IC/90/142

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

GAUGE-FIELD TOPOLOGY IN TWO DIMENSIONS: θ-VACUUM, TOPOLOGICAL PHASES AND COMPOSITE FIELDS

N. Ilieva

and

V.N. Pervushin

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1990 MIRAMARE - TRIESTE

IC/90/142

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

GAUGE-FIELD TOPOLOGY IN TWO DIMENSIONS: θ-VACUUM, TOPOLOGICAL PHASES AND COMPOSITE FIELDS *

N. Ilieva **
International Centre for Theoretical Physics, Trieste, Italy

and

V.N. Pervushin
Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office, P.O. Box 79, 101000 Moscow, USSR.

ABSTRACT

In the framework of the minimal quantization method, the residual "longitudinal" vacuum dynamics of the Abelian gauge field, that is described by a new pair of canonical variables, is revealed. This dynamics is shown to give origin to the θ-vacuum, thus providing a field analogy of the Josephson effect. The destructive interference of the topological phases - that the fermion fields are shown to acquire - is considered as a reason for the charge screening in the two-dimensional massless QED.

MIRAMARE – TRIESTE

June 1990

* To be submitted for publication.

** On leave of absence from: Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office, P.O. Box 79, 101000 Moscow, USSR.
Address after 15 August 1990: Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Lenin 72, Sofia 1784, Bulgaria.

Introduction

The description of fundamental particle interactions with the help of the gauge fields makes use of the local symmetries of the systems under consideration providing to some extent a subsidiary role for their global symmetries though the significance of the latter is by no means convincing. An important point here is the correct representation and understanding of the global features in the quantized theory that concerns the quantization procedure chosen. Minimal quantization method based on the explicit solution of the constraint equations[1,2,3], takes into account the vacuum state global symmetries, topological properties of the gauge field configuration space etc. in a natural and physically reasonable way.

In the present paper we would like to illustrate this statement by the example of some two-dimensional gauge models. In the first section the residual "longitudinal" vacuum dynamics of the Abelian gauge field is discussed. In this context a new point of view on the θ-vacuum origin and some consequences for the scalar fields are considered in the second section. The interference of the topological phases the fermion fields are shown to acquire is the subject of the third section. Finally, some concluding remarks complete this paper.

1. *Topological vacuum dynamics of the Abelian gauge field*

Let us consider the free electromagnetic field in two-dimensional space-time

$$\mathcal{L}_o(x) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \qquad \mu,\nu = 0,1 \tag{1}$$

from the position of the minimal quantization.

On the explicit solution of the constraint equation (the equation of motion for the component A_o)

$$\frac{\delta S}{\delta A_o} = 0 \quad \Rightarrow \quad A_o = \frac{1}{\partial_1^{\,2}} \partial_1\partial_o A_1, \tag{2}$$

Lagrangian (1) simply vanishes, which reflects the absence of transverse degrees of freedom in two dimensions. In minimal quantization method solution (2) should be used for defining the functional which performs a transition to physical (transverse) variables

$$\mathrm{v}[A] = exp\left\{-ie \int^{x_o} dx_{o'} \frac{1}{\partial_1^{\,2}} \partial_1\partial_{o'} A_1 \right\} = exp\left\{-ie\, \partial_1^{-1} A_1 \right\} \tag{3}$$

In terms of such variables

$$A_1^{\,\mathrm{T}}[A] = \mathrm{v}[A](A_1 + \frac{i}{e}\partial_1)\mathrm{v}[A]^{-1} \tag{4}$$

Lagrangian can be formally rewritten as

$$\mathcal{L}_o(x) = \frac{1}{2}\,(A_1^{\,\mathrm{T}}[A])^2 = 0 \tag{5}$$

However, the transverse variable $A_1^{\,\mathrm{T}}[A]$ is still not properly defined because of a possible nontrivial zero modes of the inverse

operator in eq.(2), hence, in definition (3). With homogeneous equation solutions taken into account we get for A_o

$$A_o = \frac{1}{\partial_1^{\,2}} \partial_1\partial_o A_1 + M(x), \qquad \partial_1^{\,2} M(x) = 0 \tag{6}$$

