. Mathematik*, 3 Teile, Berlin, 1765-72. Caspar Wessel mentioned this in his surveying report from 1779. From the valuation list of the books he left when he died we know that he owned the book. Papers in connection with estate no. 28. Behandlingsprotokol No. 2, 1818-19, L.

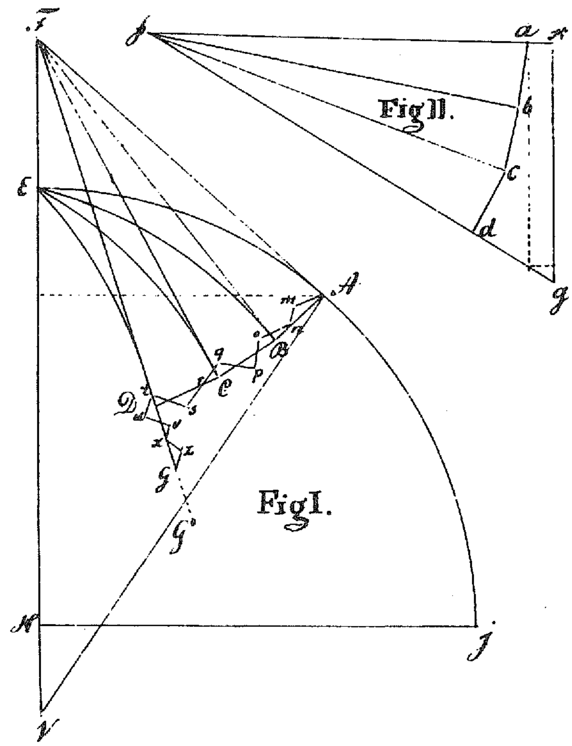


Figure 13. Caspar Wessel's figures from Trigonometriske Beregninger for 1779 *(transcript), p. 69. Kort- & Matrikelstyrelsen. Figure I illustrates a planar section through the Observatory A and the cone over the parallel circle through A, formed by the tangents to meridians at points along this circle; moreover, a chain AmnopqrstuvxzG in the primary triangular net, of which the first part is close to the parallel circle through A and the second part close to the meridian through G. Points of support along the parallel circle are B, C and D, with D on the meridian through G. Figure II illustrates the unfolded cone and the determination of the coordinates of g, the projected point in the plane corresponding to the point G on the ellipsoid. The projection is defined so that the distance DG along the meridian is equal to the distance dg along the tangent at D to the meridian through D and G.*

meridian around the given latitude. After having converted these numbers into Danish cubits he could then compare the lengths with those calculated from the triangular net. The zig-zag line *AmnopqrstuvxzG* illustrates a sequence of edges in the (primary) triangular net which from *A* to *u* almost follows the parallel circle through *A* from *A* to *D*, and from *t* to *G* almost follows the meridian through *D* from *D* to *G*.

The second of his figures, Fig II, shows the unfolded cone. The trigonometrical point *G* is marked as *g*, and the Observatory *A* as *a*. The zig-zag line *AmnopqrstuvxzG* consists of edges for which he had calculated the length; since he also knew the angles in the triangular net he was able to determine the coordinates of *g*. Knowing this and the length of *af=AF=df* he could then determine *dg* and transform this length into degrees of the meridian, using the estimated value for one degree of a meridian at the latitude of the Observatory mentioned above.

Finally he read off the angle *afg*, and the latitude and longitude of *G* which in the cone projection corresponded to the coordinates in the Observatory coordinate system.

At the beginning of Wessel's first year as a trigonometrical surveyor, on the 5th of February 1779, Bugge had presented to the Academy a superior plan by Wessel for the remaining trigonometrical surveying and cartography.[49] The plan was approved by the Academy, but has unfortunately been lost since. Following the plan Wessel chose Hesselbjerg as a good trigonometrical point that could be used as an overall reference point. By this choice he demonstrated in a convincing way his ability to combine practical and theoretical insight. Hesselbjerg was excellent since its latitude is almost the same as the Observatory and its meridian could be used as a line of reference for maps of Funen and Jutland as well as the duchies of Schleswig – Holstein. Wessel referred to the plan in his surveying reports, in particular the one in 1787.

6 The changing eighties

In 1780 Caspar Wessel married Cathrine Elisabeth Müller.[50] She had been a widow since 1776 when her first husband Marturin Brinch died.[51] Cathrine Elisabeth Müller and her five year-old daughter Anna Elisabeth Brinch moved in with Wessel.[52]

6.1 The negotiator

In 1781 Wessel was in Jutland performing trigonometrical operations. When he came back to Copenhagen by the end of September[53] Bugge asked him if he would be willing to move to the Duchy of Oldenburg to perform the trigonometrical operations in the surveying that was planned to start there the year after. The Governor of the County Georg Christian Oeder had contacted Bugge and asked him to be the leader of the project himself. It was natural for Oeder to ask the Danish Academy for help. He was educated in Göttingen, but had spent several years in Copenhagen, where among other things he had been responsible for the publication of the botanical series of books *Flora Danica*. In 1773 he moved to Oldenburg, which at the time was still partly under the rule of the Danish King. Shortly afterwards it became an independent duchy.

It was out of the question that Bugge could take upon him the trigonometrical surveying of Oldenburg. Instead he recommended Caspar Wessel or Hans Skanke. Bugge wrote in a letter to Oeder in their favour:

Both have been engaged in the Danish trigonometrical surveying for several years, and they are not only in possession of the necessary theoretical knowledge, but have also a lot of experience performing trigonometrical and astronomical observations. They are both excellent cartographers. They are highly estimated here, and it is out of the question that they will go to Oldenburg with a salary of less than 800 to 900 Rd per annum, in addition to free travel. [54]

49 Protokol for 1779. KDVS

50 Caspar Wessel: *Autobiography* (transcript), KB, NKS 4° 1977 b.

51 The year of his death is erroneously reported as 1773 in Dansk Biografisk Leksikon, Gyldendal under Caspar Wessel. The correct year is found in papers in connection with the estate of Marturin Brinch, Forseglingsprotokol 3A, No. 86, 1776, L.

52 We only know the age of the daughter from the census paper in 1787, S.

53 Caspar Wessel: *Journal over de Trigonometriske Observationer for Aar 1781*, K&M.

54 "Beyde Herren haben mehrere Jahren buy den Dänishen trigonometrishen Operationen gearbeitet, und besizen nicht nur die nothige theoretishe Kenntniss, sondern auch bereits viele Ubung in trigonometrishen

Bugge demanded a salary that was more than the double of their salary in Denmark. Perhaps this was done to make room for haggling, but it also indicates that he thought their salary in Denmark was too low.

Skanke was not interested at all, and Wessel was not very eager either. Wessel's financial situation was poor, and for this reason he was only interested if he got an advance. He could then get rid of some of his debt[55] and therefore he made it absolutely clear that he was not willing to move unless he was given an advance of 400 Rd.[56] From Oeder's point of view this was out of the question, and by mid-October the negotiations broke down. Caspar Wessel then took his family along on a journey to Norway. Bugge proceeded with the negotiations as an intermediary, and finally on the 25th of December he reported back to Oeder, after Oeder had agreed to an advance, that

I have the honour to communicate that Mr. Wessel is delighted to accept the engagement in Oldenburg, as he has informed me in writing in the latest mail from Norway, so this affair is at last put to right.[57]

In the same letter Bugge gave Oeder some information about Caspar Wessel, so that Oeder knew what kind of a man Wessel was.

He is a man of average height, lean and the colour [of his hair] is black. His face carries the impressions of profound thinking, and it has the earnest features which have to set in for a man often occupied with observations and calculations. He possesses a lot of theoretical knowledge of algebra, trigonometry and mathematical geometry, and as far as the last point is concerned, he has come up with some new and beautiful solutions to the most difficult problems in geographical surveying. He is a man of few but good words, a bit reserved, but is not in any way cross or hard to please. He is modest and has not high opinions of his own talent or his work, and he may, even when he is presenting real masterpieces containing diligence, art and insight, tell you that there is not much to it.[58]

Wessel was supposed to start the work in early May, if possible. During winter and early spring neither Bugge nor Niels Ryberg (Oeder's business connection in Copenhagen)

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und astronomishen Observationen. Beyde zeichnen Charten meisterhaft. Beyde stehen hier so gut, dass Sie nicht unter 800 bis 900 Rd in Jahre, und bey freyer Reise, nach Oldenburg gehen könnten." Letter from Thomas Bugge to Georg Christian Oeder (German transcript), 25 August 1781. NS, 31-4-36-1.

55 Letter from Georg Christian Oeder to Herr Graf, 6 February 1785, NS, 31-4-36-1.

56 Wessel had a lot of financial problems, but he did not go bankrupt by the end of his stay in Oldenburg as claimed by Viggo Brun in *Regnekunsten i det gamle Norge*. The man who was broke was actually named Caspar Rötger Wessels and lived "vor den Heiligen Geist Thor" in Oldenburg town. Oldenburgische wöchentliche Anzeigen Nr. 22 von Montag, den 29 Mai 1786.

57 "Ich kann nur die Ehre haben, zu melden, daß H. Wessel mit Vergnügen das engagement in Oldenburg annimmt, welches er mir mit lezter Norwegischen Post schriftlich gemeldet hat, und ist also diese Sache endlich zu Richtigkeit gebracht." Letter from Thomas Bugge to Georg Christian Oeder (German transcript), 25 December 1781. NS, 31-4-36-1.

58 "Er ist ein Mann von mittlerer Länge, mager und schwarzer Farbe. Sein Gesicht trägt Spuren scharfen Denkens und zeigt die ernsthaften Züge, welche bei einem Mann, der viel observiert und rechnet, sich einfinden müssen. Er besitzt viel theoretische Kenntnis besonders in der Algebra, Triginometrie und mathematischen Geometrie, in welcher letzten er neue und scöne Auflösungen der schwersten, die geographische Landmessung betreffenden Aufgaben gegeben hat. Er spricht wenig, aber sehr wohl, er ist in seinem Wesen zurückhaltend, aber dabei weder arg noch Sonderling. Er ist bescheiden und denkt immer von seinen eigenen Einsichten und Arbeiten gering und ist im Stande, indem er wirkliche Meisterstücke des Fleißes, der Kunst und der Einsicht vorlegt, Ihnen zu sagen, daß es nicht viel auf sich habe." Ibid.

heard a word from him, and by mid-April they worried about him. Then on the 23rd of April Ryberg could at last inform Oeder that Wessel was back in Copenhagen. When the contract was signed there were however still some problems with the advance. Oeder had stated that the 400 Rd should be paid when Wessel arrived in Oldenburg, but Wessel will

not move his feet an inch before he has received the 400 Rd, which according to article 7 in the contract, he is entitled to as a gratification, I have then paid him this in return of his receipt and where Professor Bugge had to sign as a surety.[59]

6.2 *In Oldenburg*

Wessel arrived in Oldenburg at the end of May. The surveying started in the area around Oldenburg town. The main trigonometrical point in his triangulation of Oldenburg was a wooden observatory placed on the rampart of the town.[60] It had been built under his instruction in order for him to perform the necessary astronomical observations, see figure 14. Today this position is marked by a stone and a plaque to commemorate Wessel's triangulation. The surveying was more difficult than in Denmark due to the topology. The landscape was even flatter than in Denmark, and the vegetation was much thicker. As a result the triangles had to be smaller than in Denmark and the progress was slower than expected. This improved when they moved to other parts of the Duchy.[61] By November the surveying in 1782 came to an end, and Wessel started his work on the first triangular map of Oldenburg.

By the end of the autumn of 1784 the whole Duchy of Oldenburg had been triangulated, and the work was of course as accurate as usual. Chamber Assessor Christian Friedrich Mentz examined in 1788 the accuracy by which the surveying was done. He used Wessel's results to evaluate the distance between two trigonometrical points and compared the distance with what he obtained when it was measured by rods. Wessel's results gave him a distance of 2693.78 feet, while the measured distance was 2693.53 feet. A result that *"is beyond what one could have expected, and which gives Wessel great honour."*[62]

Wessel's triangulation also played a role in 1824, when Carl Friedrich Gauss was planning his triangulation in the northern part of Oldenburg. In his working-diary he wrote that he had all the data from the old triangulation done in Oldenburg, where he could find the positions of a large number of churches, windmills, etc.[63]

However, Wessel was not pleased with his *astronomical* observations in Oldenburg, and for this reason he suggested that the triangular net in Oldenburg should be connected to the triangular net in Denmark. Astronomical observations were necessary to determine the latitude and longitude of the trigonometrical points in Oldenburg. If one could connect the two triangular nets, Wessel could rely on the observations performed in Copenhagen. Wessel suggested[64] in January 1785 in a letter to Bugge that Skanke the following

59 "nicht eher den Fuß von der Stelle sezen, ehe Ihm die 400 Rd ausbezahlt wären, welche Er nach den 7ten Punct des Contracts als Gratification haben soll, die ich Ihm auch gegen sein Quittung und des Prof: Bugge Caution habe fourniren müßen." Letter from Niels Ryberg to Georg Christian Oeder (German transcript), 23 April 1782. NS, 31-4-36-1.

60 Letter from Georg Christian Oeder to Herr Graf, 27 October 1782, NS 31-4-36-1.

61 Letter from Georg Christian Oeder to Thomas Bugge, 30 June1782, Oldenburger Jahrbuch Bd. 60, 1961.

62 Oldenburger Jahrbuch, Oldenburg, Bd. 60, 1961, pp. 11-12.

63 Gaede: *Beiträge zur Kenntniss von Gauss' praktisch-geodätischen Arbeiten.* Zeitschrift für Vermessungswesen, Heft 7, 1885, p. 135.

64 Letter from Caspar Wessel to Thomas Bugge (German transcript), 2 January 1785. NS, 31-4-36-1.

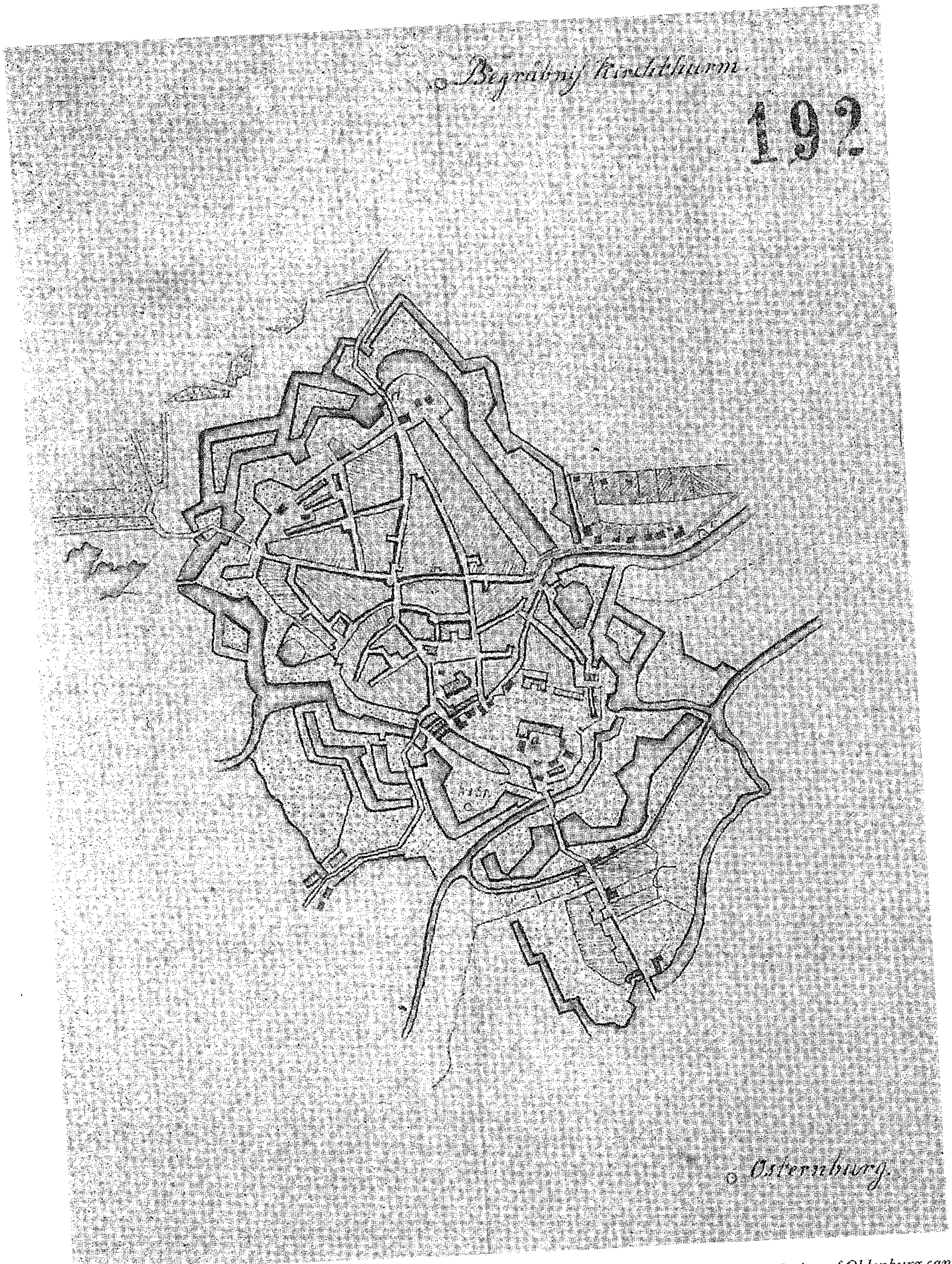

Figure 14. Caspar Wessel's map of Oldenburg town. The main trigonometrical point in his triangulation of Oldenburg can be identified as no. 131 on the rampart. On this location a wooden observatory had been built under Wessel's guidance in order for him to perform astronomical observations. The observatory was torn down when Wessel left Oldenburg and it had fulfilled its purpose. Today there is instead a memorial plaque in honour of Wessel. The building marked by **a** *is the palace which still exists. From Niedersächsisches Staatsarchiv in Oldenburg, 31-2-36-4.*

summer should continue the triangular net from Husum to the Elbe. In the spring of 1785 it was approved by the authorities in Oldenburg and Denmark to do so; Wessel continued the triangulation from Oldenburg via the estuaries of the Weser and the Elbe and up to Holstein, while Skanke at the same time continued the triangulation in Holstein to connect the two.

6.3 The family – diminished

Back in Copenhagen Wessel on the 16th of July reported to Luxdorph, who was then President of the Academy, that his work in Oldenburg was finished. During their conversation Wessel also told him that he wanted to be separated from his wife; this happened officially on the 2nd of September of that year.[65]

In the middle of September Johan Herman and Caspar Wessel received a letter from their mother[66] telling them that their father had died on the 4th of September. Ole Christopher Wessel renounced any inheritance after their father and suggested that the married sisters and the two brothers in Copenhagen should beforehand give up a part corresponding to their marriage portion respectively their previous support for studies. Johan Herman and Caspar agreed to this in November, and as a result the inheritance was divided only among the three unmarried sisters who each received 165 Rd.[67]

In the following month Caspar lost another one of his close family members, Johan Herman died on the 29th of December. He was buried in the cemetery of the Trinitatis church in Copenhagen, the church built together with the Round Tower.

During the winter of 1785 – 86 Wessel stayed in Copenhagen and continued to draw triangular maps of Oldenburg. Skanke had substituted for him as trigonometrical surveyor during the years 1782 – 85. But from the 1st of May of 1786[68] Wessel was again fully responsible for the trigonometrical surveying under the auspices of the Academy, and in the summer of 1786 he was back in Jutland performing trigonometrical measurements. By the end of the season he did not return to Copenhagen, but settled in Jutland, where he stayed at different places,[69] until after the death of his wife.

She died in December 1791 and was buried on the 30th of December.

Wife of the surveyor Kaspar Wæsel, separated, Catrine Elisabet Möller, aged 42, from No 335 in Tvergaden died from phthisis.[70]

7 The mathematician

Perhaps Wessel had already in Oldenburg elaborated on how to give an analytical representation of directions in the plane. In his Trigonometrical Surveying Report of 1787 we find directions in the tangent plane of the Observatory expressed as $T(\cos w + \sqrt{-1}\sin w)$,

65 Bolle W. Luxdorph: *Dagbøger*, vol. II, 16 July 1785. G.E.C.Gad, København 1915-1930, and Danske Kancelli, Siellandske Register, 1784-85, No. 703, R.

66 Papers in connection with the estate of Johan Herman Wessel no. 787. Forseglingsprotokol No. 2C, 1785, L.

67 Øvre Borgesyssels geistelige skifteprotokoll, No. 2 (1780-1824), p. 69, SO.

68 Letter from Thomas Bugge to Georg Christian Oeder, 17 October 1785, NS, 31-4-36-1.

69 Surveying Accounts, KDVS.

70 "Kasper Wæsel, Landmaalers Hustru Seperert Catrine Elisabet Möller, 42 Aar gl. fra N: 335 i Tvergaden af Tærende syge." Parish register for burials in the Trinitatis church, Kiøbenhavn 1791, R.

where T is the length of the direction and w the angle from the tangent of the meridian through the Observatory to the direction, measured positively against the sun, see figure 15. There is no explicit explanation of how this should be understood. Usually Wessel was very careful when he described new ideas and methods; it is possible that he had given some of the explanations earlier in one of the trigonometrical reports from Oldenburg.

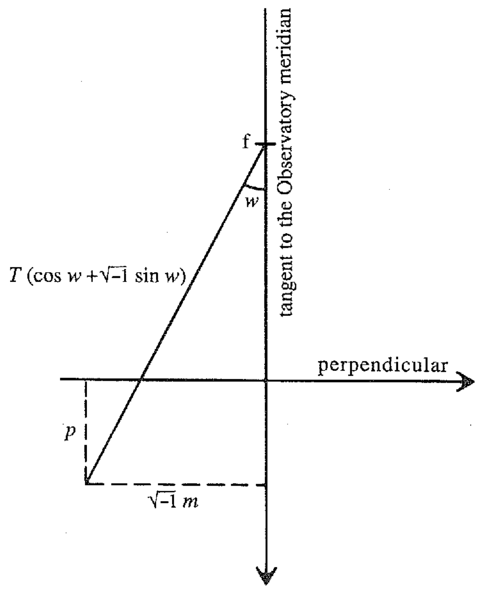

Figure 15. A Cartesian coordinate system formed by the tangent to the meridian at the Observatory, corresponding to the real axis, and its perpendicular, corresponding to the imaginary axis. Wessel denoted the coordinates of a point in the plane by p and $\sqrt{-1}m$, where p was the distance to the perpendicular and m the distance to the tangent to the meridian. Compare with Fig II in Figure 13.

He expressed the coordinates of a point by p and $\sqrt{-1}m$ in the Observatory coordinate system, see also figure 15. From the context it is clear that the tangent of the meridian corresponded to the real axis, and the perpendicular corresponded to the imaginary axis. Let us look more closely at the surveying problem that led him to this formulation.

7.1 *Wessel's trigonometrical calculations in 1787*

In 1779 Wessel had explained how to determine the latitude and longitude of a trigonometrical point G from its coordinates in the Observatory coordinate system (see figure 13). The deduction was derived under the assumption that G was reached from the Observatory A via a zig-zag line of edges in the triangular net in such a way that the sequence approximately followed first a part of the parallel circle through A and then a part of the meridian of G. He had also explained that the angle w between the tangents of the meridians of A and G is determined by $w = \lambda \sin B$, where B denotes the latitude of A and λ the difference in longitudes between A and G.

The way Wessel connected the observatory in Oldenburg to the Round Tower Observatory through edges in the triangular net was certainly more complicated than the above. The path could not be considered as well approximated by parts of only one meridian and one parallel circle.

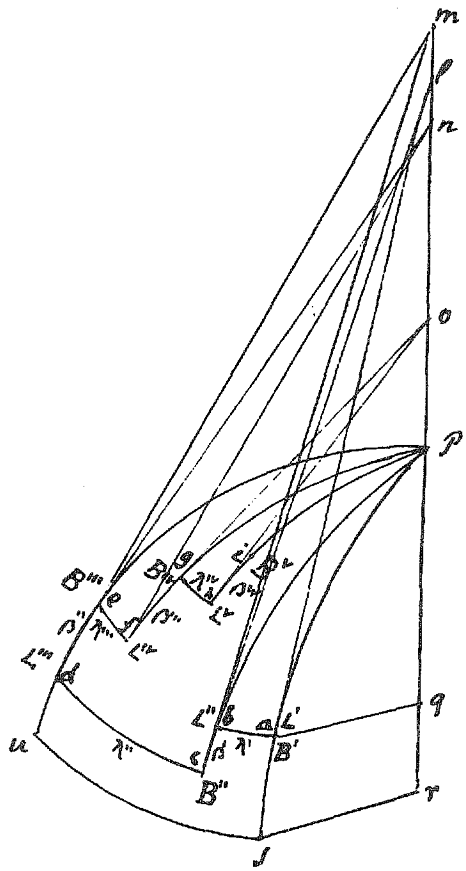

Figure 16. Caspar Wessel's figure from Trigonometriske Beregninger for 1787, *p. 49. Kort- & Matrikelstyrelsen. The figure illustrates parts of different cones, formed by the tangents to meridians at points along parallel circles of different latitude denoted by B', B'', etc. Each cone is bounded by a pair of tangents to meridians of longitude denoted by (L', L''), (L'' , L'''), etc.*

In the report of the trigonometrical calculations of 1786[71] he explained that when he determined the latitude and the longitude of the last trigonometrical point (in Jutland) of that year from its coordinates, he did not only use the method explained in 1779, but he also developed a new method, which gave a result that was in better agreement with the astronomical observations than the first one. He used a route from the Round Tower Observatory to the last trigonometrical point which was in closer accordance with the actual one, namely by following alternately parts of parallel circles and parts of meridians, keeping fixed alternately the latitudes B', B'', ... *and the longitudes* L', L'', ..., see his own illustration of this in figure 16.

In the report of the trigonometrical calculations of 1787 he explained this in more detail. He deduced formulas to determine the correspondence between latitude and longitude and given coordinates, based on the three figures (Fig 1, 2 and 3) shown in figure 17. The first of his figures shows a route that alternately follows parallel circles and meridians, together with tangents of the meridians, which in pairs meet at a point on the ex-

71 Caspar Wessel: *De Trigonometriske Beregninger for 1786*, K&M.

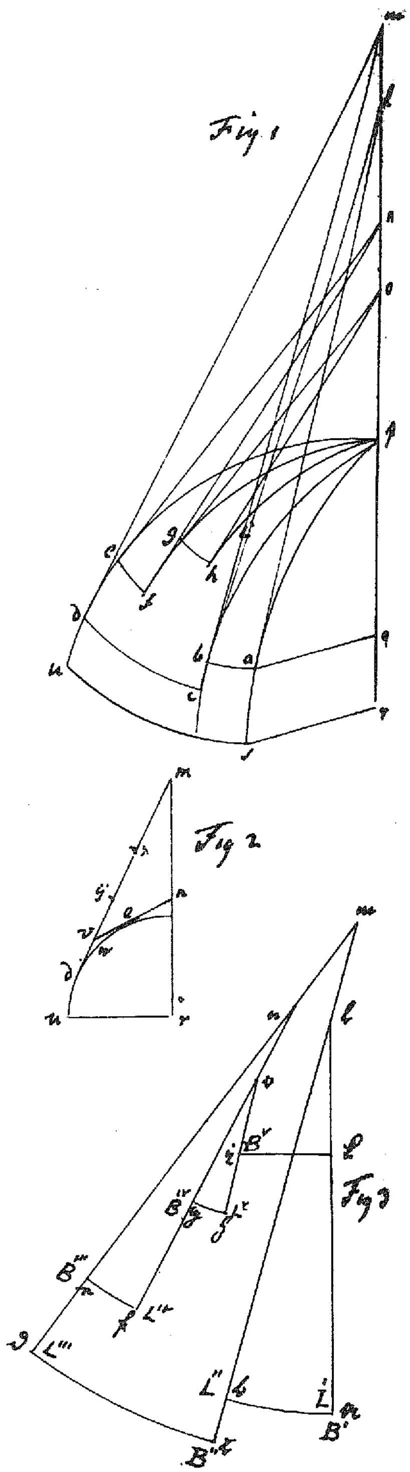

Figure 17. Caspar Wessel's figures from Trigonometriske Beregninger for 1787, *p. 50. Kort- & Matrikelstyrelsen. His Fig 1 is a repetition of the figure from page 49, shown here as figure 16. His Fig 3 illustrates an unfolding of the different cones, pieced together in the plane in the following natural way: tangents to the same meridian, but belonging to different cones, are identified isometrically. Each point on the ellipsoid is hereby projected into a unique point in the plane.*

tended axis of the earth. Note that when he mentioned (in the quote below) the length of a tangent at a point, he meant the length of the line segment on the tangent from that point to the intersection point with the extended axis of the earth. His third figure shows the unfolding of these parts of different cones, pieced together isometrically along tangents representing the same meridian. In this way points on two different tangents that are identified correspond to the same point on the earth. We shall not dwell on the deduction of any of the formulas, but only emphasise the paragraph in which Wessel introduced the notion of directions. In the text Fig 3 refers to the third of his figures shown here in figure 17.

If one imagines that all tangents Fig 3, their direction and length unchanged, are drawn from the same point l as the meridian of the point a or tangent $T' = la$, then it is clear that the sum of the first n angles, which are included by the same number of pairs of tangents, is as large as the angle that the last tangent of the last pair makes with the meridian of the point a, hence $w^n = \lambda' \sin B' + \lambda'' \sin B'' + \ldots + \lambda^n \sin B^n$. Therefore when the meridian of the point a is counted positive from l to a, so $la = T'$, and the angle w^n is counted positive from T' against the Sun and negative with the Sun, then T^n ($\cos w^n + \sqrt{-1} \sin w^n$) expresses the direction and the length of the last tangent in the pair with index n, and T^{n+1} ($\cos w^n + \sqrt{-1} \sin w^n$) is the length and the direction of the first in the pair with index n+1; since those 2 tangents have in the plane the same direction.[72]

We show this important paragraph in his own handwriting in figure 18.

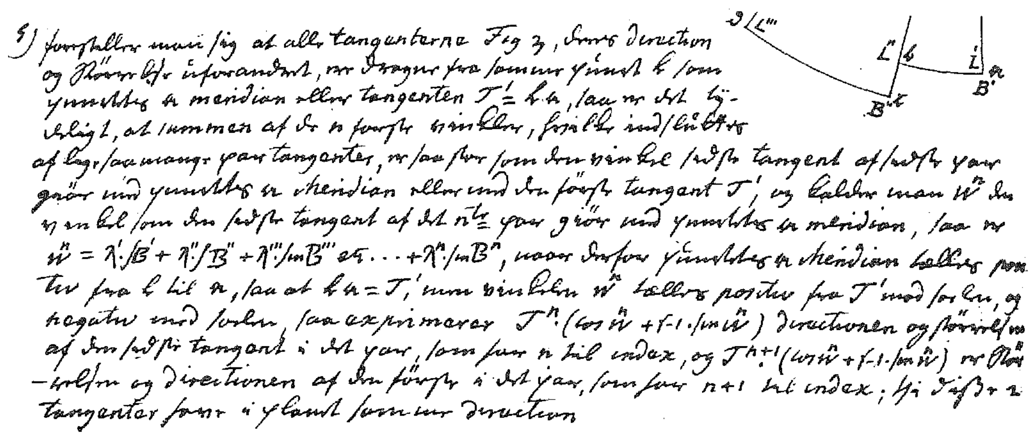

Figure 18. Caspar Wessel's section five from Trigonometriske Beregninger for 1787*, p. 50, where he used the complex numbers to represent directions in a plane. A transcript is found in footnote 72. Wessel's Fig 3 (in figure 17) illustrates the content of this section.*

72 "Forestiller man sig at alle tangenterne Fig 3, deres Direction og störrelse uforandret, ere dragne fra samme Punct l som Punctets a meridian eller tangent $T' = la$, saa er det tydeligt, at summen af de n förste vinkler, hvilke indsluttes af lige saa mange Par Tangenter, er saa stor som den vinkel sidste tangent af sidste par giör med Punctets a meridian, saa er $w^n = \lambda \sin B' + \lambda'' \sin B'' + \ldots + \lambda^n \sin B^n$ naar derfor Punctets a Meridian tælles positiv fra l til a, saa at $la = T'$, men vinkelen w^n tælles positiv fra T' mod Solen og negativ med Solen, saa exprimerer T^n ($\cos w^n + \sqrt{-1} \sin w^n$) directionen og störrelsen af den sidste tangent i det Par, som har n til index, og T^{n+1}($\cos w^n + \sqrt{-1} \sin w^n$) er störrelsen og directionen af den förste i det Par, som har $n + 1$ til index; thi disse 2 tangenter have i Planet samme direction." Caspar Wessel: *De Trigonometriske Beregninger for 1787*, K&M.

To emphasise the idea and the notion we have in figure 19 drawn two directions $T^n(\cos w^n + \sqrt{-1}\sin w^n)$ and $T^{n+1}(\cos w^n + \sqrt{-1}\sin w^n)$ from the same point, exactly as he suggested.

We see that already in 1787 Wessel had the geometrical interpretation of complex numbers as directions in a plane. There is no trace of the product rule in the report. But knowing that he was a master in handling trigonometrical formulas and noticing how he wished to change from the direction given by $(\cos w^n + \sqrt{-1}\sin w^n)$ to the direction given by $(\cos w^{n+1} + \sqrt{-1}\sin w^{n+1})$ through a turn of the angle of $(w^{n+1} - w^n)$, it is very possible that he arrived at the geometrical interpretation of the product rule by pursuing this further, and at least he had it all figured out when he wrote this treatise.

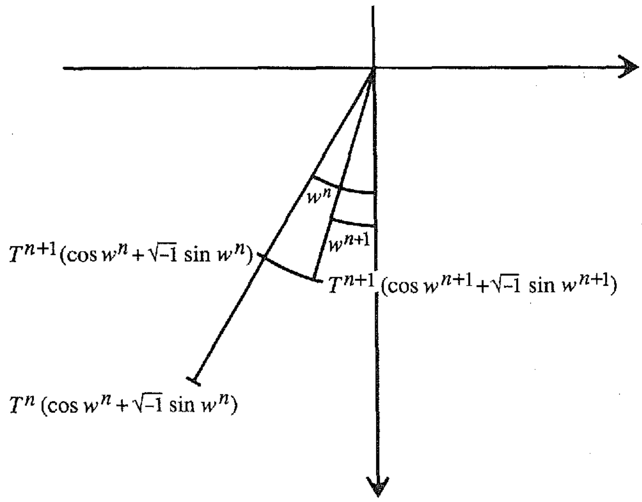

Figure 19. Part of Wessel's Fig 3 (in figure 17) reconstructed – as he encouraged the reader to do – so that directions start from the same point. Note that he wanted to change from a direction given by $(\cos w^n + \sqrt{-1}\sin w^n)$ *to a direction given by* $(\cos w^{n+1} + \sqrt{-1}\sin w^{n+1})$ *through a turn of the angle of* $(w^{n+1} - w^n)$*. In the 1787 trigonometrical surveying report he did not formulate the geometrical interpretation of the product rule for complex numbers, but it is very possible that he arrived at it by pursuing the idea in section five further.*

7.2 *Encouragement*

The question of founding a university in Norway was put on the agenda in 1793. On the 27th of March Jacob Nicolai Wilse sent out an invitation to a meeting in Christiania on the 4th of June.[73] During the spring Christen H. Pram followed up on this by publishing an article in Minerva with the title "*On the occasion of the proposal of a University in Norway*". Minerva was the most significant literary and political periodical in Copenhagen at the time, with Pram as one of its two editors. In this article he dealt with the requirements that ought to be fulfilled to establish such a university. Pram also discussed the possibilities of having Norwegians appointed as professors.

73 Supplement to the newspaper Norske Intelligenz Sedler, 27 March, Christiania, 1793.

To think that many of the present Norwegian scholars would be suitable for an appointment to a professorship, and be useful and worthy of their chair, is without insulting anyone, not likely.[74]

After this realistic view of the situation he reminded readers that some Norwegians had already been worthy of a professorship. He put an emphasis on Norwegians like Johan Ernst Gunnerus, Gerhard Schøning, Jens Kraft, Jacob Edvard Colbiørnsen

that with great honour, have been or still are teachers at the university, and also Strøm, Treschow, Monrad, C:Vessel, and who would take upon himself the task to name all those who could or ought to be counted in.[75]

These were names of leading persons in Denmark-Norway, and among them we find Caspar Wessel. Pram was without doubt thinking of Wessel as worthy of a professorship in mathematics. This was still some years prior to Wessel's mathematical treatise, but his reputation as a mathematician seems already to have been so well-known that Pram mentioned him together with the most outstanding Norwegians of the time. Wessel's recognition among his contemporary Norwegians was quite high.

During the meeting in Christiania on the 4th of June a committee was set up. The members were Ole Christopher Wessel, Bernt Anker, Jacob Nicolai Wilse, Johan Randulf Bull, Johannes Müller, Haagen Mathiesen and Jacob Rosted.[76] The committee was to work out a plan for how a university in Norway could be organised and present a draft to the authorities in Copenhagen. Their first initiative was to announce a competition where people were invited to write an article on the matter. This competition was won one year later by Pram, his contribution was called *"An attempt to establish a University in Norway"*.[77] Ole Christopher Wessel, who had been the General Advocate (generalauditeur) in Norway since 1776, was of vital importance to the work of the committee.[78] It was therefore a setback when he died on the 26th of December 1794, after having been ill for several months. In 1795 the work of the committee was completed and their proposal was sent to Copenhagen – without any significant result, and the debate regarding a Norwegian university faded away in 1795.

Caspar Wessel's reputation as a mathematician may, however, have been strengthened by the university debate, and it is not unlikely that his friends exhorted him to write down some of his results and to publish them. This may also be the reason why Tetens encouraged Wessel to write about the new mathematical theory he had elaborated on in his work as a surveyor. Tetens' request was probably made in 1795 – 96 since Wessel stated

[in 1796] I wrote a treatise, where I made an attempt to give an analytic representation of the direction of straight lines, and then used these to solve plane and spherical polygons.[79]

74 "At mange af Norges nærværende Lærde skulde være skikkede til for det første at giøres til Professorer, og kunde blive ret nyttige og deres Post værdige, er uden Fornærmelse, neppe at formode." Christen H. Pram: *I anledning af forslaget om en Højskoles oprettelse i Norge.* Minerva, Kiøbenhavn, May 1793, pp. 255-256.

75 "der med synderlig udmærkede Hæder, have været eller ere endnu Universitetslærere, som og Strøm, Treschow, Monrad, C:Vessel, og hvem kan eller vil foretage sig at opregne alle dem, der kunde og burde regnes med." Ibid.

76 Letter from the committee, 21 June 1793, Gunnerus bibliotek in Trondheim, VS III j.nr.1793-06-21

77 *"Forsøg om en Højskoles Anlegg i Norge"*. Discussed in the newspaper Norske Intelligenz Sedler 1794, nr.15.

78 Hermoder, No. 2 1795, Christiania, 1795, p. 98.

7.3 The mathematical treatise

As recounted in the introduction, on the 10th of March 1797 the treatise was then presented to the Academy and deemed worthy to be published in the collection of papers of the Academy.

In addition to Tetens two of the other persons present at the meeting probably understood that the treatise stood out compared to most of the papers published by the Academy. One of them was Christian Høyer, a Lieutenant-Captain at the Naval Academy, where he had been a teacher of mathematics since 1782. In the treatise only Tetens was given credit, but in Wessel's autobiography he mentioned Høyer as well.

The treatise was, after favourable judgements by Councillor of State Tetens and Lieutenant-Captain Höyer, incorporated in the 5th volume 3rd part of the publications of the Royal Academy.[80]

In addition to these two we can also assume that Niels Morville realised that the treatise was significant. In his private account of treatises published by the Academy he wrote about Wessel's contribution:

An original piece of its kind, which shows that the author has a command of the analytical methods with excellent dexterity.[81]

Wessel arranged with the publisher Johan Rudolph Thiele to have special prints made of the mathematical treatise for his own use before it was to appear in the collected papers of the Academy. Two other authors, Niels Morville and Anders Gamborg, made a similar agreement (Morville had two treatises). For this reason Wessel received some prints of his treatise in 1798. On a bill from Thiele to the Academy we find this described

Moreover changed and separately printed four treatises:
Morville's, Wessel's, and Gamborg's: *12 Rdl*[82]

The number of copies was probably not great. We know that Wessel gave a copy to Bugge[83] and that he kept one for himself.[84] Officially the treatise "*Om Directionens analytiske Betegning ...*" was published in November 1799 in the 5th volume of the collected papers of the Academy.

79 "[i 1796] skrev jeg en Afhandling, hvori jeg forsögte at betegne rette Liniers Direktion ved analytiske Tegn, og anvendte disse til sphæriske og retlinede Polygoners Oplösning." Caspar Wessel: *Autobiography* (transcript), KB, NKS 4° 1977 b.

80 "Afhandlingen blev, efter Etatsraad Tetens' og Kapitainlieutenant Höyers gunstige Omdömme optaget i 5te Dels 3 H. af det Kongel. Vidensk. Skrifter." Ibid.

81 "Et originalt Stykke i sit Slags, der bevidner at Forfatteren har Analytikens Anvendelse i sin Magt med udmærket Færdighed." Niels Morville: *Oversigt over artiklerne i Selskabets skrifter Vte Deel, 3die Hæfte.* Manuskriptsamling, KDVS.

82 "Desuden forandret og separat aftrykket 4 Afhandlinger: Morvilles, Wessels og Gamborgs: 12 Rdl." Receipt no. 22, Surveying Accounts 1799, KDVS.

83 Catalogue of Thomas Bugge's books for an auction after his death, Kiøbenhavn, 1815, p. 29.

84 Valuation list over Caspar Wessel's books. Papers in connection with estate no. 28. Behandlingsprotokol No. 2, 1818-19, L.

7.4 *Promotion and retirement*

In 1798 Wessel was appointed Surveying Superintendent. His career as an active surveyor was, however, slowly coming to an end. In 1796 the triangulation of Denmark and the duchies of Schleswig and Holstein was finished (except for the island of Bornholm), and he once again went back to the geographical surveying which he continued to do until he retired in 1805.

In 1805 he had been in the surveying corps for 41 years, and this fact as well as his poor health gave him boldness enough to apply for a retirement pension equivalent to his full salary.[85] This was approved by the Treasury on the 28th of February 1806, and the Academy was asked to inform Wessel of the decision. In a letter dated the 22th of March Bugge did so and moreover expressed gratitude for Wessel's devotion throughout the years.

[The Academy can] not fail to show Mr. Surveying Superintendent Wessel its complete satisfaction with the accuracy and diligence which through so many years You have shown during Your long and faithful service. Although the Academy takes great pleasure in the fact that You may now enjoy a well deserved and most honourable retirement, it constantly regrets the loss of a man so capable and active.[86]

8 The working pensioner

The loss of Wessel was so pronounced that the Academy on several occasions after he had retired asked him to do them a favour. Although he was weakened, he agreed in 1808 to draw maps of the triangular net in Schleswig and Holstein. He also added some remarks in French, explaining the calculations since the maps were requested by the French Emperor Napoleon.[87] They were sent to General Sanson, head of the Dépôt général de la Guerre in Paris. At this time Denmark-Norway was an ally of France. This had the severe consequence that after the peace treatise in Vienna the dual monarchy of Denmark-Norway ceased to exist. From 1814 Sweden and Norway formed a union under the rule of the Swedish King.

The Academy wanted to pay Wessel for his extra work, but he did not wish to receive any payment. In the records of the Academy we find

Wessel has been offered payment for the job, but he will under no circumstances accept this. It should be investigated whether he instead could be offered

1) The published geographical maps and their continuation.
2) A set of the publications of the Academy.
3) A silver medal.[88]

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85 Letter from Caspar Wessel to the King, 19 February 1805. Finanskollegiet, Sekretariatet, Journalsager 1805, No. 672.

86 "[Videnskabernes Selskab kan] ey unlade at bevidne Hr. Landmaalings-Inspecteur Wessel sin fuldkommenste Tilfredshed med den Nöiagtighed og Fliid, som Deres Velædelhed i saa mange Aar har udviist i Deres lange og troe Tieneste. Vel glæder det Selskabet, at Deres Velædelhed kan hensætte sig i en saa vel fortient og hæderlig Roelighed, men det beklager dog al Tiid Savnet af en saa duelig og virksom Mand." Letter from Thomas Bugge to Caspar Wessel. Papers in connection with estate no. 28. Behandlingsprotokol No. 2, 1818-19, L.

87 Letter from Thomas Bugge to T.C.Bruun Neergaard, 30 December 1808, KB, NKS 287e 4°, No. 3.

This was accepted, and on the 3rd of February 1809 Wessel received all of the above.

In 1812 Wessel was so tormented by rheumatism that he accepted an offer from Councillor of State Peder Andreas Kolderup Rosenvinge to move in with him and his family. Kolderup Rosenvinge was a friend of Wessel's, and at the time managing director of the Royal Mail in Denmark.[89] Here Wessel received medical treatment from one of the best known doctors of the time, Councillor of State Frederik Ludvig Bang, a former Senior Physician of Frederik's Hospital and a professor at the University.

Slowly Wessel recovered under the doctor's treatment. In 1815 he received two more maps and the latest publications of the Academy. He expressed his gratitude in a letter to Hans Christian Ørsted, who had just succeeded Thomas Bugge (after his death in January 1815) as Secretary of the Academy. On this letter we find Wessel's seal, see figure 20.

In 1815 Wessel was well enough to take upon him the task of drawing four triangular maps of Holstein. The geographical surveying of Holstein was still not finished, so no maps of Holstein had been produced. Wessel also wrote a report on the methods he had used in the cartography.

Since the projected meridian of Hesselbjerg is common to all of the engraved maps of Jutland and Schleswig, I thought I ought to give an account of how I have determined it, along with the longitude and latitude of the same station.[90]

The explanations were based on the calculations explained in the trigonometrical surveying reports of 1779, 1786, and 1787. There is no trace of complex numbers and only in words did he touch upon the possibility of letting the calculations be based on a path alternating between parts of parallel circles and meridians, in better agreement with the actual path of triangles. He argued that if he made the calculations following a more complicated path, then there was no difference in the latitude and less than half a second in the longitude, however the angle that the tangent of Hesselbjerg's meridian made with the tangent of the Observatory differed by $17''$.

In spite of this difference I have after all on this map of the remaining part of Holsteen assumed $w = -2°\,38'\,23''.56$, *because I thought the Heslebjerg meridian ought to be determined by the same rule as the other meridians and parallels are determined by.*[91]

We have seen that his development of directions using complex numbers had little impact on his work as a surveyor; it was clearly a mathematical abstraction inspired by his work as a surveyor.

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88 "Man har anbedet Wessel betaling derfor, men han vil paa ingen Maade modtage den. Det forespörges om ham ikke i det Sted kunde gives
1) De udgivne geografiske korter og deres Fortsættelse.
2) Et Exemplar af Selskabets Skrivter.
3) En Sölvmedaille." Forhandlingsprotokol for 1809, No 644, KDVS.

89 Dansk Biografisk leksikon, Tredie udgave, vol. 8, Gyldendal, København, 1981.

90 "Da Hesselbjergs Stations projekterete Meridian er fælleds for alle de over Jylland og Schlesvig stukne Kaarter, saa har jeg troet at burde gjöre Regnskab for, hvordan jeg har bestemt den, tillige med samme Stations Længde og Brede." Caspar Wessel: Untitled report (transcript), 1815, K&M.

91 "Uagtet denne Forskjel har jeg dog paa dette Kaart over den resterende Deel af Holsteen antaget $w = -2°\,38'\,23''.56$, fordi Heslebjerg Meridian burde som mig synes, bestemmes efter samme Regel, som de övrige Meridianer og Paralleler ere bestemte." Ibid.

Figure 20. Caspar Wessel's seal. The squirrel was a family symbol. It was also used on seals of Ole Wessel, Jonas Wessel (Caspar's father), Ole Christopher Wessel and Jonas Wessel (his nephew). The illustration is a reconstruction by Torbjørn Sætra, the University of Oslo, of the seal found on a letter from Caspar Wessel to Hans Christian Ørsted in 1815, in the Academy.

Wessel was once again offered money for his extra work, and this time he accepted a pay of 400 Rd, no doubt because he wanted to pay off his debt of 320 Rd to Kolderup Rosenvinge. By mid-August in 1815 he moved out from Kolderup Rosenvinge at Kongens Nytorv 2 (today no. 15) and into a flat in Studiestræde 65 (today no. 29).

On the 31st of June 1815 Caspar Wessel was made a knight of the Dannebrog, no doubt in recognition of his exceptional contribution to the surveying. The Cross of the order of the Dannebrog was handed over during an audience on the 4th of August that took place at the Royal Palace of Amalienborg,[92] and as requested he wrote the short autobiography that we have referred to several times.

In 1816 he felt that he had made his last contribution to the Academy. He therefore handed over to the Surveying Commission the copies of the surveying reports he had at home as well as the maps. He expected that the Academy would *"dislike seeing those in foreign hands at his death."*[93] To the end of his life he was a very honest and conscientious man.

Two years later, Caspar Wessel died on the 25th of March 1818. He was buried in the Assistens graveyard in Copenhagen on the 29th of March at site no. 190.[94]

From the valuation list of the books he left when he died we know that besides the publications of the Academy he owned about 200 books. They were mainly related to mathematics and surveying, but there were also some concerning religion, philosophy and literature, moreover dictionaries of Latin, German and French.[95]

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92 Register of receivers of decorations, Ordenskapitlet, Amalienborg, København.
93 Forhandlingsprotokol 1816, No 1966. KDVS.
94 Register of burials at Assistenskirkegaarden, S.
95 Papers in connection with estate no. 28. Behandlingsprotokol No. 2, 1818-19, L.

9 Acknowledgements

We have visited many archives and museums and spoken with many different people in order to unravel as much as possible about Caspar Wessel's life and work. Everywhere we have been met with extraordinary assistance and interest.

The following places have been the most important in the search:

In Denmark: Det Kongelige Danske Videnskabernes Selskab,
Kort- & Matrikelstyrelsen,
Det Kongelige Bibliotek,
Rigsarkivet,
Landsarkivet for Sjælland, Lolland-Falster og Bornholm,
Stadsarkivet, all in Copenhagen;

in Norway: Universitetsbiblioteket in Oslo,
Riksarkivet in Oslo,
Statsarkivet in Oslo,
Norske Selskab in Oslo,
The library of the Cathedral School in Oslo,
Gunnerus Bibliotek in Trondheim,
Statens Kartverk in Hønefoss;

in Oldenburg: Niedersächsisches Staatsarchiv.

We wish to express our gratitude to everyone, in these places and elsewhere, who has helped us.

There are however, a few we would like to thank explicitly for their exceptional support and encouragement. Our special thanks go to Pia Grüner, Det Kongelige Danske Videnskabernes Selskab; without her the project could not have begun. In Kort- & Matrikelstyrelsen we in particular thank Sigvard Stampe Villadsen, who made it possible to start the search for Caspar Wessel's surveying reports, and Ulla Thomassen, who went out of her way to follow up on this. Without the help of Kari Høgevold, retired from the University of Oslo and a specialist in decoding Gothic handwriting, we would not have transcripts of Oeder's letters in German. His Gothic handwriting is the most difficult we have encountered.

Finally, we wish to thank Jesper Lützen and Kirsti Andersen for many valuable comments on an earlier version of this paper.

10 Index of names

B: British, D: Danish, F: French, G: German, Gr: Greek, N: Norwegian, S: Swedish.

Johan Ahl (1729 – 1795), instrument maker. (S)
Peder Kofod Ancher (1710 – 1788), lawyer. Member of KDVS. (D)
Bernt Anker (1746 – 1805), merchant. (N)
Frederik Ludvig Bang (1747 – 1820), physician. (D)
Albert Peter Bartholin (1724 – 1798), teacher and civil servant. (D)
Anna Elisabeth Brinch (1775 – ?), stepdaughter of Caspar Wessel. (D)
Marturin Brinch (1740 – 1776), student. Father of Anna Elisabeth Brinch. (D)
Thomas Bugge (1740 – 1815), astronomer, mathematician, surveyor. Member of KDVS. (D)
Johan Randulf Bull (1749 – 1829), lawyer. (N)
Jacob Edvard Colbiørnsen (1744-1802), lawyer. Member of KDVS. (N)
Daniel Ekström (1711 – 1755), instrument maker. (S)
Anders Gamborg (1753 – 1833), philosopher. Member of KDVS. (D)
Carl Friedrich Gauss (1777 – 1855), mathematician, astronomer, surveyor. Foreign member of KDVS. (G)
Johan Ernst Gunnerus (1718 – 1773), bishop and natural science philosopher. Member of KDVS. (N)
Christen Hee (1712 – 1781), mathematician. Member of KDVS. (D)
Henrik Henricksen Hielmstierne (1715 – 1780), civil servant. Member of KDVS. (D)
Jørgen Nicolai Holm (1727 – 1769), philosopher and mathematician. (N)
Christian Horrebow (1718 – 1776), astronomer and mathematician. Member of KDVS. (D)
Christian Høyer (1758 – 1809), officer. Member of KDVS. (D)
Anna Catharina Juul (1738 – 1809), café proprietor. (D)
Niels Juul (1720 – 1788) café proprietor, husband of Anna Catharina Juul. (D)
Peder de Koefoed (1728 – 1760), surveyor. (N)
Peder Andreas Kolderup Rosenvinge (1761 – 1824), managing director of the Royal Mail. (N)
Jens Kraft (1720 – 1765), philosopher and mathematician. Member of KDVS. (N)
Heinrich Johannes Krebs (1742 – 1804), officer and mathematician. Member of KDVS. (D)
Johann Heinrich Lambert (1728 – 1777), mathematician. (F)
Bolle Willum Luxdorph (1716 – 1788), civil servant. Member of KDVS. (D)
Paul Løvenørn (1751 – 1826), naval officer. Member of KDVS. (D)
Haagen Mathiesen (1759 – 1842), merchant. (N)
Christian Friedrich Mentz (1765 – 1832), Surveyor. (G)
Søren Monrad (1744 – 1798), teacher and headmaster. (D)
Niels Morville (1743 – 1812), surveyor. Member of KDVS. (D)
Cathrine Elisabeth Müller (1749 – 1791), wife of Caspar Wessel. (D)
Johannes Müller (1736 – 1796), physician. (D)
Isaac Newton (1643 – 1727), mathematician and physicist. (B)
Christen Henriksen Pram (1756 – 1821), poet and political economist. (N)
Ptolemy (ca.140 AD), astronomer, mathematician and geographer. (Gr)
Knud Lyne Rahbek (1760 – 1830), poet and editor. (D)
Johan Jacob Rick (1749 – 1801), surveyor and military man. (N)
Jacob Rosted (1750 – 1833), teacher and headmaster. (N)
Niels Ryberg (1725 – 1804), merchant. (D)
Helene Marie Schumacher (1715 – 1789). Mother of Caspar Wessel. (N)
Gerhard Schøning (1722 – 1780), historian. Member of KDVS. (N)
Hans Skanke (1744 – 1787), surveyor. (N)
Edvard Storm (1749 – 1794), poet. (N)
Hans Strøm (1726 – 1797), parson and natural science philosopher. Member of KDVS. (N)
Johan Nicolai Tetens (1738 – 1807), philosopher, mathematician, political economist. Member of KDVS. (G)
Johan Rudolph Thiele (1736 – 1815), publisher. (D)
Peter Wessel Tordenskiold (1690 – 1720), vice-admiral. Brother of Caspar Wessel's grandfather. (N)
Niels Treschow (1751 – 1833), statesman and philosopher. Member of KDVS. (N)
Christian Braunmann Tullin (1728 – 1765), poet and alderman. (N)
Caspar von Wessel (1693 – 1768), vice-admiral. Brother of Caspar Wessel's grandfather. (N)
Caspar Wessel (1745 – 1818), surveyor and mathematician. (N)
Gjertrud Marie Wessel (1739 – 1829), sister of Caspar Wessel. (N)

Johan Herman Wessel (1742 – 1785), poet. Brother of Caspar Wessel. (N)
Jonas Wessel (1707 – 1785), parson. Father of Caspar Wessel. (N)
Ole Wessel (1687 – 1748), parson. Brother of Caspar Wessel's grandfather. (N)
Ole Christopher Wessel (1744 – 1794), surveyor and general advocate. Brother of Caspar Wessel. (N)
Ditlev Wibe (1751 – 1834), surveyor, commisioner general of war (generalkrigskommisær). (N)
Johan Wibe (1748 – 1782), poet. (N)
Niels Andreas Wibe (1759 – 1814), surveyor, commisioner general of war (generalkrigskommisær). (N)
Jacob Nicolai Wilse (1735 – 1801), parson. (D)
Georg Christian Oeder (1728 – 1791), botanist and political economist. (G)
Hans Christian Ørsted (1777 – 1851), physicist and chemist. Member of KDVS. (D)

11 References

11.1 Abbreviations

We use the following abbreviations for the main archives:
KB: Det Kongelige Bibliotek (The National Library of Denmark),
NKS: Ny Kongelig Samling (New Royal Collection);
KDVS: Det Kongelige Danske Videnskabernes Selskab (The Royal Danish Academy of Sciences and Letters);
K&M: Kort- & Matrikelstyrelsen (National Survey and Cadastre);
L: Landsarkivet for Sjælland, Lolland-Falster og Bornholm (Provincial Archives of Zealand, Lolland-Falster and Bornholm);
NS: Niedersächsisches Staatsarchiv in Oldenburg;
R: Rigsarkivet (The Danish National Archives);
RO: Riksarkivet (The National Archives of Norway);
S: Stadsarkivet (Copenhagen City Archive);
SO: Statsarkivet (The Regional State Archives of Oslo).

11.2 Caspar Wessel's work

Related to Surveying:

Reports of geographical surveying:
1764, 1765 (Ole Christopher Wessel's reports, CW his assistant).
1766, 1768, 1769.
Reports of trigonometrical surveying:
Observations: 1779, 1780, 1781, 1786, 1787, 1789, 1790, 1791, 1792, 1793, 1794, 1795, 1796.
Calculations: 1779 (transcript), 1780, 1781, 1786, 1787, 1788 (transcript), 1789, 1790, 1791, 1792, 1793, 1794, 1795, 1796

Report in 1815 (transcript) (handed in together with four triangular maps of Holstein).

Geographical maps (conceptkort):
1767/68 (together with OCW), 1769, 1772, 1773, 1774, 1775, two in 1776 (one together with Johan B. Cimber), 1777, two in 1797, 1798, 1799, 1800, 1801, 1802, two in 1803, two in 1804.

Triangular maps:
four of Oldenburg,
one connecting Oldenburg to the Elbe,
one connecting the Round Tower Observatory to Holstein,
all in Niedersächsisches Staatsarchiv (NS) in Oldenburg.

Maps:
1766: Kiøbenhavns Amt (Ole Christopher W. with help from CW);
1768: Den Nord-østlige Fierdedeel af Siælland;
1770: Den syd østlige Fierdedeel af Siælland;
1771: Den Nord Vestlige Fierdedeel af Siælland;
1772: Den Sydvestlige Fierdedeel af Siælland;
1777: Kort over Siælland og Møen (together with Hans Skanke)
1780: Kort over den nordlige Deel af Fyen.

1825: The map "Den sydlige Deel af Hertugdømmet Schleswig samt Øen Femern" that came out in 1825 was based on Wessel's constructions.

The above reports and maps (except the triangular maps) are kept in Kort- & Matrikelstyrelsen (K&M), Copenhagen.

The mathematical treatise:

In Danish:
Om Directionens analytiske Betegning, et Forsøg anvendt fornemmelig til plane og sphæriske Polygoners Opløsning.

– Special print, J.R.Thiele, Kiøbenhavn, 1798.

– Det Kongelige Danske Videnskabernes Selskabs Skrifter, Nye Samling, V, Kiøbenhavn, 1799, pp. 469-518.

– Reprinted with a preface by Sophus Lie, Archiv for Mathematik og Naturvidenskab, vol. 18 (1895), pp. 5-69.

– Reprint of the first 10 paragraphs in Jørgen Ebert: *Komplekse Tal – en historisk og aksiomatisk introduktion.* Forlaget Minor, Sønderborg, 1995.

– Facsimile of the special print from 1798 with a biography by Nils Voje Johansen, Wesselakademiets Småskrifter, Nr. 1, Vestby, 1998.

Translations:

– *Essai sur La représentation analytique de la direction.* The treatise translated by H.-G. Zeuthen, and with prefaces by H. Valentiner and T.-N. Thiele, L'Académie Royale des Sciences et des Lettres de Danmark, Copenhague 1897.

– *On the Analytic Representation of Direction; An Attempt, Applied Chiefly to the Solution of Plane and Spherical Polygons.* The first 16 paragraphs of Wessel's treatise translated by Martin A. Nordgaard and published in David E. Smith: *A Source Book in Mathematics*, New York, 1929, reprinted New York, 1959, pp. 55-66, and reprinted in H.O.Midonick, ed.: *The Treasury of Mathematics,* New York, 1965, pp. 805-814.

– *On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons.* The complete treatise translated by Flemming Damhus. This volume, 1999, pp. 101-143.

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Transcript of letter to Thomas Bugge 5 January 1785. NS, 31-4-36-1.
Letter to Thomas Bugge 28 May 1785. KB, NKS 1299e 2°, No. 365.
Transcript of letter to Georg Christian Oeder 24 January 1786. NS, 31-4-36-1.

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Apartment lease with Johan Martin Qvist, 14 August 1814. L, Papers in connection with estate no. 28. Behandlingsprotokol No. 2, 1818-19.
Autobiography (transcript) when he was made a knight of the Dannebrog, 14 August 1815. KB, NKS 4° 1977b.
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Om

Directionens analytiske Betegning,

et Forsøg,

anvendt fornemmelig

til

plane og sphæriske Polygoners Oplösning.

Af

Caspar Wessel,

Landmaaler.

Kiøbenhavn 1798.
Trykt hos Johan Rudolph Thiele.

Title page of the special print Wessel had made of his paper in 1798.

Wessel's Work On Complex Numbers and its Place in History

by

Kirsti Andersen

Introduction

Today Caspar Wessel's name is associated with the first geometric representation of complex numbers, but this has not always been the case, for his work was only acknowledged in the second half of the 1890s. The aim of the present chapter is to present Wessel's achievements, put them in a historical perspective, and gain some understanding of why they did not become known during his own time.

The term complex numbers was first introduced in 1831 by Carl Friedrich Gauss, nevertheless I use it throughout this chapter. Before Gauss the expressions "impossible" and "imaginary" quantities were the most common, and the latter kept being used after 1831, for instance by Augustin Louis Cauchy.

This chapter begins with an account of the birth of complex numbers in the Renaissance and of how mathematicians before Wessel tried to come to terms with the "impossible" quantities. Wessel's work is treated in section two; here it is shown that he started with a geometric approach and then deduced algebraic rules that led him to a geometric interpretation of the complex numbers and of the algebraic operations with them, where in particular his understanding of the product is new. In surveying the time after Wessel I have focused on ideas similar to Wessel's and on how some of the influential mathematicians treated complex numbers. I include, perhaps, more than is necessary for appreciating Wessel's contribution. However, I do find it relevant to pursue the two aspects of his work just mentioned, namely the geometric interpretation of complex numbers and of their product. The first issue brings me to touch upon the theory of complex integration, and the second induces me to look at some mathematicians' introduction of the product of complex numbers. It is natural to stop the account around the middle of the nineteenth century, because by then the geometric representation of the complex numbers was generally accepted.

1 Complex numbers before Wessel

1.1 The theory of equations and complex numbers

In the Renaissance, when all numbers occurring in mathematics were positive, the complex numbers suddenly entered the field. This happened in Rafael Bombelli's *L'algebra*, published in 1572, and has the following prehistory. In *Ars magna* (1545) Girolamo Cardano presented an algorithm for finding a (positive) root of the equation which we today write as

$$x^3 = ax + b. \tag{1}$$

The algorithm corresponds to the formula

$$x = \sqrt[3]{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{2}b)^2 - (\tfrac{1}{3}a)^3}} + \sqrt[3]{\tfrac{1}{2}b - \sqrt{(\tfrac{1}{2}b)^2 - (\tfrac{1}{3}a)^3}}. \tag{2}$$

Cardano moreover formulated, though he did not prove, some rules concerning the numbers of roots of various equations. These rules were forerunners of Descartes's sign rule and stated among other things that (1) has precisely one positive root. Cardano gave some examples of how to apply (2), but carefully avoided the so-called *casus irreducibilis* in which $(\frac{1}{2}b)^2 - (\frac{1}{3}a)^3 < 0$ and (2) gets the form

$$x = \sqrt[3]{c + \sqrt{-d}} + \sqrt[3]{c - \sqrt{-d}},\ c > 0 \text{ and } d > 0. \tag{3}$$

In his *L'algebra* Bombelli treated the example

$$x^3 = 15x + 4. \tag{4}$$

He had clearly constructed this equation so that it has the root $x = 4$. When he applied Cardano's algorithm corresponding to (2) he obtained

$$x^3 = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Since Bombelli was convinced that (3) only has one (positive) root, he got the idea that it is possible that

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 4. \tag{5}$$

In order to convey meaning to this relation Bombelli looked at the quantities $+\sqrt{-q}$ and $-\sqrt{-q}$ for $q > 0$. It is interesting to note that rather than considering these as new numbers he conceived of them as ordinary numbers provided with new operations. Thus he remarked that $+\sqrt{-q}$ and $-\sqrt{-q}$ are very different from usual square roots because they can neither be symbolized by plus (*più*) nor by minus (*meno*), but need new characteristics [Bombelli 1572, 169]. He decided on the expressions *più di meno* and *meno di meno* (or *men di meno*) and treated these as a kind of operations; I symbolize these by +* and –*. To be able to calculate Bombelli enlarged the common list of combining operations,

plus times plus is plus, plus times minus is minus, etc.,

with a set of new laws

più times più di meno is più di meno, meno times più di meno is meno di meno,
più times meno di meno is meno di meno, meno times meno di meno is più di meno,

più di meno times più di meno is meno, più di meno times men di meno is più, meno di meno times più di meno is più, meno di meno times men di meno is meno.

He took the distributive law for granted and was then able to carry out multiplications of the form $(\pm a \pm^* b)(\pm c \pm^* d)$. Furthermore he assumed a rule corresponding to

$$+^*\sqrt{q^2} = q(+^*1) \text{ or } \sqrt{-q^2} = q\sqrt{-1}, \text{ for } q > 0,$$

and then presented many examples of how to deal with the new operations. In one of these he showed that

$$\sqrt[3]{2 +^* \sqrt{121}} = 2 +^* 1,$$

and similarly

$$\sqrt[3]{2 -^* \sqrt{121}} = 2 -^* 1.$$

By adding he established the relation (5).