This modifies both functional (3)

$$\mathrm{v}[A] \longrightarrow \mathrm{v}[A]\, m(x) = \mathrm{v}^{\lambda}[A], \qquad m(x) = exp(-i\lambda\ ), \tag{7}$$

where for convenience we have taken $M(x)$ in the form

$$M(x) = \frac{1}{e}\,\partial_o\lambda(x), \tag{8}$$

and the transverse variable $A_1^{\,\mathrm{T}}[A]$

$$A_1^{\,\mathrm{T}}[A] \longrightarrow A_1^{\,\mathrm{T},\lambda}[A] = \mathrm{v}[A]m(x)(A_1 + \frac{i}{e}\partial_1)m(x)^{-1}\mathrm{v}[A]^{-1} \tag{9}$$

As there are no sources in the space, the phase $m(x)$ must have no singularities that has to be reflected by the boundary conditions needed for solving eq.(6). Putting

$$\lim_{x\to\pm\infty} M(x) = 0,$$

we obtain only trivial solutions and turn back to Lagrangian (5).

However, boundary condition which ensures the coincidence of functionals $A_1^{\,\mathrm{T}}[A]$ and $A_1^{\,\mathrm{T},\lambda}[A]$ at the space boundaries, i.e. condition on the phase $m(x)$,

$$\lim_{x\to\pm\infty} m(x) = 1, \tag{10}$$

leads to a completely different situation. Condition (10) provides for eq.(6) a family of solutions $m^n(x)$ with functions $\lambda^n(x)$ for which the following relation takes place

$$\lambda^n(x = \infty) - \lambda^n(x = -\infty) = 2\pi n, \qquad n \in \mathbb{N}$$

Phase factor $m(x)$ with boundary condition (10) defines a map of the space $R(1)$ onto the group manifold $U(1)$ which is characterized by an integer n - the degree of mapping. Thus, gauge fields are divided in topological classes with respect to the value of n, their configuration space obtaining the topology of a ring as

$$\Pi_1(U(1)) = \mathbb{Z} \qquad (11)$$

In such a theory the existence of constant vacuum electric fields without any external sources is possible due to a purely quantum effect which may be considered as a field analogy of the Josephson effect. Remind that it consists in the existence of un-damped currents as a result of the inhomogeneity of the wave function phase (for example, the phase shift of the electron wave function in a superconducting ring arround a thin solenoid

$$\Psi(x + 2\pi R) = exp(i\theta)\,\Psi(x),$$

where R is the radius of the ring and θ is the phase shift).

In the theory of interest an analogous condition on the gauge field wave function takes place

$$\Psi[A^{(n+1)}(x)] = exp(i\theta)\,\Psi[A^{(n)}(x)] \qquad (12)$$

because the points $A_1^{T(1)}$, $A_1^{T(2)}$, ..., $A_1^{T(n)}$, ...,

$$A_1^{T(n)}[A] = v[A]m^n(x)(A_1 + \frac{i}{e}\partial_1)(m^n(x))^{-1}v[A]^{-1} \qquad (13)$$

in the nonsimply connected gauge field configuration space are physically identical.

It is easy to get convinced that condition (12) gives rise to a nonvanishing constant vacuum field without any external sources.

Let us consider the Schrödinger equation

$$\left[\frac{1}{2}\int dx\,\hat{E}^2\right]\Psi = \varepsilon\Psi, \qquad \hat{E} = \frac{1}{i}\frac{\delta}{\delta A_1}$$

Its solution which is invariant under topologically trivial gauge transformations (with $n = 0$)

$$\partial_1\hat{E}\,\Psi = 0,$$

is a plane wave

$$\Psi = exp\,(\,-i\,p\,N[A]\,), \qquad p = \frac{2\pi E}{e} \qquad (14)$$

with respect to the variable $N[A]$

$$N[A] = \frac{e}{2\pi}\int dx\,A_1^{T}[A(x)], \qquad (15)$$

the latter being covariantly transformed under topologically non-trivial gauge transformations (13)

$$N[A_1^{T(n)}] = N[A_1^{T}] + \frac{\lambda^n(\infty) - \lambda^n(-\infty)}{2\pi} = N[A_1^{T}] + n$$

and representing by itself a continuous generalization of the Pontryagin index[4]

$$\nu = \frac{e}{4\pi}\int dx\,\varepsilon_{\mu\nu}F^{\mu\nu} = \int dx_0\,\partial_0 N[A] = N[A]\big|_{x_0=\infty} - N[A]\big|_{x_0=-\infty}$$