Bombelli himself found his new way of calculating more *sofistica* than real, but necessary. He might have taken the expression "sophisticated" from one of Cardano's examples, to which I return. It is important to notice that for Bombelli complex numbers were in general only a means to obtain a real and positive root, namely one of the form (3).[1] He did, though, once mention complex solutions in connection with a quadratic equation. He stated as a rule that if $(\frac{1}{2}a)^2 < b$ in the equation $x^2 + b = ax$, then an impossible problem has been formulated [Bombelli 1572, 262]. However, in the single case of $x^2 + 20 = 8x$ he went further, claiming that this equation can only be solved in a sophisticated manner which gives the roots 4 *più di meno* 2 and 4 *men di meno* 2, corresponding to $4 \pm 2\sqrt{-1}$. Analogously, Cardano had earlier in one exceptional example, concerning the system of equations $x + y = 10$ and $xy = 40$, considered the complex solutions $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ and called them truly sophisticated [Cardano 1545, chapter 37, rule 2]. For systems of linear equations there are similarly a few untypical cases in which negative solutions were taken into account. In general, however, European pre-1600 mathematicians only searched for positive solutions.

Bombelli's considerations became known, but not accepted by all mathematicians, for instance not by Simon Stevin. He considered expressions of the form (3) useless [Stevin 1585, 309-310]. Yet, it did not take long before complex numbers went from the status of occurring in formulae for real roots to the level of being counted as roots – in the same process negative roots were also included. The change was caused by the fundamental theorem of algebra concerning equations (by which I here and in the following mean polynomial equations).

1 In general it is not possible to reduce (3) by algebraic means. However, in treating quantities of the form $\sqrt[3]{a + b\sqrt{-1}}$ Bombelli chose a and b so that it is not difficult to determine p and q from the equation $p + q\sqrt{-1} = \sqrt[3]{a + b\sqrt{-1}}$ [Gericke 1970, 62].

The fundamental theorem was formulated by Albert Girard in 1629 in a way equivalent to: an equation of degree n has n roots [Girard 1629, E4^{r,v}]. Girard was not genuinely interested in the complex roots, which he called impossible. For him they had the following functions. They made his theorem true and confirmed his observation that if the coefficient to the term of highest degree is 1, then the other coefficients are equal to plus or minus the expressions which later were called the elementary symmetric functions of the roots (the constant term being equal to ± the product of the roots, etc.). Finally they gave a limit to the number of real roots: for example, if it is known that a certain fourth degree equation has two complex roots, there is no need to search for more than two real roots. Rather than attempting to prove the fundamental theorem Girard presented examples in which it applies – including cases with complex roots. He did not reveal whether he thought that all the non-real roots are complex – nor did he use the expression real root, which was first introduced in 1637 by René Descartes.

Commenting upon the fundamental theorem Descartes made the following, often quoted, remark in his *La géométrie*:

les... racines... ne sont pas tousiours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut tousiours en imaginer autant que iay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine.[2] [Descartes 1637, 380]

Thus Descartes basically claimed that an equation of degree n has n roots, of which some exist in reality and others are only to be found in our imagination. Among the latter are included roots of the type

$$a + b\sqrt{-1},\ a \text{ and } b \text{ real}, \tag{6}$$

but Descartes himself did not comment upon the form of the non-real roots.

Descartes's choice of words was so influential that it became common to use the terms real and imaginary. Gradually, it was assumed that the roots in a polynomial of any degree have the form (6). This hypothesis, or the equivalent assumption that every polynomial can be written as a product of polynomials of first and second degree, became the standard version of the fundamental theorem of algebra. It turned out that there was no obvious proof for this result. In fact, several gifted mathematicians failed in their attempts to prove the theorem, and it remained an assumption until Gauss gave a proof in 1799 which – though it built upon intuition in several steps – seems to have been convincing. Gauss was so much drawn to the fundamental theorem that he returned to it three times, providing two new proofs, and in 1849 a revision of his first proof.

1.2 Extended use of complex numbers

The theory of equations had so to speak forced mathematicians to work with complex numbers. They treated these new numbers formally and had no problems as long as they only added and multiplied them – the rules for these operations had, as we saw, already been formulated by Bombelli. They did, however, face challenges when they went further

2 the... roots... are not always real, but sometimes only imaginary. This means that for each equation one can always imagine as many roots as I have indicated, but that sometimes there is no quantity which corresponds to the imagined ones.

in their investigations. Gottfried Wilhelm Leibniz, for instance, at some point doubted that

$$\sqrt{\sqrt{-1}}$$

can be written on the form (6) [Leibniz 1702, (359-60)]. He discussed this matter in a study on integrating rational functions by splitting them up into partial fractions; in particular he wanted to write

$$x^4 + a^4 = (x + a\sqrt{\sqrt{-1}})(x - a\sqrt{\sqrt{-1}})(x + a\sqrt{-\sqrt{-1}})(x - a\sqrt{-\sqrt{-1}})$$

as a product of two real second-degree polynomials [Gericke 1994, 89-90]. In this connection he also touched upon the nature of complex numbers and called them an amphibium between existence and non-existence [Leibniz 1702, (357)]. Leibniz's doubt implies that he did not believe that all roots in a four-degree equation are of the form (6).

The mathematicians might have had less problems if they had had a more concrete approach to complex numbers and to operations with them. It would have been natural to search for this in geometry, in particular because Descartes's *La géométrie* had given rise to, although not introduced, a conception of real numbers as points on a line. However, there is no evidence that mathematicians in the seventeenth century were deeply engaged in finding out how $a + b\sqrt{-1}$ can be represented geometrically. John Wallis did take up the problem of constructing complex numbers geometrically, but not from a very general point of view [Wallis 1685, chapters 66-69]. After having interpreted a negative area as a missing area (taken by the sea) he took up special problems concerning complex numbers. He conceived algebraically of $\sqrt{-bc}$ (b and c positive) as a mean proportional between $-b$ and c and suggested a geometric construction of this mean proportional, but not a very fortunate one. He furthermore presented some rather special geometric constructions of complex solutions to quadratic equations [Eneström 1906-07, 263-269; Coolidge 1924, 13-16; Gericke 1970, 72-74]. Although Wallis's suggestions were not successful, they are interesting from a historical point of view, because they show that it was far from obvious how the geometrical interpretation should be obtained.

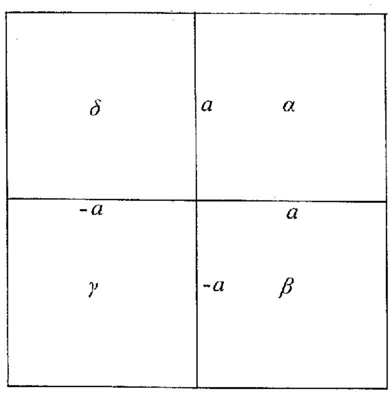

Figure 1. Adaption of figure in Kühn 1753.

After Wallis there are a few other known examples touching upon geometric interpretations of complex numbers [Cajori 1912, Schubring forthcoming]. One stems from Heinrich Kühn and is particular. From the origin (figure 1) he constructed the line segments $\pm a$ along the axes and considered the four squares α, β, γ and δ. He assigned the area $-a^2$ to β and interpreted $\sqrt{-a^2}$ as the "*latus seu radix*" of this area, similarly he interpreted $-\sqrt{-a^2}$ as the side of δ [Kühn 1753, 172].[3] Kühn's work was published in the journal of the St. Petersburg Academy despite the fact that Leonhard Euler had been very critical about it and advised against its publication [Juschkewitsch 1983, 35].

Euler's own writings contain implicitly an understanding of the complex plane, but not one which he worked out [Kline 1972, 629]. Euler had the philosophy that methods applied in analysis should be analytic or algebraic, so it is not to be expected that he would elaborate on a geometric understanding of complex numbers; he just treated them formally.

In his famous *Vollständige Anleitung zur Algebra* (1770) he introduced them in the following way, which reflects views he also expressed in his earlier works.

And since all numbers which it is possible to conceive are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers which from their nature are impossible; and therefore they are usually called imaginary quantities, because they exist merely in the imagination... But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them... [Euler 1770, §§ 143, 145. English translation from Euler 1840]

Euler was a veritable master in handling and deducing results concerning these expressions created by the mind. He used complex numbers to solve a long debated question of how to conceive of $\log(-1)$ [Kline 1972, 408-409]. In a paper published in 1751 he extended the elementary transcendental expressions to among others the following [Euler 1751, §§ 96, 100, 106].

$$(a+b\sqrt{-1})^{m+n\sqrt{-1}}, \quad \log(a+b\sqrt{-1}),$$

$$\sin(a+b\sqrt{-1}), \quad \cos(a+b\sqrt{-1}), \quad \mathrm{tg}(a+b\sqrt{-1}).$$

His work on the complex exponential function led him to the result [Euler 1751, §97]

$$\sqrt{-1}^{\sqrt{-1}} = e^{2\pi\lambda - \frac{1}{2}\pi}, \lambda \text{ being an integer.} \tag{7}$$

Moreover, his influential *Introductio* contains the relation [Euler 1748, §133]

$$(\cos\varphi + \sqrt{-1}\sin\varphi)^n = \cos n\varphi + \sqrt{-1}\sin n\varphi \tag{8}$$

which is now known as De Moivre's formula, and another one [*ibid.*, §138] which implies the former and still carries Euler's name:

3 For another discussion of Kühn's work see [Coolidge 1924, 16-18].

$$\cos\varphi + \sqrt{-1}\sin\varphi = e^{\sqrt{-1}\varphi}. \tag{9}$$

Several of Euler's results can be found – presented in other forms – in the works by some of his predecessors, but it was through Euler's publications that they became known. The idea of using the letter i for $\sqrt{-1}$ also originates with Euler, but Gauss was instrumental in making the use common [Cartan 1908, 342].

Although very succesful in treating complex numbers Euler failed at one point, namely in his attempt to prove the fundamental theorem of algebra. In his 1751 paper he presented a general proof, but this is not valid for equations of higher degree than four. Perhaps Euler himself had some doubts about his proof because at the end of his paper he stated – after having shown that the complex numbers are closed under several algebraic and transcendental operations – that he saw no reason why not all imaginary roots should be of the form $a + b\sqrt{-1}$ [Euler 1751, §124].

Euler's approach to analysis was followed by his successors and did not invite anybody to work on the problem of a geometric representation of complex numbers. The first who is reported to have presented the complex plane explicitly is the Frenchman Henri Dominique Truel. Cauchy is the source of this information, and he only revealed that in 1786 Truel had found a way of representing complex numbers in a plane [Cauchy 1847_3, (175)]. Thus it is unknown whether Truel also attempted to interpret the algebraic operations with complex numbers geometrically. The latter was done by the Norwegian Wessel, who is the next person entering the story about complex numbers.

2 Wessel's *On the Analytical Representation of Direction*

2.1 Wessel's aim

To have a short title for Wessel's *On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons* I call it his *Attempt.* He opened this by giving the impression that it was born of a wish to find expressions for line segments from which their lengths and their directions can be determined and to find a way of calculating with such expressions. This wish seems very natural for a surveyor who had solved an endless number of problems concerning triangles (p. 48 in Branner and Voje Johansen 1999). Wessel added, however, another motivation: he had been seeking a method whereby he could avoid "the impossible operations". He also stated that this had paradoxically shown that the possible sometimes has to be sought by "impossible means" [Wessel 1799, introduction]. He did not explain what he meant by "impossible operations", but it is likely that he thought of extracting the square root of –1, and generally operations with impossible *i.e.* complex numbers. If this is the case, as several commentators – starting with H. Valentiner – assume, it appears that complex numbers were also a source of inspiration for Wessel [Valentiner in Wessel 1897, VII; Coolidge 1924, 18; Crowe 1967, 6].

Irrespective of whether Wessel originally started with complex numbers or with line segments, he structured his *Attempt* as a pursuit of algebraic operations for directed line segments. This idea might be seen as an extension of what Descartes did in *La géométrie,* namely to show how the operations +, –, ·, :, and $\sqrt{\ }$ can be interpreted for traditional line segments. Wessel did not refer to Descartes, but he spent more than a page arguing that it is allowed to conceive of the algebraic operations in a broader sense so that they can be applied to an extended field of quantities. He remarked that as far as he was aware nobody had attempted this before him, but added in a note that a prize essay on *calculus situs*, by Magister Gilbert in Halle, might contain some explanation on applying algebra to directed line segments.

The way Wessel expressed himself indicates that he had not seen Gilbert's essay. He might have read about it in the dissertation *De natura, constitutione et historia matheseos primae...* (1795) by which Ludwig Wilhelm Gilbert (1769-1824) obtained his doctorate in 1794. In the introduction of this work Gilbert mentioned that he had answered the prize subject proposed by the *Fürstliche Jablonowskische Gesellschaft der Wissenschaften* (the Jablonowsky Society of Sciences) in Leipzig, and had received the prize [Gilbert 1795, 7]. The theme of the essay was the "*situs geometria et calculus" (Geometrie und Calcul der Lage)*. In his dissertation Gilbert wrote more about his prize essay but nothing which really reveals its content. My impression is, though, that Gilbert treated philosophical aspects of geometry rather than calculation with directed line segments. Being so engaged in his line segments Wessel might by reading the text of the prize subject[4] have thought that these would solve the problem and that Gilbert had the same idea. Unfortunately, I have not been able to find more information on the Gilbert's prize essay. He did not refer to it in his book on geometry [Gilbert 1798], nor is it mentioned in Ludwig Choulant's comprehensive biography of Gilbert [Choulant 1824].

2.2 *Wessel's introduction of addition and multiplication*

Wessel's line segments were defined by their directions and their lengths. He himself just called them lines, whereas I use the expression directed line segments and sometimes only line segments. His conception involves that he considered all parallel line segments which have the same length and the same orientation to be equivalent though he did not use the term.[5] When operating with line segments Wessel in fact chose convenient representants from the equivalence classes. In the following I assume this convention to be understood when I write *the* directed line segment.

Wessel started by defining addition and multiplication geometrically. His definition of addition is in accordance with the tradition of adding velocities by the parallelogram rule and corresponds to addition of vectors. In figure 2 I have illustrated how he defined the sum of ab and bc to be ac which he wrote as $ab + bc = ac$ [Wessel 1799, §1]. In a geometrical language he expressed that his addition is commutative and associative. He furthermore emphasized a result concerning polygons which turned out to be essential for his trigonometric applications of his new algebra. For a quadrangle with vertices a, b, c, and d his theorem states that

4 The question proposed by the Jablonowsky Society of Sciences, as reproduced by Gilbert: *Ist die neuere Analysis in Verbindung mit der ebnen und sphärischen Trigonometrie hinreichend den Mangel der Geometrie und des Calculs der Lage, wie sie LEIBNITZ nannte, und die noch von niemand sind bearbeitet worden, zu ersetzen? Wie weit erstrecken sich die Gränzen der leztern? Giebt es nicht etwa Lücken in der Geometrie der Größen, die sich auf keine andre Art, als durch Anwendungen von Sätzen aus der Geometrie der Lage ausfüllen und ergänzen lassen? Anwendung analytisch-trigonometrischer Kentnisse auf die wichtigsten Grundbegriffe und Sätze der Geometrie und des Calcus* [sic] *der Lage.* [Gilbert 1795, 7].

5 The Italian mathematician Giusto Bellavitis, who worked with the same concept as Wessel, introduced the technical term equipollent around 1832 [Cartan 1908, 344, note 62].

$$ab + bc + cd + da = 0. \tag{10}$$

Wessel did not spend many words on subtraction, but noticed that $-ba = ab$.

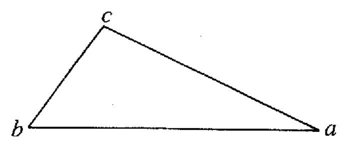

Figure 2. Illustration to Wessel's definition of addition.

The real novelty in Wessel's *Attempt* is his geometric definition of a product of two directed line segments, and here he took quite an abstract step. First he introduced a unit, 1, as Descartes had done for his geometric product of two (non-directed) line segments. Wessel did not illustrate his definition, but I have done so in figure 3, in which *oe* is the unit on a chosen, fixed, and orientated line of reference (from his examples it becomes clear that Wessel orientated the unit in the opposite direction of what is commonly done today). While Wessel's definition of the sum of two directed line segments applies in the

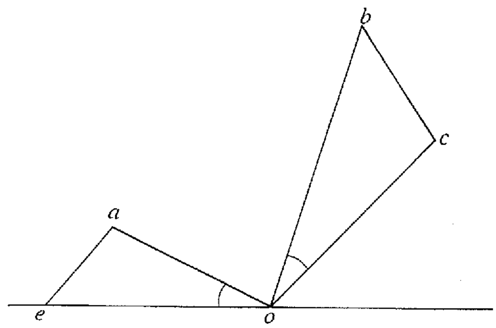

Figure 3. Illustration to Wessel's definition of multiplication.

three-dimensional space, he only defined multiplication for line segments that are situated in a plane containing the unit. Let *oa* and *ob* be two such segments and let *oc* be the, not yet defined, product. Wessel required that *oc* shall be situated in the plane determined by *oa* and *ob*, and moreover that

the product of two straight lines should in every respect be formed from the one factor, in the same way as the other factor is formed from the... unit [Wessel 1799, §4. English translation from Wessel 1999]

If we conceive of *oa* to be formed from *oe* by letting the latter be rotated a certain angle and by letting its length be multiplied by a certain number, then a geometrical interpretation of Wessel's requirement leads to the conclusion that the triangles *oea* and *obc* must be similar and similarly orientated. This implies when || signifies length – that

$$\angle eoc = \angle eoa + \angle eob \tag{11}$$

and

$$|oc|:|ob|=|oa|:|oe| \tag{12}$$

or

$$|oc|=|oa|\cdot|ob|.$$

Reckoning angles with signs Wessel used precisely the relations (11) and (12) to define the product oc (though he did not apply the sign $||$). It should be noticed, as Wessel himself did, that when oa and ob have the same direction as the unity, then the product oc corresponds to a usual product of two line segments.

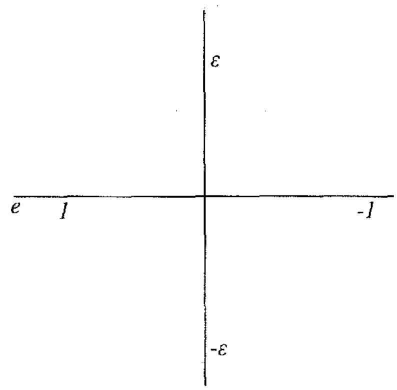

Figure 4. Illustration to Wessel's introduction of the unit ε.

To be able to work with his line segments algebraically Wessel took a step which today seems very natural, but which was quite remarkable at his time. He introduced a second unit which he let be defined by having the length 1 and the direction 90° and which he denoted ε (figure 4). Applying definition (11) Wessel found that the direction of $\varepsilon\cdot\varepsilon$ is 180°, that is the direction of -1, and from (12) he concluded that $\varepsilon\cdot\varepsilon$ has the length 1, that is the same length as -1, so that

$$\varepsilon\cdot\varepsilon=-1\,.$$

By similar applications of (11) and (12) Wessel constructed a multiplication table for 1, -1, ε, and $-\varepsilon$ in §5 of his *Attempt* and then laconically remarked "from this it follows that ε becomes $=\sqrt{-1}$ ". This observation shows quite clearly that he was aware that he had given a geometric interpretation of $\sqrt{-1}$, but he did not make any point out of this achievement.

From multiplying the units Wessel proceeded towards obtaining general algebraic expressions for directed line segments and an algebraic multiplication rule. He did this in two steps; first he dealt with directed line segments of length 1 and secondly with general line segments [Wessel 1799, §§ 6-10]. In my presentation the two steps are combined. By applying his definition of addition Wessel straightforwardly found that a general directed line segment can be written as $a+\varepsilon b$ and that when it has direction v and length r, then

$$a=r\cos v,\ b=r\sin v, \text{ or}$$

$$a+\varepsilon b = r(\cos v + \varepsilon \sin v). \tag{13}$$

Multiplying $a+\varepsilon b$ by

$$c+\varepsilon d = r'(\cos u + \varepsilon \sin u) \tag{14}$$

Wessel obtained according to (11) and (12)

$$(a+\varepsilon b)\cdot(c+\varepsilon d) = rr'[\cos(v+u)+\varepsilon \sin(v+u)]. \tag{15}$$

He wanted a more direct way to calculate the left-hand side and found this by applying the addition formulae for cosine and sine:

$$\cos(v+u) = \cos v \cos u - \sin v \sin u, \quad \sin(v+u) = \cos v \sin u + \sin v \cos u. \tag{16}$$

These together with (13) and (14) allowed him to rewrite the right-hand side of (15) as

$$(r\cos v \cdot r'\cos u - r\sin v \cdot r'\sin u) + \varepsilon(r\cos v \cdot r'\sin u + r\sin v \cdot r'\cos u) =$$

$$ac - bd + \varepsilon(ad + bc),$$

thereby he had proved the formula

$$(a+\varepsilon b)\cdot(c+\varepsilon d) = ac - bd + \varepsilon(ad + bc).$$

Starting with geometric definitions of addition and multiplication Wessel deduced that algebraically his multiplication follows the same rule as the product of complex numbers (when ε is treated as $\sqrt{-1}$). Moreover, he had implicitly shown how complex numbers as well as a sum and a product of complex numbers can be interpreted geometrically. This is quite an accomplishment. Wessel was too modest to express this in strong words, but he did remark that he had shown "that the direction of all lines in the same plane may be expressed as analytically as their length, and this without burdening the memory by new symbols or rules" [Wessel 1799, introduction]. Although satisfied with this result he did not hesitate to mention its limitations, namely that the two factors and 1 have to be in the same plane [Wessel 1799, §10].

Wessel continued with division and root extraction and confirmed the usual results. In dealing with root extractions, he first noticed that his definition of multiplication implies that

$$\left(\cos\tfrac{v}{m} + \varepsilon \sin\tfrac{v}{m}\right)^m = \cos v + \varepsilon \sin v. \tag{17}$$

This is, in fact, another deduction of De Moivre's formula (8). It is difficult to say whether this formula, which Wessel had undoubtedly known for a long time, played any role in his first investigations. It contains the germ of the idea that the direction of the product of two directed lines should be the sum of the factors' directions, so Wessel might have found an inspiration for his definition of a product in this formula and then later turned the order so that the formula became a consequence.

By taking the mth root on both side of (17) Wessel concluded that

$$(\cos v + \varepsilon \sin v)^{\frac{1}{m}} = \cos \tfrac{v}{m} + \varepsilon \sin \tfrac{v}{m}$$

and added the $m-1$ other roots, $\cos \frac{2\pi\lambda + v}{m} + \varepsilon \sin \frac{2\pi\lambda + v}{m}$ for $\lambda = 1, 2, \ldots, m-1$ [Wessel 1799, §§13-15]. In listing these roots Wessel broke with a tradition, for he defined π to be 360° (I have kept to the usual convention).

Wessel was also acquainted with Euler's formula (9) and with his results concerning complex exponential and logarithm functions. Manipulating with infinite series Wessel noticed that $(1+x)^m$ can be written in the form

$$(1+x)^m = e^{ma + mb\sqrt{-1}},$$

where e^{ma} is the length of $(1+x)^m$ and mb its direction [Wessel 1799, §16]. He promised to give detailed proofs for these statements on another occasion, but as far as it is known Wessel never wrote another mathematical paper.

2.3 Wessel's universal trigonometric formula

To demonstrate that his directed line segments could be of use Wessel sophisticatedly showed how the theorem of Cotes follows from his calculation rules [Wessel 1799, §18]. His main concern, however, was to solve trigonometric problems. For this purpose he derived a universal formula from which unknown elements in a polygon can be determined. He introduced a system of abbreviations which is quite elegant and handy for intense work with his formula, but less adequate for a survey of its content; I shall therefore keep to a more traditional notation. Wessel's result applies to any polygon, but he exemplified it – and so do I – by looking at a quadrangle [Wessel 1799, §22]. Let this be *abcd* (figure 5) having the inner angles *A, B, C,* and *D* (he actually considered the outer angles). Wessel placed this quadrangle so that the *ab* has the direction of the unit 1, and applied the earlier mentioned result

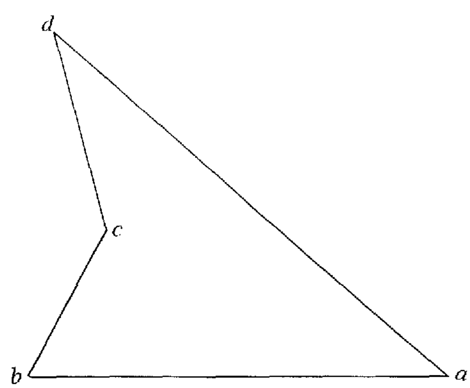

Figure 5. Figure 2 in Wessel 1799 modified.

$$ab + bc + cd + da = 0. \tag{10}$$

He then expressed the directed line segments in the form $r\,(\cos v{+}\varepsilon\sin v)$ and got an equation equivalent to the following.

$$\begin{aligned}&|ab|+|bc|\,[\cos(180°-B) + \varepsilon\sin(180°-B)]\\&+|cd|\,[\cos(360°-(B+C)) + \varepsilon\sin(360°-(B+C))]\\&+|ad|\,[\cos(180°+A) + \varepsilon\sin(180°+A)] = 0.\end{aligned} \tag{18}$$

This is his universal formula, from which he obtained unknown elements in the quadrangle, applying the relation

$$A + B + C + D = 360°. \tag{19}$$

In an example Wessel assumed that all the sides and angles apart from $|cd|$, A, and B are given, and showed how the angle B can be determined [*ibid.*, §23]. His strategy was to rewrite (18) in such a way that the coefficient to ε only contains one unknown element. Since $|cd|$ is unknown, he wanted to get rid of this term as a coefficient to ε, and hence multiplied (18) by $\cos(B{+}C){+}\varepsilon\sin(B{+}C)$. By using (19) he then got

$$\begin{aligned}&|ab|[\cos(B+C) + \varepsilon\sin(B+C)] + |bc|\,[\cos(180°+C) + \varepsilon\sin(180°+C)]\\&+|cd|+|ad|\,[\cos(180°-D) + \varepsilon\sin(180°-D)] = 0.\end{aligned}$$

This equation can only be satisfied if the coefficient to ε is 0, that is if

$$|ab|\sin(B+C) - |bc|\sin C + |ad|\sin D = 0,$$

or

$$\sin(B+C) = \frac{|bc|\sin C - |ad|\sin D}{|ab|}.$$

From the latter relation B can be determined. Wessel indicated how to proceed in the cases in which two sides and an angle, or three angles are unknown. Unfortunately for surveyors and others who had to solve triangles, Wessel's new approach did not reduce the actual calculations involved in determining angles and sides, but his universal formula (18) did make it easier to realize how the calculations should be carried out.

2.4 Wessel's treatment of spherical triangles

Wessel was also interested in solving spherical triangles so he was naturally led to extend his calculations with directed line segments to three dimensions. For this purpose he considered a sphere with radius r and introduced so to speak two complex planes in this. He let the planes be perpendicular to each other sharing the real axis and termed the second imaginary unit η, *i.e.* $\eta^2 = -1$. In figure 7 op is the real axis, $p\pi$ one of the imaginaries, $p\pi$ being equal to ηr and similarly $pq = -\varepsilon r$. Quite easily Wessel found that any directed radius in the sphere can be written in the form

$$x + \eta y + \varepsilon z. \tag{20}$$

His definition of the sum of directed line segments covered the three-dimensional situation and resulted in an obvious rule for how two of these expressions should be added. Multiplication, however, was a problem. In introducing the product of two line segments Wessel stressed, as we saw, the fact that he had not been able to introduce an algebraic product in three dimensions. Undoubtedly, he had searched for a meaning of $\varepsilon \cdot \eta$ and had found none.

Wessel did not let himself stop by the lack of a general product, but used the product which he had introduced in a plane and linked this to rotations. This step actually turned out to be sufficient for solving spherical triangles, as will be shown. From his definition of a product of two line segments coplanar with 1 and ε, it follows that rotating the directed line segment $x + \varepsilon z = r(\cos u + \varepsilon \sin u)$ the angle v around the origin is the same as multiplying it by $\cos v + \varepsilon \sin v$. He applied this result in three dimensions in the following way. First he looked at how a directed radius $x + \eta y + \varepsilon z$ is changed when it is rotated the angle v around the η-axis. Since the y-coordinate remains unchanged, he concluded that this turning process can be considered as a rotation in a horizontal plane, and that therefore the rotated radius is given by

$$\eta y + x' + \varepsilon z' = \eta y + (\cos v + \varepsilon \sin v) \cdot (x + \varepsilon z).$$

Wessel introduced the composition ,, [Wessel 1799, §30] defined by

$$(x + \eta y + \varepsilon z) \,,, (\cos v + \varepsilon \sin v) = \eta y + (\cos v + \varepsilon \sin v) \cdot (x + \varepsilon z). \tag{21}$$

Completely analogously he considered a rotation around the ε-axis and the operation

$$(x + \eta y + \varepsilon z) \,,, (\cos u + \eta \sin u) = \varepsilon z + (\cos u + \eta \sin u) \cdot (x + \eta y). \tag{22}$$

When $\cos v = \cos u = -1$ the two definitions give different results; however, Wessel avoided this confusion by introducing special symbols for the angles so that the axis of rotation was indicated. As earlier I do not take his special notation into account.

Wessel applied the rotation process to solve spherical triangles by an ingenious deduction of another universal formula [*ibid.*, §37]. In fact, he deduced his result for any spherical polygon, but I present it for a triangle. Let this (figure 6, where the notation is mine) have the sides a, b, and c and the outer angles A, B, and C – it turns out to be an advantage that Wessel considered these rather than the inner angles. To follow Wessel's procedure let us imagine (figure 7) that the position of the sphere is given with respect to an equator oq and a meridian $o\pi$ which are fixed in space – meaning that they do not take part in the rotations.

First Wessel placed the sphere so that the vertex A of the triangle is at the north pole π and the prolongation of its side b lies on the meridian $o\pi$. Then he rotated the sphere al-

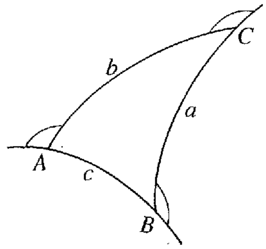

Figure 6. A spherical triangle.

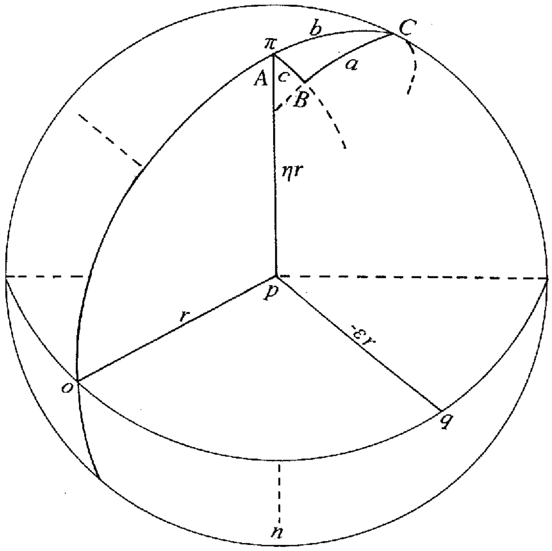

Figure 7. Adaption, with some changed letters, of figure 6 in Wessel 1799.

ternatively around the η-axis and the ε-axis, all in all six times in such a way that the rotations around the η-axis are determined by the outer angles of the triangle and the rotations around the ε-axis by its sides. Thus he started by turning the sphere the angle A around the η-axis, thereby obtaining that the side c lies on $o\pi$ (figure 8). Next he turned it the angle c around the ε-axis so that B falls in π, and proceeded in this way until the sphere had been turned the angle b around the ε-axis. Finally he made the fundamental observation that after this process the sphere has ended in its starting position. Hence for any point on the sphere he had the following relation [Wessel 1799, §37, section 6]

$$(x + \eta y + \varepsilon z) \,,, (\cos A + \varepsilon \sin A) \,,, (\cos c + \eta \sin c) \,,,\ldots,, (\cos b + \eta \sin b) = (x + \eta y + \varepsilon z). \quad (23)$$

This is Wessel's universal spherical formula which serves the same function as the plane formula (18). In the relation it is not important to start with $\cos A + \varepsilon \sin A$, but it is essential to keep the order. The sphere could first have been placed as shown in figure 8, then the first factor would be $\cos c + \eta \sin c$ and the last $\cos A + \varepsilon \sin A$. The order should be kept because as – Wessel was well aware – the rotation of A degrees around the η-axis does not commute with the rotation of c degrees around the ε-axis [*ibid.*, §37 introduction and section 10].

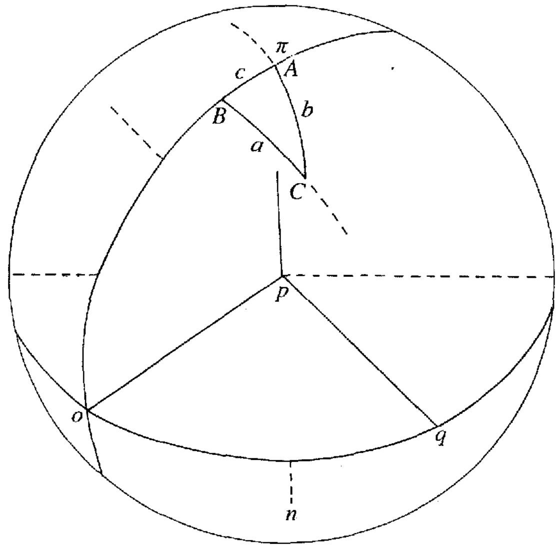

Figure 8. Adaption, with some changed letters, of figure 7 in Wessel 1799.