The variable N is a new topological variable in the theory which describes the residual *"longitudinal"* dynamics of the Abelian gauge field in two-dimensional space-time with an action

$$S_{top} = \frac{1}{2}\int d^2x\left[\frac{\partial_1\partial_0\lambda}{e}\right]^2 = \frac{1}{2}\,I\int dx_0\,\dot{N}^2, \qquad (16)$$

where

$$I = \frac{1}{V} \left(\frac{2\pi}{e} \right)^2 , \qquad V = \int d^2x$$

and a finite energy density

$$\frac{\varepsilon}{V} = \frac{1}{2} \left[\frac{e}{2\pi} (2\pi k + \theta) \right]^2$$

As it follows from (16), topological variable N has no consistent classical interpretation. However, classical approximation in such a theory is unacceptable because the validity of quantum theory is expanded over the entire space volume V ($\sim I^{-1}$). This observation strongly fixes the infrared regularization to be removed at the end.

Quantization of topological action (16) is straightforward

$$p = \frac{\delta S_{top}}{\delta \dot{N}} = \dot{N} I, \qquad [p, N] = i$$

and the topological momentum spectrum is easily obtained when the equivalence of the states

$$< p \mid N > = exp(-ipN) \quad \text{and} \quad < p \mid N+n > = exp(-ip(N+n))$$

is taken into account. The real state represents itself a Bloch wave - an average over this degeneration with a weight $exp(in\theta)$:

$$< p \mid N > = \lim_{l \to \infty} \frac{1}{l} \sum_{n=-l/2}^{n=l/2} exp(in\theta)\, exp(-ip(N+n)) =$$

$$= \begin{cases} exp(-i(2\pi k + \theta)), & p = 2\pi k + \theta \\ 0, & p \neq 2\pi k + \theta \end{cases}$$

$$k = 0, \pm 1, \pm 2, \ldots\ldots ; \ |\theta| < \pi$$

This means that the operator $\hat{E}$ is a constant electric field operator with a spectrum

$$\hat{E}\, \Phi = \frac{e}{2\pi}\, p\, \Phi, \qquad p = 2\pi k + \theta \qquad (17)$$

Minimal (in modulo) value of this field $E_{min} = e\theta/(2\pi)$ coincides with Coleman's constant electric field[5] which he had proposed in order to reproduce the results of the θ-vacuum introduction in the Schwinger model. However, he had motivated the existence of this field with the $R(1)$-space properties and considered θ as a parameter. In our approach θ is an intrinsic characteristic of the theory, connected with its topological structure and with the spectrum of the new vacuum physical variable which describes the collective rotation of the gauge field in the configuration space. This allows one to consider such a vacuum degeneration as a field analogy of the Josephson effect, which also takes place in the Schwinger model[1,6].

2. θ-vacuum and composite scalar fields in the Schwinger model

The Schwinger model provides a manifestation of the residual vacuum dynamics of the Abelian gauge field in two dimensions when considered in the frames of the minimal quantization method. The Lagrangian

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} + e\gamma^{\mu}A_{\mu})\psi \qquad (18)$$

contains a constraint

$$\frac{\delta S}{\delta A_0} = 0 \quad \Rightarrow \quad \partial_1^{\,2} A_0 = \partial_1 \partial_0 A_1 + e j_0, \qquad j^{\mu} = \bar{\psi} \gamma^{\mu} \psi \tag{19}$$

After defining variables $A_1^{T,\lambda}[A]$ by eq.(9) and $\psi^{T,\lambda}[A,\psi]$ as

$$\psi^{T,\lambda}[A,\psi] = v^{\lambda}[A]\, \psi, \qquad \bar{\psi}^{T,\lambda}[A,\psi] = \bar{\psi}\, v^{\lambda}[A]^{-1} \tag{20}$$