In his calculations Wessel made frequent use of the relations

$$\eta r \,,, (\cos v + \varepsilon \sin v) = \eta r \quad \text{and} \quad \varepsilon r \,,, (\cos u + \eta \sin u) = \varepsilon r \tag{24}$$

which are immediate consequences of (21) and (22). He described a general procedure for determining unknown elements in a spherical triangle from (23) [Wessel 1799, §37, sections 7-12], which I illustrate by assuming that a, b, and C are given and that B has to be determined. He got rid of the unknown A by setting $(x,y,z)=(0,r,0)$ in (23) and concluding from (24) that $\eta r \,,, (\cos A + \varepsilon \sin A) = \eta r$, thereby he obtained

$$\eta r =$$
$$\eta r \,,, (\cos c + \eta \sin c) \,,, (\cos B + \varepsilon \sin B) \,,, (\cos a + \eta \sin a) \,,, (\cos C + \varepsilon \sin C) \,,, (\cos b + \eta \sin b).$$

To avoid the unknown c he isolated it on the right-hand side. First he "multiplied" the two sides by $(\cos b - \eta \sin b)$ using that

$$\begin{aligned}&(\cos b + \eta \sin b) \,,, (\cos b - \eta \sin b) = (\cos b + \eta \sin b) \cdot (\cos b - \eta \sin b)\\&= (\cos b + \eta \sin b) \cdot (\cos(-b) + \eta \sin(-b)) = 1,\end{aligned}$$

then by $(\cos C - \varepsilon \sin C)$, etc., and obtained:

$$\begin{aligned}&\eta r \,,, (\cos b - \eta \sin b) \,,, (\cos C - \varepsilon \sin C) \,,, (\cos a - \eta \sin a) \,,, (\cos B - \varepsilon \sin B)\\&= \eta r \,,, (\cos c + \eta \sin c).\end{aligned}$$

Now, there is no ε on the right-hand side, hence the coefficient to ε on the left-hand side has to be 0. This implies that

$$\cos B\ (\sin b \sin C) + \sin B\ (\cos a \sin b \cos C + \sin a \cos b) = 0$$

from which B can be determined.

Wessel proceeded to demonstrate how the universal formula (23) leads to important formulae for spherical triangles. In his first example he set $(x,y,z)=(0,0,1)$, applied the just described multiplication procedure and found that

$$\varepsilon\ ,,\ (\cos A + \varepsilon \sin A)\ ,,\ (\cos c + \eta \sin c) =$$
$$\varepsilon\ ,,\ (\cos b - \eta \sin b)\ ,,\ (\cos C - \varepsilon \sin C)\ ,,\ (\cos a - \eta \sin a)\ ,,\ (\cos B - \varepsilon \sin B).$$

By setting the coefficients to η and ε on the two sides equal he very elegantly obtained two of the standard formulae for spherical triangles [Wessel 1799, §38]

$$\cos A = \cos B \cos C - \sin B \sin C \cos a \ \text{ and } \ \sin A = \frac{\sin a \sin C}{\sin c}.$$

Wessel seems to have been seduced by the power of his universal formula, for he used it to derive a great number of trigonometric formulae and some general results concerning spherical triangles [Wessel 1799, §§ 38-63]. He furthermore mentioned the possibility of applying it to plane triangles by assuming that the radius of the sphere is infinitely large and the sides of a triangle are infinitely small parts of circles on the sphere [Wessel 1799, §37, section 13]. He ended the *Attempt* by indicating how his theory can be applied to a rectilinear polygon having sides in different planes [*ibid.*, §§ 64-71]. Altogether his *Attempt* is a remarkable piece of mathematics, although it was not appreciated as such in its own time.

3 Complex numbers from Wessel to the mid-nineteenth century

3.1 Geometric representations

From the end of the nineteenth century every historical exposition touching upon complex numbers has mentioned Wessel's ideas, but they were not, as told earlier, noticed at the time of their publication. In fact, Wessel's *Attempt* only became known after Sophus Andreas Christensen had mentioned it in his thesis [Christensen 1895, 245-46]. Christensen did not pay much attention to the contents, but his remarks inspired the Danish mathematician Christian Juel to study Wessel's work and publish a paper on it [Juel 1895]. Sophus Lie also got interested and had Wessel's *Attempt* reprinted in 1896. It was moreover translated into French the following year [Wessel 1897], and part of it into English three decades later [Wessel 1929].

The belated fame of Wessel's achievement has been explained by the fact that it was presented in Danish – a language read by few. In my opinion the reason was rather that around 1800 the problem of representing the complex numbers geometrically was not considered important in the leading circle of mathematicians, and I shall give three arguments to support this view. First, if the problem really had been considered an essential issue, then some mathematicians able to read Danish would have been familiar with it and would have called attention to Wessel's solution. En passant it can be remarked that Niels Henrik Abel belonged to the group who could have made Wessel's result known. Viggo Brun has documented that in August 1822 Abel borrowed the issue of the *Kongelige Danske Videnskabernes Selskabs Skrifter* which contains Wessel's *Attempt* [Brun 1962, 110]. However, as Brun has pointed out, Abel's interest was presumably not caught by Wessel's paper but by one on the theory of equations written by Carl Ferdinand Degen.

Secondly, it is essential to observe that at the beginning of the nineteenth century none of the influential mathematicians published on the geometrical representation of complex numbers. This was done by a number of other persons – to whom I return – who in general did not work on mainstream mathematics and who did not write any other really remarkable piece of mathematics. Moreover, at first their publications were not noticed by the first-rank mathematicians.

The third argument is slightly more complicated; it involves Gauss, and his motives are often difficult to make out. A drawing in one of his notebooks indicates that by 1805 he worked with the complex plane [Gauss, *Werke*, vol.8, 105], and in 1811 he explicitly referred to this in a letter to Friederich Wilhelm Bessel (cf. section 3.3). He did, however, not publish anything about his understanding of complex numbers until 1831. This took place in a paper which is described in section 3.3 and which contains the claim that already when he composed his first proof of the fundamental theorem of algebra (published in 1799) he had an understanding of the complex plane [Gauss 1831, (175)]. Irrespective of whether Gauss remembered correctly in 1831 he could, as just shown, have published much earlier on his interpretation of complex numbers. So the question becomes why he did not. He is known for having kept some of his results unpublished for a long time; in the case of non-Euclidean geometry he later maintained that he feared the philosophical reactions to his insights. I do not think that he saw similar troubles in introducing the complex plane and believe that if he really had found his understanding important he would have published it earlier. That he later would indicate that he had a kind of priority to the result was not atypical of his behaviour.

Among the half dozen who published on the geometric representation of complex numbers at the beginning of the nineteenth century only one was remembered, namely Argand – about whom very little is known (Schubring forthcoming). In 1806 he anonymously and probably at his own expense published the booklet *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.* At first it seems that this work was also going to be neglected, but seven years later attention was drawn to it by a strange incident [Jones 1970].

In volume four of *Annales des mathématiques* the professor of military art at the artillery school at Metz, Jacques Frédéric Français, published a paper on the geometric interpretation of "imaginary symbols" [Français 1813]. At the end of his paper he honestly told that the origin of his ideas was to be found in a letter which his deceased brother had received from Adrien Marie Legendre and which appeared to contain a paraphrase of another person's work. Français concluded by expressing the hope that this person would let himself and his work become known. The reaction to Français's appeal, dated July, 6, 1813, came very quickly. By November Argand had written a paper which also appeared in volume four of the *Annales* [Argand 1813]. In this he identified himself as the author of Français's ideas, gave a summary of his *Essai,* and added a few more reflections. Français's and Argand's papers gave rise to a debate on complex numbers which ran in the *Annales* until 1815. François Joseph Servois reacted upon the first papers, Français upon Argand's paper, and the two authors answered Servois.[6]

The editor of the *Annales*, Joseph Diaz Gergonne, had the habit of commenting upon the various papers in his journal, and he made several remarks to the discussion on complex numbers. One of his notes to Français's first paper was that a couple of years earlier he himself had the idea of representing $n\sqrt{-1}$ on a line perpendicular to a line on which the usual numbers n were represented [Gergonne 1813]. He showed a great interest in the polemic, but nevertheless it was not kept alive for long. It is remarkable that so few – and none of the famous – French mathematicians took part in it; in particular it is striking that Legendre, who was familiar with Argand's work and indirectly had been instrumental in bringing it to light, kept silent. Gergonne included a note by the well known mathematician Sylvestre François Lacroix, but not one revealing any opinion on geometrical representation of complex numbers [Lacroix 1814]. It just contained the fact that in the same year as Argand's *Essai* was published, another paper on the same topic, written by Abbé Buée, had appeared in *Philosophical Transactions* [Buée 1806]. Buée had indeed the idea of representing $\sqrt{-1}$ as a unit line segment perpendicicular to the usual unit, but if I have understood him correctly he did not show how $a + b\sqrt{-1}$ should be interpreted geometrically.

Shortly after the debate in the *Annales* was closed, yet another publication on complex numbers appeared, namely Benjamin Gompertz's *The Principles and Application of Imaginary Quantities* [Gompertz 1817-18]. This work was mentioned by George Peacock [1834, 234] and William Rowan Hamilton [1853, (136)], but otherwise left out of the history of complex numbers, Cartan's exposition being an exception [Cartan 1908, 337-338]. In his first book Gompertz presented a very special algebraic interpretation of complex numbers, and in his second an even more particular geometric interpretation. He referred to Wallis (calling him Wallace) and Buée, admitting that he had not understood what they were after, but nevertheless they had inspired him to work with complex numbers geometrically [Gompertz 1818, iv and vi].

Because so little attention was paid to the geometrical representation writers could continue to publish on the subject without being aware of predecessors. In 1828 this happened again – even on both sides of the Channel – with the publications of John Warren, *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, and C.V. Mourey, *La vraie théorie des quantités negatives et des quantités prétendues imaginaires.* Warren saw Mourey's book after having written his own and was inspired to work on quantities of the form

$$(a + b\sqrt{-1})^{m + n\sqrt{-1}}$$

[Warren 1829_1, 254 and Warren 1829_2, 339]. Warren's own work was studied and appreciated by Hamilton [Hamilton 1853, (135)]. Basically Mourey and Warren had the same ideas as Wessel, namely to work with directed line segments and introduce addition and multiplication formally – although Warren let multiplication be preceded by proportionality.

After having worked on complex numbers for a couple of decades Cauchy and Hamilton described the history of interpreting them geometrically [Cauchy 1847_3, (175); Ham-

6 All the relevant papers are reprinted in Hoüel 1874.

ilton 1853, (135-137, 150)]. Besides the persons already mentioned Cauchy brought up the names Faure and Vallès – to whose work I have paid no attention. From then and until the end of the century it seems to have been common to ascribe the first satisfactory publication on geometric representation of the complex numbers to Argand, with the result that this representation was frequently called an Argand diagram.

It is outside the scope of this chapter to go into details with all the mentioned literature on geometric representation. With Wessel in focus I find it most relevant to compare his work with the best known of the other early publications, namely Argand's. In presenting the latter I touch upon the contributions by Français and Servois. Moreover, to give an impression of what status geometric representation got in the treatment of complex numbers I indicate how the three influential mathematicians, Gauss, Cauchy, and Hamilton, approached the complex numbers.

3.2 Argand's ideas

Like Wessel, Argand worked with directed line segments calling them *lignes en direction* or *lignes dirigées.* While Wessel aimed at being able to calculate with these quantities, Argand used them to give meaning to complex numbers. He began his *Essai* by stating that sometimes negative numbers only seem to exist in the imagination, but can be made real by an appropriate interpretation. He found that the way to make negative numbers real was to consider both their absolute value and their direction, and took this as a starting point for getting $\sqrt{-1}$ out of the realm of imagination. He considered $x = \sqrt{-1}$ to be defined by the relation

$$1 : x = x : (-1) \tag{25}$$

and wanted to interpret this proportion geometrically.

As we saw in section 1.2, Wallis had earlier followed a similar path, but had not found a satisfactory geometrical interpretation of $a\sqrt{-1}$. For the classical geometrical "calculation" with line segments, in which directions were not taken into account, the Greek mathematicians applied a construction of a mean proportional which is presented in Euclid's *Elements* (figure 9). In *La géométrie* Descartes took up this construction to show how $\sqrt{a}$ ($a > 0$) can be constructed as a mean proportional between 1 and a [Descartes 1637, 298]. It is very likely that Argand realized that if directions are considered, then this construction also provides the solution to how to obtain $\sqrt{-1}$ geometrically. At any rate, he introduced a unit circle with centre K (figure 10) and two points A and I so that, in his own

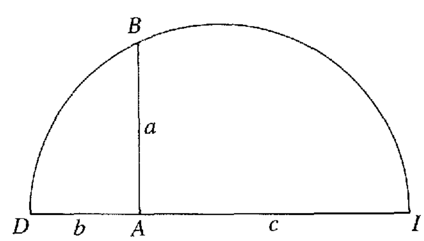

Figure 9. Given b *and* c*, construct* a *so that* b:a=a:c. *In theorem VI.13 of the* Elements *Euclid solved this problem in the following way (the notation is mine). He constructed a circle with diameter* DI *equal to* b+c*, drew a perpendicular to the diameter at the point* A *determined by* DA=b*, found the point of intersection* B *of the perpendicular and the circle, and claimed that* AB=a.

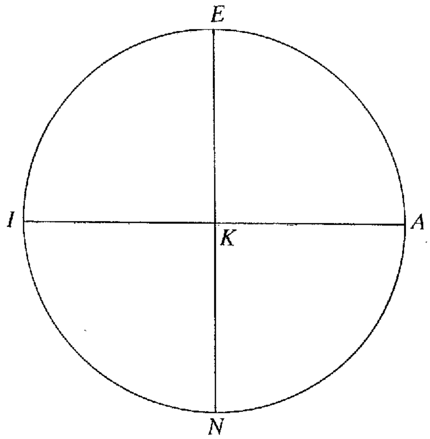

Figure 10. Redrawing, with changed letters, of part of figure 4 in Argand 1806.

notation, $\overline{KA} = 1$ and $\overline{KI} = -1$, and then claimed that the radius $\overline{KE}$ perpendicular to KA fulfils (25). Accordingly he set $\overline{KE} = \sqrt{-1}$ and similarly $\overline{KN} = -\sqrt{-1}$. He proceeded to show that any line segment parallel to line KA can be written as $\pm a$ and any segment parallel to KE as $\pm b\sqrt{-1}$. Apparently applying the parallelogram rule of addition, which he explicitly took up later, he concluded that any directed line segment can be written on the form $\pm a \pm b\sqrt{-1}$ [Argand 1806, (12)]. He then used the inverse result (any expression $\pm a \pm b\sqrt{-1}$ represents a directed line segment) and claimed that $\pm a \pm b\sqrt{-1}$ is real, because directed line segments have this quality.

If Argand had only wanted to show how $a + b\sqrt{-1}$ can be interpreted geometrically he could have stopped here, but he moved on to a geometrical introduction of a product of directed line segments. He focused upon the fact that he had obtained $\overline{KE}$ by halving the angle AKI and then maintained the following. For any given radius $\overline{KP}$ on the unit circle (figure 11) the radius $\overline{KQ}$ determined by $\angle AKQ = \angle QKP$ is a mean proportional between $\overline{KA}$ and $\overline{KP}$. Generalizing once more he claimed (figure 12) that if for two given radii $\overline{KB}$ and $\overline{KC}$ on the unit circle, the radius $\overline{KD}$ is determined by

$$\angle CKD = \angle AKB \tag{26}$$

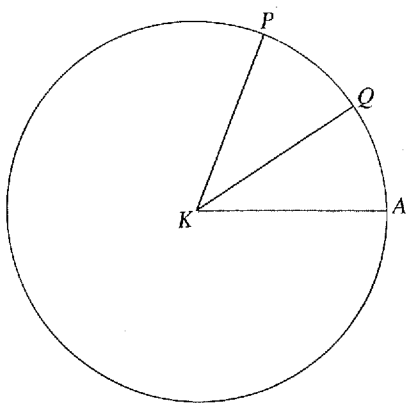

Figure 11. Illustration to Argand.

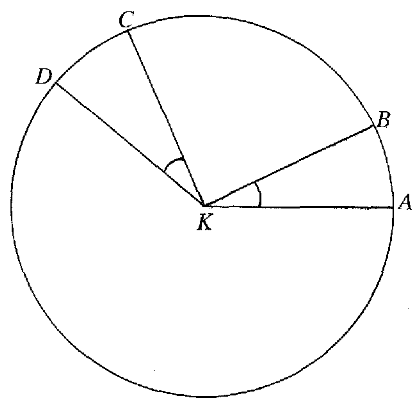

Figure 12. Redrawing, with changed letters, of part of figure 7 in Argand 1806.

then

$$\overline{KA}:\overline{KB} = \overline{KC}:\overline{KD}. \tag{27}$$

He used the latter relation to introduce a product [Argand 1806, (21)], simply by setting

$$\overline{KD} = \overline{KB} \times \overline{KC}. \tag{28}$$

The relations (26) and (28) show that for directed line segments of length 1, Argand's definition of a product is the same as Wessel's. Argand extended his product to general directed line segments in an obvious manner, and got the same product as the one presented by Wessel.

There is, however, an essential difference between Wessel's and Argand's approaches. Wessel gave a formal definition, whereas Argand attempted to show that the product had to be the one he introduced. Thus Argand called his definition a principle which in some sense extends the geometrical relation between positive and negative, and mentioned that his deductions of this principle and the principle of addition "do not possess a sufficient degree of evidence". Hence he conceived of the principles as hypotheses whose legitimation remains to be established or rejected [Argand 1806, (9, 60)]. Apparently Argand was hoping that future investigation would bring arguments for a logical necessity of his geometric definition of the product.

Argand and Wessel did not only have different views upon the status of the definition of the product, they also applied the product differently. As we have seen, Wessel used his geometrical product to deduce the usual algebraic rule for multiplying expressions of the form $a + b\sqrt{-1}$. Argand took this rule for granted and used his product to deduce well known theorems in a new manner. An illustrative example is their treatment of the equality

$$(\cos v + \sqrt{-1}\sin v)(\cos u + \sqrt{-1}\sin u) = \cos(v+u) + \sqrt{-1}\sin(v+u). \tag{29}$$

The relation itself is a consequence of their geometrical introduction of the product of two directed line segments. As we have seen, Wessel founded his proof of the algebraic multiplication rule on this relation, applying the addition formulae (16) for cosine and sine. Argand on the other hand applied (29) to obtain these formulae.

Among Argand's other results is a proof of Ptolemy's theorem concerning the diagonals in a quadrangle which can be inscribed in a circle [Argand 1806, (58)]. He also sketched a proof of the fundamental theorem of algebra [*ibid.*, (58-59)] which was later criticized by Servois [1814, 231].

When Argand returned to the work on directed line segments in 1813, he considered the possibility of extending his calculations to three dimensions. With arguments that are not so easy to follow he suggested that the unit of the third dimension should be [Argand 1813, 146]

$$\sqrt{-1}^{\sqrt{-1}}.$$

Servois protested against this proposal with a reference to Euler's result (7) [Servois 1814, 231], but did not manage to convince Argand completely [Argand 1815, 198-199].

In his work, derived from Legendre's presentation of Argand's *Essai*, Français took several shortcuts. He introduced the notation a_α for a directed line segment having the length a and the direction angle α, and started with a formal introduction of a proportion [Français 1813, 62]. Undoubtedly inspired by Argand's definition (27), he let

$$a_\alpha : b_\beta = c_\gamma : d_\delta \text{ when } b:a = d:c \text{ and } \beta - \alpha = \delta - \gamma\,[\bmod 2\pi]. \tag{30}$$

For $a = a_0$, this gave him that

$$1{:}1_\alpha = a : a_\alpha;$$

based on this relationship he implicitly introduced the product

$$a_\alpha = a \cdot 1_\alpha. \tag{31}$$

Français assumed that his new concept of proportionality included the relation

$$a:(a\sqrt{-1}) = (a\sqrt{-1}):(-a)$$

and then found that $a\sqrt{-1} = a_{\frac{\pi}{2}}$ [Français 1813, 65]. Moreover, he took the relation $e^{\pi\sqrt{-1}} = -1$ for granted, and concluded by an argument concerning mean proportionals that

$$1_{\frac{m}{n}\pi} = e^{\frac{m}{n}\pi\sqrt{-1}}$$

which he generalized to $1_\alpha = e^{\alpha\sqrt{-1}}$. Finally, he combined the last equality with (31) and got [Français 1813, 67]

$$a_\alpha = ae^{\alpha\sqrt{-1}}.$$

Without knowing about Argand's suggestion of how to advance to three dimensions, Français proposed to do this by considering imaginary angles. A large part of the debate in the *Annales* actually concerns the extension to three dimensions. Servois was as sceptical about Français's approach as he was about Argand's. Without giving any details Servois himself suggested a path to follow [Servois 1814, 235] which was later praised by Hamilton [1853, (150)].

In general, Servois was far from enthusiastic about Français's and Argand's papers. He appreciated their attempts but found them insufficient, claiming that they were based on deductions by analogy – *i.e.* generalizations. In Servois's opinion the directed line segments had provided the complex numbers with a "*masque géométrique*" [Servois 1814, 230]. He did not see any use of this mask, nor was he convinced that the derived analytical expressions for line segments were uniquely determined. To a directed line segment which has length a and direction angle α Servois assigned the function $\varphi(a, \alpha)$. He listed a number of obvious conditions that this function has to fulfil. He admitted that $\varphi(a, \alpha) = ae^{\alpha\sqrt{-1}}$ satisfies these conditions, but was far from convinced that no other function would do the same [Servois 1814, 233-234]. In fact, he would only accept the geometric interpretation of complex numbers when it had been shown that there only exists one φ.

If Servois had known Wessel's work he would presumably have acknowledged its logical deductions – a quality he missed in Argand's and François's papers. It is, however, not likely that he would have appreciated Wessel's definition of a product of directed line segments, because this definition did not contain any arguments pointing to the uniqueness of his φ.

In a note to Servois's paper Gergonne dissociated himself from Servois's views [Servois 1814, note page 229]. Gergonne pointed out that new ideas in mathematics have often lacked foundation in the beginning and mentioned particularly the infinitesimal calculus. Thus it appears that, despite Servois's criticism, Gergonne was convinced about the geometric representation of the complex numbers and the geometric interpretation of addition and multiplication. Some decades later several other mathematicians were also convinced, and they presumably found that the geometric interpretation justified the formal treatment of $\sqrt{-1}$. However, as we shall see illustrated in sections 3.4 and 3.5, mathematicians working on the foundation of complex numbers were in general more interested in finding an algebraic justification than in following up the geometrical approach.

3.3 Gauss and complex numbers

As indicated earlier, the first time Gauss explicitly mentioned an interpretation of the complex numbers in a publication was in 1831. Although he claimed that a geometrical understanding of the complex numbers is implicitly present in his proof of the fundamental theorem of algebra from 1799 (cf. section 3.1), he did not refer directly to complex numbers in 1799. Gauss gave his 1831 paper the Latin title *Theoria residuorum biquadraticorum, Commentatio secunda*, which is rather misleading because it is written in German and contains very little on biquadratic residues. His objective was to enlarge number theory to the field of complex integers, *i.e.*

$m + ni$, where m and n are integers. (32)

He expressed the view that for too long the dealing with complex numbers had been "*ein an sich inhaltleeres Zeichenspiel*" (an *an sich* empty game with symbols) [Gauss 1831, (175)]. He wanted to give them a "*Bürgerrecht*" (citizenship) [*ibid.*, (171)] and mentioned in this connection a representation in a plane. Restricting his considerations to complex numbers of the form (32) he introduced an infinite grid of squares in the plane and conceived of these numbers as the vertices of the squares. He mentioned that passing to a neighbouring point meant, according to direction, to add +1, −1, $+i$, or $-i$, and furthermore that i can be considered as a mean proportional between 1 and −1 [*ibid.*, (177)]. Gauss's paper is very short and does not describe the entire complex plane, nevertheless it presumably contributed significantly to the acceptance of this plane.

Gauss ended his 1831 paper by promising to come back to the issue of complex numbers and in this connection to deal with the question why the usual rules of arithmetic cannot be respected in higher dimensions than two. Gauss never published the announced paper, but his brief remark indicates that he – like Wessel and several others – had investigated the possibility of extending complex numbers to three dimensions, and that he had come to the conclusion that this cannot be done.[7] A formal proof of this impossibility was later given by several mathematicians, among them Karl Weierstrass and Hermann Hankel [Bottazzini 1986, 180, note 38].

Gauss's possible interest in expanding calculations with complex numbers to three dimensions was far from academic, but related to his work in various fields. Among other places, he dealt with directed line segments in a three-dimensional space in his famous contribution to differential geometry, *Disquisitiones generales circa superficies curvas* (1828). Here he assigned to a directed line segment a length and a radius of given direction, namely that radius in a unit sphere which is parallel to and orientated in the same direction as the line segment [Gauss 1828, §1]. By trigonometrical calculations he found a number of results which later became part of vector calculus. This kind of enterprise is likely to have inspired Gauss to search for an algebra of line segments in a three-dimensional space.

Gauss did not keep all his ideas on complex numbers to himself until 1831. In fact, already twenty years earlier he revealed his ideas about complex numbers and complex integration in a letter to Bessel:

What should one understand by $\int \varphi x \cdot dx$ for $x=a+bi$? Obviously, if we want to start from clear concepts, we have to assume that x passes from the value for which the integral has to be 0 to $x=a+bi$ through infinitely small increments (each of the form $x=\alpha + \beta i$), and then to sum all the $\varphi x \cdot dx$. Thereby the meaning is completely determined. However, the passage can take place in infinitely many ways: Just like the realm of all real magnitudes can be conceived as an infinite straight line, so can the realm of all magnitudes, real and imaginary, be made meaningful by an infinite plane, in which every point, determined by abscissa=a and ordinate=b, as it were represents the quantity a+bi. The continuous passage from one value of x to another a+bi then happens along a curve and is therefore possible in infinitely many ways. I claim now that after two different passages the integral $\int \varphi x \cdot dx$ acquires the same value when φx never becomes equal to ∞ in the region enclosed by the two

7 Dedekind was of the opinion that Gauss meant something else [Lützen, forthcoming].

curves representing the two passages. This is a very beautiful theorem whose not exactly difficult proof I shall give at a suitable occasion. [Translation of Gauss 1811, (90-91)]

The "suitable occasion" never turned up. The quote in itself, however, shows that by 1811 Gauss had a clear idea of the complex plane and that this was essential for his concept of a complex integral. He had, in fact, used the complex plane to realize one of the fundamental theorems in complex integration theory – often called Cauchy's integral theorem.

Evidently, the geometrical representation of complex numbers played an important role in Gauss's understanding of these. Whether he also considered a geometric interpretation of the product of two complex numbers cannot, as far as I am aware, be decided.

3.4. Cauchy and complex numbers.

Cauchy was a veritable master of complex function theory [Bottazzini 1986, 151-172; Bottazzini 1990]. For a very long time, however, he worked in this field without acknowledging complex numbers as such, but considering them as formal expressions that gave rise to relations between real numbers (for more on Cauchy and the ontological problem concerning complex numbers, see [Bottazzini 1990, CII-CIII]). In his influential textbook *Cours d'analyse* Cauchy wrote that

an imaginary equation is only a symbolic representation of two equations between real quantities. [Translation of Cauchy 1821, iv]

He claimed for instance that the relation (29) makes no sense, and is only a way of expressing the additions formulae (16) for cosine and sine [Cauchy 1821, (154-55)]. Soon afterwards he composed several tracts on complex integration and derived, among other things, simple versions of the result which Gauss had formulated in his letter to Bessel, the Cauchy integral theorem, but he did not express it explicitly in terms of paths in the complex plane [Kline 1972, 636-639]. The theorem also occurs in Cauchy's essay "*Mémoire sur les intégrales definies pris entre des limites imaginaires*" which he published in 1825 and which has been particularly praised by historians of mathematics. Here Cauchy considered the integral

$$\int_{x_0 + y_0\sqrt{-1}}^{X + Y\sqrt{-1}} f(z)dz. \tag{33}$$

He set $z = x + y\sqrt{-1}$, introduced two monotonous and continuous functions $x = \varphi(t)$ and $y = \chi(t)$, let $dz = (\varphi'(t) + \sqrt{-1}\chi'(t))dt$, and then formulated "his" integral theorem by stating that (33) is independent of the choice of the functions $x = \varphi(t)$ and $y = \chi(t)$ when $f(z)$ remains finite for the considered x's and y's [Cauchy 1825, (44)]. He did add an interpretation in terms of curves, but curves in the real plane between the points (x_0,y_0) and (X,Y) [Cauchy 1825, (56); on this point see also Bottazzini 1990, CXVII].

Cauchy continued his work on complex numbers with what could be called "a horror of $\sqrt{-1}$", and for a long time he held his view from 1821. As late as 1847 he praised the formal conception claiming that thereby he had avoided "the torture of finding out what is represented by the symbol $\sqrt{-1}$, for which the German geometers substitute the letter i" [1847_2, (313)]. This remark is part of Cauchy's introduction to a paper in which he wanted to show that "the letter i can be reduced to a real quantity" [*ibid.*]. His means to do this is very interesting as it is quite advanced for its time, though natural for modern mathematicians. With a reference to Gauss and Ernst Eduard Kummer, Cauchy set

$$\varphi(x) \equiv \chi(x)\,[\bmod\,\omega(x)] \tag{34}$$

when $\varphi(x)$ and $\chi(x)$ are two polynomials that have the same remainder after having being divided by the polynomial $\omega(x)$. By introducing i as a "symbolic letter" he rewrote (34) as [Cauchy 1847_2, (314)]

$$\varphi(i) = \chi(i). \tag{35}$$

This implies in particular that $\omega(i) = 0$, and he called i a "symbolic root" in this equation. For the case in which $\omega(x) = x^2 + 1$ he got that i is a symbolic root in $i^2 + 1 = 0$.

Based on this interpretation Cauchy deduced among other results the usual product rule for complex numbers as follows. The relation

$$(a+bx)(c+dx) \equiv ac - bd + (ad+bc)x\ [\bmod\, x^2+1]$$

together with (34) and (35) shows that

$$(a+bi)(c+di) = ac - bd + (ad+bc)i.$$

In another paper from 1847 Cauchy returned to the role of complex numbers as symbolic expressions. Here he advocated a new way of avoiding $\sqrt{-1}$, namely by "replacing the theory of imaginary expressions by the theory of quantities which I call geometric" [Cauchy 1847_3, (176)]. He stated that he had come to this solution after "mature reflections" and that he had been inspired by a paper by Saint Venant published in *Comptes rendus* in 1845. Cauchy's "geometric quantities" are exactly the same as Wessel's directed line segments, and interestingly enough he also treated them as Wessel had done. Contrary to Argand, Cauchy defined sum and product geometrically without any attempt at justifying his definition. In his choice of notation Cauchy might have been inspired by Français, because he let r_p represent a "geometric quantity" when this has the length r and forms the angle p with a given, fixed line. He followed up his first paper on "geometric quantities" with many more in which he reformulated parts of his complex function theory [Bottazzini 1986, 169]. In the second paper he set $i = 1_{\frac{\pi}{2}}$ and then got his quantities written on the form $x{+}iy$ [Cauchy 1847_4].

By 1847 Cauchy eventually accepted the geometric representations of complex numbers and of their products. It is remarkable that by then he had independently of a geometric understanding made most of his important contributions to complex function theory – or at least without admitting a geometric interpretation.

3.5 Hamilton and complex numbers

Cauchy's early treatment of complex numbers corresponds to some extent to considering them as pairs, or as he called them, couples of real numbers. In a wish to give the complex numbers a *raison d'être* Hamilton explicitly suggested this interpretation [Hamilton 1837, (76-84)]. In a later presentation of his theory he referred to Cauchy's *Cours d'analyse*, but claimed that he had developed his ideas independently of this work [Hamilton 1853, (123)]. In these later comments he also stated that he had wished to give square roots of negatives a meaning

without introducing considerations so expressly geometrical, *as those which involve the conception of an* angle. [Hamilton 1853, (117); emphasis original]

His introduction of the operations for pairs of real numbers consists of a mixture of intuition, postulates and rigorous deductions [Gericke 1970, 83-85]. In an abbreviated and slightly modernized version his ideas can be described as follows. He naturally defined the sum by

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2).$$

He knew, of course, that in order to obtain complex numbers the rule for multiplication should be

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1), \tag{36}$$

but he did not want to take this definition out of the blue [Hamilton 1837, (82)]. He explicitly formulated some rules and tacitly assumed others, so that at his disposal were the distributive law, the associative law, the commutative law, and the following results

$$\begin{aligned}
&(a,0) \cdot (1,0) = (a,0),\\
&(a,0) \cdot (0,1) = (0,a),\\
&(a,0) \cdot (b_1, b_2) = (a,0) \cdot (b_1,0) + (a,0) \cdot (0, b_2) = (ab_1,0) + (0, ab_2) = (ab_1, ab_2).
\end{aligned} \tag{37}$$

He then found

$$\begin{aligned}
&(a_1, a_2) \cdot (b_1, b_2) = (a_1,0) \cdot (b_1, b_2) + (0, a_2) \cdot [(b_1,0) + (0, b_2)] =\\
&(a_1b_1, a_1b_2) + (0, a_2b_1) + (0, a_2) \cdot (0, b_2) =\\
&(a_1b_1, a_1b_2 + a_2b_1) + (0, a_2) \cdot (0, b_2).
\end{aligned}$$

So Hamilton's remaining problem was to define $(0, a_2) \cdot (0, b_2)$, for which according to (37) it was sufficient to define $(0,1)$. $(0,1)$. He set the latter product to be (γ_1, γ_2) and wanted to determine γ_1 and γ_2 . For this he looked at the relation

$$(a_1,a_2) \cdot (b_1,b_2) = (c_1,c_2), \tag{38}$$

in which $(b_1, b_2) \neq (0,0)$ and (c_1, c_2) are supposed to be given, and then required that the relation (38) should define (a_1, a_2) uniquely. He found that a necessary and sufficient condition for this is that

$$b_1(b_1 + b_2\gamma_2) - b_2^2\gamma_1 \neq 0,$$

or

$$(b_1 + \tfrac{1}{2}b_2\gamma_2)^2 - (\gamma_1 + \tfrac{1}{4}\gamma_2^2)\,b_2^2 \neq 0.$$

Since this relation has to apply for all (b_1, b_2) Hamilton concluded that a necessary and sufficient condition is that

$$(\gamma_1 + \tfrac{1}{4}\gamma_2^2) < 0.$$

This can, Hamilton wrote, simply be obtained by setting

$$\gamma_1 = -1, \qquad \gamma_2 = 0. \tag{39}$$

Hereby he obtained the multiplication rule (36), but he had not shown that the choice $(\gamma_1, \gamma_2) = (0,-1)$ is the only possible one. Had he allowed geometrical arguments, as for instance that the modulus (length) of a product of two complex numbers should be equal to the product of the moduli, he could easily have concluded that (39) is the only solution. By insisting on an entirely algebraic treatment Hamilton ran into the same problem as Servois had earlier pointed to in the geometrical approach – he could not prove uniqueness.

Having introduced complex numbers as pairs of real numbers, Hamilton spent much time in finding an algebra for triples. Like Wessel and others, he had to give up. By going on to a fourth dimension he found a satisfactory solution – in which he, however, had to accept that multiplication is not commutative – and thereby created the quaternions. Some of Wessel's first commentators compared his treatment of triples with Hamilton's algebra for quaternions. Thus T.N.Thiele remarked that if Wessel had considered a rotation around the real axis, he could have been led to introduce a third imaginary quantity and would have come to the same fundamental relations as Hamilton [Thiele in Wessel 1897, XIII]. He also remarked, as Juel had done earlier, that Wessel's rotation product (21) can be written in the following way

$$(x + \eta y + \varepsilon z)\,,,\,(\cos v + \varepsilon \sin v) = (\cos\tfrac{v}{2} + \varepsilon \sin\tfrac{v}{2}) \cdot (x + \eta y + \varepsilon z) \cdot (\cos\tfrac{v}{2} + \varepsilon \sin\tfrac{v}{2})$$

where $\cdot$ is the quaternion product, and (22) similarly [Juel 1895, 35; Thiele in Wessel 1897, XIV]. Such observations are mathematically interesting; historiographically, however, they are rather irrelevant because there are no indications that Wessel thought of including the fourth dimension.

Concluding remarks

The material presented in this chapter will hopefully have shown that among the early treatises on representing complex numbers by line segments – or vice versa – Wessel's was by far the most elegant. Moreover, I hope that it has become clear that when Wessel, and others, thought that the imaginary could be given a meaning by geometry, the influential mathematicians – perhaps Gauss excluded – were of the opinion that analysis, including complex numbers, should be arithmetized. A geometrical interpretation of the complex numbers and their product could at most be considered an illustration, not a foundation.[8] In 1829 Warren reported some of the reactions he had received to his book from the previous year, one of them being

it is improper to introduce geometric considerations into questions purely algebraic; and that the geometric representation, if any exists, can only be analogical, and not a true algebraic representation of the roots. [Warren 1829_1, 250].

So the timing of Wessel's *Attempt* was so to speak bad and explains why his elegant approach had no impact.

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8 Also Peacock criticized the geometrical interpretation of the complex numbers [Peacock 1834, 229-233].

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Om

Directionens analytiske Betegning,

et Forsøg,

anvendt fornemmelig

til

plane og sphæriske Polygoners Opløsning.

Af

Caspar Wessel, Landmaaler.

Nærværende Forsøg angaaer det Spørgsmaal, hvordan Directionen analytisk bør betegnes, eller hvordan rette Linier burde udtrykkes, naar af een eneste Ligning mellem een ubekiendt og andre givne Linier skulde kunne findes et Udtryk, der forestillede baade den ubekiendtes Længde og dens Direction.

For nogenledes at kunne besvare dette Spørgsmaal, lægger jeg til Grundvold to Sætninger, der synes mig unegtelige. Den første er: at den Directionens Forandring, der ved algebraiske Operationer kan frembringes, ogsaa bør ved deres Tegn at forestilles. Den anden: at Direction er ingen Gienstand for Algebra, uden for saavidt den ved algebraiske Operationer kan forandres. Men da den ved disse ei kan forandres (i det mindste efter den sædvanlige Forklaring), uden til den modsatte, eller fra positiv til privativ, og omvendt: saa skulde disse to Directioner alene kunne betegnes paa den bekiendte Maade, og i Hensigt til de øvrige Problemet være uopløseligt. Dette er vel

The first page of Wessel's paper as it appeared in Det Kongelige Danske Videnskabernes Selskabs Skrifter, Nye Samling, V, *1799.*

On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons.

By
Caspar Wessel, Surveyor.

Translated from

**Om Direktionens analytiske Betegning,
et Forsøg anvendt fornemmelig til plane og sphæriske Polygoners Opløsning**

by
Flemming Damhus

Translator's note: This translation is an attempt to represent Wessel's work as truly as possible. This implies that the sentence structure often is rather complicated and involved. To a large extent Wessel's punctuation and division into paragraphs is preserved. Thus the semicolon is often kept, where a period would be more natural today. In Wessel's work the trigonometric functions cosine, sine, tangent, and cotangent are written cos., sin., tan., and cot.. In this presentation the period is omitted. In § 34 Wessel even abbreviates cosine to c, and sine to Gothic type s. Here I also write cos and sin without the period. Similarly, the multiplication dot, which does not seem to appear according to any particular rule, is often left out. Finally, please note that in §§ 15 and 18 Wessel uses $\pi = 360°$.

On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons.

By
Caspar Wessel, Surveyor.

The present attempt deals with the question of how to represent the direction analytically, or, how one ought to express straight lines, if from a single equation in one unknown line and some given lines one is to be able to find an expression representing both the length and the direction of the unknown line.

In order to answer this question reasonably well, I base the argument on two propositions, which I consider indisputable. The first one is: the change in direction produced by algebraic operations should be represented by their symbols. The second one is: the direction is only a subject of algebra to the extent that it can be changed by algebraic operations. But since the direction cannot be changed by these (at least according to the usual explanation), except to the opposite, i.e. from positive to negative, and conversely, then only these two directions should be denoted in the well-known way, and for the other directions the problem should be unsolvable. This is probably also the reason why nobody else has treated it[1]. Without doubt it has been considered impermissible to change anything in the explanation of the operations once agreed upon. And there will be no objection to this as long as the explanation is applied to quantities in general; but probably it should not be called impermissible in certain cases, where the nature of the quantities seems to invite a more precise definition of the operations and to allow a useful application of it, because in going from arithmetic to geometric analysis, or from operations with abstract numbers to those with straight lines, one meets quantities that permit the same, as well as many more relations than the numbers can have to each other; so, if one takes the operations in a wider sense, and does not, as before, restrict them to be used only on segments of the same or opposite direction, but extends their formerly restricted concept somewhat, so that it becomes applicable not only in the former cases, but also in infinitely many more cases; I say, that, if one takes this liberty and yet does not violate the usual rules of operation, then one will not contradict the basic theory of numbers; but one will carry it further, adapting it to the nature of the quantities and observing the rule of method that requires making a difficult theory understandable little by little. Thus, it is not an unreasonable requirement to take the operations used in geometry in a broader sense than the one given to them in arithmetic; one may also easily admit that in this way it will be possible to produce infinitely many changes in the directions of the lines. But hereby one obtains the result (as will be proved later on) not only that all impossible operations can be avoided and light will be shed upon the paradoxical statement that the possible may sometimes be sought by impossible means, but also that the direc-

1 An exception might be Master Gilbert in Halle, whose paper on *Calculus Situs* may contain some explanation on this subject

tion of all lines in the same plane may be expressed as analytically as their length, and this without burdening the memory by new symbols or rules. Since it seems beyond doubt that the general validity of geometrical theorems is often seen with greater ease, when the direction is represented analytically and subjected to the algebraic rules of operation, than when represented pictorially, and that only in a few cases, then it seems not only permissible, but even useful to apply operations that are extended to other lines than those of the same and of the opposite direction. On account of this I attempt

I. First to determine the rules governing such operations;
II. Next, by means of a couple of examples, to show their application to lines in the same plane;
III. Thereafter, to determine the direction of lines in different planes by a new method of operation that is not algebraic;
IV. By means of this to solve plane and spherical polygons in general;
V. Lastly, to derive in the same manner the known formulas in spherical trigonometry.

This is the main content of this treatise. The occasion for its composition was that I sought a method whereby the impossible operations could be avoided, and once it was found, I used it to convince myself of the generality of some known formulas. Mr. Tetens, Councillor of State, had the patience to read these first investigations, and I owe to the encouragement, advice, and guidance of this distinguished scholar that the present paper now appears less incomplete and has been deemed worthy of publication in the collection of papers of The Royal Danish Academy of Sciences and Letters.

I.

How to Form New Straight Lines from Given Ones by the Algebraic Operations, and especially, to Determine Which Directions and Signs They Should Have.

There are homogeneous quantities which will only increase or decrease one another, like increments or decrements, when associated to the same subject.

There are other quantities that in the same situations may change one another in numerous other ways. Straight lines are of this kind.

Thus the distance of a point from a plane can change in numerous ways, when the point describes a more or less inclined straight line outside the plane.

If this line is *perpendicular*, i.e. the path of the point makes a right angle with the axis of the plane, then the point remains in a parallel to the plane, and its motion has no influence on its distance from the plane.

If the described line is *indirect*, i.e. it makes a skew angle with the axis of the plane, then it contributes a smaller segment than its own length to the extension or reduction of the distance, and may increase or decrease the distance in infinitely many ways.

If the line is *direct*, i.e. collinear with the distance, it adds or subtracts to the distance by its full length, in the first case it is positive, otherwise negative.

Thus, all the straight lines described by a point are, with respect to their effect on the distance of the point from a plane outside the lines either *direct, indirect,* or *perpendicular*[2], depending on whether they add or subtract all of, part of, or none of their own length.

Since a quantity is called absolute if given by itself and not relative to another quantity, then in the previous definitions the distance may be called the absolute line, and the contribution of the relative to the extension or reduction of the absolute may be called the effect of the relative.

There are more quantities than straight lines that may admit of the mentioned relations. So, it might be useful to explain such relations in general, and include their general concept in the explanation of the operations; but since both advice from experts, the content of this paper, and the plainness of the presentation require that I do not burden the reader with such abstract concepts, I give only the geometric explanations, and hence I say

§ 1.

Two straight lines are added together, when one joins them together so that one begins where the other one ends, and next one draws a straight line from the first to the last point of the joined lines, and takes this to be their sum.

If, for example, a point moves forward 3 feet and then backwards 2 feet, then the sum of these two paths is not the first 3 and the last 2 feet together, but the sum is one foot forward, because this path has the same effect as the other two paths.

Similarly, when one side of a triangle extends from a to b, and the other from b to c, then the third from a to c must be called the sum, and should be denoted $ab + bc$, so that ac and $ab + bc$ have the same meaning, or $ac = ab + bc = -ba + bc$, if ba is the opposite of ab. If the added lines are direct then the definition agrees completely with the usual one. If they are not direct it does not disagree with the analogy to call a straight line the sum of two other joined lines, in so far as it has the same effect as the other two. The meaning I have given to the sign + is not so unusual either; for example, in the expression

$$ab + \frac{ba}{2} = \frac{1}{2}ab$$

the term $\frac{ba}{2}$ is no part of the sum. Thus one may write $ab + bc = ac$ without thinking of bc as part of ac; $ab + bc$ is only the sign representing ac.

§ 2.

When more than two straight lines are to be added the same rule is followed; they are joined together so that the last point of the first is joined to the first one of the second, the last point of the second to the first point of the third, etc. Finally, a straight line is drawn from the beginning of the first to the end of the last, and this is called the sum of all of them.

Which line one chooses to be the first, and which the second, the third, etc. is immaterial, because wherever a point describes a straight line within three mutually or-

2 *Indifferent* would be more suitable if it did not jar on unaccustomed ears.

thogonal planes, the segment has the same effect on the distance of the point from each of the three planes; consequently, one of several added lines contributes the same to the position of the last point of the sum, whether it is the first, the last, or has any other number among the addends; thus the order in the addition of straight lines is immaterial, and the sum always remains the same, because the initial point is assumed to be given, and the last point always attains the same position.

Hence, in this case one may also denote the sum by inserting the sign + between the lines to be added. For instance, when in a quadrilateral the first side is drawn from a to b, the second from b to c, the third from c to d, and the fourth from a to d: then one can write $ad = ab + bc + cd$.

§ 3.

If the sum of several lengths, widths, and heights = 0, then the sum of the lengths, that of the widths, and that of the heights, each sum separately = 0.

§ 4.

The product of two straight lines should in every respect be formed from the one factor, in the same way as the other factor is formed from the positive or absolute unit line that is set = 1, that is:

First, the factors must have such directions, that they can both be included in the same plane as the positive unit.

Next, concerning the length of the product, it must be to the one factor as the other is to the unit; and

Finally, if the positive unit, the factors, and the product are given a common initial point, then the product, with respect to the direction must lie in the plane of the unit and the factors, and the product must deviate as many degrees from the one factor, and to the same side, as the other factor deviates from the unit, so that the directional angle of the product or its deviation from the positive unit is the sum of the directional angles of the factors.

§ 5.

Let +1 denote the positive, rectilinear unit, and $+\varepsilon$ a certain different unit, perpendicular to the positive unit, and with the same initial point; then the directional angle of +1 is 0, of −1 it is 180°, of $+\varepsilon$ it is 90°, and of $-\varepsilon$ it is −90° or 270°; and according to the rule that the directional angle of the product is the sum of those of the two factors, one gets

$$(+1)\cdot(+1) = +1,\ (+1)\cdot(-1) = -1,\ (-1)\cdot(-1) = +1,\ (+1)\cdot(+\varepsilon) = +\varepsilon,\ (+1)\cdot(-\varepsilon) = -\varepsilon,$$

$$(-1)\cdot(+\varepsilon) = -\varepsilon,\ (-1)\cdot(-\varepsilon) = +\varepsilon,\ (+\varepsilon)\cdot(+\varepsilon) = -1,\ (+\varepsilon)\cdot(-\varepsilon) = +1,\ (-\varepsilon)\cdot(-\varepsilon) = -1.$$

From this it follows that ε becomes $= \sqrt{-1}$, and the deviation of the product is determined so that not a single one of the usual rules of operation is violated.

§ 6.

Cosine to a circular arc that begins at the endpoint of its radius +1, is the piece of +1, or the opposite radius, starting from the centre and ending at the perpendicular from the last point of the arc. Sine to the same arc is drawn perpendicular to cosine from its last point to the last point of the arc.

From § 5 it follows that sine to a right angle is $= \sqrt{-1}$. Let us put $\sqrt{-1} = \varepsilon$; let v denote any angle, and $\sin v$ a straight line of length sine of the angle v, but positive when the measure of the angle ends in the first semi-circular circumference, and negative when it ends in the second semi-circular circumference; then it follows from §§ 4 and 5, that $\varepsilon \sin v$ expresses the sine of the angle v in direction as well as in length.

§ 7.

The radius that starts at the centre and deviates the angle v from the absolute or positive unit is, according to §§ 1 and 6, equal to $\cos v + \varepsilon \sin v$. But the product of two factors, of which one deviates the angle v from the unit, and the other the angle u from the unit, must itself deviate the angle $v + u$ from the unit, according to § 4. So, when the segment $\cos v + \varepsilon \sin v$ is multiplied by the segment $\cos u + \varepsilon \sin u$, then the product becomes a straight line, whose directional angle is $v + u$. Therefore, following §§ 1 and 6, the product may be denoted $\cos(v + u) + \varepsilon \sin(v + u)$.

§ 8.

This product $(\cos v + \varepsilon \sin v)(\cos u + \varepsilon \sin u)$ or $\cos(v + u) + \varepsilon \sin(v + u)$ may be expressed in still another way by adding in one sum the partial products that results when each of the added lines making up the one factor is multiplied by each of the lines, whose sum makes up the second factor. Thus

$$(\cos v + \varepsilon \sin v)(\cos u + \varepsilon \sin u) =$$

$$\cos v \cdot \cos u - \sin v \cdot \sin u + \varepsilon(\cos v \cdot \sin u + \cos u \cdot \sin v)$$

which follows from the well-known trigonometric formulas

$$\cos(v + u) = \cos v \cdot \cos u - \sin v \cdot \sin u, \text{ and } \sin(v + u) = \cos v \cdot \sin u + \cos u \cdot \sin v.$$

These two formulas may be proved precisely and without much trouble for all the cases whether both of the angles v and u, or just one of them, are positive, negative, greater than or less than a right angle. Consequently, the theorems that one derives from them become valid in general.

§ 9.

According to § 7 $\cos v + \varepsilon \sin v$ is a radius in a circle and of length = 1; its deviation from $\cos 0°$ is the angle v; from this it follows that $r \cos v + r\varepsilon \sin v$ denotes a straight line whose length is r, and whose directional angle is $= v$, for when the smaller sides of a right triangle are increased r times, then so is the hypotenuse, and the angles are unchanged; but the sum of the smaller sides is, according to § 1, equal to the hypotenuse, that is $r\cos v + r\varepsilon\sin v = r(\cos v + \varepsilon\sin v)$. So, this is a general expression for a straight line coplanar with the lines $\cos 0°$ and $\varepsilon\sin 90°$, of length r and deviating from $\cos 0°$ by v degrees.

§ 10.

Let a, b, c, d denote direct lines of any lengths whatsoever, positive or negative, and assume that the two indirect lines $a+\varepsilon b$ and $c+\varepsilon d$ are coplanar with the absolute unit; then their product can be found, even when their deviation from the absolute unit is unknown; all one needs to do is to multiply each of the added lines in the one sum with each of those whose sum is the second factor; adding up all these products one gets the required product, its length as well as its direction, namely
$(a+\varepsilon b)(c+\varepsilon d) = ac - bd + \varepsilon(ad+bc)$.

P r o o f : Let the line $a+\varepsilon b$ have length A and deviate v degrees from the absolute unit, and let the line $c+\varepsilon d$ have length $= C$ and deviation $= u$; then according to § 9:

$$a+\varepsilon b = A\cos v + A\varepsilon\sin v \quad\text{and}\quad c+\varepsilon d = C\cos u + C\varepsilon\sin u, \quad\text{so that}$$

$$a = A\cos v,\quad b = A\sin v,\quad c = C\cos u,\quad d = C\sin u \quad (\S\ 3),$$

but according to § 4 $(a+\varepsilon b)(c+\varepsilon d) = AC[\cos(v+u) + \varepsilon\sin(v+u)] =$
$AC[\cos v\ \cos u - \sin v\ \sin u + \varepsilon(\cos v\ \sin u + \cos u\ \sin v)]$, § 8. Consequently, by replacing $AC\cos v\ \cos u$ by ac, and $AC\sin v\ \sin u$ by bd, etc., we get what was to be proved.

From this it follows that even if the added lines of the sum are not all direct there is no need for an exception to the known rule, on which the theory of equations and the theory of integral functions and their *Divisores simplices* are based, namely, when two sums are to be multiplied, then each of the added quantities in the one sum must be multiplied by every term in the second sum. Therefore one may be assured that when an equation is about straight lines and its root is of the form $a+\varepsilon b$, then one is dealing with an indirect line. But if one wanted to multiply two lines that are not both in the same plane as the absolute unit, then the above mentioned rule would have to be abandoned. This is the reason why I omit multiplication of such lines. A different method of denoting their changed direction will appear in the following, §§ 24 – 35.

§ 11.

The quotient multiplied by the divisor must be equal to the dividend. Thus it need not be proved that these lines must lie in the same plane with the absolute unit, because it follows immediately from the definition in § 4. Similarly, it is easily seen that the quotient must deviate from the absolute unit by the angle $v-u$, if the dividend deviates by the angle v, and the divisor by the angle u, both from the unit.

Consider for instance the case, when $A(\cos v + \varepsilon\sin v)$ is to be divided by
$B(\cos u + \varepsilon\sin u)$; then the quotient is $= \frac{A}{B}[\cos(v-u) + \varepsilon\sin(v-u)]$, because
$\frac{A}{B}[\cos(v-u) + \varepsilon\sin(v-u)]\cdot B(\cos u + \varepsilon\sin u) = A(\cos v + \varepsilon\sin v)$ according to § 7. This is

the case since $\frac{A}{B}[\cos(v-u)+\varepsilon\sin(v-u)]$ multiplied by the divisor $B(\cos u+\varepsilon\sin u)$ is equal to the dividend $A(\cos v+\varepsilon\sin v)$, and hence the quotient we are looking for is $\frac{A}{B}[\cos(v-u)+\varepsilon\sin(v-u)]$.

§ 12.

If a, b, c, and d are direct lines, and the indirect $a+\varepsilon b$ and $c+\varepsilon d$ are coplanar with the absolute unit, then $\frac{1}{c+\varepsilon d}=\frac{c-\varepsilon d}{c^2+d^2}$, and the quotient
$\frac{a+\varepsilon b}{c+\varepsilon d}=(a+\varepsilon b)\cdot\frac{1}{c+\varepsilon d}=(a+\varepsilon b)\cdot\frac{c-\varepsilon d}{c^2+d^2}=[ac+bd+\varepsilon(bc-ad)]:(c^2+d^2)$,
because according to § 9 one may substitute $a+\varepsilon b=A(\cos v+\varepsilon\sin v)$ and
$c+\varepsilon d=C(\cos u+\varepsilon\sin u)$, and hence $c-\varepsilon d=C(\cos u-\varepsilon\sin u)$ according to § 3, and because $(c+\varepsilon d)(c-\varepsilon d)$ is $=c^2+d^2=C^2$ (§ 10), it follows $\frac{c-\varepsilon d}{c^2+d^2}=\frac{1}{C}(\cos u-\varepsilon\sin u)$, § 10, or
$\frac{c-\varepsilon d}{c^2+d^2}=\frac{1}{C}(\cos(-u)+\varepsilon\sin(-u))=\frac{1}{c+\varepsilon d}$, § 11; when this is multiplied by
$a+\varepsilon b=A(\cos v+\varepsilon\sin v)$ one gets $(a+\varepsilon b)\cdot\frac{c-\varepsilon d}{c^2+d^2}=\frac{A}{C}(\cos(v-u)+\varepsilon\sin(v-u))=\frac{a+\varepsilon b}{c+\varepsilon d}$, § 11.

Indirect quantities of this kind have this in common with the direct ones, that when the dividend is a sum of several quantities, then each one of these divided by the divisor gives several quotients, whose sum is the quotient wanted.

§ 13.

If m is an integer, then $\cos\frac{v}{m}+\varepsilon\sin\frac{v}{m}$ multiplied by itself m times produces the power $\cos v+\varepsilon\sin v$, (§ 7); hence $(\cos v+\varepsilon\sin v)^{\frac{1}{m}}=\cos\frac{v}{m}+\varepsilon\sin\frac{v}{m}$; but from § 11
$\cos(-\frac{v}{m})+\varepsilon\sin(-\frac{v}{m})=\frac{1}{\cos\frac{v}{m}+\varepsilon\sin\frac{v}{m}}=\frac{1}{(\cos v+\varepsilon\sin v)^{\frac{1}{m}}}=(\cos v+\varepsilon\sin v)^{-\frac{1}{m}}$.

So, whether m is positive or negative we always have
$\cos\frac{v}{m}+\varepsilon\sin\frac{v}{m}=(\cos v+\varepsilon\sin v)^{\frac{1}{m}}$ and therefore, when m and n are both integers
$(\cos v+\varepsilon\sin v)^{\frac{n}{m}}=\cos\frac{n}{m}v+\varepsilon\sin\frac{n}{m}v$.

From this one may find the value of expressions like $\sqrt[n]{(b+c\sqrt{-1})}$ or
$\sqrt[m]{[a+\sqrt[n]{(b+c\sqrt{-1})}]}$; for instance $\sqrt[3]{(4\sqrt{3}+4\sqrt{-1})}$ designates a straight line, whose length is $=2$, and whose angle with the absolute unit is $10°$.

§ 14.

When two angles have equal sines and equal cosines, then their difference is either 0 or ± 4 right angles, or a multiple of ± 4 right angles; and conversely, when the difference between two angles is 0 or ± 4 right angles, taken once or several times, then their sines as well as their cosines are equal.

§ 15.

If m is an integer, and $\pi = 360^\circ$, then $(\cos v + \varepsilon \cdot \sin v)^{\frac{1}{m}}$ attains only the following m different values $\cos\frac{v}{m} + \varepsilon \cdot \sin\frac{v}{m}$, $\cos\frac{\pi+v}{m} + \varepsilon \cdot \sin\frac{\pi+v}{m}$, $\cos\frac{2\pi+v}{m} + \varepsilon \cdot \sin\frac{2\pi+v}{m}$,...., $\cos\frac{(m-1)\pi+v}{m} + \varepsilon \cdot \sin\frac{(m-1)\pi+v}{m}$, because the numbers by which π is multiplied in the preceding sequence are in an arithmetic progression 1, 2, 3, 4,...., $m-1$. Therefore the sum of any two $= m$, when the one is as far from 1 as the other is from $m-1$, and if their number is odd, then twice the middle one $= m$; hence, when one adds $\frac{(m-n)\pi+v}{m}$ to $\frac{(m-u)\pi+v}{m}$, and the former in the sequence is as far from $\frac{\pi+v}{m}$ as $\frac{(m-n)\pi+v}{m}$ is from $\frac{(m-1)\pi+v}{m}$, then the sum $= \frac{2m-u-n}{m}\pi + \frac{2v}{m} = \pi + \frac{2v}{m}$. But to add $\frac{(m-n)\pi}{m}$ is the same as subtracting $\frac{(m-n)(-\pi)}{m}$; and since the difference is π, then $\frac{(m-n)(-\pi)+v}{m}$ has the same cosine and sine as $\frac{(m-u)\pi+v}{m}$, according to § 14; similarly, $\frac{(m-u)(-\pi)+v}{m}$ and $\frac{(m-n)\pi+v}{m}$ have the same cosine and sine; thus $-\pi$ does not give any other values than $+\pi$. But that none of these are equal follows from the fact that the difference between two of the angles in the sequence is always less than π, and never $= 0$. Neither does one get more values by continuing the sequence, because then one gets the angles $\pi + \frac{v}{m}$, $\pi + \frac{\pi+v}{m}$, $\pi + \frac{2\pi+v}{m}$, etc., so according to § 14 the values of their cosine and sine are the same as before. If the angles were to fall outside the sequence, then π was not multiplied by an integer in the numerator, and the angles taken m times could not produce an angle which subtracted from v gave 0, or $\pm\pi$, or a multiple of $\pm\pi$; hence, neither could the m'th power of such an angle have cosine and sine $= \cos v + \varepsilon \sin v$.

§ 16.

Without knowing the angle that the indirect line $1 + x$ makes with the absolute, one finds, when the length of x is less than 1, the power $(1+x)^m = 1 + \frac{mx}{1} + \frac{m}{1} \cdot \frac{m-1}{2} x^2 +$ etc. and if this series is rearranged after the powers of m, it keeps its value and is changed into $1 + \frac{ml}{1} + \frac{m^2l^2}{1\cdot 2} + \frac{m^3l^3}{1\cdot 2\cdot 3} +$ etc., where $l = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$ etc., and is a sum of a direct line a and a perpendicular $b\sqrt{-1}$, then b is the least measure of the angle that $1+x$ makes with $+1$, and if one puts $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} +$ etc. $= e$, then $(1+x)^m$ or $1 + \frac{ml}{1} + \frac{m^2l^2}{1\cdot 2} + \frac{m^3l^3}{1\cdot 2\cdot 3} +$ etc. is denoted $e^{ma + mb\sqrt{-1}}$, that means $(1+x)^m$ has the length e^{ma}, and a directional angle whose measure is mb, m being assumed to be positive or negative. Thus the direction of lines in the same plane may be expressed in still another way, namely by means of the natural logarithms. I shall present complete proof of these theorems at some other time, if so permitted. Now that I have accounted for how to find the sum, the product, the quotient, and the power of straight lines, I shall only give a couple of examples of applications of the method.

II.

Proof of Cotes's Theorem.

§ 17.

I assume it to be known that when the equation $x^n + px^{n-1} + qx^{n-2} + \cdots + sx + t = 0$ has the n roots $a, b, c, \ldots g$; then the entire function $z^n + pz^{n-1} + qz^{n-2} + \cdots + sz + t$ has the *Divisores simplices* $z - a$, $z - b$, $z - c, \ldots$, $z - g$, and it is the product of all of them.

§ 18.

The theorem invented by Cotes is the following:
Let the n arcs ab, bc, cd, de, ea (Fig. 1) all be of size $\frac{360^\circ}{n} = \frac{\pi}{n}$, and call the radius $oa = r$, $ao = -r$, $op = z$, $po = -z$, and let p be the last point of the lines ap, bp, cp, dp, ep: then

$$ap \cdot bp \cdot cp \cdot dp \cdot ep = z^n - r^n;$$

For from § 1 and § 9 it follows that

$$ap = z - r$$
$$bp = z - r(\cos\tfrac{\pi}{n} + \varepsilon \sin\tfrac{\pi}{n})$$
$$cp = z - r(\cos\tfrac{2\pi}{n} + \varepsilon \sin\tfrac{2\pi}{n})$$
$$dp = z - r(\cos\tfrac{3\pi}{n} + \varepsilon \sin\tfrac{3\pi}{n})$$
$$ep = z - r(\cos\tfrac{4\pi}{n} + \varepsilon \sin\tfrac{4\pi}{n}), \text{ or}$$
$$ep = z - r(\cos\frac{(n-1)\pi}{n} + \varepsilon \sin\frac{(n-1)\pi}{n}),$$

and from §15 it follows that the roots of the equation $x^n - r^n = 0$ are

$$r,\ r(\cos\frac{\pi}{n} + \varepsilon \sin\frac{\pi}{n}),\ r(\cos\frac{2\pi}{n} + \varepsilon \sin\frac{2\pi}{n}), \cdots, r(\cos\frac{(n-1)\pi}{n} + \varepsilon \sin\frac{(n-1)\pi}{n});$$

thus, according to § 17 we get $z^n - r^n = ap \cdot bp \cdot cp \cdot dp \cdot ep$. Consequently the length of $z^n - r^n$ is the product of the lengths of ap, bp, cp, dp, ep, § 4.

On the Solution of Plane Polygons.

§ 19.

I list without proof the following well-known formulas from trigonometry:

a) $\sin(a+b) = \sin a \cdot \cos b + \sin b \cdot \cos a$

b) $\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$

c) $\sin 2a = 2 \sin a \cdot \cos a$

d) $1 + \cos 2a = 2\cos^2 a$

e) $1-\cos 2a = 2\sin^2 a$

f) $\tan a = \frac{\sin 2a}{1+\cos 2a}$

g) $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$

h) $\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$

i) $\cos b - \cos a = 2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$

k) $\frac{\sin a+\sin b}{\cos a+\cos b} = \tan\frac{a+b}{2}$

l) $\frac{\sin a-\sin b}{\cos a+\cos b} = \tan\frac{a-b}{2}$

§ 20.

In solving polygons it may also be useful to remember that when a problem has been brought so far that one has $a = b\cos u + c\sin u$, and u is the only unknown, then one may avoid the quadratic equation, from which $\sin u$ or $\cos u$ can be found by substituting $\frac{b}{c} = \tan\varphi$, or $\frac{b}{c} = \cot\psi$. From this φ or ψ can be found as positive or negative and not greater than 90°. Once φ or ψ is found one seeks u from one of the following equations:

$$\sin(u+\varphi) = \frac{a\sin\varphi}{b} = \frac{a\cos\varphi}{c}, \text{ and } \cos(u-\psi) = \cos(\psi-u) = \frac{a\cos\psi}{b} = \frac{a\sin\psi}{c};$$

for when the terms in the given equation $a = b\cos u + c\sin u$ are divided by c or b, and one substitutes $\frac{b}{c} = \tan\varphi$, or $\frac{b}{c} = \cot\psi$, and $\frac{c}{b} = \cot\varphi$ or $\tan\psi$, then one gets

$\frac{a}{c} = \tan\varphi\cos u + \sin u = \cot\psi\cos u + \sin u$, or

$\frac{a}{b} = \cos u + \cot\varphi\cdot\sin u = \cos u + \tan\psi\cdot\sin u.$

Hence, when the terms in the first of these equations are multiplied by $\cos\varphi$, in the second by $\sin\psi$, in the third by $\sin\varphi$, and in the fourth by $\cos\psi$, one gets:

$\frac{a}{c}\cos\varphi = \sin\varphi\cos u + \cos\varphi\sin u,$

$\frac{a}{c}\sin\psi = \cos\psi\cos u + \sin u\sin\psi$

$\frac{a}{b}\sin\varphi = \sin\varphi\cos u + \cos\varphi\sin u,$

$\frac{a}{b}\cos\psi = \cos\psi\cos u + \sin u\sin\psi;$

thus according to §19, a, b,

$$\frac{a\cos\varphi}{c} = \sin(u+\varphi),\ \frac{a\sin\psi}{c} = \cos(u-\psi) = \cos(\psi-u),$$

and

$$\frac{a\sin\varphi}{b} = \sin(u+\varphi),\ \frac{a\cos\psi}{b} = \cos(u-\psi) = \cos(\psi-u).$$

§ 21.

When in a plane polygon all the angles and all the sides, but three, are given, then the polygon is undetermined. This is clear, if all three unknown sides follow one another; for then one can draw parallels with the one unknown side meeting the other two unknown sides in several points, and so these three sides may have numerous values; but from this it also follows that the three unknown sides may have equally many values even if they do not follow one another; because with respect to their sum the order of the sides is arbitrary (§ 2), and therefore from any polygon one can always construct another, in which the lengths and the directions of the sides are the same, but their order alone is different.

§ 22.

In a polygon *abcd* (Fig. 2) we assume the side *ab* to be absolute, and counted from *a* to *b*, the second *bc* to be counted from *b* to *c*, *cd* from *c* to *d*, and *da* from *d* to *a*. The even numbers *II, IV, VI, VIII* denote the lengths of the sides, the odd *I, III, V, VII* are their deviations from the prolongation of the previous side, counted positive or negative, e.g. positive with the Sun, negative against the Sun.

I', III', V', VII' denote $\cos I + \varepsilon \cdot \sin I$, $\cos III + \varepsilon \cdot \sin III$, $\cos V + \varepsilon \cdot \sin V$, etc.

$I^{-\prime}, III^{-\prime}, V^{-\prime}, VII^{-\prime}$ denote $\cos(-I) + \varepsilon \cdot \sin(-I)$, $\cos III - \varepsilon \cdot \sin III$, $\cos V - \varepsilon \cdot \sin V$, etc.

Assume this and draw from *a* parallels with *bc, cd, da,* and it follows:

I) The first parallel deviates from *ab* by *III* degrees, the second parallel from *ab* by $III+V$ degrees, the third parallel or the prolongation *ae* of the last side *da* deviates from *ab* by $III + V + VII$ or $-I$ degrees. Thus all angles added together = 0, and the angular measure of their sum is 0 or ± 4 right angles, or a multiple of this.

II) $II + IV \cdot III' + VI \cdot III' \cdot V' + VIII \cdot III' \cdot V' \cdot VII' = 0$, because $ab + bc + cd + da = 0$ (§ 2); but $ab = II$, $bc = IV \cdot III'$ (§ 9), $cd = VI[\cos(III + V) + \varepsilon \cdot \sin(III + V)]$, according to number I and § 9, and hence by § 7 $cd = VI \cdot III' \cdot V'$, and likewise it is shown that $da = VIII \cdot III' \cdot V' \cdot VII'$.

III) $II \cdot III' \cdot V' \cdot VII' + IV \cdot V' \cdot VII' + VI \cdot VII' + VIII = 0$; for when the terms in the previous equation II are divided by $III' \cdot V' \cdot VII'$, one gets from § 12 :
$II \cdot III^{-\prime} \cdot V^{-\prime} \cdot VII^{-\prime} + IV \cdot V^{-\prime} \cdot VII^{-\prime} + VI \cdot VII^{-\prime} + VIII = 0$, and since every term but the last in this equation is multiplied by a cosine and a sine, (the first term is for instance $= II[\cos(III + V + VII) - \varepsilon \cdot \sin(III + V + VII)]$); but the sum of all the direct terms is = 0, as well as the sum of the terms multiplied by sine, § 3. Therefore the sum is = 0, even if each sine gets the opposite direction, and when this happens, the expression is transformed into the one to be proved.

IV) $III' + III^{-\prime} = 2\cos III,\ III' \cdot V' + III^{-\prime} \cdot V^{-\prime} = 2\cos(III + V),$

$III' \cdot V^{-\prime} + III^{-\prime} \cdot V' = 2\cos(III - V),\ III' - III^{-\prime} = 2\varepsilon \cdot \sin III,$

$III' \cdot V' - III^{-\prime} \cdot V^{-\prime} = 2\varepsilon \cdot \sin(III + V),\ III' \cdot V^{-\prime} - III^{-\prime} \cdot V' = 2\varepsilon \cdot \sin(III - V),$

$$\frac{(III')^2 - 1}{(III')^2 + 1} = \varepsilon \cdot \tan III = \frac{1 - (III^{-\prime})^2}{1 + (III^{-\prime})^2},$$

$$\frac{(III')^2 + 1}{(III')^2 - 1} = -\varepsilon \cdot \cot III = \frac{1 + (III^{-\prime})^2}{1 - (III^{-\prime})^2}.$$

The truth of these formulas is easily seen by substituting instead of $III', III^{-\prime}, V', V^{-\prime}$ their values $\cos III + \varepsilon \cdot \sin III, \cos III - \varepsilon \cdot \sin III, \cos V + \varepsilon \cdot \sin V$, etc.

§ 23.

Two equations of the type in II and III in the previous section are sufficient for solving any polygon, when only three angles, or two angles and one side, or one angle and two sides are unknown; for in the last case the unknown angle has the same cosine and sine as the opposite of the sum of the rest of the angles (§ 22 no. I); in the other two cases one of the unknown angles will drop out of the equations, when it is denoted by I as in § 22 no. II and no. III: Consequently, the equations contain only two unknowns. So, from the one equation one can find how one unknown is a function of the other; inserting the function into the other equation will liberate it from this unknown, and thereby one finally finds the value of the other unknown.

For example, in the polygon Fig. 2 let I, III, VI be unknowns, and we look for III; then according to § 22 no. II and no. III:

$$II + IV \cdot III' + VI \cdot III' \cdot V' + VIII \cdot III' \cdot V' \cdot VII' = 0 =$$
$$II \cdot III' \cdot V' \cdot VII' + IV \cdot V' \cdot VII' + VI \cdot VII' + VIII.$$

From the first equation one finds

$$-II \cdot III^{-\prime} \cdot V^{-\prime} - IV \cdot V^{-\prime} - VIII \cdot VII' = VI,$$

and when this value of VI is inserted into the second equation and its terms have been divided by VII', one gets

$$II \cdot III' \cdot V' - II \cdot III^{-\prime} \cdot V^{-\prime} + IV \cdot V' - IV \cdot V^{-\prime} + VIII \cdot VII^{-\prime} - VIII \cdot VII' = 0.$$

So according to § 22 no. IV

$$II \cdot \varepsilon \cdot 2\sin(III + V) + IV \cdot \varepsilon \cdot 2\sin V - VIII \cdot \varepsilon \cdot 2\sin VII = 0, \text{ or}$$
$$\sin(III + V) = \frac{VIII \cdot \sin VII - IV \cdot \sin V}{II}.$$

III.

How to Denote the Direction of the Radii of a Sphere.

§ 24.

I assume that two horizontal radii of a sphere are orthogonal and both perpendicular to a third radius in the sphere. One of the horizontal radii I draw from the centre to the left and $= r$; the other horizontal radius goes from the centre and forward and is $= \varepsilon r$; but the vertical one from the centre and upwards I put $= \eta r$, and the opposite ones I put equal $-r$, $-\varepsilon r$, $-\eta r$. By the letter r we denote the length of the radius; the units ε and η are both perpendicular to $+1$, and compared to this η^2 as well as ε^2 must $= -1$, according to § 5.

§ 25.

If one draws a plane through the four radii $r, -r, r\eta, -\eta r$, and another through $r, -r, \varepsilon r, -\varepsilon r$, then these planes are orthogonal and cut the sphere in two great circles, of which I call the one through r and $r\eta$ the vertical circle, and the one through the horizontal radii r and εr the horizon.

The arcs in the vertical circle and its parallels are counted from the point on the left hand, where they are intersected by the horizon, positive upwards, negative downwards. The horizontal arcs are counted from the vertical, positive with the Sun, and negative against the Sun. For instance, when $ow\gamma hpi$ (Fig. 3) denotes the horizon, $kfs\beta$ its parallel, $ok\pi qnv$ the vertical, π and n the poles of the horizon, and p and γ the poles of the vertical: then it is assumed that $co = r$, $ct = -r$, $c\gamma = \varepsilon r$, $cp = -\varepsilon r$, $c\pi = \eta r$, $cn = -\eta r$, $o\gamma = +90°$, $op = -90°$, $o\pi = +90°$, $on = -90°$; and the arcs in the parallel are counted from k, positive to the left, and negative to the right.

§ 26.

Draw from the centre c of the sphere (Fig. 3) a line cd to a point d in the common radius of the horizon and the vertical, and from d another line de, parallel to the axis πn of the horizon, and again draw a third line ef, parallel to the axis $p\gamma$ of the vertical: then these three lines are co-ordinates to the point f, where the last line ef ends. The first cd is the abscissa of the point f, and is denoted by x; it is either in the same direction as the radius $+r$, or it is negative like the radius $-r$. The second and the third lines de and ef are the ordinates of the point f; the second de represents the distance of the point f from the plane of the horizon; it is denoted ηy, because it is parallel to ηr or to $-\eta r$. The third ef is the distance of the point f from the vertical plane, and is denoted by εz, because it is parallel to the radius εr or $-\varepsilon r$. The third $ef (= \varepsilon z)$ makes a right angle with the second $de (= \eta y)$, and this ηy makes a right angle with the first $cd (= x)$.

§ 27.

A radius whose outermost point has the co-ordinates $x, \eta y, \varepsilon z$, I shall denote by the sum $x + \eta y + \varepsilon z$ (§ 2). $x + \eta y$ is multiplied by $a + \eta b$, and $x + \varepsilon z$ by $a + \varepsilon b$ like $c + d\sqrt{-1}$ with $a + b\sqrt{-1}$; because the directional angles of η and of ε are both counted from the same radius +1 (§ 25), and then according to § 5, η^2 as well as ε^2 must $= -1$, and thus the products $(x + \eta y) \cdot (a + \eta b)$ and $(x + \varepsilon z) \cdot (a + \varepsilon b)$ are found according to the rule § 10.

§ 28.

If a point moves forward or backward in a circumference βkfs of a horizontal circle (Fig. 3) by a certain number of degrees fs $(= III)$, and its co-ordinates were $cd (= x')$, $de (= \eta y')$, and $ef (= \varepsilon z')$: then the ordinate $\eta y'$ remains unchanged, because the point keeps the same distance from the horizon; but the abscissa $cd (= ue)$ or x' changes to $ul (= x'')$, and the ordinate $ef (= \varepsilon z')$, changes to $ls (= \varepsilon z'')$, and the sum of the two new co-ordinates $ul + ls (= x''+\varepsilon z'')$ becomes $(x'+\varepsilon z') \cdot (\cos III + \varepsilon \sin III)$; for let the radius uk be named ρ, let the measure of the angle kuf be I, so that the measure of the angle $kus = I + III$: then according to § 9 $ue + ef (= x'+\varepsilon z') = \rho \cdot (\cos I + \varepsilon \sin I)$; similarly,
$ul + ls (= x''+\varepsilon z'') = \rho \cdot [\cos(I + III) + \varepsilon \sin(I + III)] =$
$\rho \cdot (\cos I + \varepsilon \sin I) \cdot (\cos III + \varepsilon \sin III)$, § 8; and hence, when $\rho \cdot (\cos I + \varepsilon \sin I)$ is replaced by $x'+\varepsilon z'$, one gets $x''+\varepsilon z'' = (x'+\varepsilon z') \cdot (\cos III + \varepsilon \sin III)$.

§ 29.

When a point describes an arc of the vertical, or its parallel, of a certain number II degrees, then the sum of its previous co-ordinates x' and $\eta y'$ are changed into $(x'+\eta y') \cdot (\cos II + \eta \cdot \sin II)$, but the third co-ordinate $\varepsilon z'$ remains unchanged, because the distance from the vertical cannot change as long as the point stays in the parallel of the vertical. According to § 27, the rest of the proof is just like the preceding one.

§ 30.

If the radius of the sphere has the co-ordinates $x, \eta y$, and εz, then according to § 27 we denote this radius by $x + \eta y + \varepsilon z$. But if its direction is changed so that its point farthest out is moved I horizontal degrees then it becomes $= \eta y + (x + \varepsilon z) \cdot (\cos I + \varepsilon \sin I) =$
$\eta y + x \cos I - z \sin I + \varepsilon x \sin I + \varepsilon z \cos I$ (§ 28) and is denoted by
$(x + \eta y + \varepsilon z),,(\cos I + \varepsilon \sin I)$.

§ 31.

On the other hand, if the direction of a radius $x + \eta y + \varepsilon z$ is changed by moving its last point II vertical degrees, then it becomes $= \varepsilon z + (x + \eta y) \cdot (\cos II + \eta \sin II) =$
$\varepsilon z + x \cos II - y \sin II + \eta x \sin II + \eta y \cos II$ (§ 29) and is denoted by
$(x + \eta y + \varepsilon z),,(\cos II + \eta \sin II)$.

§ 32.

From this it follows that $(x+\eta y+\varepsilon z)\text{,,}(\cos I+\varepsilon\sin I)\text{,,}(\cos III+\varepsilon\sin III)$ is the same as $(x+\eta y+\varepsilon z)\text{,,}(\cos(I+III)+\varepsilon\sin(I+III))$; for either the last point of the radius $x+\eta y+\varepsilon z$ first moves forward I horizontal degrees and next III horizontal degrees, or it moves in one sweep the whole arc $I+III$; then the radius from the centre c to the last point in the arc III becomes the same. Likewise it follows that

$$(x+\eta y+\varepsilon z)\text{,,}(\cos II+\eta\sin II)\text{,,}(\cos IV+\eta\sin IV)=$$
$$(x+\eta y+\varepsilon z)\text{,,}(\cos(II+IV)+\eta\sin(II+IV))$$

and hence

$$x+\eta y+\varepsilon z=$$
$$(x+\eta y+\varepsilon z)\text{,,}(\cos I+\varepsilon\sin I)\text{,,}(\cos I-\varepsilon\sin I)=$$
$$(x+\eta y+\varepsilon z)\text{,,}(\cos II+\eta\sin II)\text{,,}(\cos II-\eta\sin II).$$

§ 33.

Let a point move, partly parallel to the horizon, partly parallel to the vertical, describing alternately a horizontal and a vertical arc, and let the traversed arcs in degrees, in the order they follow one another, be denoted I, II, III, IV, V, VI; let radius from the centre of the sphere (Fig. 4) to the first point in the first arc be called s; let radius from the centre to the last point in the last arc be S; and let $\cos I+\varepsilon\sin I$, be denoted by I', $\cos II+\varepsilon\sin II$ by II', $\cos III+\varepsilon\sin III$ by III', etc., and $\cos I-\varepsilon\sin I$ by $I^{-\prime}$, $\cos II-\varepsilon\sin II$ by $II^{-\prime}$, etc., then according to §§ 30 and 31 $S=s\text{,,}I'\text{,,}II'\text{,,}III'\text{,,}IV'\text{,,}V'\text{,,}VI'$, and in this equation the last units IV', V', VI', or as many as one wishes, immediately following one another, may be removed, if their reciprocals joined by the sign (,,) in the reversed order are adjoined to the the first term; i.e. according § 32 one has $S\text{,,}VI^{-\prime}\text{,,}V^{-\prime}\text{,,}IV^{-\prime}=s\text{,,}I'\text{,,}II'\text{,,}III'$, or $s\text{,,}I'\text{,,}II'=S\text{,,}VI^{-\prime}\text{,,}V^{-\prime}\text{,,}IV^{-\prime}\text{,,}III^{-\prime}$, etc.

§ 34.

Assume $s=S$, and let $s=\eta r$, then $s\text{,,}I'=\eta r$.

I) $$s\text{,,}I'\text{,,}II'=r\cdot(\eta\cos II-\sin II)=S\text{,,}VI^{-\prime}\text{,,}V^{-\prime}\text{,,}IV^{-\prime}\text{,,}III^{-\prime}=$$
$$r\cdot\begin{Bmatrix}\cos III\cos IV\cos V\sin VI+\cos III\sin IV\cos VI-\sin III\sin V\sin VI\\ +\eta(\cos IV\cos VI-\sin IV\cos V\sin VI)\\ -\varepsilon(\sin III\cos IV\cos V\sin VI+\sin III\sin IV\cos VI+\cos III\sin V\sin VI)\end{Bmatrix}.$$

II) $$s\text{,,}I'\text{,,}II'\text{,,}III'=r\cdot(\eta\cos II-\sin II\cos III-\varepsilon\sin II\sin III)=S\text{,,}VI^{-\prime}\text{,,}V^{-\prime}\text{,,}IV^{-\prime}=$$
$$r\cdot\begin{Bmatrix}\cos IV\cos V\sin VI+\sin IV\cos VI\\ -\varepsilon\sin V\sin VI\\ +\eta(\cos IV\cos VI-\sin IV\cos V\sin VI)\end{Bmatrix}.$$

III) $s,, I',, II',, III',, IV' =$

$$r \cdot \begin{Bmatrix} \eta(\cos II \cos IV - \sin II \cos III \sin IV) \\ -(\cos II \sin IV + \sin II \cos III \cos IV) \\ -\varepsilon \sin II \sin III \end{Bmatrix} =$$

$S,, VI^{-\prime},, V^{-\prime} = r \cdot (\cos V \sin VI - \varepsilon \sin V \sin VI + \eta \cos VI)$.

IV) $s,, I',, II',, III',, IV',, V' =$

$$r \cdot \begin{Bmatrix} \eta(\cos II \cos IV - \sin II \cos III \sin IV) \\ -(\cos II \sin IV \cos V + \sin II \cos III \cos IV \cos V - \sin II \sin III \sin V) \\ -\varepsilon(\cos II \sin IV \sin V + \sin II \cos III \cos IV \sin V + \sin II \sin III \cos V) \end{Bmatrix}$$

$= S,, VI^{-\prime} = r \cdot (\eta \cos VI + \sin VI)$.

§ 35.

If we let $s = S = \varepsilon r$, we get the following equations:

I) $s,, I' = r \cdot (\varepsilon \cos I - \sin I) = S,, VI^{-\prime},, V^{-\prime},, IV^{-\prime},, III^{-\prime},, II^{-\prime} =$

$$r \cdot \begin{Bmatrix} \varepsilon(\cos III \cos V - \sin III \cos IV \sin V) \\ +\cos II \sin III \cos V + \cos II \cos III \cos IV \sin V - \sin II \sin IV \sin V \\ -\eta(\sin II \sin III \cos V + \sin II \cos III \cos IV \sin V + \cos II \sin IV \sin V) \end{Bmatrix}.$$

II) $s,, I',, II' = r \cdot (-\sin I \cos II - \eta \sin I \sin II + \varepsilon \cos I) =$
$S,, VI^{-\prime},, V^{-\prime},, IV^{-\prime},, III^{-\prime} =$

$$r \cdot \begin{Bmatrix} \varepsilon(\cos III \cos V - \sin III \cos IV \sin V) \\ -\eta \sin IV \sin V \\ +\sin III \cos V + \cos III \cos IV \sin V \end{Bmatrix}.$$

III) $s,, I',, II',, III' =$

$$r \cdot \begin{Bmatrix} -(\sin I \cos II \cos III + \cos I \sin III) \\ -\varepsilon(\sin I \cos II \sin III - \cos I \cos III) \\ -\eta \sin I \sin II \end{Bmatrix} =$$

$S,, VI^{-\prime},, V^{-\prime},, IV^{-\prime} = r \cdot (\varepsilon \cos V + \cos IV \sin V - \eta \sin IV \sin V)$.

IV) $s,, I',, II',, III',, IV' =$

$$r \cdot \begin{Bmatrix} \varepsilon(\cos I \cos III - \sin I \cos II \sin III) \\ -(\sin I \cos II \cos III \cos IV + \cos I \sin III \cos IV - \sin I \sin II \sin IV) \\ -\eta(\sin I \sin II \cos IV + \sin I \cos II \cos III \sin IV + \cos I \sin III \sin IV) \end{Bmatrix}$$

$= S,, VI^{-\prime},, V^{-\prime} = r \cdot (\varepsilon \cos V + \sin V)$.

IV.

On Solving Spherical Polygons.

§ 36.

A spherical polygon is the figure which one gets on the surface of a sphere by joining more than two arcs of great circles so that the one following begins where the preceding one ends, and the last ends where the first begins. The sides of the polygon are the great circular arcs forming the polygon; the measure of the angles is the number of degrees that each plane of a side differs from the plane of the prolongation of the preceding side. When the radius = 1, we denote the sides and the angles of the polygon in the order they follow each other (Fig. 5) by *I, II, III, IV, V, VI,* etc.; the odd numbers denote the angles, and the even numbers the sides; for instance, *II* is the side between *I* and *III*, and *III* is the angle that the side *IV* deviates from the extension of *II*.

§ 37.

If the angles and the sides of the polygon are given, except one angle and two sides, or except two angles and one side, or three sides, or three angles: then the unknowns may be determined by the following equation,

$$s,, I',, II',, III',, IV',, V',, \ldots,, N' = s,$$

in which s is undetermined and may be assumed to be either the common radius r of the horizontal and the vertical circles, or s may be put $= \varepsilon r$, which is the horizontal radius of the sphere, perpendicular to r, or s may $= \eta r$, which is the vertical radius, perpendicular to r and εr. ε^2 as well as η^2 is $= -1$, by § 27.

$$I' = \cos I + \varepsilon \sin I, \quad II' = \cos II + \eta \sin II,$$
$$III' = \cos III + \varepsilon \sin III, \ldots, N' = \cos N + \eta \sin N,$$
$$I^{-\prime} = \frac{1}{\cos I + \varepsilon \sin I}, \; II^{-\prime} = \frac{1}{\cos II + \eta \sin II}, \text{ etc.}$$

The sign (,,) whereby s, I', II', etc. are connected means that first s must be multiplied by I', next $s,, I'$ by II', then $s,, I',, II'$ by III', etc.; but with the restriction that the one line among the added lines in the multiplicand which lies outside the plane of the circular arc in the marked multiplier, remains unchanged, so that

$$\eta,, (\cos I + \varepsilon \sin I) = \eta, \; \varepsilon,, (\cos II + \eta \sin II) = \varepsilon,$$
$$(x + \eta y + \varepsilon z),, (\cos III + \varepsilon \sin III) = \eta y + (x + \varepsilon z) \cdot (\cos III + \varepsilon \sin III)$$

as already observed §§ 28 – 32.

How the above equation $s,, I',, II',, III',, IV',, V',, \ldots,, N' = s$, may serve in solving a spherical polygon is seen from the following:

Assume that the sphere *qhvw* (Fig. 6) can be turned over about the axis πcn of the horizontal great circle *hpow*, and about the axis $pc\gamma$ of the vertical great circle $q\pi ov$, without the position of these two circles being changed, and let the same sphere

1) Be placed so that in the polygon *I II III IV V VI* the last point of the last side is in the pole π of the horizon, and the extension of the same side is in the quadrant πo of the vertical Fig. 6.

2) Next let the sphere be turned about the axis πcn of the horizon *I* horizontal degrees, then the side *II* falls in the vertical between π and *o*, as shown in Fig. 7.

3) When the sphere is now turned *II* vertical degrees about the axis $pc\gamma$ of the vertical, then side *II* passes through the pole π of the horizon, and the sphere gets the position of Fig. 8.

4) Now turn the sphere again *III* horizontal degrees about the axis of the horizon, and the side *IV* will lie in the vertical arc between *o* and π (Fig. 9).

5) And if one continues like this, turning the sphere *IV* vertical degrees, *V* horizontal degrees, *VI* vertical, etc.: the sphere will finally obtain the same position as it had at first, no. 1, Fig. 6.
   By turning the sphere alternately about the axes of the horizon and the vertical, an arbitrary point of the sphere will describe first a horizontal arc measured by the first angle of the polygon; next a vertical arc of as many degrees as the first side of the polygon; then again a horizontal, measuring the second angle, etc., until the sphere has returned to the first position, and each of its points is back to where it came from, having described as many horizontal arcs as there are angles in the polygon, and as many vertical arcs as there are sides in the polygon.

6) Consequently, when in a polygon the angles and the sides all together are of number *N*, and when in the first position of the sphere any point whatsoever had as co-ordinates the three lines with sum $x+\eta y+\varepsilon z(=s)$: then according to § 33 $s = s,, I',, II',, III',, IV',, V',, \ldots,, N'$. Furthermore it should be noted
   a) that on the surface of the sphere, while it is turned about, the fixed point *p* will draw a polygon in which the first side is = the angle *I*, the following angle = the side *II*, the following side = the angle *III*, etc.; for when the sphere is turned about the axis of the horizon, and its surface passes by the point *p*, then the same point draws the sides of the polygon on the surface of the sphere, and when it is turned about the axis of the vertical, each former side will get its inclination to the extension of the following side, all of which is not difficult to imagine, even though the polygon had to be left out of the Figs. 6, 7, etc., to avoid lines falling on top of one another and everything becoming indistinct.
   b) By the fixed point *o* another polygon is drawn, the angles of which are alternately $-90°$, and $+90°$; the sides are *I, II, III, IV, ... ,N,* and the equation of the polygon is $s,, I',, (-\varepsilon),, II',, \varepsilon,, III',, (-\varepsilon),, IV',, \varepsilon,, \ldots,, (-\varepsilon),, N',, \varepsilon = s$.
   Enough said about this, since this equation will not be used in the following. I

must now again return to the formula that I have once made the basis of all the rest, namely:

7) $s,, I',, II',, III',, IV',, V',, \ldots,, N' = s$. This formula may be changed in many ways; since s is the sum of the co-ordinates of a certain point, one may replace s by any line whatsoever, consequently also by $\varepsilon, \eta, \eta r$, or εr.

8) In the first terms of the formula any one of the units that follow s may be chosen as the first, if the following is chosen as the second, the next following as the third, etc., the preceding as the last, the next preceding as the last but one, etc. For instance, suppose the first term begins with $s,, III$, then Fig. 8 shows the first position of the sphere, Fig. 9 the second, Fig. 7 the last but one, and Fig. 8 the last, so that according to § 33, and in the same way as shown before, the equation must become $s,, III',, IV',, \ldots,, N',, I',, II' = s$.

9) The two just mentioned changes of the equation $s,, I',, II',, III',, IV',, V',, \ldots,, N' = s$ are useful in the sense, that any of the units $I', II', III', \ldots, N'$ may be eliminated; for instance, if III' is to be removed, then I change the equation to $s,, III',, IV',, V',, \ldots,, N',, I',, II' = s$, after which I let $s = \eta r$, whereby $s,, III'$ becomes $= \eta r$ by § 28. If IV' is to be left out, the equation is changed to $s,, IV',, V',, \ldots,, N',, I',, II',, III' = s$, and let $s = \varepsilon r$, whereby, according to § 29, $s,, IV' = \varepsilon r$.

10) Since $s,, I',, II',, III',, IV',, V',, \ldots,, N' = s$, then according to § 33 $s,, N^{-\prime},, \ldots,, VI^{-\prime},, V^{-\prime} = s,, I',, II',, III',, IV'$, or, in general one may remove from the first terms of the equation as many units as one pleases, provided the reciprocal of the removed terms in the reversed order, joined by (,,), are connected by (,,) to the second term s.

11) Hereby one may make any unit the last one in one of the terms of the equation, and consequently bring out an equation in which this unit does not occur. For instance, if in the equation $s,, I',, II',, III',, IV' = s,, N^{-\prime},, \ldots,, VI^{-\prime},, V^{-\prime}$ the entire first term is $= x + \eta y + \varepsilon z$, while the second is $= x + \eta \mathfrak{y} + \varepsilon \mathfrak{z}$: then according to § 3 $\varepsilon z = \varepsilon \mathfrak{z}$; in this equation the IV' does not occur, because $IV' = \cos IV + \eta \sin IV$, i.e. since there was multiplication by IV' in the first term, the εz was unchanged, § 29.

12) The equation that we find for the unknown u, when we have eliminated the other two unknowns which are not sought in the manner mentioned above, has the form $a = b \cos u + c \sin u$; for one easily sees that it can never contain $\cos u \cdot \sin u$, or powers of $\cos u$ and $\sin u$. In order to solve this equation we let, as shown before (§ 20),

$$\tfrac{b}{c} = \cot \psi, \text{ and } \cos(u - \psi) = \tfrac{a \sin \psi}{c} = \tfrac{a \cos \psi}{b}.$$

13) If the radius r of the sphere is infinitely large, and the sides of the spherical polygon are infinitely small parts of the periphery, then the spherical polygon is turned into

a plane polygon, the sides of which are the sines of the spherical polygon multiplied by the radius of the sphere. Thus the solution fits both spherical and plane polygons.

V.

Now I will try to derive from the same equation (§ 37 No. 6)

The Most Important Properties of the Spherical Triangles.

§ 38.

Since the equation of the triangle is $s,, I',, II',,\ldots,, VI' = s$ (§ 37 No. 6), and the beginning of the progression is undetermined (§ 37 No. 8), it may start with I' or III' or V', namely if it is assumed that the first unit lies in the plane of the horizon, or that it be cos+ ε sin. For this reason I denote the lines in the progression following VI' by $VII', VIII', IX', X', XI'$ etc., so that $I', VII', XIII'$ become synonymous, as well as $II', VIII', XIV'$ etc. This way of counting should not confuse anybody; because the number of the line, not counting beyond VI, is found by subtracting VI as often as possible from the numbers exceeding this number. Next I set cos $+\,\varepsilon$ sin of the angle where the progression begins to be $(n+I)'$ and let n be undetermined, except that it does denote either 0 or an even number. According to this convention and § 37 No. 8 the general equation of the triangle becomes:

$$s,, (n+I)',, (n+II)',, (n+III)',, (n+IV)',, (n+V)',, (n+VI)' = s.$$

If this equation is changed in accordance with § 33 to

$$\varepsilon,, (n+I)',, (n+II)' = \varepsilon,, (n+VI)^{-\prime},, (n+V)^{-\prime},, (n+IV)^{-\prime},, (n+III)^{-\prime}$$

then from § 35 No. II, and § 3,

$$\text{I)}\ \cos(n+I) = \cos(n+III)\cdot\cos(n+V) - \sin(n+III)\cdot\cos(n+IV)\cdot\sin(n+V).$$

$$\text{II)}\ \sin(n+I) = \frac{\sin(n+IV)\cdot\sin(n+V)}{\sin(n+II)}.$$

If it is changed into

$$\varepsilon,, (n+I)',, (n+II)',, (n+III)',, (n+IV)' = \varepsilon,, (n+VI)^{-\prime},, (n+V)^{-\prime},$$

then an equation appears like the one in § 35, No. IV, except that n has been added to the numbers I, II, III, etc., and r is assumed = 1. Thus when the terms of this equation containing η are divided by $\sin(n+I)$, then one gets

$$\text{III)}\ -\cot(n+I) = \frac{\cot(n+IV)\cdot\sin(n+II)}{\sin(n+III)} + \cot(n+III)\cdot\cos(n+II).$$

If it is changed into

$$\eta_{,,}(n+I)'_{,,}(n+II)'_{,,}(n+III)'=\eta_{,,}(n+VI)^{-\prime}{}_{,,}(n+V)^{-\prime}{}_{,,}(n+IV)^{-\prime},$$

then it gives, according to § 34 No. II,

$$\text{IV)}\ \cos(n+II)=\cos(n+IV)\cdot\cos(n+VI)-\sin(n+IV)\cdot\cos(n+V)\cdot\sin(n+VI)$$

$$\text{V)}\ \sin(n+II)=\frac{\sin(n+V)\cdot\sin(n+VI)}{\sin(n+III)}.$$

And finally, if it is changed into

$$\eta_{,,}(n+I)'_{,,}(n+II)'_{,,}(n+III)'_{,,}(n+IV)'_{,,}(n+V)'=\eta_{,,}(n+VI)^{-\prime},$$

one gets from the term containing ε, § 34 No. IV,

$$\text{VI)}\ -\cot(n+II)=\frac{\cot(n+V)\cdot\sin(n+III)}{\sin(n+IV)}+\cot(n+IV)\cdot\cos(n+III).$$

§ 39.

In the previous six equations it is assumed that n is zero or any positive even number; but comparing the first three with the last three, one will find that in the first three one may replace n by $n+I$ or any positive odd number; so, in the first three equations n may denote zero or any positive integer. One may also, instead of n, write $n+$ any positive integer whatsoever; for instance, one may write, instead of n: $0+3$, $1+3$, $2+3$, $3+3$, $4+3$, etc., in short $n+3$. From this it follows that when in the equation III, § 38, we replace n by $n+III$, then it is changed into

$$-\cot(n+IV)=\frac{\cot(n+VII)\cdot\sin(n+V)}{\sin(n+VI)}+\cot(n+VI)\cdot\cos(n+V).$$

Consequently

$$-\cot(n+I)=\frac{\cot(n+IV)\cdot\sin(n+VI)}{\sin(n+V)}+\cot(n+V)\cdot\cos(n+VI),$$

and comparing this equation with equation III, § 38, we get the following double expression for $-\cot(n+I)$,

$$\text{I)}\ -\cot(n+I)=\frac{\cot(n+IV)\cdot\sin\begin{pmatrix}n+II\\n+VI\end{pmatrix}}{\sin\begin{pmatrix}n+III\\n+V\end{pmatrix}}+\cot\begin{pmatrix}n+III\\n+V\end{pmatrix}\cdot\cos\begin{pmatrix}n+II\\n+VI\end{pmatrix},$$

a formula from which $-\cot(n+I)$ gets the same value whether one uses only the top entries or only the bottom ones of the double entries.

Likewise, if in equation II, § 38 one writes $n + II$ instead of n, one gets

$$\sin(n + III) = \frac{\sin(n+VI)\cdot\sin(n+I)}{\sin(n+IV)}, \text{ hence } \sin(n + I) = \frac{\sin(n+III)\cdot\sin(n+IV)}{\sin(n+VI)},$$

and from this equation and the one in § 38 No. II it follows

$$\text{II)}\ \sin(n + I) = \frac{\sin(n+IV)\cdot\sin\begin{pmatrix} n + III \\ n + V \end{pmatrix}}{\sin\begin{pmatrix} n + VI \\ n + II \end{pmatrix}}.$$

If in equation I, § 38 the number n is replaced by $n + III$, one gets:

$$\text{III)}\ \cos(n + I) = \frac{\cos(n+VI)\cdot\cos(n+II)-\cos(n+IV)}{\sin(n+VI)\cdot\sin(n+II)}.$$

§ 40.

By the same substitution one gets from the same equation

$$\cos(n + IV) = \cos(n + VI)\cdot\cos(n + II) - \sin(n + VI)\cdot\cos(n + I)\cdot\sin(n + II),$$

or

$$-\cos(n + IV) + \frac{\cos(n+VI)\cdot\cos(n+II)}{\sin(n+IV)}\cdot\sin(n + IV) = \sin(n + VI)\cdot\sin(n + II)\cdot\cos(n + I).$$

So, if the last term of this equation is called a, $\cos(n + IV)$ is put $= \cos u$, $-1 = b$, and

$$\frac{\cos(n+VI)\cdot\cos(n+II)}{\sin(n+IV)} = c:$$

then according to § 20 we may assume

$$\tan\psi = -\frac{\cos(n+VI)\cdot\cos(n+II)}{\sin(n+IV)}, \text{ and}$$

$$\cos(n + I) = -\frac{\cos[(n+IV)-\psi]}{\cos\psi\cdot\sin(n+VI)\cdot\sin(n+II)}.$$

§ 41.

The equation I § 38 is

$$\overbrace{\underbrace{\cos(n+I)}_{a}}^{a} = \underbrace{\cos\overbrace{(n+III)}^{u}}_{b}\cdot\overbrace{\cos\underbrace{(n+V)}_{u}}^{b} - \underbrace{\sin\overbrace{(n+III)}^{u}\cdot\mathrlap{\overbrace{}^{c}}\cos(n+IV)}_{c}\cdot\sin\underbrace{(n+V)}_{u},$$

and when its terms are denoted by the letters a, b, c, u written above or below, then we find $\cos(n + I)$ from § 20 by the following formulas:

$$-\cos(n+IV)\cdot\tan\binom{n+V}{n+III}=\cot\binom{\varphi'}{\varphi}, \text{ and}$$

$$\cos(n+I)=\frac{\sin\binom{(n+III)+\varphi'}{(n+V)+\varphi}\cdot\cos\binom{n+V}{n+III}}{\sin\binom{\varphi'}{\varphi}}.$$

§ 42.

According to § 39 No. I

$$-\cot(n+I)=\frac{\cot(n+IV)\cdot\sin\binom{n+VI}{n+II}}{\sin\binom{n+V}{n+III}}+\cot\binom{n+V}{n+III}\cdot\cos\binom{n+VI}{n+II},$$

and when

$$-\cot(n+I)=a,\ \frac{\cot(n+IV)}{\sin\binom{n+V}{n+III}}=c,\ \binom{n+VI}{n+II}=u,\ \cot\binom{n+V}{n+III}=b,$$

and the equation is compared with the one in § 20, then it is easy to verify the following formulas:

$$\tan\binom{\varphi'}{\varphi}=\tan(n+IV)\cdot\cos\binom{n+V}{n+III}.$$

$$-\cot(n+I)=\frac{\sin\binom{(n+VI)+\varphi'}{(n+II)+\varphi}}{\sin\binom{\varphi'}{\varphi}}\cdot\cot\binom{n+V}{n+III}.$$

§ 43.

If, in the last two equations of § 41, n is replaced by $n+II$, then

$$\cot\varphi=-\cos(n+VI)\cdot\tan(n+V), \text{ and}$$

$$\sin((n+I)+\varphi)=\frac{\cos(n+III)\cdot\sin\varphi}{\cos(n+V)}.$$

But if n is replaced by $n+IV$, then

$$\cot\varphi'=-\cos(n+II)\cdot\tan(n+III), \text{ and}$$

$$\sin((n+I)+\varphi')=\frac{\cos(n+V)\cdot\sin\varphi'}{\cos(n+III)}.$$

So

$$\sin\begin{pmatrix}(n+I)+\varphi'\\(n+I)+\varphi\end{pmatrix}=\frac{\cos\begin{pmatrix}(n+V)\\(n+III)\end{pmatrix}}{\cos\begin{pmatrix}(n+III)\\(n+V)\end{pmatrix}}\cdot\sin\begin{pmatrix}\varphi'\\\varphi\end{pmatrix},\text{ and}$$

$$\cot\begin{pmatrix}\varphi'\\\varphi\end{pmatrix}=-\cos\begin{pmatrix}n+II\\n+VI\end{pmatrix}\cdot\tan\begin{pmatrix}n+III\\n+V\end{pmatrix}.$$

§ *44.*

By replacing n by $n+V$ in the last two equations of § 42, one gets

$$\sin[(n+I)+\varphi]=-\cot(n+VI)\cdot\tan(n+II)\cdot\sin\varphi,\text{ and}$$

$$\tan\varphi=\tan(n+III)\cdot\cos(n+II);$$

But by replacing n by $n+I$ one gets:

$$\sin[(n+I)+\varphi']=-\cot(n+II)\cdot\tan(n+VI)\cdot\sin\varphi',\text{ and}$$

$$\tan\varphi'=\tan(n+V)\cdot\cos(n+VI);$$

from this it follows:

$$\sin\begin{pmatrix}(n+I)+\varphi'\\(n+I)+\varphi\end{pmatrix}=-\cot\begin{pmatrix}n+II\\n+VI\end{pmatrix}\cdot\tan\begin{pmatrix}n+VI\\n+II\end{pmatrix}\sin\begin{pmatrix}\varphi'\\\varphi\end{pmatrix},\text{ and}$$

$$\tan\begin{pmatrix}\varphi'\\\varphi\end{pmatrix}=\tan\begin{pmatrix}n+V\\n+III\end{pmatrix}\cos\begin{pmatrix}n+VI\\n+II\end{pmatrix}.$$

§ *45.*

$$\sin^2\tfrac{1}{2}(n+I)=\frac{\sin\frac{1}{2}[(n+II)+(n+IV)+(n+VI)]\cdot\sin\frac{1}{2}[(n+II)+(n+VI)-(n+IV)]}{\sin(n+II)\cdot\sin(n+VI)};$$

because

$$\cos(n+I)=\frac{\cos(n+VI)\cdot\cos(n+II)-\cos(n+IV)}{\sin(n+VI)\cdot\sin(n+II)},\ \text{§ 39 No. III, and}$$

$$2\sin^2\tfrac{1}{2}(n+I)=1-\cos(n+I),\ \text{§ 19 e. So}$$

$$2\sin^2\tfrac{1}{2}(n+I)=1-\frac{\cos(n+VI)\cdot\cos(n+II)-\cos(n+IV)}{\sin(n+VI)\cdot\sin(n+II)},\text{ or}$$

$$2\sin^2\tfrac{1}{2}(n+I) = \frac{\sin(n+VI)\cdot\sin(n+II) - \cos(n+VI)\cdot\cos(n+II) + \cos(n+IV)}{\sin(n+VI)\cdot\sin(n+II)},$$

or, according to § 19 b,

$$2\sin^2\tfrac{1}{2}(n+I) = \frac{\cos(n+IV) - \cos[(n+VI)+(n+II)]}{\sin(n+VI)\cdot\sin(n+II)},$$

and since $\cos b - \cos a = 2\sin\tfrac{1}{2}(a+b)\cdot\sin\tfrac{1}{2}(a-b)$, § 19 i : then

$$\sin^2\tfrac{1}{2}(n+I) = \frac{\sin\tfrac{1}{2}[(n+IV)+(n+VI)+(n+II)]\cdot\sin\tfrac{1}{2}[(n+VI)+(n+II)-(n+IV)]}{\sin(n+VI)\cdot\sin(n+II)}.$$

§ 46.

$$\cos^2\tfrac{1}{2}(n+I) = \frac{\sin\tfrac{1}{2}[(n+IV)+(n+II)-(n+VI)]\cdot\sin\tfrac{1}{2}[(n+IV)-(n+II)+(n+VI)]}{\sin(n+II)\cdot\sin(n+VI)};$$

because $1+\cos(n+I) = 2\cos^2\tfrac{1}{2}(n+I)$, § 19 d; but according to equation III § 39

$$1+\cos(n+I) = \frac{\sin(n+II)\cdot\sin(n+VI) + \cos(n+II)\cdot\cos(n+VI) - \cos(n+IV)}{\sin(n+II)\cdot\sin(n+VI)}, \text{ so}$$

$$2\cos^2\tfrac{1}{2}(n+I) = \frac{\cos[(n+II)-(n+VI)] - \cos(n+IV)}{\sin(n+II)\sin(n+VI)}, \text{ § 19 b.}$$

Hence, since

$$\cos b - \cos a = 2\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b), \text{ §19 i, then}$$

$$\cos^2\tfrac{1}{2}(n+I) = \frac{\sin\tfrac{1}{2}[(n+II)+(n+IV)-(n+VI)]\sin\tfrac{1}{2}[(n+IV)-(n+II)+(n+VI)]}{\sin(n+II)\cdot\sin(n+VI)}.$$

§ 47.

$$-\tan\tfrac{1}{2}[(n+I)-(n+III)] = \frac{\sin\tfrac{1}{2}[(n+IV)-(n+VI)]}{\sin\tfrac{1}{2}[(n+IV)+(n+VI)]}\cdot\tan\tfrac{1}{2}(n+V), \text{ and}$$

$$-\tan\tfrac{1}{2}[(n+I)+(n+III)] = \frac{\cos\tfrac{1}{2}[(n+IV)-(n+VI)]}{\cos\tfrac{1}{2}[(n+IV)+(n+VI)]}\cdot\tan\tfrac{1}{2}(n+V),$$

which one can prove this way:
I) By adding and subtracting $\sin(n+I)$, the equation

$$\sin(n+III) = \frac{\sin(n+I)\sin(n+VI)}{\sin(n+IV)}, \text{ § 39 I,}$$

is changed into the following two equations

$$\text{a) } \sin(n+I)-\sin(n+III)=\frac{\sin(n+I)\cdot\sin(n+IV)-\sin(n+I)\cdot\sin(n+VI)}{\sin(n+IV)}, \text{ and}$$

$$\text{b) } \sin(n+I)+\sin(n+III)=\frac{\sin(n+I)\cdot\sin(n+IV)+\sin(n+I)\cdot\sin(n+VI)}{\sin(n+IV)}.$$

By replacing n by $n+II$ in equation I, § 39, one gets

$$-\cot(n+III)=\frac{\cos(n+IV)\cdot\cos(n+V)}{\sin(n+V)}+\frac{\sin(n+IV)\cdot\cos(n+VI)}{\sin(n+V)\cdot\sin(n+VI)}.$$

When the terms of this equation are multiplied by the terms of the following equation

$$\sin(n+III)=\frac{\sin(n+I)\sin(n+VI)}{\sin(n+IV)}, \text{ § 39 I, one gets}$$

$$-\cos(n+III)=\frac{\cos(n+IV)\cdot\sin(n+VI)\cdot\sin(n+I)\cdot\cos(n+V)}{\sin(n+IV)\cdot\sin(n+V)}+\frac{\cos(n+VI)\cdot\sin(n+I)}{\sin(n+V)};$$

but since

$$-\cos(n+I)=\frac{\cos(n+VI)\cdot\cos(n+V)\cdot\sin(n+I)}{\sin(n+V)}+\frac{\sin(n+VI)\cdot\cos(n+IV)\cdot\sin(n+I)}{\sin(n+IV)\cdot\sin(n+V)},$$

§ 39 I: then the sum

$$\text{c) } -\cos(n+I)-\cos(n+III)=\frac{[\cos(n+V)+1]\cdot\sin[(n+IV)+(n+VI)]}{\sin(n+IV)}\cdot\frac{\sin(n+I)}{\sin(n+V)},$$

and when the formula a is divided by the formula c, one gets

$$-\frac{\sin(n+I)-\sin(n+III)}{\cos(n+I)+\cos(n+III)}=\frac{\sin(n+V)}{1+\cos(n+V)}\cdot\frac{\sin(n+IV)-\sin(n+VI)}{\sin[(n+IV)+(n+VI)]}, \text{ but}$$

$$\frac{\sin(n+I)-\sin(n+III)}{\cos(n+I)+\cos(n+III)}=\tan\tfrac{1}{2}[(n+I)-(n+III)], \text{ § 19 l, and}$$

$$\frac{\sin(n+V)}{1+\cos(n+V)}=\tan\tfrac{1}{2}(n+V), \text{ §19 f, therefore}$$

$$-\tan\tfrac{1}{2}[(n+I)-(n+III)]=\frac{\tan\frac{1}{2}(n+V)\cdot[\sin(n+IV)-\sin(n+VI)]}{\sin[(n+IV)+(n+VI)]}, \text{ and since}$$

$$\sin(n+IV)-\sin(n+VI)=2\cos\tfrac{1}{2}[(n+IV)+(n+VI)]\cdot\sin\tfrac{1}{2}[(n+IV)-(n+VI)],$$

§ 19 h, and

$$\frac{2\cos\frac{1}{2}[(n+IV)+(n+VI)]}{\sin[(n+IV)+(n+VI)]} = \frac{1}{\sin\frac{1}{2}[(n+IV)+(n+VI)]}, \text{ § 19 c : then}$$

$$-\tan\frac{1}{2}[(n+I)-(n+III)] = \frac{\tan\frac{1}{2}(n+V)\cdot\sin\frac{1}{2}[(n+IV)-(n+VI)]}{\sin\frac{1}{2}[(n+IV)+(n+VI)]}.$$

II) In the same manner we find by dividing formula b by formula c :

$$-\frac{\sin(n+I)+\sin(n+III)}{\cos(n+I)+\cos(n+III)} = \frac{\sin(n+V)}{1+\cos(n+V)}\cdot\frac{\sin(n+IV)+\sin(n+VI)}{\sin[(n+IV)+(n+VI)]} =$$

$$-\tan\frac{1}{2}[(n+I)+(n+III)], \text{ § 19 k,}$$

and when we replace $\frac{\sin(n+V)}{1+\cos(n+V)}$ by $\tan\frac{1}{2}(n+V)$, § 19 f, and replace $\sin(n+IV)+\sin(n+VI)$ by

$$2\sin\frac{1}{2}[(n+IV)+(n+VI)]\cdot\cos\frac{1}{2}[(n+IV)-(n+VI)], \text{ § 19 g, then}$$

$$\tan\frac{1}{2}(n+V)\cdot\frac{2\sin\frac{1}{2}[(n+IV)+(n+VI)]\cdot\cos\frac{1}{2}[(n+IV)-(n+VI)]}{\sin[(n+IV)+(n+VI)]} =$$

$$-\tan\frac{1}{2}[(n+I)+(n+III)]; \text{ but since}$$

$$\frac{2\sin\frac{1}{2}[(n+IV)+(n+VI)]}{\sin[(n+IV)+(n+VI)]} = \frac{1}{\cos\frac{1}{2}[(n+IV)+(n+VI)]}, \text{ by § 19 c, then}$$

$$-\tan\frac{1}{2}[(n+I)+(n+III)] = \tan\frac{1}{2}(n+V)\cdot\frac{\cos\frac{1}{2}[(n+IV)-(n+VI)]}{\cos\frac{1}{2}[(n+IV)+(n+VI)]}.$$

§ 48.

If the three angles are given in a spherical triangle, then the sides may be found by means of any of the following formulas, if n is assumed $= 0$, II, or IV:

$$\text{I) } \cos(n+II) = \frac{\cos(n+I)\cdot\cos(n+III)-\cos(n+V)}{\sin(n+I)\cdot\sin(n+III)}, \text{ § 39 eq. III.}$$

$$\text{II) } \left.\begin{cases} \cos(n+II) = -\dfrac{\cos[(n+V)-\psi]}{\cos\psi\cdot\sin(n+I)\cdot\sin(n+III)}, \text{ and} \\ \tan\psi = -\dfrac{\cos(n+I)\cdot\cos(n+III)}{\sin(n+V)}. \end{cases}\right\} \text{ § 40.}$$

$$\text{III) } \sin^2\frac{1}{2}(n+II) = \frac{\sin\frac{1}{2}[(n+III)+(n+V)+(n+I)]\cdot\sin\frac{1}{2}[(n+III)+(n+I)-(n+V)]}{\sin(n+III)\cdot\sin(n+I)}, \text{ § 45.}$$

Substituting in the first of these three formulas

a) $n + I = 90°$, then $\cos(n+II) = -\frac{\cos(n+V)}{\sin(n+III)}$.

b) $n + III = 90°$, then $\cos(n+II) = -\frac{\cos(n+V)}{\sin(n+I)}$.

c) $n + V = 90°$, then $\cos(n+II) = \cot(n+I)\cdot\cot(n+III)$.

§ 49.

If two angles and their common side are given, then

A) the angle opposite the given side is determined by the formula IV or V, when one substitutes $n = 0$, or II, or IV.

IV) $\cos(n+I) = \cos(n+III)\cdot\cos(n+V) - \sin(n+III)\cdot\cos(n+IV)\cdot\sin(n+V)$, § 38, equation I.

$$\text{V)}\ \left\{\begin{aligned} &\cos(n+I) = \frac{\sin\begin{pmatrix}(n+III)+\varphi'\\(n+V)+\varphi\end{pmatrix}\cdot\cos\begin{pmatrix}n+V\\n+III\end{pmatrix}}{\sin\begin{pmatrix}\varphi'\\\varphi\end{pmatrix}}, \text{and}\\ &\cot\begin{pmatrix}\varphi'\\\varphi\end{pmatrix} = -\cos(n+IV)\cdot\tan\begin{pmatrix}n+V\\n+III\end{pmatrix}.\end{aligned}\right\}\ \S\ 41.$$

Equation IV gives

a) $n + III = 90°$: $\cos(n+I) = -\cos(n+IV)\cdot\sin(n+V)$.

b) $n + V = 90°$: $\cos(n+I) = -\cos(n+IV)\cdot\sin(n+III)$.

c) $n + IV = 90°$: $\cos(n+I) = \cos(n+III)\cdot\cos(n+V)$.

B) The two remaining sides are found using any of the following formulas, in which n can be assumed $= 0$, or II, or IV.

$$\text{VI)}\ -\cot(n+II) = \frac{\cot(n+V)\cdot\sin\begin{pmatrix}n+III\\n+I\end{pmatrix}}{\sin\begin{pmatrix}n+IV\\n+VI\end{pmatrix}} + \cot\begin{pmatrix}n+IV\\n+VI\end{pmatrix}\cdot\cos\begin{pmatrix}n+III\\n+I\end{pmatrix},\ \S\ 39,\text{ eq. I}$$

$$\text{VII)}\ \left\{\begin{aligned} &-\cot(n+II) = \frac{\sin\begin{pmatrix}(n+I)+\varphi'\\(n+III)+\varphi\end{pmatrix}}{\sin\begin{pmatrix}\varphi'\\\varphi\end{pmatrix}}\cdot\cot\begin{pmatrix}n+VI\\n+IV\end{pmatrix}, \text{and}\\ &\tan\begin{pmatrix}\varphi'\\\varphi\end{pmatrix} = \tan(n+V)\cdot\cos\begin{pmatrix}n+VI\\n+IV\end{pmatrix}.\end{aligned}\right\}\ \S\ 42.$$

$$\text{VIII)}\left\{\begin{aligned}
&-\tan\tfrac{1}{2}[(n+II)-(n+IV)] = \tan\tfrac{1}{2}(n+VI)\cdot\frac{\sin\frac{1}{2}[(n+V)-(n+I)]}{\sin\frac{1}{2}[(n+V)+(n+I)]}.\\
&-\tan\tfrac{1}{2}[(n+II)+(n+IV)] = \tan\tfrac{1}{2}(n+VI)\cdot\frac{\cos\frac{1}{2}[(n+V)-(n+I)]}{\cos\frac{1}{2}[(n+V)+(n+I)]}.
\end{aligned}\right\}\ \S\ 47.$$

If in the formula VI

d) $n + III = 90°$, then $-\cot(n+II) = \dfrac{\cot(n+V)}{\sin(n+IV)}$.

e) $n + IV = 90°$, then $-\cot(n+II) = \cot(n+V)\cdot\sin(n+III)$.

f) $n + V = 90°$, then $-\cot(n+II) = \cot\begin{pmatrix} n+VI \\ n+IV \end{pmatrix}\cdot\cos\begin{pmatrix} n+I \\ n+III \end{pmatrix}$.

g) $n + I = 90°$, then $-\cot(n+II) = \dfrac{\cot(n+V)}{\sin(n+VI)}$.

h) $n + VI = 90°$, then $-\cot(n+II) = \cot(n+V)\cdot\sin(n+I)$.

§ 50.

Given one angle and two sides, of which one side is opposite the given angle, then

A) the third side is found from the following formula no. IX, substituting $n = 0$, *II*, or *IV*.

$$\text{IX)}\left\{\begin{aligned}
&\sin\begin{pmatrix}(n+II)+\varphi' \\ (n+II)+\varphi\end{pmatrix} = \frac{\cos\begin{pmatrix} n+VI \\ n+IV\end{pmatrix}\cdot\sin\begin{pmatrix}\varphi' \\ \varphi\end{pmatrix}}{\cos\begin{pmatrix} n+IV \\ n+VI\end{pmatrix}},\text{and}\\
&\cot\begin{pmatrix}\varphi' \\ \varphi\end{pmatrix} = -\cos\begin{pmatrix} n+III \\ n+I\end{pmatrix}\cdot\tan\begin{pmatrix} n+IV \\ n+VI\end{pmatrix}.
\end{aligned}\right\}\ \S\ 43.$$

If	then	by	replacing n by
a) $n + III = 90°$,	$\cos(n+II) = \dfrac{\cos(n+VI)}{\cos(n+IV)}$,	§ 49 c,	$n + V$
b) $n + I = 90°$,	$\cos(n+II) = \dfrac{\cos(n+IV)}{\cos(n+VI)}$,	§ 49 c,	$n + III$
c) $n + IV = 90°$,	$\sin(n+II) = -\dfrac{\cos(n+VI)}{\cos(n+III)}$,	§ 49 b,	$n + V$
d) $n + VI = 90°$,	$\sin(n+II) = -\dfrac{\cos(n+IV)}{\cos(n+I)}$,	§ 49 a,	$n + III$

B) The angle included by the two given sides is found from the following formula X, substituting $n = 0$, *II*, or *IV*.

$$\text{X)}\left\{\begin{array}{l} \sin\begin{pmatrix}(n+I)+\varphi' \\ (n+I)+\varphi\end{pmatrix} = -\cot\begin{pmatrix}n+II \\ n+VI\end{pmatrix}\cdot\tan\begin{pmatrix}n+VI \\ n+II\end{pmatrix}\cdot\sin\begin{pmatrix}\varphi' \\ \varphi\end{pmatrix}, \text{and} \\ \tan\begin{pmatrix}\varphi' \\ \varphi\end{pmatrix} = \tan\begin{pmatrix}n+V \\ n+III\end{pmatrix}\cdot\cos\begin{pmatrix}n+VI \\ n+II\end{pmatrix}. \end{array}\right\}\ \S\ 44.$$

If	then	by	replacing n by
e) $n+III=90^\circ$,	$\cos(n+I)=-\cot(n+VI)\cdot\tan(n+II)$,	§ 49 f,	$n+IV$.
f) $n+V=90^\circ$,	$\cos(n+I)=-\cot(n+II)\cdot\tan(n+VI)$,	§ 49 f.	
g) $n+VI=90^\circ$,	$\sin(n+I)=-\cot(n+II)\cdot\tan(n+V)$,	§ 49 h.	
h) $n+II=90^\circ$,	$\cot(n+I)=-\frac{\cot(n+V)}{\cos(n+VI)}$,	§ 49 f,	$n+III$.

C) The opposite angle to the second side is found from the formula XI.

$$\text{XI)}\ \sin(n+I) = \frac{\sin(n+IV)\cdot\sin\begin{pmatrix}n+III \\ n+V\end{pmatrix}}{\sin\begin{pmatrix}n+VI \\ n+II\end{pmatrix}},\ \S\ 39,\ \text{equation II.}$$

§ 51.

When one replaces sides by angles, and angles by sides in the three previous examples, § 48, 49, 50, then one can solve the problems by assuming n to be $= I$, or III, or V in the formulas given there, § 39.

§ 52.

If in a spherical triangle the sides are less than two right angles and positive, then the angles may be assumed to have the same properties; for the first angle may be counted positive and less than two right angles, as seen from § 37, Fig. 6; and that the remaining two angles are of the same kind as the first one is seen by the formula

$$\sin I = \frac{\sin IV\cdot\sin\begin{pmatrix}III \\ V\end{pmatrix}}{\sin\begin{pmatrix}VI \\ II\end{pmatrix}},\ \S\ 50\ \text{XI.}$$

In the following it is assumed that angles and sides are less than 180°.

§ 53.

When the sides of a spherical triangle are less than two right angles, then it is completely determined by three given angles, by three given sides, by two angles and their common side, or by two sides and the included angle. This is seen from the formulas §§ 48, 49, and the theorem § 51.

§ 54.

From the previous formulas it also follows that equal sides are opposite equal angles, and conversely; for instance, if one puts $I = III$ in the formula IV § 49, then $IV = VI$; because

$$\cos I = \cos III \cdot \cos V - \sin III \cdot \cos IV \cdot \sin V, \text{ and}$$

$$\cos III = \cos V \cdot \cos I - \sin V \cdot \cos VI \cdot \sin I.$$

(The last equation is obtained from the first one by increasing the numbers by II).

§ 55.

Opposite a greater angle is a smaller side, and conversely. It follows from § 47:

$$-\tan\tfrac{1}{2}(I - III) = \tan\tfrac{1}{2}V \cdot \frac{\sin\frac{1}{2}(IV - VI)}{\sin\frac{1}{2}(IV + VI)};$$

because when $I - III$ is negative, then $IV - VI$ must be positive.

§ 56.

The sum of any two sides is greater than the third side, and the sum of all three is less than four right angles; because according to §§ 45 and 46

$$\cos^2\tfrac{1}{2}(I) = \frac{\sin\frac{1}{2}(IV + II - VI)\cdot\sin\frac{1}{2}(IV - II + VI)}{\sin II \cdot \sin VI}, \text{ and}$$

$$\sin^2\tfrac{1}{2}(I) = \frac{\sin\frac{1}{2}(IV + II + VI)\cdot\sin\frac{1}{2}(VI + II - IV)}{\sin VI \cdot \sin II};$$

but if in the first equation $VI > IV + II$ then II must be greater than $IV + VI$; because otherwise $\cos^2\frac{1}{2}(I)$ would be negative; but it is impossible that VI should be greater than $IV + II$, when $II > IV + VI$; neither could VI be $= IV + II$; for in that case $\cos^2\frac{1}{2}(I) = 0$. Consequently VI must be $< IV + II$, and in general the sum of two sides is greater than the third side; therefore, in the second equation $\sin\frac{1}{2}(VI + II - IV)$ is positive; and therefore also $\sin\frac{1}{2}(IV + II + VI)$ is positive; consequently $\frac{IV + II + VI}{2} < 180°$ (§ 52), and $IV + II + VI$ is less than $360°$.

§ 57.

In a similar way it is shown that the sum of two angles is greater than the third angle, and the sum of all three is less than four right angles, which is also a consequence of § 37, no. 6a, and § 56.

§ 58.

On a semi-sphere choose a point C (Fig. 12) between the pole P of the base-circle and its circumference; and draw a great circular arc CB to the periphery of the base[3], then CB is least when it ends in r, where the perpendicular PC extended meets the circumference of the base-circle; from here it increases from Cr to become $= 90° = rQ = CQ$, and still more, until it becomes $= 180° - Cr\,(= Cq)$, so that B falls below QR, when CB is obtuse; but if CB is acute then B lies above QR.

For let CBq be denoted by I, the hypotenuse BC by II, the side Cr by IV, and rB by VI: then according to § 49 c (when n is replaced by I) $\cos II = \cos IV \cdot \cos VI$, or $\cos BC = \cos Cr \cdot \cos rB$, from which the truth of the claim is easily seen.

While rB increases from 0 to 90°, and CB from Cr to CQ (= 90°), the angle CBq increases from 90° to 180°$-Cr$; but thereafter, when rB increases from 90° to 180°, and CB from 90° to 180°$-Cr$, then CBq decreases from 180°$-Cr$ to 90°; indeed, when we replace n by V in § 49 h, then $V = 90°$, and $-\cot I = \cot IV \cdot \sin VI$, or $-\cot CBq = \cot Cr \cdot \sin rB$, a formula from which the proof of the theorem is easy to derive.

§ 59.

Let us assume that the sides (II, IV, VI) of a triangle are less than two right angles, and that in the equation $\sin I = \frac{\sin IV \cdot \sin V}{\sin II}$ (§ 50 XI) the arcs IV, V, II are oblique: then the following table shows in which cases the angle I we are looking for, is acute, obtuse, or ambiguous. Namely,

$$\left.\begin{matrix}1\\2\end{matrix}\right\} \; V \text{ obtuse}, \quad IV \text{ obtuse}, \quad \text{and } II \begin{pmatrix} < 180° - IV \\ > 180° - IV \end{pmatrix}, \quad \text{then } I \begin{pmatrix} \text{ambiguous} \\ \text{acute} \end{pmatrix}.$$

$$\left.\begin{matrix}3\\4\end{matrix}\right\} \; V \text{ acute}, \quad IV \text{ acute}, \quad \text{and } II \begin{pmatrix} > 180° - IV \\ < 180° - IV \end{pmatrix}, \quad \text{then } I \begin{pmatrix} \text{ambiguous} \\ \text{obtuse} \end{pmatrix}.$$

$$\left.\begin{matrix}5\\6\end{matrix}\right\} \; V \text{ obtuse}, \quad IV \text{ acute}, \quad \text{and } II \begin{pmatrix} < IV \\ > IV \end{pmatrix}, \quad \text{then } I \begin{pmatrix} \text{ambiguous} \\ \text{obtuse} \end{pmatrix}.$$

$$\left.\begin{matrix}7\\8\end{matrix}\right\} \; V \text{ acute}, \quad IV \text{ obtuse}, \quad \text{and } II \begin{pmatrix} > IV \\ < IV \end{pmatrix}, \quad \text{then } I \begin{pmatrix} \text{ambiguous} \\ \text{acute} \end{pmatrix}.$$

3 *Fundamentals of Mathematics*, by Kästner. The second theorem of spherical trigonometry.

$$\left.\begin{array}{l} 9 \\ 10 \end{array}\right\} \begin{array}{l} II = IV\text{, and then } I = V. \\ II + IV = 180°\text{, and then } I = 180° - V. \end{array}$$

P r o o f : Since none of the sides of triangle *ABC* (Figs. 13-18) is greater than 180°, all of the triangle lies within the surface of one of the hemispheres cut out by the plane of the side *AB* (= *VI*), and since the sides *II* and *IV* are oblique, they meet in a point *C* outside the pole *P* of the base circle *ABD*. Thus, from the pole *P* one may draw the great circular arc *PC* in either direction, and likewise its perpendicular *QPR*, until they both reach the circumference of the base circle. These two semicircles are represented in the figures 13-18 by the two straight lines *rq* and *QR*.

1) Since *IV* is obtuse, and *II* is acute, the point *A* (Fig. 13) falls below *QR*, and *B* as well as the farthest point *D* of the semicircle *ACD* lies above *QR*, § 58. But since the arc 180°– *IV* (= *CD*) is assumed to be greater than *II* (= *CB*), this must fall between *Cr* and *CD*, or else between *Cr* and *CQ*, § 58, and must be able to have the same size in both places. In the first case *I* becomes acute, in the second case obtuse; thus *I* is ambiguous.

2) Since the arc *II* is assumed > 180°–*IV*, or *II* > *CD* (Fig. 14), then *CDR* < *CBA* (§ 55), or *V* < 180°–*I* ; but *V* is assumed to be obtuse, hence *I* is acute.

3) *IV* is acute and *II* is obtuse, consequently *A* (Fig.15) falls above, but *B* below *QR*. And since the arc *II* > 180°– *IV*, it may according to § 58 have the same size between *CD* and *Cq*, as when deviating by the same amount from *Cq*, but in the opposite direction. Thus *I* may be acute or obtuse, § 58.

4) Since *II* < 180°–*IV*, or *II* < *CD* (Fig. 16); then in Δ*CBD* we have *CDQ* > *CBA*, or *V* > 180°–*I* (§ 55); but *V* is acute, hence *I* is obtuse.

5) Both *II* and *IV* are assumed to be acute; hence *B*, *A*, and *C* (Fig. 17) are on the same side of *QR* (§ 58); and since it is assumed *II* < *IV*, then *B* may fall on either side of *Cr*; to the one side *I* becomes obtuse, to the other side acute (§ 58), therefore *I* is ambigous.

6) *II* > *IV*, hence *V* < *I* (§ 55); but *V* is obtuse, hence *I* is obtuse.

7) Since *II* and *IV* are both obtuse, then they intersect the perpendicular *QR* (Fig.18) between the points *Q* and *R* (§ 58). And since *II* > *IV*, then the arc *II* must lie either between *IV* and *Cq*, or on the other side of *Cq*. Hence the angle *I* must either be acute or obtuse (§ 58).

8) When *II* < *IV*, then *V* > *I* (§ 55), hence when *V* is acute, then so is *I*.

9) *II* = *IV* gives *V* = *I* (§ 44).

10) $II + IV = 180°$, gives $I = 180°-V$, because the supplements to II and IV form together with AB another $\Delta ABC'$, whose angles and sides are equal to those of ΔABC (§ 53).

§ 60.

We assume as in the previous § 59 that the sides (II, IV, VI) of the triangle are $< 180°$, but that in the equation $\sin I = \frac{\sin IV \cdot \sin V}{\sin II}$ (§ 50, XI) two of the given (V, IV, II) are right angles: then

$I = 180°-IV$, if V and II are right, (Fig. 19)

$I = 90° = II$, if V and IV are right.

$I = 90° = V$, if IV and II are right.