Lagrangian (18) takes the following form on the explicit solutions

$$A_0 = \frac{1}{\partial_1^{\,2}} (\partial_1 \partial_0 A_1 + e j_0) + M(x) \tag{21}$$

of the constraint equation (19):

$$L = \int dx \, [\, \tfrac{1}{2} (\partial_0 A_1^{T})^2 + i\bar{\psi}^{T}\gamma^{\mu}\partial_{\mu}\psi^{T} - \tfrac{e^2}{2} (\partial_1^{-1} j_0)^2 \,] =$$

$$= \int dx \, [i\bar{\psi}^{T}\gamma^{\mu}\partial_{\mu}\psi^{T} - \tfrac{e^2}{2} (\partial_1^{-1} j_0)^2 + \tfrac{Ie^2}{2\pi} (N j_1 - \dot{N}\partial_1^{-1} j_0)] + \tfrac{1}{2} I \dot{N}^2$$

The first two terms are what is usually obtained, for example, in the Coulomb gauge[5]. Note, however, that this resemblance is just formal because of the transformation properties of the fields involved since the main property of variables (4),(9),(20) is their invariance under gauge transformations of the initial fields A_{μ}, ψ, and $\bar{\psi}$. The second two terms describe the fermion interaction with the topological mode N that has nontrivial consequences.

As bosonization provides an adequate description of two-dimensional field models with fermions[8,9] let us turn to the equivalent bosonic formulation of the Schwinger model. For the fermion currents the correspondence relation reads

$$\varepsilon^{\mu\nu} j_{\nu}(x) = \frac{1}{\sqrt{2\pi}} \partial^{\mu}\varphi \tag{22}$$

and for the fermions themselves[3,10] -

$$\psi(x) = \exp\{i\sqrt{\pi}\,\gamma_5 [\varphi(x) + \sigma(x)]\}\, \chi_0(x)$$
$$\psi^{+}(x) = \chi_0^{+}(x) \exp\{-i\sqrt{\pi}\,\gamma_5 [\varphi(x) + \sigma(x)]\} \tag{23}$$

where χ is a free massless fermion field and φ and σ are massive and massless scalar fields respectively, the latter being quantized with an indefinite metric.

The effective bosonic Hamiltonian then reads

$$H_{eff}^{T} = \frac{1}{2} \int dx \, [\, \Pi^2 + (\partial_1 \tilde{\varphi})^2 + m^2 \tilde{\varphi}^2 \,], \qquad \Pi = \partial_0 \tilde{\varphi} \tag{24}$$

where the scalar field is a combination of the "primary" field from (22),(23) and the topological momentum p

$$\tilde{\varphi}(x) = \varphi(x) - \frac{p}{2\sqrt{\pi}} \tag{25}$$

that is a result of the nontrivial vacuum dynamics of the gauge field.

Hamiltonian (24) is not invariant under ordinary chiral transformations but under simultaneous transformations of the observable fields

$$\eta^{n} H_{eff}^{T} \eta^{-n} = H^{T}$$

$$\eta^{n} = \exp\{in\sqrt{\pi}\,(Q_5 - 2N)\}, \qquad Q_5 = \int dx \, j_{50}(x),$$

which might be considered as chiral transformations for the new composite field $\tilde{\varphi}$ in the topologically nontrivial theory.

From such a point of view it is just the composite field $\tilde{\varphi}$ and not the "primary" one φ that should have a vanishing vacuum expectation value. Therefore, in the Schwinger model we find the following vacuum structure[10]

$$|vac\rangle = exp(-\frac{1}{2}\, i\, p\, Q_5\,)|0\rangle \qquad (26)$$

with $|0\rangle$ being the Fock vacuum for the field φ.