If, on the other hand, only one of the given (V, IV, II) is right, then either

$\left.\begin{matrix}1\\2\end{matrix}\right)$ V right and $IV \begin{pmatrix} > II \\ < II \end{pmatrix}$, hence I is $\begin{pmatrix}\text{acute}\\ \text{obtuse}\end{pmatrix}$, or

$\left.\begin{matrix}3\\4\end{matrix}\right)$ V acute, IV right, and $II \begin{pmatrix}\text{acute}\\ \text{obtuse}\end{pmatrix}$, hence I is $\begin{pmatrix}\text{impossible}\\ \text{ambiguous}\end{pmatrix}$,

$\left.\begin{matrix}5\\6\end{matrix}\right)$ V obtuse, IV right, and $II \begin{pmatrix}\text{obtuse}\\ \text{acute}\end{pmatrix}$, hence I is $\begin{pmatrix}\text{impossible}\\ \text{ambiguous}\end{pmatrix}$,

$\left.\begin{matrix}7\\8\end{matrix}\right)$ V acute, $IV \begin{pmatrix}\text{obtuse}\\ \text{acute}\end{pmatrix}$, and II right, hence I is $\begin{pmatrix}\text{acute}\\ \text{obtuse}\end{pmatrix}$,

$\left.\begin{matrix}9\\10\end{matrix}\right)$ V obtuse, $IV \begin{pmatrix}\text{obtuse}\\ \text{acute}\end{pmatrix}$, and II right, hence I is $\begin{pmatrix}\text{acute}\\ \text{obtuse}\end{pmatrix}$.

P r o o f. Nos. 1 and 2 follow from the fact that the greater side is opposite the lesser angle (§ 55).

Nos. 3 and 5 are impossible; for when IV is right, then V and II can neither both be acute, nor both be obtuse, because $-\cot II = \cot V \cdot \sin III$ (§ 49 e).

In 7, 8, 9, 10 I and IV are of different kinds, because when $II = 90°$ then $-\cot IV = \cot I \cdot \sin III$, which follows from § 49 h, assuming $n = II$.

Nos. 4 and 6 may be proved as follows: Either II is obtuse and V is acute, as in No. 4, and in ΔACB Fig. 20, or II is acute and V is obtuse, as No. 6, and in $\Delta ACB'$: then one can make out of II together with the supplements to IV and VI another $\Delta A'BC$, or $A'B'C$, in which II, IV, and V keep their sizes ; but the angle I is changed to its supplement. Consequently, one may make from the same data II, IV, V two different triangles.

§ 61.

According to § 37 No. 6 a any triangle may be changed into another one, in which the angles are the sides of the former triangle, and the sides are the angles of the former, the order being unchanged. Hence, if from given *VI, V, III* we are to find *II* by the formula $\sin II = \frac{\sin V \cdot \sin VI}{\sin III}$: then the given and the wanted parts may be denoted as in §§ 59, 60, and the rules mentioned there may be applied in this case, too.

§ 62.

Since the formulas IX and X § 50 are derived from an equation containing both sine and cosine of the wanted arc, then it might be presumed that they should not give, as in eq. XI, § 50, a positive value less than 180° of the arc to be found, and would not agree with the data of the triangle, when these were all positive and less than two right angles. But to be completely convinced about this I assume:

1) that $n^- + II^-$ is positive, less than two right angles, and a value of $n + II$, computed by means of eq. IX, § 50 from the data $n + III, n + IV, n + VI$. Then I conclude that in a triangle with given $n^- + II^-, n + III$, and $n + IV$, and in which the opposite to $n + III$ is called $n^- + VI^-$, the equation IV in § 49 yields
$\cos(n^- + VI^-) = \cos(n^- + II^-) \cdot \cos(n + IV) - \sin(n^- + II^-) \cdot \cos(n + III) \cdot \sin(n + IV)$;
but the same value is obtained for $\cos(n + VI)$, when in the equation
$\sin[(n^- + II^-) + \varphi'] = \frac{\cos(n + VI)}{\cos(n + IV)} \sin\varphi'$ (§ 50 IX) the sine of the sum is expressed by cosine and sine of the parts, and dividing by $\sin\varphi'$, and finally substituting $-\cos(n + III) \cdot \tan(n + IV)$ for $\cot\varphi'$. Hence we have $n + VI = n^- + VI^-$, and the calculated value $n^- + II^-$, and the given $n + III, n + IV, n + VI$ belong to one and the same triangle.

2) Likewise I assume in eq. X of § 50 that $n^- + I^-$ is a value of $n + I$, furthermore that it is positive, less than 180°, and calculated from the data $n + II, n + V, n + VI$; next I conclude from § 49 eq. VI, that in a triangle with given $n^- + I^-, n + V, n + VI$, and in which the opposite to $n + V$ is denoted $n^- + II^-$, we have

$$-\cot(n^- + II^-) = \frac{\cot(n + V) \cdot \sin(n^- + I^-)}{\sin(n + VI)} + \cot(n + VI) \cdot \cos(n^- + I^-), \text{ or}$$

dividing by $\cot(n + VI)$,

$$-\cot(n^- + II^-) \cdot \tan(n + VI) = \frac{\cot(n + V) \cdot \sin(n^- + I^-)}{\cos(n + VI)} + \cos(n^- + I^-); \text{ but this same}$$

formula is the result for $\cot(n + II)$ when we express $\sin[(n^- + I^-) + \varphi']$ in the equation X § 50 by cosine and sine of $n^- + I^-$ and of φ', next divide by $\sin\varphi'$ and finally replace $\cot\varphi'$ by its value; consequently $n^- + II^- = n + II$, and hence $n^- + I^-, n + II, n + V, n + VI$ belong to the same triangle.

§ 63.

In the same manner one may also construct a triangle whose sides and angles are less than two right angles, from two given parts in each of the equations c, d, g, § 50 together with the value of the unknown, when this is positive and less than 180°; furthermore it may be shown, that the third given part also belongs to the constructed triangle, and hence that the calculated value of the unknown part can always coexist with the given parts; e.g. according to § 50 c $n + IV = 90°$, and $\sin(n + II) = -\frac{\cos(n+VI)}{\cos(n+III)}$. Now let $n^- + II^-$ be the value of the unknown, and construct a triangle from $n^- + II^-, n + IV$, and $n + III$, in which the opposite to $n + III$ is called $n^- + VI^-$: then $\cos(n^- + VI^-) = -\sin(n^- + II^-)\cdot\cos(n + III)$ § 49 b; but $\cos(n + VI)$ is also $= -\sin(n^- + II^-)\cdot\cos(n + III)$, § 50 c; hence $n + VI = n^- + VI^-$.

I further add the following, in order to show how the symbols of direction proposed in §§ 30 and 31 may be applied to express an equation for rectilinear polygons, whose sides extend in different planes.

§ 64.

A polygon with the properties mentioned is undetermined when it has four sides of unknown lengths.

Proof.

1) Let the four sides of unknown lengths follow one another, and be denoted by *ab, bc, cd, de,* Fig. 21. If the points *a, b, c* are on a straight line, then *ab* may be shortened, and *cb* prolonged by the same amount, without the direction of these two lines or the direction and length of the rest of the lines being changed in any way. Thus those two sides of the polygon are undetermined in this case.
If neither *abc,* nor *cde* is on a straight line, but *abc* is in the same plane as *cde*; then one may draw, in this plane, outside *c* parallels to *bc* and *cd,* intersecting *ab* and *de.* Hence in this case the polygon is also undetermined.
If the triangle *abc* is not in the same plane as the triangle *cde,* then the two planes intersect in a straight line through *c,* and from points on this line away from *c,* one can draw parallels to *cd* and *cb,* intersecting *ab* and *de.* Thus the polygon is undetermined.
2) The sides of any rectilinear polygon may be given an arbitrary order without changing their sum, direction and length, § 2. That is, if *ab, bc, cd, de* do not follow each other in uninterrupted order, as assumed in the preceding proof, then one can imagine another polygon, whose sides are the same, but the four unknowns are in a connected sequence. And since some of these four in this order could have infi-

nitely many values, according to the first proof, then they may have just as many, when they are rearranged in the previous order, § 2.

§ 65.

In any rectilinear polygon, whose sides do not all belong to the same plane, it is assumed that each side begins where the preceding ends, so that also the sum of all of them = 0 according to § 2. Next we assume that the length of the first, the second, the third,..., the m'th or last side is denoted by a *mark* of the same order in the sequence $\overline{)I^{\cup}}, \overline{)III^{\cup}}, \overline{)V^{\cup}}, \overline{)VII^{\cup}}, \ldots, \overline{)(2m-I)^{\cup}}$, and the sides themselves, in the order they follow one another, by the odd numbers $I^{\cup}, III^{\cup}, V^{\cup}, VII^{\cup}, \ldots, (2m-I)^{\cup}$ with an added upper *tittle* at the right hand corner to distinguish the side from the angle between the plane through the side and the preceding one, and the plane through the side and the following one; because the angles between these planes are also denoted by the numbers $I, III, V, VII, \ldots, (2m-I)$ so that I (Fig. 22) is the angle between the two planes intersecting in the side $I^{\cup}$, or the angle between the planes *CDA* and *DAB*; III the angle between those intersecting in the side $III^{\cup}$, or the angle between the planes *DAB* and *ABC*, etc. $(2m-I)$ is the angle that the plane through the last and the first side makes with the plane through the last and the next to the last side.

Furthermore we assume that the angle that each side deviates from the prolongation of the preceding is denoted by the even number $II, IV, VI, \ldots$, or $2m$, which is one unit greater than the number of the preceding side; namely II is the angle, that $III^{\cup}$ deviates from the prolongation of $I^{\cup}$; IV is the angle that $V^{\cup}$ deviates from the prolongation of $III^{\cup}$, etc., $2m$ is the angle that the first side $I^{\cup}$ deviates from the prolongation of the last side $(2m-I)^{\cup}$.

§ 66.

All these angles between the planes, as well as between the sides, one may assume to be positive and decide for oneself whether the deviation of a side from the prolongation of the preceding is to be greater than or less than two right angles. But once this is decided, it is no longer unimportant how the skewness of the planes is to be measured, if the rules for the solution of these polygons are to be valid in all cases.

§ 67.

If we are to measure the mutual slope of the planes by one and the same rule, we must consider three of the sides of the polygon in a row, as in Fig. 23, with the line from a to b, the line from b to c, and the one from c to d; next draw from the endpoint c of the middle side bc a parallel cf to the preceding side ab; draw with the same point c as centre a circular arc fg from the parallel cf to the extension of the middle side cg, measuring the angle cbe, that the middle side deviates from the prolongation be of the preceding side; similarly, draw with the same centre and the same radius a circular arc gi from the prolongation cg of the middle side to the following side cd. The spherical angle igh, by which the last mentioned arc gi deviates from the prolongation gh of the first arc fg, will then be as big as the angle that the plane through the middle and the following side deviates from the plane through the middle and the preceding side, or as big as the deviation of the plane *bcd* from the plane *abc*. And this angle is measured by following on the sphere the arc *fg*, coming from *f* to *g*, then the measure of the angle is from the prolongation of *fg* to the left. In this manner these angles can be determined when one wants to know some of them in order to calculate the rest.

§ 68.

But if the directions of the sides in a polygon (Fig. 22) are almost all known, then its angles can be more clearly presented, when from the centre $\mathfrak{c}$ of the sphere *wphv* (Fig. 24) we draw the radii $\mathfrak{c}\mathfrak{A}$, $\mathfrak{c}\mathfrak{B}$, $\mathfrak{c}\mathfrak{C}$, and $\mathfrak{c}\mathfrak{D}$ each one in a direction such that it remains parallel to the side of the same order in the sequence $I^{\cup}$, $III^{\cup}$, $V^{\cup}$, $VII^{\cup}$, Fig. 22; for in drawing the great circular arcs $\mathfrak{A}\mathfrak{B}$, $\mathfrak{B}\mathfrak{C}$, $\mathfrak{C}\mathfrak{D}$, and $\mathfrak{D}\mathfrak{A}$ one gets a spherical polygon $\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$, in which the sides measure the angles *II, IV, VI, VIII* of the rectilinear polygon, and the spherical angles are the same as the angles *I, III, V, VII* of the planes in the rectilinear figure 22[4]. Thus, for the angles of such a polygon one has the same equation as for a spherical polygon, that is $s,, I',, II',, III',, IV',,\ldots,,(2m)' = s$ (§ 37). Here s may denote any line, and $2\ m$ is the last angle of the rectilinear polygon, or the deviation of the first side from the m^{th} (that is, the prolongation of the last side).

§ 69.

Now I assume, that $\mathfrak{A}wyhp$ (Fig. 24) is the horizon, $\mathfrak{A}\pi qv$ the vertical circle, $\mathfrak{A}$ is the common zero of both circles; the horizontal circles are counted positive to the left, and the vertical ones positive upwards; the radius $\mathfrak{c}\mathfrak{A} = +1, \mathfrak{c}\gamma = \varepsilon, \mathfrak{c}\pi = \eta$, and any two of these radii include a right angle, as assumed earlier in §§ 24 and 25. I assume furthermore that the vertex of the first angle I of the polygon $\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$ falls in the common zero $\mathfrak{A}$ of the horizon and the vertical, and that the prolongation of the last side *VIII* falls in the vertical $\mathfrak{A}\,v$ below the horizon. Under this assumption the radius
$\mathfrak{c}\,V = \eta,, IV^{-\prime},, III^{-\prime},, II^{-\prime},, I^{-\prime},,(-\eta)$, and in general, if the last side $2m$ of the spherical polygon is vertical, and is made to end in the zero point $\mathfrak{A}$, but prolonged goes under the horizon, and if in this position of the sphere the radius $\mathfrak{c}(n+I)$ is drawn to the vertex of the angle $(n+I)$, or to the last point of the side n of the polygon, then the same radius is

$$\mathfrak{c}(n+I) = \eta,, n^{-\prime},,(n-I)^{-\prime},,(n-II)^{-\prime},,\ldots,, II^{-\prime},, I^{-\prime},,(-\eta).$$

In order to prove this theorem, I assume the horizontal and the vertical circles to be immobile as in § 37, and let the sphere turn from the position mentioned (Fig. 24), first 90 vertical degrees, next I horizontal degrees, then II vertically, next III horizontally, etc., finally n vertical degrees. This way the vertex of the angle $(n+I)$ has been moved as many degrees as the sphere has been turned, and according to § 33 the radius $\mathfrak{c}(n+I)$ is

4 According to the drawing Fig. 24 and the rule § 67, the angles *III* and *VII* are greater than, but the angles *I* and *V* less than 180°. The fact that *V* falls below the plane of projection means that the side *VI* seems to be to the right, even though it does fall to the left, when one follows, on the sphere, the arc *IV* from $\mathfrak{B}$ to $\mathfrak{C}$

changed, first to $\mathfrak{c}(n+I),,\eta$, next to $\mathfrak{c}(n+I),,\eta,,I'$, next to $\mathfrak{c}(n+I),,\eta,,I',,II'$, and then to $\mathfrak{c}(n+I),,\eta,,I',,II',,III'$, etc., finally it is changed to $\mathfrak{c}(n+I),,\eta,,I',,II',,III',,\ldots,,(n-I)',,n'$ and becomes equal to η, because the last point of the side n, and hence the last point of the radius $\mathfrak{c}(n+I)$, now falls in the pole π of the horizon. From the equation $\mathfrak{c}(n+I),,\eta,,I',,II',,III',,\ldots,,(n-I)',,n'=\eta$ we conclude, according to § 33: $\mathfrak{c}(n+I)=\eta,,n^{-\prime},,(n-I)^{-\prime},,(n-II)^{-\prime},,\ldots,,II^{-\prime},,I^{-\prime},,(-\eta)$, which was to be proved.

§ 70.

According to the preceding formula we therefore have in Fig. 24 $\mathfrak{c}I=+1$, $\mathfrak{c}III=\eta,,II^{-\prime},,I^{-\prime},,(-\eta)$, and $\mathfrak{c}V=\eta,,IV^{-\prime},,III^{-\prime},,II^{-\prime},,I^{-\prime},,(-\eta)$, etc. Furthermore we have from the condition § 68, that $\mathfrak{c}I$ is parallel to $I^{\cup}$, $\mathfrak{c}III$ parallel to $III^{\cup}$, $\mathfrak{c}V$ to $V^{\cup}$, etc., Fig. 22. Hence

$$I^{\cup}=\mathfrak{c}I\cdot\overline{)I^{\cup}},\ III^{\cup}=\mathfrak{c}III\cdot\overline{)III^{\cup}},\ V^{\cup}=\mathfrak{c}V\cdot\overline{)V^{\cup}},\ \text{etc. and}$$

$$(2m-I)^{\cup}=\mathfrak{c}(2m-I)\cdot\overline{)(2m-I)^{\cup}},\ \S\ 65,$$

(by $(2m-I)^{\cup}$ we mean the m^{th} and last side of the rectilinear polygon). Furthermore, since

$$I^{\cup}+III^{\cup}+V^{\cup}+\ldots+(2m-I)^{\cup}=0,\ \S\ 2\text{: then also}$$

$$\overline{)I^{\cup}}+\overline{)III^{\cup}}\cdot\mathfrak{c}III+\overline{)V^{\cup}}\cdot\mathfrak{c}V+\ldots+\overline{)(2m-I)^{\cup}}\cdot\mathfrak{c}(2m-I)=0,$$

and if we replace in this equation the radii $\mathfrak{c}III, \mathfrak{c}V, \mathfrak{c}VII, \ldots, \mathfrak{c}(2m-I)$ by their values according to § 69, and thereupon move the last point of each radius 90 vertical degrees, then we arrive at the equation

$$\overline{)I^{\cup}}\cdot\eta+\overline{)III^{\cup}}\cdot\eta,,II^{-\prime},,I^{-\prime}+\overline{)V^{\cup}}\cdot\eta,,IV^{-\prime},,III^{-\prime},,II^{-\prime},,I^{-\prime}+$$

$$\overline{)VII^{\cup}}\cdot\eta,,VI^{-\prime},,V^{-\prime},,IV^{-\prime},,III^{-\prime},,II^{-\prime},,I^{-\prime}+\ldots+$$

$$\overline{)(2m-I)^{\cup}}\cdot\eta,,(2m-II)^{-\prime},,(2m-III)^{-\prime},,\ldots,,II^{-\prime},,I^{-\prime}=0,$$

in which one may omit $I^{-\prime}$ in the manner described in § 33.

§ 71.

For any rectilinear polygon, in which the sides are not coplanar, one has the following two equations:

A) $s,, I',, II',, III',, IV',, V',, \ldots,, (2m)' = s$, § 68, and

B) $\overline{)I^{\cup}} \cdot \eta + \overline{)III^{\cup}} \cdot \eta,, II^{-\prime},, I^{-\prime} + \overline{)V^{\cup}} \cdot \eta,, IV^{-\prime},, III^{-\prime},, II^{-\prime},, I^{-\prime} +$

$\overline{)VII^{\cup}} \cdot \eta,, VI^{-\prime},, V^{-\prime},, IV^{-\prime},, III^{-},, II^{-\prime},, I^{-\prime} + \ldots +$

$\overline{)(2m-I)^{\cup}} \cdot \eta,, (2m-II)^{-\prime},, (2m-III)^{-\prime},, \ldots,, II^{-\prime},, I^{-\prime} = 0$, § 70.

In order that these equations can be understood without help from the preceding sections, I want to repeat here the meaning of the symbols.

The sides are numbered so that preceding one ends where the following begins.

The first, the second, the third,..., the m^{th} or last side of the polygon is denoted in sequence by $I^{\cup}, III^{\cup}, V^{\cup}, VII^{\cup}, \ldots, (2m-I)^{\cup}$, Fig. 22.

The lengths of the sides by $\overline{)I^{\cup}}, \overline{)III^{\cup}}, \overline{)V^{\cup}}, \overline{)VII^{\cup}}, \ldots, \overline{)(2m-I)^{\cup}}$.

The deviation of each side from the prolongation of the preceding side by an even number $II, IV, VI, \ldots$, or $2m$, which is one unit greater than the odd one that marks the preceding side.

The deviation of the angle that the plane through the middle and the following side of three consecutive sides makes with the plane through the middle and the preceding side is denoted by the odd number I, III, V, etc., or $(2m-I)$, belonging to the middle side.

All angles are positive. Whether they are to be greater than or less than two right angles is best seen from §§ 66 and 67.

The angles $II, IV, VI, \ldots, 2m$ are measured in the vertical, or in a circle perpendicular to the horizon, in which the angles $I, III, V, VII, \ldots, (2m-I)$ are measured, § 25. Both circles intersect in the radius +1.

Sine of 90 degrees, or $\sqrt{-1}$ (§ 6) is denoted by η in the vertical circle, in the horizontal circle by ε; and ε^2 as well as η^2 is $= -1$, according to § 5.

Substituting $n = II, IV, \ldots$, or $2m$, then $\cos n + \eta \sin n$ is denoted n', and $\frac{1}{\cos n + \eta \sin n}$ by $n^{-\prime}$, § 7.

Substituting $n = I, III, V, \ldots$ or $(2m-I)$, then n', means the same as $\cos n + \varepsilon \sin n$, and $n^{-\prime}$ the same as $\frac{1}{\cos n + \varepsilon \sin n}$, § 7.

$\cos n$ and $\sin n$ are in the same direction in the first and the third quadrant, but in the second and the fourth they are opposite, § 6.

The sign ,, has only halfway the meaning of the ordinary sign of multiplication, because the line in the expression of the multiplicand, which is outside the plane of the circular arc in the mark of the multiplicator stays unchanged by the operation; for instance, when 2, 3ε, and 4η are straight lines, then $(2+3\varepsilon+4\eta),, II'$ is the same as $3\varepsilon+(2+4\eta)\cdot(\cos II+\eta\sin II)$; similarly, $(2+3\varepsilon+4\eta),, I'$ is the same as $4\eta+(2+3\varepsilon)\cdot(\cos I+\varepsilon\sin I)$.

Furthermore it should be observed that the operation is carried out in the order of the factors from left to right; thus, for instance, if we want to find the value of $(2+3\varepsilon+4\eta),, I',, II'$, we must first look for the value of $(2+3\varepsilon+4\eta),, I'$ $(=4\eta+2\cos I+2\varepsilon\sin I-3\sin I+3\varepsilon\cos I)$, and next the value of $(4\eta+2\cos I+2\varepsilon\sin I-3\sin I+3\varepsilon\cos I),, II'$.

s may denote a straight line of any length and direction; thus, in equation A) one may replace s by a term of equation B) and hence change the expression of the term; for instance, if $s=\overline{)III^{\cup}}\cdot\eta,, II^{-\prime},, I^{-\prime}=$ the second term in B), then the equation A) is changed into

$$\overline{)III^{\cup}}\cdot\eta,, IV',, V',, VI',, \ldots,, (2m)'=\overline{)III^{\cup}}\cdot\eta,, II^{-\prime},, I^{-\prime}, \text{§ } 32.$$

I have not pursued the investigation of these polygons any further.

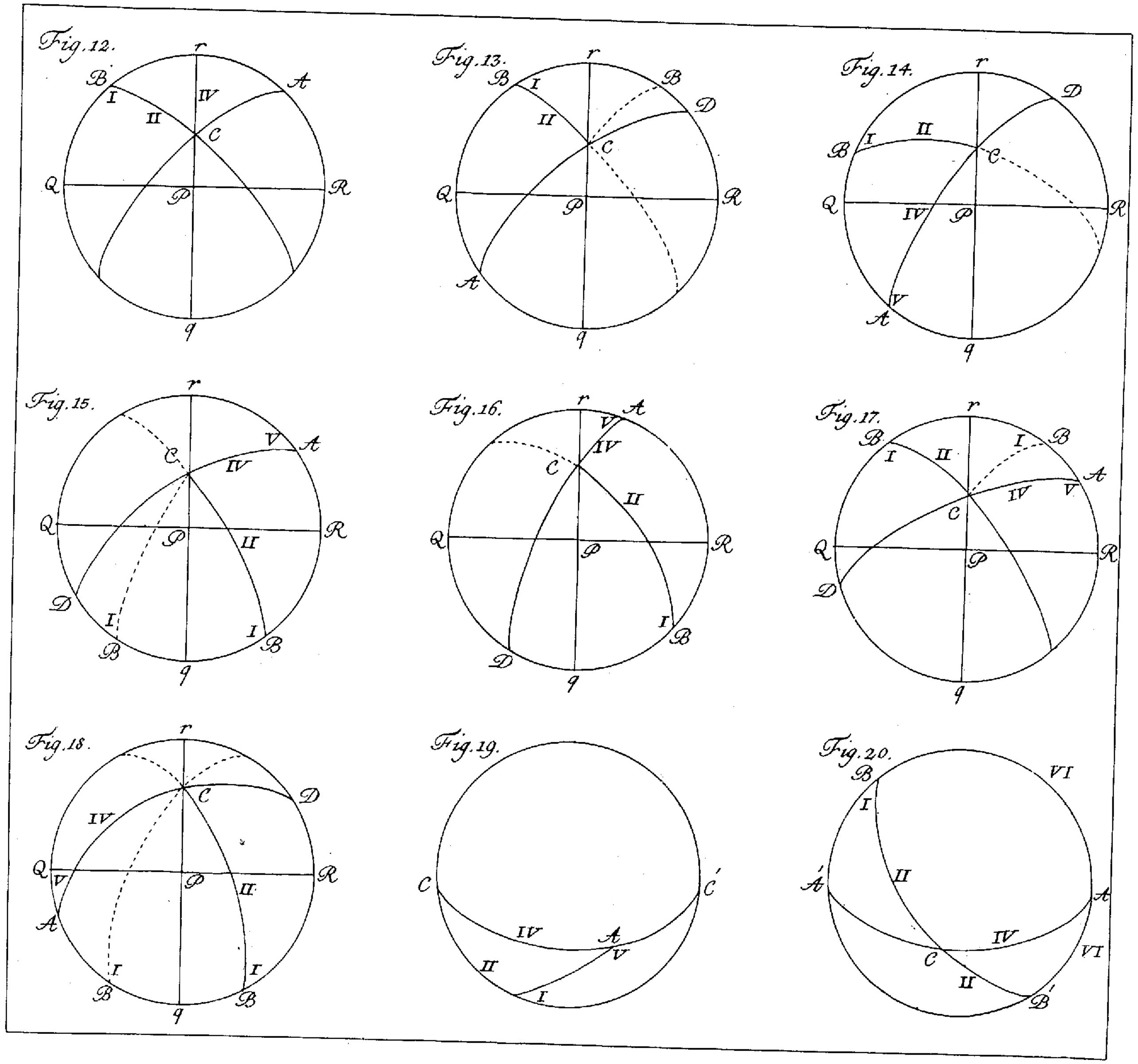
Fig. 12.
Fig. 13.
Fig. 14.
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.
Fig. 20.

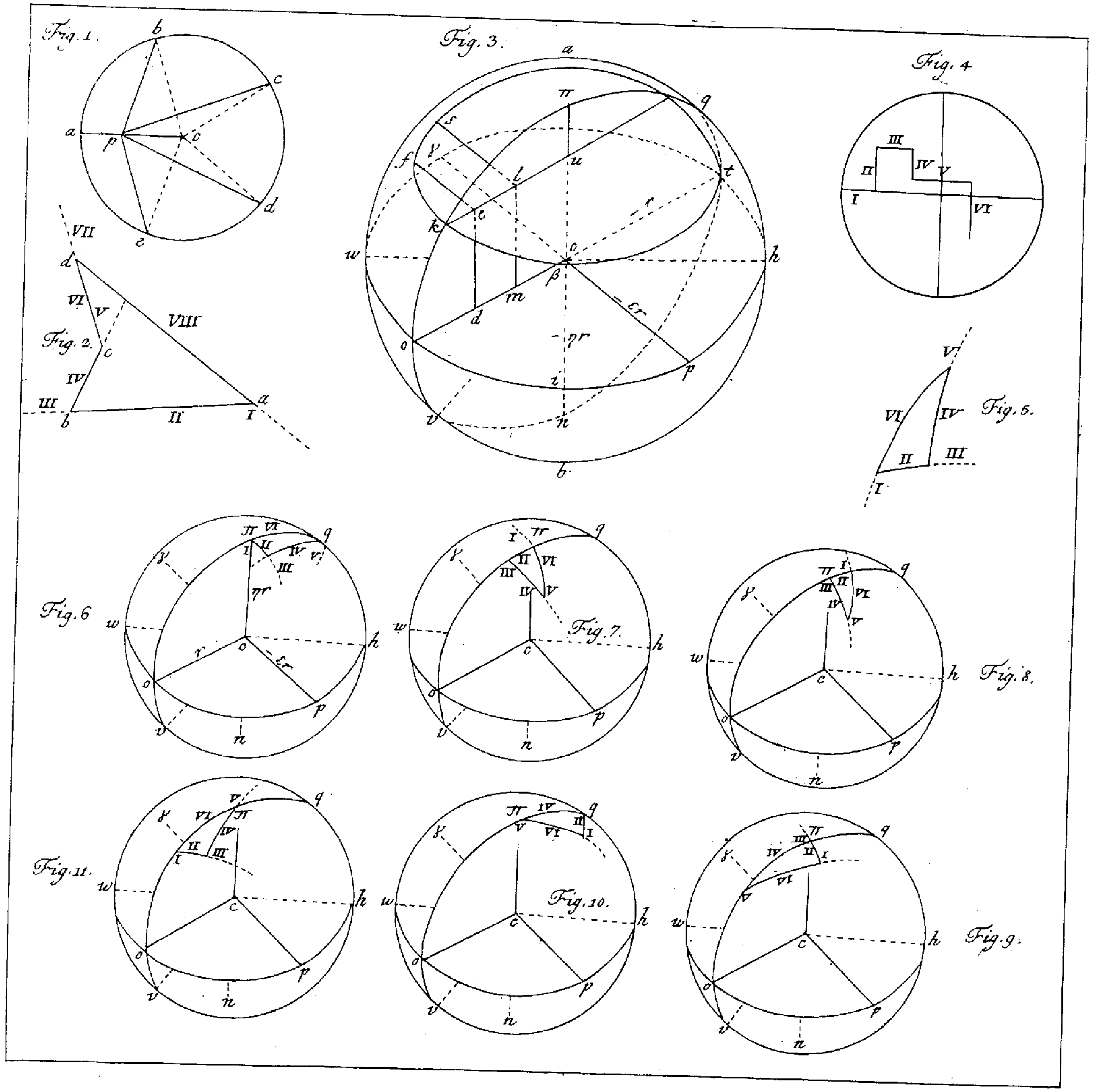
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4
Fig. 5.
Fig. 6
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.

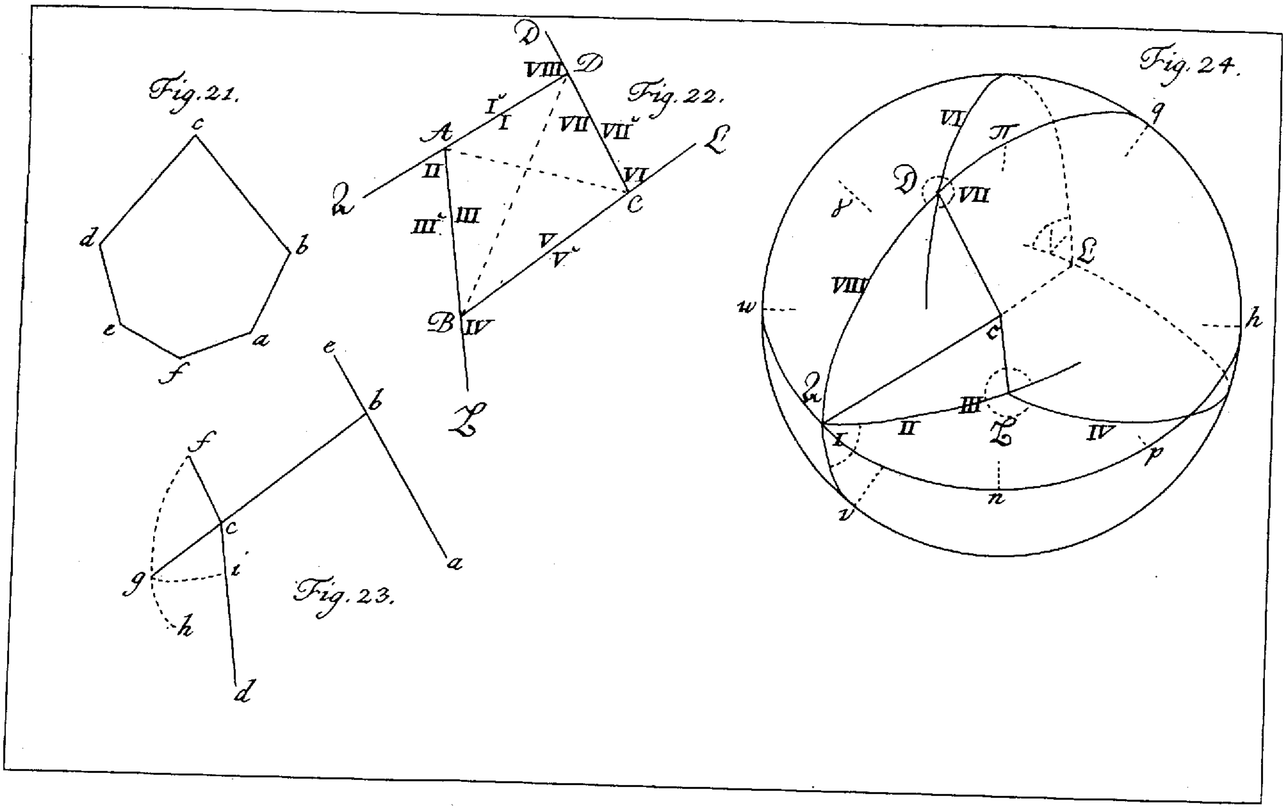

Fig. 21.

Fig. 22.

Fig. 23.

Fig. 24.