Vacuum (26) is a coherent state of the observable fields. It differs from the usual θ-vacuum, the latter providing an averaging over the vacuum degeneration just at the state level. Contrary to this in the case (26) an average over the topological degeneration is taken only when various physical quantities calculated (i.e. *"in brackets"*). Vacuum structure (26) leads to nonvanishing, in principle, fermion condensates which depend on θ, preserves clasterization property[11] and restricts Wilson expansion only to the zero-Brilluen zone. Note that in this context any relation of the vacuum degeneration with the chiral symmetry breaking is missing.

3. The topological phases destructive interference of two-dimensional fermion fields

Due to relations (20),(23) the Green functions of the model will be generated by a functional Z with an action

$$S[\eta^\lambda,\eta^\lambda,J] = \int d^2x\, [\, \frac{1}{2}\,(\partial_\mu\varphi)^2 - \frac{1}{2}\,(\partial_\mu\sigma)^2 - \frac{2}{2}\,\varphi + i\chi\gamma^\mu\partial_\mu\chi + J\varphi + $$

$$+ \bar{\eta}^\lambda\, exp(i\sqrt{\pi}\,\gamma_5(\varphi + \sigma))\,\chi_o + \bar{\chi}_o\, exp(-i\sqrt{\pi}\,\gamma_5(\varphi+\sigma))\,\eta^\lambda\,]$$

where

$$\eta^\lambda(x) = m(x)\,\eta^T(x), \qquad \bar{\eta}^\lambda(x) = m(x)^{-1}\,\bar{\eta}^T(x) \qquad (27)$$

$$\eta^T(x) = v[A]\,\eta(x), \qquad \bar{\eta}^T(x) = \bar{\eta}(x)\, v[A]^{-1}$$

The source functions are modified as compared with the initial ones $\eta(x)$, $\bar{\eta}(x)$ because of the transition to gauge invariant fermion fields through eq.(20),(7). Note, that the $v[A]$ factor has been included into the "transverse" source function, which is possible because of the absence of a dynamical term for the field A in the resulting Lagrangian.

Thus, only topological phases $m(x)$, $m(x)^{-1}$ enter explicitly the generating functional and cannot be "hidden". When vacuum structure (26) taken into account, this phase factors significantly change the two-point fermion Green function which now reads

$$G(x-y) = exp(-i\%[\Delta_m(x-y) - \Delta_o(x-y)])\; G_o(x-y)\, \langle m(x)^{-1} m(y)\rangle_\theta, \quad (28)$$

where $G_o(x)$, $\Delta_m(x)$ and $\Delta_o(x)$ are the Green function of a free massless fermion field, massive and massless free scalar fields respectively (note, that the last two ones extinguish their infrared divergencies) and by the subscript θ an average over the vacuum degeneration is denoted.

The Green function thus derived differs from the known one by the last factor concerning vacuum degeneration. However, when this factor omitted, at small p the standard Green function has the following assymptotic behaviour

$$\tilde{G}(p)_2 \xrightarrow[p\to 0]{} \frac{\hat{p}}{(p^2+i\varepsilon)^{5/4}}\,,$$

which allows the existence of a particle with the free fermion quantum numbers in the excitation spectrum that contradicts the conclusion about a charge screening in the model.

To consider the effect of the last multiplier in (28) let us introduce an infrared regularization by restricting the space-time volume

$$-\frac{T}{2} \leqslant x_o \leqslant \frac{T}{2}, \qquad -\frac{R}{2} \leqslant x_1 \leqslant \frac{R}{2}$$

Then the solution of eq.(6) with boundary condition (10) can be written explicitly

$$\lambda(x_o | x_1) = 2\pi N(x_o) \frac{x_1}{R}$$

$$\Rightarrow \quad m((x_o | x_1) = \exp(-i\lambda(x_o | x_1)x)) \qquad (29)$$

where $N(x_o)$ is a smooth function with integer boundary values

$$N(\pm \frac{T}{2}) = n_\pm$$

Though $m(x)$ itself is a smooth function (29), after taking an average over degeneration it describes, a singularity appears

$$\langle m(x_o | x_1) \rangle_\theta = \lim_{l \to \infty} \frac{1}{l} \sum_{n=-l/2}^{n=l/2} \langle n \mid n + \frac{x_1}{R} \rangle = \delta_{x_1/R,0}$$

where $\delta_{x_1/R,0}$ is the Croneker symbol.

This singularity does not affect the two-current correlator structure because the phases extinguish each other. There remains the pole at the point $p^2 = e^2/m$, representing the existence of a massive scalar particle in the spectrum.

At the same time the fermion Green function becomes

$$G(p) = \lim_{R,T \to \infty} \lim_{l \to \infty} \frac{1}{l} \int d^2x\, d^2y\, \exp(ip(x-y))\, \tilde{G}(p)$$

$$\sum_{n=-l/2}^{n=l/2} \sum_{s=-\infty}^{s=\infty} \langle n \mid n + s - \frac{1}{R} \rangle \langle n - s + \frac{1}{R} \mid n \rangle =$$

$$= \lim_{R,T \to \infty} \int d^2x\, d^2y\, \exp(ip(x-y))\, \tilde{G}(p)\, \delta_{x_1, y_1} = 0$$

Note that replacing the boundary procedures (removing of the infrared regularization and taking an average over the degeneration) one is left with the old Green function $\tilde{G}$. However, this replacement is inconsistent because of the quantum nature of topological variable N, as has been mentioned in the previous section.

The identical vanishing of the fermion Green function due to the destructive interference of the fermion field topological phases can be considered as a manifestation of the charge screening in the Schwinger model.

Concluding remarks

We have considered some two-dimensional field models in the frames of the minimal quantization method. As a result, the residual vacuum dynamics of the Abelian gauge field in two dimensions has been found out and described with a new pair of canonical variables (topological coordinate and momentum).

It has been shown that θ-vacuum originates in this nontrivial topological vacuum dynamics of the Abelian gauge field and has nothing to do with the chiral symmetry breaking that provides the existence of a field analogy of the Josephson effect.

As fermion fields in this case acquire topological phase factors, the fermion Green function is changed due to the destructive interference of these phases. The identical vanishing of the latter in the momentum space is considered as a manifestation of the charge screening in two-dimensional massless electrodynamics. It is interesting to note that topological structure of the Schwinger model is similar to the one of the Yang-Mills field in four-dimensional space-time. Therefore, the topological mechanism considered might work in this case too, providing a basis for the hadronization picture[7].

ACKNOWLEDGMENTS

One of the authors (N.I.) would like to thank Professor Abdus Salam, International Atomic Energy Agency and UNESCO for kind hospitality at the International Centre for Theoretical Physics - Trieste, where this work has been completed. Fruitful discussions with I.Ya.Aref'eva, M.Asorey, H.Grosse, D.I.Kazakov, H.Nielsen, V.Pavlov, A.S.Schwartz, S.Randjbar-Daemi and D.Stoyanov are kindly acknowledged.

References

1. N.Ilieva, V.N.Pervushin, Sov.J.Nucl.Phys. **39** (1984) 1011.
   N.Ilieva, Int.J.Mod.Phys. **A4** (1989) 4567.
2. Nguyen S.H., V.N.Pervushin, Fortschr.Phys. **37** (1989) 611.
3. N.Ilieva, V.N.Pervushin, Sov.J.Part.Nucl. **22**,no.3 (1991).
4. A.Belavin et al., Phys.Lett. **59B** (1975) 85.
5. S.Coleman, Ann.Phys.(N.Y.) **101** (1976) 239.
6. J.Kiskis, Phys.Rev. **D31** (1985) 2008.
7. V.N.Pervushin, Riv.Nuovo Cim. **8**, no.10 (1985).
8. J.Kogut, L.Susskind, Phys.Rev. **D11** (1975) 3594.
   S.Coleman, R.Jackiw, L.Susskind, Ann.Phys.(N.Y.) **93** (1975) 267.
9. B.Klaiber, Lectures in Theoretical Physics, Boulder Lectures, 1967. Gordon and Breach, New York, 1968, p.141.
10. N.Ilieva, in Proc.7th Warsaw Symposium on Elementary Particle Physics (Kazimierz, May 1984), Warszawa, 1984, p.65.
11. A.N.Tavkhelidze, V.F.Tokarev, Sov.J.Part.Nucl. **16** (1985) 973.

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica