CHAPTER

8

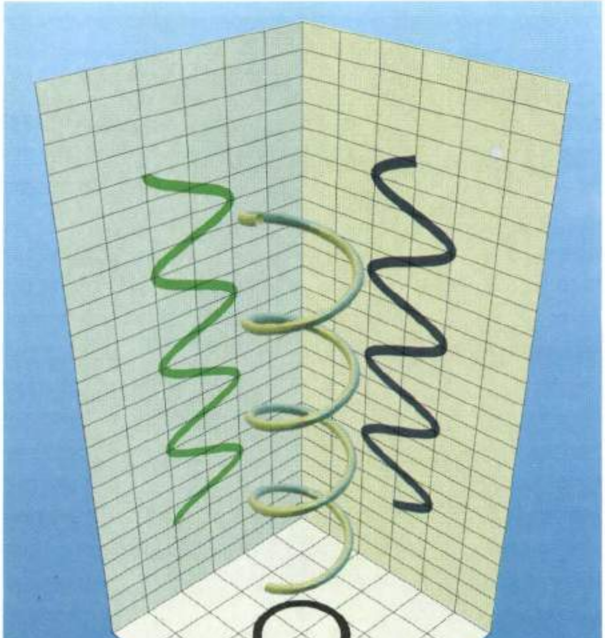
INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

INTRODUCTION

The chapter begins with a review of the inverse trigonometric functions that are studied in trigonometry courses. We next apply methods of calculus to obtain formulas for derivatives and integrals. The fact that function values may be regarded as angles allows us to consider applications such as measuring the rate of change in the angle of elevation as an observer tracks an object in flight, finding the rate at which a searchlight is rotating, and determining an angle that minimizes energy loss as blood flows through a blood vessel.

The hyperbolic functions, defined in Section 8.3, are used in the physical sciences and engineering to describe the shape of a flexible cable that is supported at each end, to find the velocity of an object in a resisting medium such as air or water, and to study the diffusion of radon gas through a basement wall.

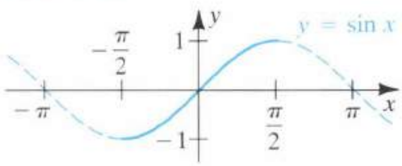
The chapter closes with a discussion of the inverse hyperbolic functions. These functions are used primarily for evaluating certain types of integrals.



8.1 INVERSE TRIGONOMETRIC FUNCTIONS

Since the trigonometric functions are not one-to-one, they do not have inverse functions (see Section 7.1). By restricting their domains, however, we may obtain one-to-one functions that have the same values as the trigonometric functions and that *do* have inverses over these restricted domains.

FIGURE 8.1



Let us first consider the graph of the sine function, whose domain is $\mathbb{R}$ and range is the closed interval $[-1, 1]$ (see Figure 8.1). The sine function is not one-to-one, since a horizontal line such as $y = \frac{1}{2}$ intersects the graph in more than one point. Thus, numbers such as $\pi/6$, $5\pi/6$, and $-7\pi/6$ yield the same function value, $\frac{1}{2}$. If we restrict the domain to $[-\pi/2, \pi/2]$, then, as illustrated by the solid portion of the graph in Figure 8.1, we obtain an increasing function that takes on every value of the sine function once and only once. This new function, with domain $[-\pi/2, \pi/2]$ and range $[-1, 1]$, is continuous and increasing and hence, by Theorem (7.6), has an inverse function that is continuous and increasing. The inverse function has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. This leads to the following definition.

Definition **(8.1)**

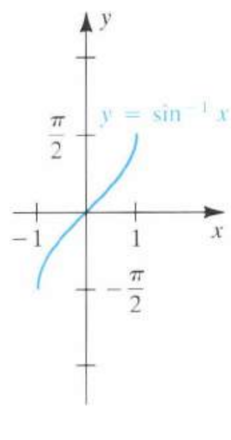
> The **inverse sine function**, denoted $\sin^{-1}$, is defined by
> $$y = \sin^{-1} x \quad \text{if and only if} \quad x = \sin y$$
> for $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.

The inverse sine function is also called the **arcsine function**, and arcsin x is often used in place of $\sin^{-1} x$. The -1 in $\sin^{-1}$ is not to be regarded as an exponent, but rather as a means of denoting this inverse function. The notation $y = \sin^{-1} x$ may be read *y is the inverse sine of x*. The equation $x = \sin y$ in the definition allows us to regard y as an angle, and hence $y = \sin^{-1} x$ may also be read *y is the angle whose sine is x*. Observe that, by Definition (8.1),

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}, \quad \text{or} \quad -\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}.$$

ILLUSTRATION

FIGURE 8.2



- If $y = \sin^{-1} \frac{1}{2}$, then $\sin y = \frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Hence $y = \frac{\pi}{6}$.
- If $y = \arcsin\left(-\frac{1}{2}\right)$, then $\sin y = -\frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Hence $y = -\frac{\pi}{6}$.

Using the method we introduced in Section 7.1 for sketching the graph of an inverse function, we can sketch the graph of $y = \sin^{-1} x$ by reflecting the solid portion of Figure 8.1 through the line $y = x$. This gives us Figure 8.2. We could also use the equation $x = \sin y$ with $-\pi/2 \le y \le \pi/2$ to find points on the graph.

The relationships $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ that hold for any inverse function f^{-1} give us the following properties.

Properties of $\sin^{-1}$ **(8.2)**

(i) $\sin(\sin^{-1} x) = \sin(\arcsin x) = x$ if $-1 \le x \le 1$

(ii) $\sin^{-1}(\sin x) = \arcsin(\sin x) = x$ if $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

ILLUSTRATION

- $\sin\left(\sin^{-1}\dfrac{1}{2}\right) = \dfrac{1}{2}$ since $-1 < \dfrac{1}{2} < 1$
- $\arcsin\left(\sin\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ since $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$
- $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

Be careful when using (8.2). In the third part of the preceding illustration, $2\pi/3$ is *not* between $-\pi/2$ and $\pi/2$, and hence we cannot use (ii) of (8.2). Instead, we use properties of special angles (see Section 1.3) to first evaluate $\sin(2\pi/3)$ and then find $\sin^{-1}(\sqrt{3}/2)$.

FIGURE 8.3

We may use the other five trigonometric functions to define inverse trigonometric functions. If the domain of the cosine function is restricted to the interval $[0, \pi]$ (see the solid portion of Figure 8.3), we obtain a one-to-one continuous decreasing function that has a continuous decreasing inverse function. This leads to the next definition.

Definition **(8.3)**

The **inverse cosine function**, denoted $\cos^{-1}$, is defined by

$$y = \cos^{-1} x \quad \text{if and only if} \quad x = \cos y$$

for $-1 \le x \le 1$ and $0 \le y \le \pi$.

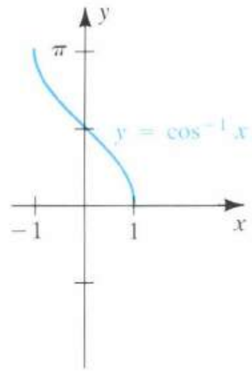
The domain of the inverse cosine function is $[-1, 1]$, and the range is $[0, \pi]$. The notation $y = \cos^{-1} x$ may be read *y is the inverse cosine of x* or *y is the angle whose cosine is x*. The inverse cosine function is also called the **arccosine function**, and the notation $\arccos x$ is used interchangeably with $\cos^{-1} x$.

ILLUSTRATION

- If $y = \cos^{-1}\dfrac{1}{2}$, then $\cos y = \dfrac{1}{2}$ and $0 \le y \le \pi$. Hence $y = \dfrac{\pi}{3}$.
- If $y = \arccos\left(-\dfrac{1}{2}\right)$, then $\cos y = -\dfrac{1}{2}$ and $0 \le y \le \pi$. Hence $y = \dfrac{2\pi}{3}$.

The graph of the inverse cosine function may be found by reflecting the solid portion of Figure 8.3 through the line $y = x$. This gives us the sketch in Figure 8.4. We could also use the equation $x = \cos y$ with $0 \le y \le \pi$ to find points on the graph.

FIGURE 8.4



Since cos and $\cos^{-1}$ are inverse functions of each other, we obtain the following properties.

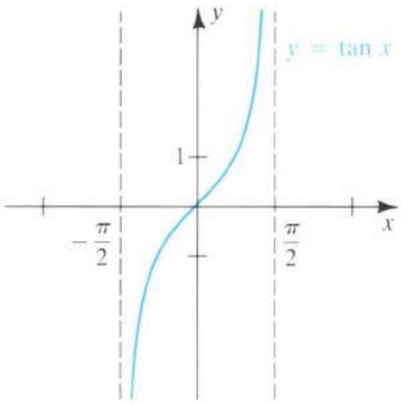
Properties of $\cos^{-1}$ **(8.4)**

(i) $\cos(\cos^{-1} x) = \cos(\arccos x) = x$ if $-1 \le x \le 1$

(ii) $\cos^{-1}(\cos x) = \arccos(\cos x) = x$ if $0 \le x \le \pi$

ILLUSTRATION

FIGURE 8.5
$y = \tan x,\ -\pi/2 \le x \le \pi/2$



- $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}$ since $-1 < -\frac{1}{2} < 1$
- $\arccos\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ since $0 < \frac{2\pi}{3} < \pi$
- $\cos^{-1}\left[\cos\left(-\frac{\pi}{4}\right)\right] = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Note that in the third part of the preceding illustration, $-\pi/4$ is *not* between 0 and π, and hence we cannot use (ii) of (8.4). Instead, we first evaluate $\cos(-\pi/4)$ and then find $\cos^{-1}(\sqrt{2}/2)$.

If we restrict the domain of the tangent function to the open interval $(-\pi/2, \pi/2)$, we obtain a continuous increasing function (see Figure 8.5). We use this *new* function to define the *inverse tangent function.*

Definition **(8.5)**

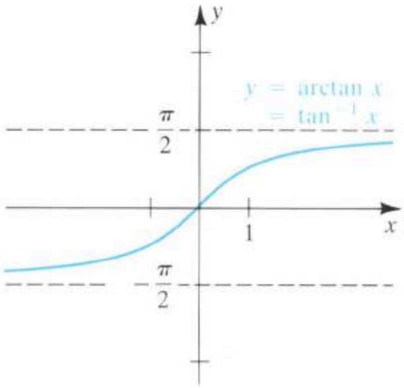
The **inverse tangent function**, or **arctangent function**, denoted by $\tan^{-1}$, or arctan, is defined by

$$y = \tan^{-1} x = \arctan x \quad \text{if and only if} \quad x = \tan y$$

for every x and $-\pi/2 < y < \pi/2$.

The domain of the arctangent function is $\mathbb{R}$, and the range is the open interval $(-\pi/2, \pi/2)$. We can obtain the graph of $y = \tan^{-1} x$ in Figure 8.6 by reflecting the graph in Figure 8.5 through the line $y = x$.

FIGURE 8.6



As with $\sin^{-1}$ and $\cos^{-1}$, we have the following.

Properties of $\tan^{-1}$ (8.6)

(i) $\tan(\tan^{-1} x) = \tan(\arctan x) = x$ for every x

(ii) $\tan^{-1}(\tan x) = \arctan(\tan x) = x$ if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

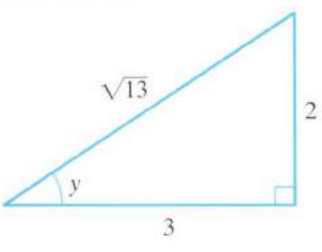
ILLUSTRATION

- If $y = \arctan(-1)$, then $\tan y = -1$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

  Hence $y = -\dfrac{\pi}{4}$.
- $\tan(\tan^{-1} 1000) = 1000$ by (8.6)(i)
- $\tan^{-1}\left(\tan \dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ since $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$
- $\arctan(\tan \pi) = \arctan 0 = 0$

EXAMPLE 1 Find the exact value of $\sec(\arctan \frac{2}{3})$.

FIGURE 8.7

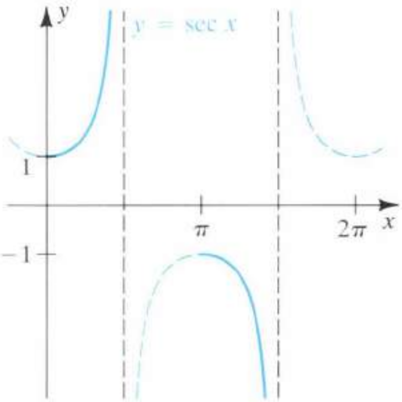


SOLUTION If we let $y = \arctan \frac{2}{3}$, then $\tan y = \frac{2}{3}$. We wish to find $\sec y$. Since $-\pi/2 < \arctan x < \pi/2$ for every x and $\tan y > 0$, it follows that $0 < y < \pi/2$. Thus, we may regard y as the radian measure of an angle of a right triangle such that $\tan y = \frac{2}{3}$, as illustrated in Figure 8.7. By the Pythagorean theorem, the hypotenuse is $\sqrt{3^2 + 2^2} = \sqrt{13}$. Referring to the triangle, we obtain

$$\sec\left(\arctan \frac{2}{3}\right) = \sec y = \frac{\sqrt{13}}{3}.$$

If we consider the graph of $y = \sec x$, there are many ways to restrict x so that we obtain a one-to-one function that takes on every value of the secant function. There is no universal agreement on how this should be done. It is convenient to restrict x to the intervals $[0, \pi/2)$ and $[\pi, 3\pi/2)$, as indicated by the solid portion of the graph of $y = \sec x$ in Figure 8.8, instead of to the "more natural" intervals $[0, \pi/2)$ and $(\pi/2, \pi]$, because the differentiation formula for the inverse secant is simpler. We show in the next section that $D_x \sec^{-1} x = 1/(x\sqrt{x^2 - 1})$. Thus, the slope of the tangent line to the graph of $y = \sec^{-1} x$ is negative if $x < -1$ or positive if $x > 1$. For the more natural intervals, the slope is always positive, and we would have $D_x \sec^{-1} x = 1/(|x|\sqrt{x^2 - 1})$.

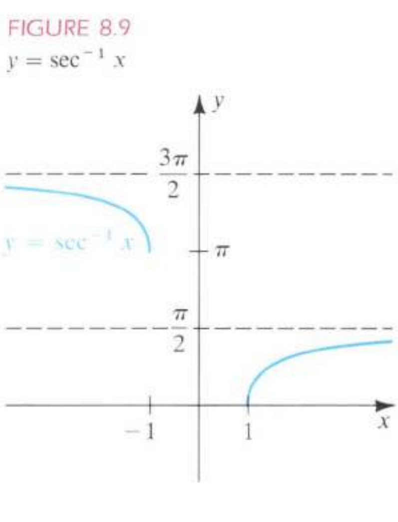
FIGURE 8.8
$y = \sec x$



Definition **(8.7)**

> The **inverse secant function**, or **arcsecant function**, denoted by $\sec^{-1}$, or arcsec, is defined by
>
> $$y = \sec^{-1} x = \operatorname{arcsec} x \quad \text{if and only if} \quad x = \sec y$$
>
> for $|x| \geq 1$ and y in $[0, \pi/2)$ or in $[\pi, 3\pi/2)$.

FIGURE 8.9
$y = \sec^{-1} x$



The graph of $y = \sec^{-1} x$ is sketched in Figure 8.9.

The inverse cotangent function, $\cot^{-1}$, and inverse cosecant function, $\csc^{-1}$, can be defined in similar fashion (see Exercises 31–32).

The following examples illustrate some of the manipulations that can be carried out with inverse trigonometric functions.

EXAMPLE 2 Find the exact value of $\sin(\arctan \frac{1}{2} - \arccos \frac{4}{5})$.

SOLUTION If we let

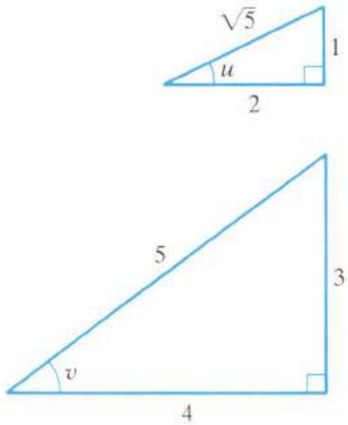
$$u = \arctan \tfrac{1}{2} \quad \text{and} \quad v = \arccos \tfrac{4}{5},$$

then

$$\tan u = \tfrac{1}{2} \quad \text{and} \quad \cos v = \tfrac{4}{5}.$$

We wish to find $\sin(u - v)$. Since u and v are in the interval $(0, \pi/2)$, they can be considered as the radian measures of positive acute angles, and

FIGURE 8.10



we may refer to the right triangles in Figure 8.10. This gives us

$$\sin u = \frac{1}{\sqrt{5}}, \qquad \cos u = \frac{2}{\sqrt{5}}, \qquad \sin v = \frac{3}{5}, \qquad \cos v = \frac{4}{5}.$$

Using the subtraction formula for the sine function, we obtain

$$\begin{aligned}\sin(u - v) &= \sin u \cos v - \cos u \sin v \\ &= \frac{1}{\sqrt{5}}\frac{4}{5} - \frac{2}{\sqrt{5}}\frac{3}{5} \\ &= -\frac{2}{5\sqrt{5}} = -\frac{2\sqrt{5}}{25}.\end{aligned}$$

EXAMPLE 3 If $-1 \le x \le 1$, rewrite $\cos(\sin^{-1} x)$ as an algebraic expression in x.

SOLUTION Let

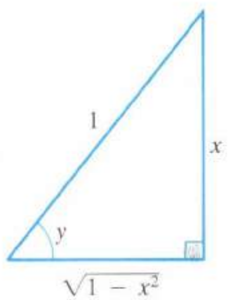
$$y = \sin^{-1} x, \quad \text{or, equivalently,} \quad \sin y = x.$$

We wish to express $\cos y$ in terms of x. Since $-\pi/2 \le y \le \pi/2$, it follows that $\cos y \ge 0$, and hence

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Consequently $$\cos(\sin^{-1} x) = \sqrt{1 - x^2}.$$

FIGURE 8.11
$y = \sin^{-1} x$



The last identity can also be seen geometrically if $0 < x < 1$. In this case $0 < y < \pi/2$, and we may regard y as the radian measure of an angle of a right triangle such that $\sin y = x$, as illustrated in Figure 8.11. (The side of length $\sqrt{1 - x^2}$ is found by using the Pythagorean theorem.) Referring to the triangle, we have

$$\cos(\sin^{-1} x) = \cos y = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

EXAMPLE 4 Find the solutions of $5 \sin^2 t + 3 \sin t - 1 = 0$ that are in the interval $[-\pi/2, \pi/2]$.

SOLUTION The equation may be regarded as a quadratic equation in $\sin t$. Applying the quadratic formula yields

$$\sin t = \frac{-3 \pm \sqrt{9 + 20}}{10} = \frac{-3 \pm \sqrt{29}}{10}.$$

Using the definition of the inverse sine function, we obtain the following solutions:

$$t = \sin^{-1} \tfrac{1}{10}(-3 + \sqrt{29}) \approx 0.2408$$

$$t = \sin^{-1} \tfrac{1}{10}(-3 - \sqrt{29}) \approx -0.9946$$

EXERCISES 8.1

Exer. 1–18: Find the exact value of the expression, whenever it is defined.

1 (a) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ (b) $\cos^{-1}\left(-\frac{1}{2}\right)$
(c) $\tan^{-1}(-\sqrt{3})$

2 (a) $\sin^{-1}\left(-\frac{1}{2}\right)$ (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
(c) $\tan^{-1}(-1)$

3 (a) $\arcsin\frac{\sqrt{3}}{2}$ (b) $\arccos\frac{\sqrt{2}}{2}$
(c) $\arctan\frac{1}{\sqrt{3}}$

4 (a) $\arcsin 0$ (b) $\arccos(-1)$
(c) $\arctan 0$

5 (a) $\sin^{-1}\frac{\pi}{3}$ (b) $\cos^{-1}\frac{\pi}{2}$
(c) $\tan^{-1} 1$

6 (a) $\arcsin\frac{\pi}{2}$ (b) $\arccos\frac{\pi}{3}$
(c) $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

7 (a) $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$ (b) $\cos\left(\arccos\frac{1}{2}\right)$
(c) $\tan(\arctan 14)$

8 (a) $\sin\left(\sin^{-1}\frac{2}{3}\right)$ (b) $\cos\left[\cos^{-1}\left(-\frac{1}{5}\right)\right]$
(c) $\tan[\tan^{-1}(-9)]$

9 (a) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ (b) $\cos^{-1}\left(\cos\frac{5\pi}{6}\right)$
(c) $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

10 (a) $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$ (b) $\arccos(\cos 0)$
(c) $\arctan\left(\tan\frac{\pi}{4}\right)$

11 (a) $\arcsin\left(\sin\frac{5\pi}{4}\right)$ (b) $\arccos\left(\cos\frac{5\pi}{4}\right)$
(c) $\arctan\left(\tan\frac{7\pi}{4}\right)$

12 (a) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ (b) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$
(c) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

13 (a) $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cos(\tan^{-1} 1)$
(c) $\tan[\sin^{-1}(-1)]$

14 (a) $\sin(\tan^{-1}\sqrt{3})$ (b) $\cos(\sin^{-1} 1)$
(c) $\tan(\cos^{-1} 0)$

15 (a) $\cot\left(\sin^{-1}\frac{2}{3}\right)$ (b) $\sec\left[\tan^{-1}\left(-\frac{3}{5}\right)\right]$
(c) $\csc\left[\cos^{-1}\left(-\frac{1}{4}\right)\right]$

16 (a) $\cot\left[\sin^{-1}\left(-\frac{2}{5}\right)\right]$ (b) $\sec\left(\tan^{-1}\frac{7}{4}\right)$
(c) $\csc\left(\cos^{-1}\frac{1}{5}\right)$

17 (a) $\sin\left(\arcsin\frac{1}{2}+\arccos 0\right)$
(b) $\cos\left[\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right]$
(c) $\tan\left(\arctan\frac{4}{3}+\arccos\frac{8}{17}\right)$

18 (a) $\sin\left[\sin^{-1}\frac{5}{13}-\cos^{-1}\left(-\frac{3}{5}\right)\right]$
(b) $\cos\left(\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{4}\right)$
(c) $\tan\left[\cos^{-1}\frac{1}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Exer. 19–22: Rewrite as an algebraic expression in x for $x > 0$.

19 $\sin(\tan^{-1} x)$ 20 $\tan(\arccos x)$

21 $\sec\left(\sin^{-1}\frac{x}{3}\right)$ 22 $\cot\left(\sin^{-1}\frac{1}{x}\right)$

Exer. 23–30: Sketch the graph of the equation.

23 $y=\sin^{-1} 2x$ 24 $y=\frac{1}{2}\sin^{-1} x$

25 $y=\cos^{-1}\frac{1}{2}x$ 26 $y=2\cos^{-1} x$

27 $y=2\tan^{-1} x$ 28 $y=\tan^{-1} 2x$

29 $y=\sin(\arccos x)$ 30 $y=\sin(\sin^{-1} x)$

31 (a) Define $\cot^{-1}$ by restricting the domain of the cotangent function to the interval $(0, \pi)$.
(b) Sketch the graph of $y=\cot^{-1} x$.

32 (a) Define $\csc^{-1}$ by restricting the domain of the cosecant function to $[-\pi/2, 0) \cup (0, \pi/2]$.

(b) Sketch the graph of $y = \csc^{-1} x$.

Exer. 33–36: (a) Use inverse trigonometric functions to find the solutions of the equation that are in the given interval. (b) Approximate the solutions to four decimal places.

33 $2\tan^2 t + 9\tan t + 3 = 0$; $(-\pi/2, \pi/2)$

34 $3\sin^2 t + 7\sin t + 3 = 0$; $[-\pi/2, \pi/2]$

35 $15\cos^4 x - 14\cos^2 x + 3 = 0$; $[0, \pi]$

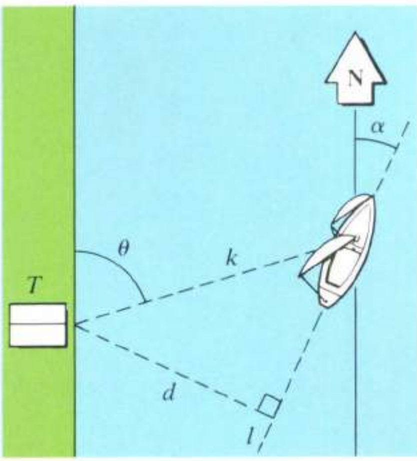
36 $3\tan^4 \theta - 19\tan^2 \theta + 2 = 0$; $(-\pi/2, \pi/2)$

37 As shown in the figure, a sailboat is following a straight-line course l. The shortest distance from a tracking station T to the course is d miles. As the boat sails, the tracking station records its distance k from T and its direction θ with respect to T. Angle α specifies the direction of the sailboat.

(a) Express α in terms of d, k, and θ.

(b) Estimate α to the nearest degree if $d = 50$ mi and $k = 210$ mi and $\theta = 53.4°$.

EXERCISE 37



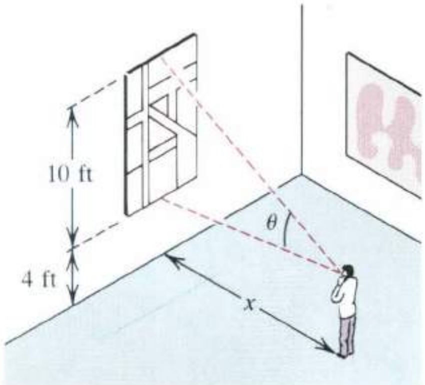
38 An art critic whose eye level is 6 feet above the floor views a painting that is 10 feet in height and is mounted 4 feet above the floor, as shown in the figure.

(a) If the critic is standing x feet from the wall, express the viewing angle θ in terms of x.

(b) Use the addition formula for the tangent to show that $\theta = \tan^{-1}\left(\dfrac{10x}{x^2 - 16}\right)$.

(c) For what value of x is $\theta = 45°$?

EXERCISE 38



39 The only inverse trigonometric function available in some computer languages is $\tan^{-1}$. In BASIC, this function is denoted by ATN(X). Express the following in terms of $\tan^{-1}$.

(a) $\sin^{-1} x$ for $|x| < 1$

(b) $\cos^{-1} x$ for $|x| < 1$ and $x \neq 0$

40 If $-1 \leq x \leq 1$, is it always possible to find $\sin^{-1}(\sin^{-1} x)$ by pressing the calculator key sequence INV SIN twice? If not, determine the permissible values of x.

8.2 DERIVATIVES AND INTEGRALS

In this section we shall concentrate on the inverse sine, cosine, tangent, and secant functions. Formulas for their derivatives and for integrals that result in inverse trigonometric functions are listed in the next two theorems, with $u = g(x)$ differentiable and x restricted to values for which the indicated expressions have meaning. You may find it surprising to learn that although we used trigonometric functions to define inverse trigonometric functions, their derivatives are *algebraic* functions.

Theorem **(8.8)**

(i) $D_x \sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}} D_x u$

(ii) $D_x \cos^{-1} u = -\dfrac{1}{\sqrt{1-u^2}} D_x u$

(iii) $D_x \tan^{-1} u = \dfrac{1}{1+u^2} D_x u$

(iv) $D_x \sec^{-1} u = \dfrac{1}{u\sqrt{u^2-1}} D_x u$

PROOF We shall consider only the special case $u = x$, since the formulas for $u = g(x)$ may then be obtained by applying the chain rule.

If we let $f(x) = \sin x$ and $g(x) = \sin^{-1} x$ in Theorem (7.7), then it follows that the inverse sine function g is differentiable if $|x| < 1$. We shall use implicit differentiation to find $g'(x)$. First note that the equations

$$y = \sin^{-1} x \quad \text{and} \quad \sin y = x$$

are equivalent if $-1 < x < 1$ and $-\pi/2 < y < \pi/2$. Differentiating $\sin y = x$ implicitly, we have

$$\cos y \, D_x y = 1$$

and hence
$$D_x \sin^{-1} x = D_x y = \frac{1}{\cos y}.$$

Since $-\pi/2 < y < \pi/2$, $\cos y$ is positive and, therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Thus,
$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

for $|x| < 1$. The inverse sine function is not differentiable at ± 1. This fact is evident from Figure 8.2, since vertical tangent lines occur at the endpoints of the graph.

The formula for $D_x \cos^{-1} x$ can be obtained in similar fashion.

It follows from Theorem (7.7) that the inverse tangent function is differentiable at every real number. Let us consider the equivalent equations

$$y = \tan^{-1} x \quad \text{and} \quad \tan y = x$$

for $-\pi/2 < y < \pi/2$. Differentiating $\tan y = x$ implicitly, we have

$$\sec^2 y \, D_x y = 1.$$

Consequently,
$$D_x \tan^{-1} x = D_x y = \frac{1}{\sec^2 y}.$$

Using the fact that $\sec^2 y = 1 + \tan^2 y = 1 + x^2$ gives us

$$D_x \tan^{-1} x = \frac{1}{1+x^2}.$$

Finally, consider the equivalent equations

$$y = \sec^{-1} x \quad \text{and} \quad \sec y = x$$

for y in either $(0, \pi/2)$ or $(\pi, 3\pi/2)$. Differentiating $\sec y = x$ implicitly yields

$$\sec y \tan y \, D_x \, y = 1.$$

Since $0 < y < \pi/2$ or $\pi < y < 3\pi/2$, it follows that $\sec y \tan y \neq 0$ and, hence,

$$D_x \sec^{-1} x = D_x \, y = \frac{1}{\sec y \tan y}.$$

Using the fact that $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$, we obtain

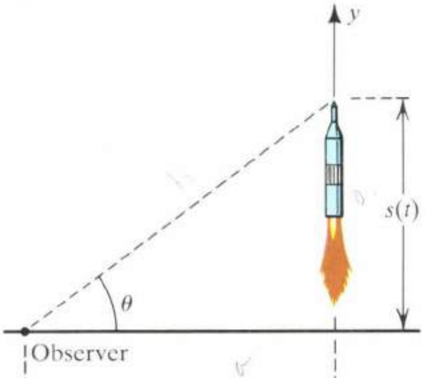
$$D_x \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$$

for $|x| > 1$. The inverse secant function is not differentiable at $x = \pm 1$. Note that the graph has vertical tangent lines at the points with these x-coordinates (see Figure 8.9).

ILLUSTRATION

$f(x)$	$f'(x)$
$\sin^{-1} 3x$	$\dfrac{1}{\sqrt{1 - (3x)^2}} D_x (3x) = \dfrac{3}{\sqrt{1 - 9x^2}}$
$\arccos (\ln x)$	$-\dfrac{1}{\sqrt{1 - (\ln x)^2}} D_x \ln x = -\dfrac{1}{x\sqrt{1 - (\ln x)^2}}$
$\tan^{-1} e^{2x}$	$\dfrac{1}{1 + (e^{2x})^2} D_x \, e^{2x} = \dfrac{2e^{2x}}{1 + e^{4x}}$
$\operatorname{arcsec} (x^2)$	$\dfrac{1}{x^2\sqrt{(x^2)^2 - 1}} D_x (x^2) = \dfrac{2}{x\sqrt{x^4 - 1}}$

FIGURE 8.12



EXAMPLE 1 A rocket is fired directly upward with initial velocity 0 and burns fuel at a rate that produces a constant acceleration of 50 ft/sec^2 for $0 \leq t \leq 5$, with time t in seconds. As illustrated in Figure 8.12, an observer 400 feet from the launching pad visually follows the flight of the rocket.

(a) Express the angle of elevation θ of the rocket as a function of t.

(b) The observer perceives the rocket to be rising fastest when $d\theta/dt$ is largest. (Of course, this is an illusion, since the velocity is steadily increasing.) Determine the height of the rocket at the moment of perceived maximum velocity.

SOLUTION

(a) Let $s(t)$ denote the height of the rocket at time t (see Figure 8.12). The fact that the acceleration is always 50 gives us the differential equation

$$s''(t) = 50,$$

subject to the initial conditions $s'(0) = 0$ and $s(0) = 0$. Integrating with respect to t, we obtain

$$\int s''(t)\,dt = \int 50\,dt$$
$$s'(t) = 50t + C$$

for some constant C. Substituting $t = 0$ and using $s'(0) = 0$ gives us $0 = 50(0) + C$, or $C = 0$. Hence

$$s'(t) = 50t.$$

Integrating again, we have

$$\int s'(t)\,dt = \int 50t\,dt$$
$$s(t) = 25t^2 + D$$

for some constant D. If we substitute $t = 0$ and use $s(0) = 0$, we obtain $0 = 25(0) + D$, or $D = 0$. Hence

$$s(t) = 25t^2.$$

Referring to Figure 8.12, with $s(t) = 25t^2$, we find

$$\tan\theta = \frac{25t^2}{400} = \frac{t^2}{16}, \quad \text{or} \quad \theta = \arctan\frac{t^2}{16}.$$

(b) By Theorem (8.8), the rate of change of θ with respect to t is

$$\frac{d\theta}{dt} = \frac{1}{1 + (t^2/16)^2}\left(\frac{2t}{16}\right) = \frac{32t}{256 + t^4}.$$

Since we wish to find the maximum value of $d\theta/dt$, we begin by finding the critical numbers of $d\theta/dt$. Using the quotient rule, we obtain

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \frac{(256 + t^4)(32) - 32t(4t^3)}{(256 + t^4)^2} = \frac{32(256 - 3t^4)}{(256 + t^4)^2}.$$

Considering $d^2\theta/dt^2 = 0$ gives us the critical number $t = \sqrt[4]{256/3}$. It follows from the first (or second) derivative test that $d\theta/dt$ has a maximum value at $t = \sqrt[4]{256/3} \approx 3.04$ sec. The height of the rocket at this time is

$$s(\sqrt[4]{256/3}) = 25(\sqrt[4]{256/3})^2 = 25\sqrt{256/3} \approx 230.9 \text{ ft}.$$

We may use differentiation formulas (i), (ii), and (iv) of Theorem (8.7) to obtain the following integration formulas:

(1) $$\int \frac{1}{\sqrt{1 - u^2}}\,du = \sin^{-1} u + C$$

(2) $$\int \frac{1}{1 + u^2}\,du = \tan^{-1} u + C$$

(3) $$\int \frac{1}{u\sqrt{u^2 - 1}}\,du = \sec^{-1} u + C$$

These formulas can be generalized as follows for $a > 0$.

Theorem (8.9)

(i) $$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1} \frac{u}{a} + C$$

(ii) $$\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

(iii) $$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$

PROOF Let us prove (ii). As usual, it is sufficient to consider the case $u = x$. We begin by writing

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a^2} \int \frac{1}{1 + (x/a)^2}\, dx.$$

Next we make the substitution $v = x/a$, $dv = (1/a)\, dx$. Introducing the factor $1/a$ in the integrand, compensating by multiplying the integral by a, and using formula (2), preceding this theorem, gives us the following:

$$\begin{aligned}
\int \frac{1}{a^2 + x^2}\, dx &= \frac{1}{a} \int \frac{1}{1 + (x/a)^2} \cdot \frac{1}{a}\, dx \\
&= \frac{1}{a} \int \frac{1}{1 + v^2}\, dv \\
&= \frac{1}{a} \tan^{-1} v + C \\
&= \frac{1}{a} \tan^{-1} \frac{x}{a} + C
\end{aligned}$$

The remaining formulas may be proved in similar fashion.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\, dx.$

SOLUTION The integral may be written as in the first formula of Theorem (8.9) by letting $a = 1$ and using the substitution

$$u = e^{2x}, \qquad du = 2e^{2x}\, dx.$$

We introduce a factor 2 in the integrand and proceed as follows:

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\, dx &= \frac{1}{2} \int \frac{1}{\sqrt{1 - (e^{2x})^2}}\, 2e^{2x}\, dx \\
&= \frac{1}{2} \int \frac{1}{\sqrt{1 - u^2}}\, du \\
&= \tfrac{1}{2} \sin^{-1} u + C \\
&= \tfrac{1}{2} \sin^{-1} e^{2x} + C
\end{aligned}$$

EXAMPLE 3 Evaluate $\int \frac{x^2}{5 + x^6}\,dx$.

SOLUTION The integral may be written as in the second formula of Theorem (8.9) by letting $a^2 = 5$ and using the substitution

$$u = x^3, \qquad du = 3x^2\,dx.$$

We introduce a factor 3 in the integrand and proceed as follows:

$$\begin{aligned}
\int \frac{x^2}{5 + x^6}\,dx &= \frac{1}{3}\int \frac{1}{5 + (x^3)^2}\,3x^2\,dx \\
&= \frac{1}{3}\int \frac{1}{(\sqrt{5})^2 + u^2}\,du \\
&= \frac{1}{3}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + C \\
&= \frac{\sqrt{5}}{15}\tan^{-1}\frac{x^3}{\sqrt{5}} + C
\end{aligned}$$

EXAMPLE 4 Evaluate $\int \frac{1}{x\sqrt{x^4 - 9}}\,dx$.

SOLUTION The integral may be written as in Theorem (8.9)(iii) by letting $a^2 = 9$ and using the substitution

$$u = x^2, \qquad du = 2x\,dx.$$

We introduce $2x$ in the integrand by multiplying numerator and denominator by $2x$ and then proceed as follows:

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^4 - 9}}\,dx &= \int \frac{1}{2x \cdot x\sqrt{(x^2)^2 - 3^2}}\,2x\,dx \\
&= \frac{1}{2}\int \frac{1}{u\sqrt{u^2 - 3^2}}\,du \\
&= \frac{1}{2}\cdot\frac{1}{3}\sec^{-1}\frac{u}{3} + C \\
&= \frac{1}{6}\sec^{-1}\frac{x^2}{3} + C
\end{aligned}$$

EXERCISES 8.2

Exer. 1–26: Find $f'(x)$ if $f(x)$ is the given expression.

1 $\sin^{-1}\sqrt{x}$

2 $\sin^{-1}\frac{1}{3}x$

3 $\tan^{-1}(3x - 5)$

4 $\tan^{-1}(x^2)$

5 $e^{-x}\,\text{arcsec}\,e^{-x}$

6 $\sqrt{\text{arcsec}\,3x}$

7 $x^2\arctan(x^2)$

8 $\tan^{-1}\sin 2x$

9 $\sec^{-1}\sqrt{x^2 - 1}$

10 $x^2\sec^{-1}5x$

11 $\dfrac{1}{\sin^{-1}x}$

12 $\arcsin\ln x$

13 $(1 + \cos^{-1}3x)^3$

14 $\cos^{-1}\cos e^x$

15 $\ln\arctan(x^2)$

16 $\arctan\dfrac{x+1}{x-1}$

17 $\cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1} x$

18 $x \arccos \sqrt{4x+1}$

19 $3^{\arcsin(x^3)}$

20 $\left(\dfrac{1}{x} - \arcsin \dfrac{1}{x}\right)^4$

21 $\dfrac{\arctan x}{x^2+1}$

22 $\dfrac{e^{2x}}{\sin^{-1} 5x}$

23 $\sqrt{x} \sec^{-1} \sqrt{x}$

24 $(\sin 2x)(\sin^{-1} 2x)$

25 $(\tan x)^{\arctan x}$

26 $(\tan^{-1} 4x)e^{\tan^{-1} 4x}$

Exer. 27–28: Find y'.

27 $x^2 + x \sin^{-1} y = ye^x$

28 $\ln(x+y) = \tan^{-1} xy$

Exer. 29–44: Evaluate the integral.

29 (a) $\displaystyle\int \frac{1}{x^2+16}\,dx$ (b) $\displaystyle\int_0^4 \frac{1}{x^2+16}\,dx$

30 (a) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1} e^x$ (b) $\displaystyle\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$

31 (a) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ (b) $\displaystyle\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^4}}\,dx$

32 (a) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx$ (b) $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$

33 $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,dx$

34 $\displaystyle\int \frac{\cos x}{\sqrt{9-\sin^2 x}}\,dx$

35 $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx$

36 $\displaystyle\int \frac{1}{e^x\sqrt{1-e^{-2x}}}\,dx$

37 $\displaystyle\int \frac{e^x}{\sqrt{16-e^{2x}}}\,dx$

38 $\displaystyle\int \frac{\sec x \tan x}{1+\sec^2 x}\,dx$

39 $\displaystyle\int \frac{1}{x\sqrt{x^6-4}}\,dx$

40 $\displaystyle\int \frac{x}{\sqrt{36-x^2}}\,dx$

41 $\displaystyle\int \frac{x}{x^2+9}\,dx$

42 $\displaystyle\int \frac{1}{x\sqrt{x-1}}\,dx$

43 $\displaystyle\int \frac{1}{\sqrt{e^{2x}-25}}\,dx$

44 $\displaystyle\int \frac{e^x}{\sqrt{4-e^x}}\,dx$

45 The floor of a storage shed has the shape of a right triangle. The sides opposite and adjacent to an acute angle θ of the triangle are measured as 10 feet and 7 feet, respectively, with a possible error of ± 0.5 inch in the 10-foot measurement. Use the differential of an inverse trigonometric function to approximate the error in the calculated value of θ.

46 Use differentials to approximate the arc length of the graph of $y = \tan^{-1} x$ from $A(0,0)$ to $B(0.1, \tan^{-1} 0.1)$.

47 An airplane at a constant altitude of 5 miles and a speed of 500 mi/hr is flying in a direction away from an observer on the ground. Use inverse trigonometric functions to find the rate at which the angle of elevation is changing when the airplane flies over a point 2 miles from the observer.

48 A searchlight located $\frac{1}{8}$ mile from the nearest point P on a straight road is trained on an automobile traveling on the road at a rate of 50 mi/hr. Use inverse trigonometric functions to find the rate at which the searchlight is rotating when the car is $\frac{1}{4}$ mile from P.

49 A billboard 20 feet high is located on top of a building, with its lower edge 60 feet above the level of a viewer's eye. Use inverse trigonometric functions to find how far from a point directly below the sign a viewer should stand to maximize the angle between the lines of sight of the top and bottom of the billboard (see Example 8 of Section 4.5).

50 The velocity, at time t, of a point moving on a coordinate line is $(1+t^2)^{-1}$ ft/sec. If the point is at the origin at $t = 0$, find its position at the instant that the acceleration and the velocity have the same absolute value.

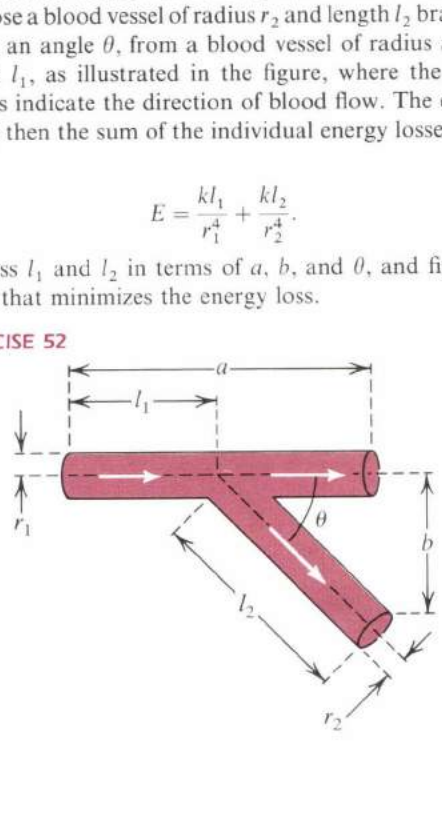
51 A missile is fired vertically from a point that is 5 miles from a tracking station and at the same elevation. For the first 20 seconds of flight, its angle of elevation changes at a constant rate of 2° per second. Use inverse trigonometric functions to find the velocity of the missile when the angle of elevation is 30°.

52 Blood flowing through a blood vessel causes a loss of energy due to friction. According to *Poiseuille's law*, this energy loss E is given by $E = kl/r^4$, where r is the radius of the blood vessel, l is the length, and k is a constant. Suppose a blood vessel of radius r_2 and length l_2 branches off, at an angle θ, from a blood vessel of radius r_1 and length l_1, as illustrated in the figure, where the white arrows indicate the direction of blood flow. The energy loss is then the sum of the individual energy losses; that is,

$$E = \frac{kl_1}{r_1^4} + \frac{kl_2}{r_2^4}.$$

Express l_1 and l_2 in terms of a, b, and θ, and find the angle that minimizes the energy loss.

EXERCISE 52



c 53 Use Simpson's rule, with $n = 4$, to approximate the arc length of the graph of $y = \arcsin x$ from $A(0, 0)$ to $B(1/2, \pi/6)$.

c 54 The graph of $y = 4 \arctan(x^2)$ from $A(0, 0)$ to $B(1, \pi)$ is revolved about the x-axis. Use the trapezoidal rule, with $n = 8$, to approximate the area of the resulting surface.

8.3 HYPERBOLIC FUNCTIONS

The exponential expressions

$$\frac{e^x - e^{-x}}{2} \quad \text{and} \quad \frac{e^x + e^{-x}}{2}$$

occur in advanced applications of calculus. Their properties are similar in many ways to those of $\sin x$ and $\cos x$. Later in our discussion, we shall see why they are called the *hyperbolic sine* and the *hyperbolic cosine* of x.

Definition **(8.10)**

The **hyperbolic sine function**, denoted by **sinh**, and the **hyperbolic cosine function**, denoted by **cosh**, are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

for every real number x.

We pronounce $\sinh x$ and $\cosh x$ as *sinch* x and *kosh* x, respectively.

The graph of $y = \cosh x$ may be found by **addition of y-coordinates**. Noting that $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$, we first sketch the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same coordinate plane, as shown with dashes in Figure 8.13. We then add the y-coordinates of points on these graphs to obtain the graph of $y = \cosh x$. Note that the range of cosh is $[1, \infty)$.

FIGURE 8.13

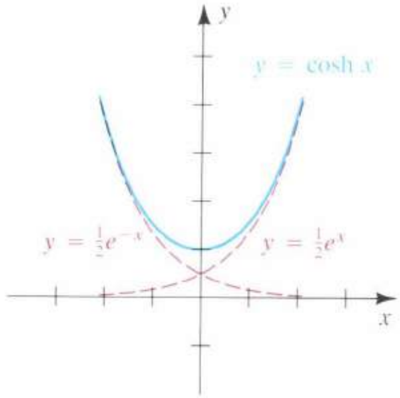
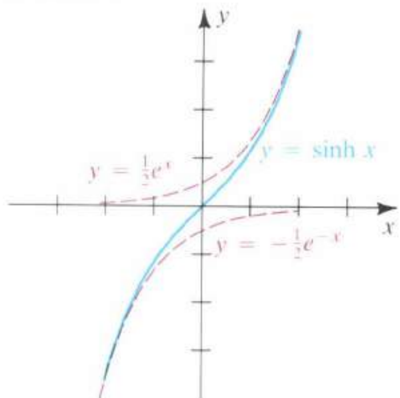


FIGURE 8.14



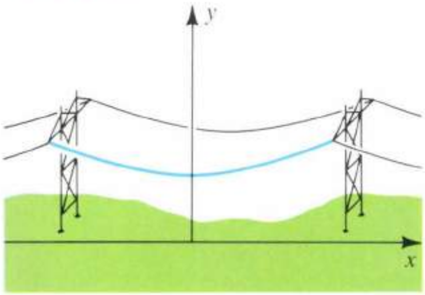
We may find the graph of $y = \sinh x$ by adding y-coordinates of the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$, as shown in Figure 8.14.

Some scientific calculators have keys that can be used to find values of sinh and cosh directly. We can also substitute numbers for x in Definition (8.10), as in the following illustration.

ILLUSTRATION

- $\sinh 2 = \dfrac{e^2 - e^{-2}}{2} \approx 3.63$
- $\cosh 0.5 = \dfrac{e^{0.5} + e^{-0.5}}{2} \approx 1.13$

FIGURE 8.15



The hyperbolic cosine function can be used to describe the shape of a uniform flexible cable, or chain, whose ends are supported from the same height. As illustrated in Figure 8.15, telephone or power lines may be strung between poles in this manner. The shape of the cable appears to be a parabola, but is actually a **catenary** (after the Latin word for *chain*). If we introduce a coordinate system, as in Figure 8.15 it can be shown that an equation corresponding to the shape of the cable is $y = a \cosh (x/a)$ for some real number a.

The hyperbolic cosine function also occurs in the analysis of motion in a resisting medium. If an object is dropped from a given height and if air resistance is disregarded, then the distance y that it falls in t seconds is $y = \frac{1}{2}gt^2$, where g is a gravitational constant. However, air resistance cannot always be disregarded. As the velocity of the object increases, air resistance may significantly affect its motion. For example, if the air resistance is directly proportional to the square of the velocity, then the distance y that the object falls in t seconds is given by

$$y = A \ln (\cosh Bt)$$

for constants A and B. Another application is given in Example 2 of this section.

Many identities similar to those for trigonometric functions hold for the hyperbolic sine and cosine functions. For example, if $\cosh^2 x$ and $\sinh^2 x$ denote $(\cosh x)^2$ and $(\sinh x)^2$, respectively, we have the following identity.

Theorem **(8.11)**

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

PROOF By Definition (8.10),

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= \tfrac{4}{4} = 1.
\end{aligned}$$

Theorem (8.11) is analogous to the trigonometric identity $\cos^2 x + \sin^2 x = 1$. Other hyperbolic identities are stated in the exercises. To verify an identity, it is sufficient to express the hyperbolic functions in terms of exponential functions and show that one side of the equation can be transformed into the other as illustrated in the proof of Theorem (8.11). The hyperbolic identities are similar to (but not always the same as) certain trigonometric identities—differences usually involve signs of terms.

FIGURE 8.16
$x^2 + y^2 = 1$

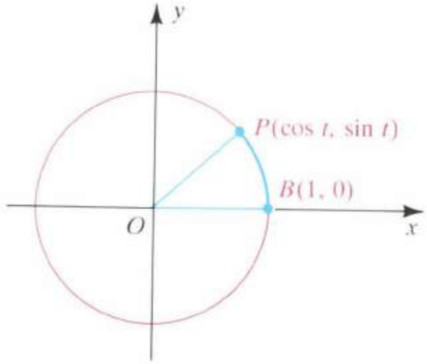
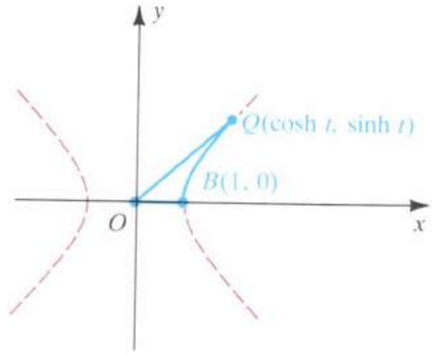


FIGURE 8.17
$x^2 - y^2 = 1$



If t is a real number, there is an interesting geometric relationship between the points $P(\cos t, \sin t)$ and $Q(\cosh t, \sinh t)$ in a coordinate plane. Let us consider the graphs of $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$, sketched in Figures 8.16 and 8.17. The graph in Figure 8.16 is the unit circle with center at the origin. The graph in Figure 8.17 is a *hyperbola*. (Hyperbolas and their properties will be discussed in detail in Chapter 12.) Note first that since $\cos^2 t + \sin^2 t = 1$, the point $P(\cos t, \sin t)$ is on the circle $x^2 + y^2 = 1$. Next, by Theorem (8.11), $\cosh^2 t - \sinh^2 t = 1$, and hence the point $Q(\cosh t, \sinh t)$ is on the hyperbola $x^2 - y^2 = 1$. These are the reasons for referring to cos and sin as *circular* functions and to cosh and sinh as *hyperbolic* functions.

The graphs in Figures 8.16 and 8.17 are related in another way. If $0 < t < \pi/2$, then t is the radian measure of angle POB, shown in Figure 8.16. By Theorem (1.15), the area A of the shaded circular sector is $A = \frac{1}{2}(1)^2 t = \frac{1}{2}t$, and hence $t = 2A$. Similarly, if $Q(\cosh t, \sinh t)$ is the point in Figure 8.17, then $t = 2A$ for the area A of the shaded hyperbolic sector (see Exercise 47).

The impressive analogies between the trigonometric and hyperbolic sine and cosine functions motivate us to define hyperbolic functions that correspond to the four remaining trigonometric functions. The **hyperbolic tangent**, **hyperbolic cotangent**, **hyperbolic secant**, and **hyperbolic cosecant functions**, denoted by tanh, coth, sech, and csch, respectively, are defined as follows.

Definition **(8.12)**

(i) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

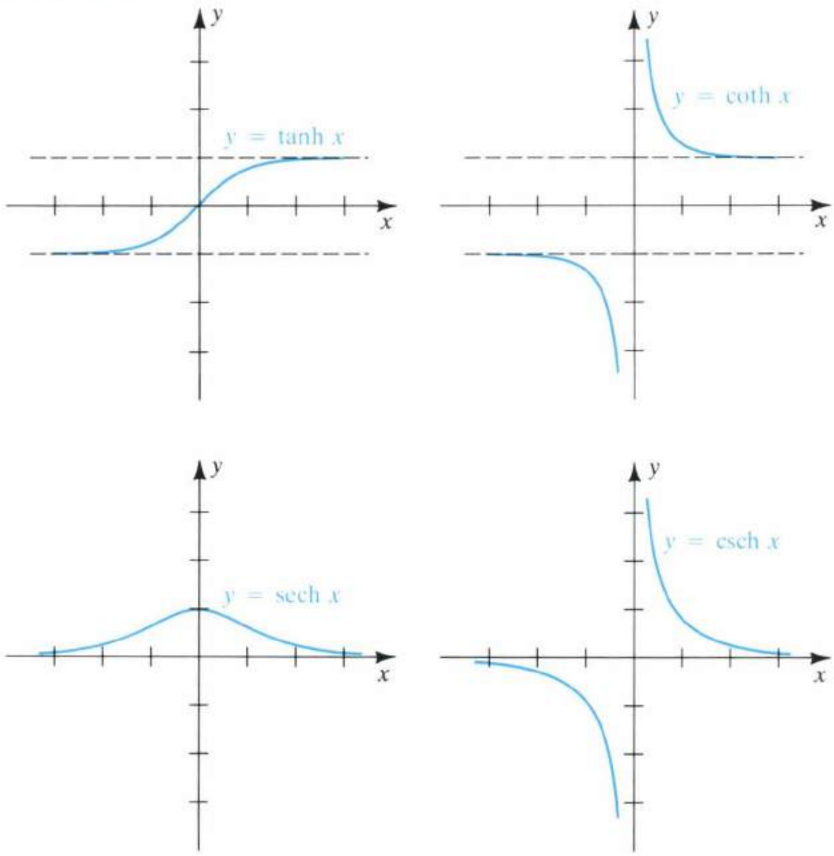
(ii) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$

(iii) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(iv) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}, \quad x \neq 0$

We pronounce the four function values in the preceding definition as *tansh x*, *cotansh x*, *setch x*, and *cosetch x*. Their graphs are sketched in Figure 8.18.

FIGURE 8.18



If we divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ (see (8.11)) by $\cosh^2 x$, we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Using the definitions of tanh x and sech x gives us (i) of the next theorem. Formula (ii) may be obtained by dividing both sides of (8.11) by $\sinh^2 x$.

Theorem **(8.13)**

(i) $1 - \tanh^2 x = \operatorname{sech}^2 x$ **(ii)** $\coth^2 x - 1 = \operatorname{csch}^2 x$

Note the similarities and differences between (8.13) and the analogous trigonometric identities.

Derivative formulas for the hyperbolic functions are listed in the next theorem, where $u = g(x)$ and g is differentiable.

Theorem **(8.14)**

(i) $D_x \sinh u = \cosh u \; D_x u$

(ii) $D_x \cosh u = \sinh u \; D_x u$

(iii) $D_x \tanh u = \operatorname{sech}^2 u \; D_x u$

(iv) $D_x \coth u = -\operatorname{csch}^2 u \; D_x u$

(v) $D_x \operatorname{sech} u = -\operatorname{sech} u \tanh u \; D_x u$

(vi) $D_x \operatorname{csch} u = -\operatorname{csch} u \coth u \; D_x u$

PROOF As usual, we consider only the case $u = x$. Since $D_x e^x = e^x$ and $D_x e^{-x} = -e^{-x}$,

$$D_x \sinh x = D_x \left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and $$D_x \cosh x = D_x \left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

To differentiate $\tanh x$, we apply the quotient rule as follows:

$$\begin{aligned} D_x \tanh x &= D_x \frac{\sinh x}{\cosh x} \\ &= \frac{\cosh x \; D_x \sinh x - \sinh x \; D_x \cosh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \end{aligned}$$

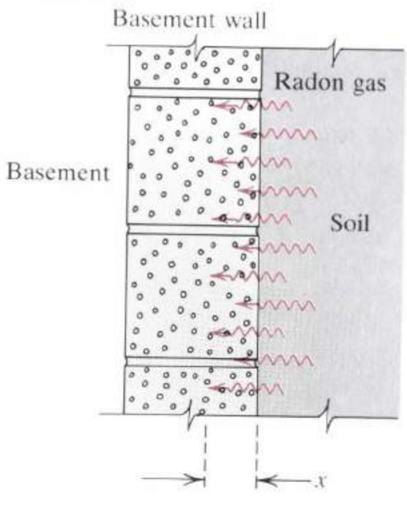
The remaining formulas can be proved in similar fashion.

EXAMPLE 1 If $f(x) = \cosh(x^2 + 1)$, find $f'(x)$.

SOLUTION Applying Theorem (8.14)(i), with $u = x^2 + 1$, we obtain

$$\begin{aligned} f'(x) &= \sinh(x^2 + 1) \cdot D_x (x^2 + 1) \\ &= 2x \sinh(x^2 + 1). \end{aligned}$$

FIGURE 8.19



EXAMPLE 2 Radon gas can readily diffuse through solid materials such as brick and cement. If the direction of diffusion in a basement wall is perpendicular to the surface, as illustrated in Figure 8.19, then the radon concentration $f(x)$ (in joules/cm^3) in the air-filled pores within the wall at a distance x from the outside surface can be approximated by

$$f(x) = A \sinh(qx) + B \cosh(qx) + k,$$

where the constant q depends on the porosity of the wall, the half-life of radon, and a diffusion coefficient; the constant k is the maximum radon concentration in the air-filled pores; and A and B are constants that depend on initial conditions. Show that $y = f(x)$ is a solution of the *diffusion equation*

$$\frac{d^2y}{dx^2} - q^2y + q^2k = 0.$$

SOLUTION Differentiating $y = f(x)$ twice gives us

$$\frac{dy}{dx} = qA \cosh(qx) + qB \sinh(qx)$$

and

$$\frac{d^2y}{dx^2} = q^2A \sinh(qx) + q^2B \cosh(qx).$$

Since $y = A \sinh(qx) + B \cosh(qx) + k$, we have

$$q^2y = q^2A \sinh(qx) + q^2B \cosh(qx) + q^2k.$$

Subtracting the expressions for d^2y/dx^2 and q^2y yields

$$\frac{d^2y}{dx^2} - q^2y = -q^2k$$

and hence

$$\frac{d^2y}{dx^2} - q^2y + q^2k = 0.$$

The integration formulas that correspond to the derivative formulas in Theorem (8.14) are as follows.

Theorem (8.15)

Ojo!

(i) $\int \sinh u \, du = \cosh u + C$

(ii) $\int \cosh u \, du = \sinh u + C$

(iii) $\int \operatorname{sech}^2 u \, du = \tanh u + C$

(iv) $\int \operatorname{csch}^2 u \, du = -\coth u + C$

(v) $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

(vi) $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

EXAMPLE 3 Evaluate $\int x^2 \sinh x^3 \, dx$.

SOLUTION If we let $u = x^3$, then $du = 3x^2 \, dx$ and

$$\begin{aligned}\int x^2 \sinh x^3 \, dx &= \tfrac{1}{3} \int (\sinh x^3) 3x^2 \, dx \\ &= \tfrac{1}{3} \int \sinh u \, du = \tfrac{1}{3} \cosh u + C \\ &= \tfrac{1}{3} \cosh x^3 + C.\end{aligned}$$

EXERCISES 8.3

c **Exer. 1–2: Approximate to four decimal places.**

1 (a) $\sinh 4$ (b) $\cosh \ln 4$ (c) $\tanh(-3)$ (d) $\coth 10$ (e) $\operatorname{sech} 2$ (f) $\operatorname{csch}(-1)$

2 (a) $\sinh \ln 4$ (b) $\cosh 4$ (c) $\tanh 3$ (d) $\coth(-10)$ (e) $\operatorname{sech}(-2)$ (f) $\operatorname{csch} 1$

Exer. 3–26: Find $f'(x)$ if $f(x)$ is the given expression.

3 $\sinh 5x$

4 $\sinh(x^2+1)$

5 $\cosh(x^3)$

6 $\cosh^3 x$

7 $\sqrt{x}\tanh\sqrt{x}$

8 $\arctan \tanh x$

9 $\coth \dfrac{1}{x}$

10 $\dfrac{\coth x}{\cot x}$

11 $\dfrac{\operatorname{sech}(x^2)}{x^2+1}$

12 $\sqrt{\operatorname{sech} 5x}$

13 $\operatorname{csch}^2 6x$

14 $x \operatorname{csch} e^{4x}$

15 $\ln \sinh 2x$

16 $\sinh^2 3x$ = 6 sinh 3x cosh 3x

17 $\cosh\sqrt{4x^2+3}$

18 $\dfrac{1+\cosh x}{1-\cosh x}$

19 $\dfrac{1}{\tanh x + 1}$

20 $\ln|\tanh x|$

21 $\coth \ln x$

22 $\coth^3 2x$

23 $e^{3x}\operatorname{sech} x$

24 $\frac{1}{2}\operatorname{sech}(x^2+1)$

25 $\tan^{-1}(\operatorname{csch} x)$

26 $\operatorname{csch} \ln x$

Exer. 27–42: Evaluate the integral.

27 $\int x^2\cosh(x^3)\,dx$

28 $\int \dfrac{1}{\operatorname{sech} 7x}\,dx$

29 $\int \dfrac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

30 $\int x\sinh(2x^2)\,dx$

31 $\int \dfrac{1}{\cosh^2 3x}\,dx$

32 $\int \operatorname{sech}^2(5x)\,dx$

33 $\int \operatorname{csch}^2(\frac{1}{2}x)\,dx$

34 $\int (\sinh 4x)^{-2}\,dx$

35 $\int \tanh 3x \operatorname{sech} 3x\,dx$

36 $\int \sinh x \operatorname{sech}^2 x\,dx$

37 $\int \cosh x \operatorname{csch}^2 x\,dx$

38 $\int \coth 6x \operatorname{csch} 6x\,dx$

39 $\int \coth x\,dx$

40 $\int \tanh x\,dx$

41 $\int \sinh x\cosh x\,dx$

42 $\int \operatorname{sech} x\,dx$

43 Find the area of the region bounded by the graphs of $y=\sinh 3x$, $y=0$, and $x=1$.

44 Find the arc length of the graph of $y=\cosh x$ from $(0,1)$ to $(1,\cosh 1)$.

45 Find the points on the graph of $y=\sinh x$ at which the tangent line has slope 2.

46 The region bounded by the graphs of $y=\cosh x$, $x=-1$, $x=1$, and $y=0$ is revolved about the x-axis. Find the volume of the resulting solid.

47 If A is the region shown in Figure 8.17, prove that $t=2A$.

48 Sketch the graph of $x^2-y^2=1$ and show that as t varies, the point $P(\cosh t, \sinh t)$ traces the part of the graph in quadrants I and IV.

49 The Gateway Arch in St. Louis has the shape of an inverted catenary (see figure). Rising 630 feet at its center and stretching 630 feet across its base, the shape of the arch can be approximated by

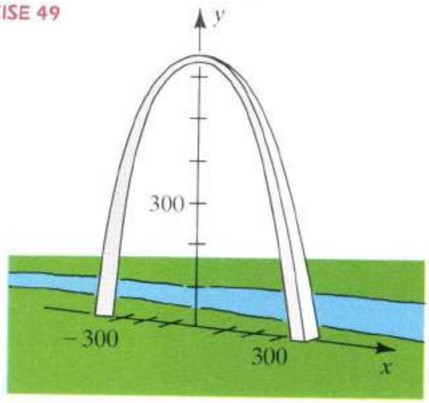
$$y = -127.7\cosh(x/127.7) + 757.7$$

for $-315 \le x \le 315$.

(a) Approximate the total open area under the arch.

(b) Approximate the total length of the arch.

EXERCISE 49



50 If a steel ball of mass m is released into water and the force of resistance is directly proportional to the square of the velocity, then the distance y the ball travels in t seconds is given by

$$y = km \ln\cosh\left(\sqrt{\frac{g}{km}}\,t\right),$$

where g is a gravitational constant and $k>0$. Show that y is a solution of the differential equation

$$m\frac{d^2y}{dt^2} + \frac{1}{k}\left(\frac{dy}{dt}\right)^2 = mg.$$

51 If a wave of length L is traveling across water of depth h (see figure on next page), the velocity v, or *celerity*, of the wave is related to L and h by the formula

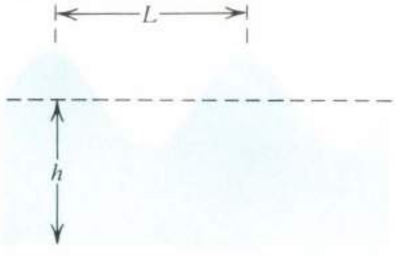
$$v^2 = \frac{gL}{2\pi}\tanh\frac{2\pi h}{L},$$

where g is a gravitational constant.

(a) Find $\lim_{h \to \infty} v^2$ and conclude that $v \approx \sqrt{gL/(2\pi)}$ in deep water.

(b) If $x \approx 0$ and f is a continuous function, then, by the mean value theorem (4.12), $f(x) - f(0) \approx f'(0)\,x$. Use this fact to show that $v \approx \sqrt{gh}$ if $h/L \approx 0$. Conclude that wave velocity is independent of wave length in shallow water.

EXERCISE 51



52 A soap bubble formed by two parallel concentric rings is shown in the figure. If the rings are not too far apart, it can be shown that the function f whose graph generates this surface of revolution is a solution of the differential equation $yy'' = 1 + (y')^2$, where $y = f(x)$. If A and B are positive constants, show that $y = A \cosh Bx$ is a solution if and only if $AB = 1$. Conclude that the graph is a catenary.

EXERCISE 52

c 53 Graph, on the same coordinate axes, $y = \tanh x$ and $y = \operatorname{sech}^2 x$ for $0 \le x \le 2$.

(a) Estimate the x-coordinate a of the point of intersection of the graphs.

(b) Use Newton's method to approximate a to three decimal places.

c 54 Graph, on the same coordinate axes, $y = \cosh^2 x$ and $y = 2$.

(a) Set up integrals for estimating the centroid of the region R bounded by the graphs.

(b) Use Simpson's rule, with $n = 4$, to approximate the coordinates of the centroid of R.

Exer. 55–72: Verify the identity.

55 $\cosh x + \sinh x = e^x$

56 $\cosh x - \sinh x = e^{-x}$

57 $\sinh(-x) = -\sinh x$

58 $\cosh(-x) = \cosh x$

59 $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

60 $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

61 $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

62 $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

63 $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

64 $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

65 $\sinh 2x = 2 \sinh x \cosh x$

66 $\cosh 2x = \cosh^2 x + \sinh^2 x$

67 $\sinh^2 \dfrac{x}{2} = \dfrac{\cosh x - 1}{2}$

68 $\cosh^2 \dfrac{x}{2} = \dfrac{\cosh x + 1}{2}$

69 $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

70 $\tanh \dfrac{x}{2} = \dfrac{\sinh x}{1 + \cosh x}$

71 $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for every positive integer n (*Hint:* Use Exercise 55.)

72 $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$ for every positive integer n

8.4 INVERSE HYPERBOLIC FUNCTIONS

The hyperbolic sine function is continuous and increasing for every x and hence, by Theorem (7.6), has a continuous, increasing inverse function, denoted by $\sinh^{-1}$. Since $\sinh x$ is defined in terms of e^x, we might expect that $\sinh^{-1}$ can be expressed in terms of the inverse, ln, of the natural exponential function. The first formula of the next theorem shows that this is the case.

Theorem (8.16)

(i) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

(ii) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1$

(iii) $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}, \quad |x| < 1$

(iv) $\operatorname{sech}^{-1} x = \ln \dfrac{1 + \sqrt{1 - x^2}}{x}, \quad 0 < x \leq 1$

PROOF To prove (i), we begin by noting that

$$y = \sinh^{-1} x \quad \text{if and only if} \quad x = \sinh y.$$

The equation $x = \sinh y$ can be used to find an explicit form for $\sinh^{-1} x$. Thus, if

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

then

$$e^y - 2x - e^{-y} = 0.$$

Multiplying both sides by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

Applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}, \quad \text{or} \quad e^y = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0$ and e^y is never negative, we must have

$$e^y = x + \sqrt{x^2 + 1}.$$

The equivalent logarithmic form is

$$y = \ln\left(x + \sqrt{x^2 + 1}\right);$$

that is,

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Formulas (ii)–(iv) are obtained in similar fashion. As with trigonometric functions, some inverse functions exist only if the domain is restricted. For example, if the domain of cosh is restricted to the set of nonnegative real numbers, then the resulting function is continuous and increasing, and its inverse function $\cosh^{-1}$ is defined by

$$y = \cosh^{-1} x \quad \text{if and only if} \quad \cosh y = x, \quad y \geq 0.$$

Employing the process used for $\sinh^{-1} x$ leads us to (ii). Similarly,

$$y = \tanh^{-1} x \quad \text{if and only if} \quad \tanh y = x \quad \text{for} \quad |x| < 1.$$

Using Definition (8.12), we may write $\tanh y = x$ as

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x.$$

Solving for y gives us (iii).

Finally, if we restrict the domain of sech to nonnegative numbers, the result is a one-to-one function, and we define

$$y = \operatorname{sech}^{-1} x \quad \text{if and only if} \quad \operatorname{sech} y = x, \quad y \geq 0.$$

Again, introducing the exponential form leads to (iv).

In the next theorem $u = g(x)$, where g is differentiable and x is suitably restricted.

Theorem (8.17)

(i) $$D_x \sinh^{-1} u = \frac{1}{\sqrt{u^2+1}} D_x u$$

(ii) $$D_x \cosh^{-1} u = \frac{1}{\sqrt{u^2-1}} D_x u, \quad u > 1$$

(iii) $$D_x \tanh^{-1} u = \frac{1}{1-u^2} D_x u, \quad |u| < 1$$

(iv) $$D_x \operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}} D_x u, \quad 0 < u < 1$$

PROOF By Theorem (8.16)(i),

$$\begin{aligned} D_x \sinh^{-1} x &= D_x \ln(x + \sqrt{x^2+1}) \\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}} \\ &= \frac{1}{\sqrt{x^2+1}}. \end{aligned}$$

This formula can be extended to $D_x \sinh^{-1} u$ by applying the chain rule. The remaining formulas can be proved in similar fashion.

EXAMPLE 1 If $y = \sinh^{-1}(\tan x)$, find dy/dx.

SOLUTION Using Theorem (8.17)(i) with $u = \tan x$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{\tan^2 x + 1}} \frac{d}{dx} \tan x = \frac{1}{\sqrt{\sec^2 x}} \sec^2 x \\ &= \frac{1}{|\sec x|} |\sec x|^2 = |\sec x|. \end{aligned}$$

The following theorem may be verified by differentiating the right-hand side of each formula.

Theorem (8.18)

(i) $$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\frac{u}{a} + C, \quad a > 0$$

(ii) $$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\frac{u}{a} + C, \quad 0 < a < u$$

(iii) $$\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C, \quad |u| < a$$

(iv) $$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + C, \quad 0 < |u| < a$$

If we use Theorem (8.16), then each of the integration formulas in the preceding theorem can be expressed in terms of the natural logarithmic function. To illustrate,

$$\begin{aligned}\int \frac{1}{\sqrt{a^2+u^2}}\,du &= \sinh^{-1}\frac{u}{a} + C\\ &= \ln\left(\frac{u}{a} + \sqrt{\left(\frac{u}{a}\right)^2 + 1}\right) + C.\end{aligned}$$

We can show that if $a > 0$, then the last formula can be written as

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left(u + \sqrt{a^2+u^2}\right) + D,$$

where D is a constant. In Section 9.3 we shall discuss another method for evaluating the integrals in Theorem (8.18).

EXAMPLE 2 Evaluate $\displaystyle\int \frac{1}{\sqrt{25+9x^2}}\,dx$.

SOLUTION We may express the integral as in Theorem (8.18)(i), by using the substitution

$$u = 3x, \qquad du = 3\,dx.$$

Since du contains the factor 3, we adjust the integrand by multiplying by 3 and then compensate by multiplying the integral by $\frac{1}{3}$ before substituting:

$$\begin{aligned}\int \frac{1}{\sqrt{25+9x^2}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+(3x)^2}}\,3\,dx\\ &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+u^2}}\,du\\ &= \frac{1}{3}\sinh^{-1}\frac{u}{5} + C\\ &= \frac{1}{3}\sinh^{-1}\frac{3x}{5} + C\end{aligned}$$

EXAMPLE 3 Evaluate $\int \frac{e^x}{16 - e^{2x}}\,dx$.

SOLUTION Substituting $u = e^x$, $du = e^x\,dx$ and applying Theorem (8.18)(iii) with $a = 4$, we have

$$\begin{aligned}\int \frac{e^x}{16 - e^{2x}}\,dx &= \int \frac{1}{4^2 - (e^x)^2}\,e^x\,dx\\ &= \int \frac{1}{4^2 - u^2}\,du\\ &= \frac{1}{4}\tanh^{-1}\frac{u}{4} + C\\ &= \frac{1}{4}\tanh^{-1}\frac{e^x}{4} + C\end{aligned}$$

for $|u| < a$ (that is, $e^x < 4$).

EXERCISES 8.4

c **Exer. 1–2: Approximate to four decimal places.**

1 (a) $\sinh^{-1} 1$ (b) $\cosh^{-1} 2$ (c) $\tanh^{-1}(-\frac{1}{2})$ (d) $\operatorname{sech}^{-1}\frac{1}{2}$

2 (a) $\sinh^{-1}(-2)$ (b) $\cosh^{-1} 5$ (c) $\tanh^{-1}\frac{1}{3}$ (d) $\operatorname{sech}^{-1}\frac{3}{5}$

Exer. 3–18: Find $f'(x)$ if $f(x)$ is the given expression.

3 $\sinh^{-1} 5x$

4 $\sinh^{-1} e^x$

5 $\cosh^{-1}\sqrt{x}$

6 $\sqrt{\cosh^{-1} x}$

7 $\tanh^{-1}(-4x)$

8 $\tanh^{-1}\sin 3x$

9 $\operatorname{sech}^{-1} x^2$

10 $\operatorname{sech}^{-1}\sqrt{1-x}$

11 $x\sinh^{-1}\frac{1}{x}$

12 $\frac{1}{\sinh^{-1} x^2}$

13 $\ln\cosh^{-1} 4x$

14 $\cosh^{-1}\ln 4x$

15 $\tanh^{-1}(x+1)$

16 $\tanh^{-1} x^3$

17 $\operatorname{sech}^{-1}\sqrt{x}$

18 $(\operatorname{sech}^{-1} x)^{-1}$

Exer. 19–26: Evaluate the integral.

19 $\int \frac{1}{\sqrt{81 + 16x^2}}\,dx$

20 $\int \frac{1}{\sqrt{16x^2 - 9}}\,dx$

21 $\int \frac{1}{49 - 4x^2}\,dx$

22 $\int \frac{\sin x}{\sqrt{1 + \cos^2 x}}\,dx$

23 $\int \frac{e^x}{\sqrt{e^{2x} - 16}}\,dx$

24 $\int \frac{2}{5 - 3x^2}\,dx$

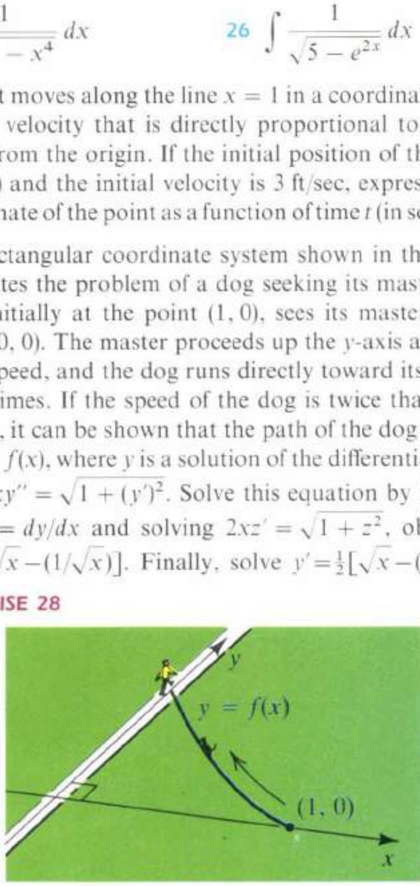
25 $\int \frac{1}{x\sqrt{9 - x^4}}\,dx$

26 $\int \frac{1}{\sqrt{5 - e^{2x}}}\,dx$

27 A point moves along the line $x = 1$ in a coordinate plane with a velocity that is directly proportional to its distance from the origin. If the initial position of the point is $(1, 0)$ and the initial velocity is 3 ft/sec, express the y-coordinate of the point as a function of time t (in seconds).

28 The rectangular coordinate system shown in the figure illustrates the problem of a dog seeking its master. The dog, initially at the point $(1, 0)$, sees its master at the point $(0, 0)$. The master proceeds up the y-axis at a constant speed, and the dog runs directly toward its master at all times. If the speed of the dog is twice that of the master, it can be shown that the path of the dog is given by $y = f(x)$, where y is a solution of the differential equation $2xy'' = \sqrt{1 + (y')^2}$. Solve this equation by first letting $z = dy/dx$ and solving $2xz' = \sqrt{1 + z^2}$, obtaining $z = \frac{1}{2}[\sqrt{x} - (1/\sqrt{x})]$. Finally, solve $y' = \frac{1}{2}[\sqrt{x} - (1/\sqrt{x})]$.

EXERCISE 28



Exer. 29–32: Sketch the graph of the equation.

29 $y = \sinh^{-1} x$

30 $y = \cosh^{-1} x$

31 $y = \tanh^{-1} x$

32 $y = \operatorname{sech}^{-1} x$

Exer. 33–38: Verify the formula.

33 $D_x \cosh^{-1} u = \dfrac{1}{\sqrt{u^2 - 1}} D_x u, \quad u > 1$

34 $D_x \tanh^{-1} u = \dfrac{1}{1 - u^2} D_x u, \quad |u| < 1$

35 $D_x \operatorname{sech}^{-1} u = -\dfrac{1}{u\sqrt{1 - u^2}} D_x u, \quad 0 < u < 1$

36 $\displaystyle\int \frac{1}{\sqrt{u^2 - a^2}}\, du = \cosh^{-1} \frac{u}{a} + C, \quad 0 < a < u$

37 $\displaystyle\int \frac{1}{a^2 - u^2}\, du = \frac{1}{a} \tanh^{-1} \frac{u}{a} + C, \quad |u| < a$

38 $\displaystyle\int \frac{1}{u\sqrt{a^2 - u^2}}\, du = -\frac{1}{a} \operatorname{sech}^{-1} \frac{|u|}{a} + C, \quad 0 < |u| < a$

Exer. 39–41: Derive the formula (see Theorem (8.16)).

39 $\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}), \quad x \geq 1$

40 $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}, \quad |x| < 1$

41 $\operatorname{sech}^{-1} x = \ln \dfrac{1 + \sqrt{1 - x^2}}{x}, \quad 0 < x \leq 1$

8.5 REVIEW EXERCISES

Exer. 1–24: Find $f'(x)$ if $f(x)$ is the given expression.

1 $\arctan \sqrt{x - 1}$

2 $\tan^{-1} (\ln 3x)$

3 $x^2 \operatorname{arcsec} (x^2)$

4 $\dfrac{1}{\cos^{-1} x}$

5 $2^{\arctan 2x}$

6 $(1 + \operatorname{arcsec} 2x)^{\sqrt{2}}$

7 $\ln \tan^{-1} (x^2)$

8 $\dfrac{1 - x^2}{\arccos x}$

9 $\sin^{-1} \sqrt{1 - x^2}$

10 $\sqrt{\sin^{-1} (1 - x^2)}$

11 $(\tan x + \tan^{-1} x)^4$

12 $\tan^{-1} \sqrt{\tan 2x}$

13 $\tan^{-1} (\tan^{-1} x)$

14 $e^{4x} \sec^{-1} e^{4x}$

15 $\cosh e^{-5x}$

16 $\dfrac{\ln \sinh x}{x}$

17 $e^{-x} \sinh e^{-x}$

18 $e^{x \cosh x}$

19 $\dfrac{\sinh x}{\cosh x - \sinh x}$

20 $\ln \tanh (5x + 1)$

21 $\sinh^{-1} (x^2)$

22 $\cosh^{-1} \tan x$

23 $\tanh^{-1} (\tanh \sqrt[3]{x})$

24 $\dfrac{1}{x} \tanh \dfrac{1}{x}$

Exer. 25–40: Evaluate the integral.

25 $\displaystyle\int \frac{1}{4 + 9x^2}\, dx$

26 $\displaystyle\int \frac{x}{4 + 9x^2}\, dx$

27 $\displaystyle\int \frac{e^{2x}}{\sqrt{1 - e^{2x}}}\, dx$

28 $\displaystyle\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx$

29 $\displaystyle\int \frac{x}{\operatorname{sech} (x^2)}\, dx$

30 $\displaystyle\int \frac{1}{x\sqrt{x^4 - 1}}\, dx$

31 $\displaystyle\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx$

32 $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$

33 $\displaystyle\int \frac{\sinh (\ln x)}{x}\, dx$

34 $\displaystyle\int \operatorname{sech}^2 (1 - 2x)\, dx$

35 $\displaystyle\int \frac{1}{\sqrt{9 - 4x^2}}\, dx$

36 $\displaystyle\int \frac{x}{\sqrt{9 - 4x^2}}\, dx$

37 $\displaystyle\int \frac{1}{x\sqrt{9 - 4x^2}}\, dx$

38 $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 9}}\, dx$

39 $\displaystyle\int \frac{x}{\sqrt{25x^2 + 36}}\, dx$

40 $\displaystyle\int \frac{1}{\sqrt{25x^2 + 36}}\, dx$

41 Find the points on the graph of $y = \sin^{-1} 3x$ at which the tangent line is parallel to the line through $A(2, -3)$ and $B(4, 7)$.

42 Find the points of inflection and discuss the concavity of the graph of $y = x \sin^{-1} x$.

43 Find the local extrema of $f(x) = 8 \sec x + \csc x$ on the interval $(0, \pi/2)$, and describe where $f(x)$ is increasing or is decreasing on that interval.

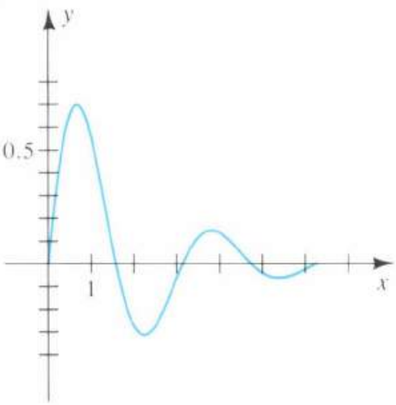
44 Find the area of the region bounded by the graphs of $y = x/(x^4 + 1)$, $x = 1$, and $y = 0$.

45 Damped oscillations are oscillations of decreasing magnitude that occur when frictional forces are considered. Shown in the figure on the next page is a graph of the damped oscillations given by $f(x) = e^{-x/2} \sin 2x$.

(a) Find the x-coordinates of the extrema of f for $0 \leq x \leq 2\pi$.

(b) Approximate the x-coordinates in part (a) to two decimal places.

EXERCISE 45



46 Find the arc length of the graph of $y = \ln \tanh \frac{1}{2}x$ from $x = 1$ to $x = 2$.

47 A balloon is released from level ground, 500 meters away from a person who observes its vertical ascent. If the balloon rises at a constant rate of 2 m/sec, use inverse trigonometric functions to find the rate at which the angle of elevation of the observer's line of sight is changing at the instant the balloon is at a height of 100 meters. (Disregard the observer's height.)

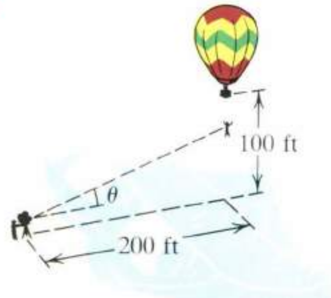
48 A square picture with sides 2 feet long is hung on a wall with the base 6 feet above the floor. A person whose eye level is 5 feet above the floor approaches the picture at a rate of 2 ft/sec. If θ is the angle between the line of sight and the top and bottom of the picture, find

(a) the rate at which θ is changing when the person is 8 feet from the wall

(b) the distance from the wall at which θ has its maximum value

49 A stuntman jumps from a hot-air balloon that is hovering at a constant altitude, 100 feet above a lake. A movie camera on shore, 200 feet from a point directly below the balloon, follows the stuntman's descent (see figure). At what rate is the angle of elevation θ of the camera changing 2 seconds after the stuntman jumps? (Disregard the height of the camera.)

EXERCISE 49

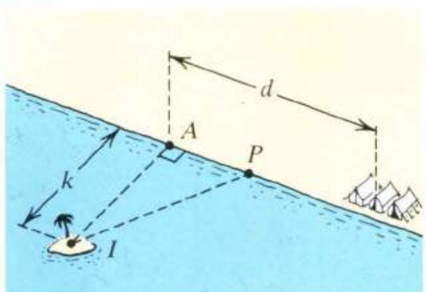


50 A person on a small island I, which is k miles from the closest point A on a straight shoreline, wishes to reach a camp that is d miles downshore from A by swimming to some point P on shore and then walking the rest of the way (see figure). Suppose the person burns c_1 calories per mile while swimming and c_2 calories per mile while walking, where $c_1 > c_2$.

(a) Find a formula for the total number c of calories burned in completing the trip.

(b) For what angle AIP does c have a minimum value?

EXERCISE 50



CHAPTER

9

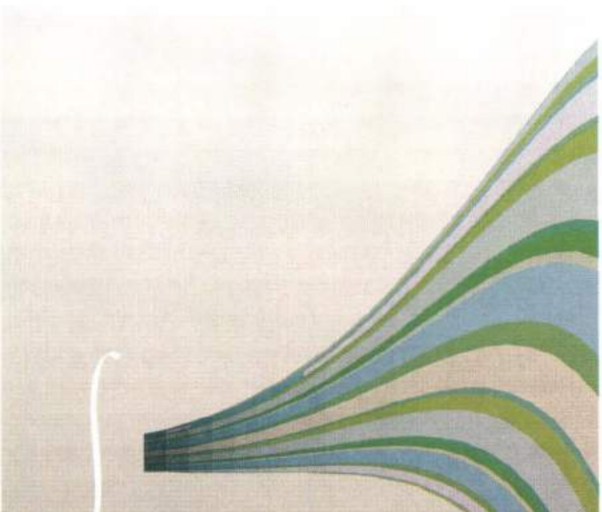
TECHNIQUES OF INTEGRATION

INTRODUCTION

In previous chapters we obtained formulas for evaluating various types of integrals. Many are listed on the inside front cover of this text. We also discussed the *method of substitution*, which is used to change a complicated integral into one that can be readily evaluated. In this chapter we consider additional ways to simplify integrals. Foremost among these is *integration by parts*, which we discuss in the first section. This powerful device allows us to obtain indefinite integrals of $\ln x$, $\tan^{-1} x$, and other important transcendental expressions. In later sections we develop techniques for simplifying integrals that contain powers of trigonometric functions, radicals, and rational expressions.

The use of a table of integrals is explained in Section 9.7. Such tables are always incomplete, and it is sometimes necessary to use skills obtained in previous sections *before* consulting a table. The same can be said for computer programs that are designed to evaluate various (but not all) indefinite integrals.

For applications involving *definite* integrals, it is often unnecessary to find an antiderivative and apply the fundamental theorem of calculus, because the trapezoidal rule or Simpson's rule can be used to obtain numerical approximations. In such cases either a computer or a programmable calculator is invaluable, since it can usually arrive at an approximation in a matter of seconds.



9.1 INTEGRATION BY PARTS

Up to this stage of our work we have been unable to evaluate integrals such as the following:

$$\int \ln x\,dx, \quad \int xe^x\,dx, \quad \int x^2 \sin x\,dx, \quad \int \tan^{-1} x\,dx$$

The next formula will enable us to evaluate not only these, but also many other types of integrals.

Integration by parts formula **(9.1)**

If $u = f(x)$ and $v = g(x)$ and if f' and g' are continuous, then

$$\int u\,dv = uv - \int v\,du.$$

PROOF By the product rule,

$$D_x\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x),$$

or, equivalently,

$$f(x)g'(x) = D_x\left[f(x)g(x)\right] - g(x)f'(x).$$

Integrating both sides of the previous equation gives us

$$\int f(x)g'(x)\,dx = \int D_x\left[f(x)g(x)\right]dx - \int g(x)f'(x)\,dx.$$

By Theorem (5.5)(i), the first integral on the right side equals $f(x)g(x) + C$. Since another constant of integration is obtained from the second integral, we may omit C in the formula; that is,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Since $dv = g'(x)\,dx$ and $du = f'(x)\,dx$, we may write the preceding formula as in (9.1).

When applying Formula (9.1) to an integral, we begin by letting one part of the integrand correspond to dv. The expression we choose for dv must include the differential dx. After selecting dv, we designate the remaining part of the integrand by u and then find du. Since this process involves splitting the integrand into two parts, the use of (9.1) is referred to as **integrating by parts**. A proper choice for dv is crucial. We usually *let dv equal the most complicated part of the integrand that can be readily integrated*. The following examples illustrate this method of integration.

EXAMPLE 1 Evaluate $\int xe^{2x}\,dx$.

SOLUTION The following list contains all possible choices for dv:

$$dx, \quad x\,dx, \quad e^{2x}\,dx, \quad xe^{2x}\,dx$$

The most complicated of these expressions that can be readily integrated is $e^{2x}\,dx$. Thus, we let

$$dv = e^{2x}\,dx.$$

The remaining part of the integrand is u—that is, $u = x$. To find v, we integrate dv, obtaining $v = \frac{1}{2}e^{2x}$. Note that a constant of integration is not added at this stage of the solution. (In Exercise 51 you are asked to prove that if a constant *is* added to v, the same final result is obtained.) If $u = x$, then $du = dx$. For ease of reference let us display these expressions as follows:

$$\begin{aligned} dv &= e^{2x}\,dx \qquad & u &= x \\ v &= \tfrac{1}{2}e^{2x} \qquad & du &= dx \end{aligned}$$

Substituting these expressions in Formula (9.1)—that is, *integrating by parts*—we obtain

$$\int xe^{2x}\,dx = x(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}\,dx.$$

We may find the integral on the right side as in Section 7.4. This gives us

$$\int xe^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C.$$

It takes considerable practice to become proficient in making a suitable choice for dv. To illustrate, if we had chosen $dv = x\,dx$ in Example 1, then it would have been necessary to let $u = e^{2x}$, giving us

$$\begin{aligned} dv &= x\,dx \qquad & u &= e^{2x} \\ v &= \tfrac{1}{2}x^2 \qquad & du &= 2e^{2x}\,dx. \end{aligned}$$

Integrating by parts, we obtain

$$\int xe^{2x}\,dx = \tfrac{1}{2}x^2e^{2x} - \int x^2e^{2x}\,dx.$$

Since the exponent associated with x has increased, the integral on the right is more complicated than the given integral. This indicates that we have made an incorrect choice for dv.

EXAMPLE 2 Evaluate

(a) $\int x \sec^2 x\,dx$ **(b)** $\int_0^{\pi/3} x \sec^2 x\,dx$

SOLUTION

(a) The possible choices for dv are

$$dx, \quad x\,dx, \quad \sec x\,dx, \quad x \sec x\,dx, \quad \sec^2 x\,dx, \quad x\sec^2 x\,dx.$$

The most complicated of these expressions that can be readily integrated is $\sec^2 x\,dx$. Thus, we let

$$\begin{aligned} dv &= \sec^2 x\,dx \qquad & u &= x \\ v &= \tan x \qquad & du &= dx. \end{aligned}$$

Integrating by parts gives us

$$\begin{aligned} \int x\sec^2 x\,dx &= x\tan x - \int \tan x\,dx \\ &= x\tan x + \ln|\cos x| + C. \end{aligned}$$

(b) The indefinite integral obtained in part (a) is an antiderivative of $x \sec^2 x$. Using the fundamental theorem of calculus (and dropping the constant of integration C), we obtain

$$\begin{aligned}\int_0^{\pi/3} x \sec^2 x \, dx &= \Big[x \tan x + \ln|\cos x|\Big]_0^{\pi/3} \\ &= \left(\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\left|\cos\frac{\pi}{3}\right|\right) - (0 + \ln 1) \\ &= \left(\frac{\pi}{3}\sqrt{3} + \ln\frac{1}{2}\right) - (0 + 0) \\ &= \frac{\pi}{3}\sqrt{3} - \ln 2 \approx 1.12.\end{aligned}$$

If, in Example 2, we had chosen $dv = x\,dx$ and $u = \sec^2 x$, then the integration by parts formula (9.1) would have led to a more complicated integral. (Verify this fact.)

In the next example we use integration by parts to find an antiderivative of the natural logarithmic function.

EXAMPLE 3 Evaluate $\int \ln x \, dx$.

SOLUTION Let

$$dv = dx \qquad u = \ln x$$
$$v = x \qquad du = \frac{1}{x}\,dx$$

and integrate by parts as follows:

$$\begin{aligned}\int \ln x \, dx &= (\ln x)x - \int (x)\frac{1}{x}\,dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C\end{aligned}$$

Sometimes it is necessary to use integration by parts more than once in the same problem. This is illustrated in the next example.

EXAMPLE 4 Evaluate $\int x^2 e^{2x}\,dx$.

SOLUTION Let

$$dv = e^{2x}\,dx \qquad u = x^2$$
$$v = \tfrac{1}{2}e^{2x} \qquad du = 2x\,dx$$

and integrate by parts as follows:

$$\int x^2e^{2x}\,dx = x^2(\tfrac{1}{2}e^{2x}) - \int (\tfrac{1}{2}e^{2x})2x\,dx$$
$$= \tfrac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx$$

To evaluate the integral on the right side of the last equation, we must again integrate by parts. Proceeding exactly as in Example 1 leads to

$$\int x^2e^{2x}\,dx = \tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + C.$$

The following example illustrates another device for evaluating an integral by means of two applications of the integration by parts formula.

EXAMPLE 5 Evaluate $\int e^x \cos x\,dx$.

SOLUTION We could either let $dv = \cos x\,dx$ or let $dv = e^x\,dx$, since each of these expressions is readily integrable. Let us choose

$$dv = \cos x\,dx \qquad u = e^x$$
$$v = \sin x \qquad du = e^x\,dx$$

and integrate by parts as follows:

$$\int e^x \cos x\,dx = e^x \sin x - \int (\sin x)e^x\,dx$$

(1)
$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

We next apply integration by parts to the integral on the right side of equation (1). Since we chose a trigonometric form for dv in the first integration by parts, we shall also choose a trigonometric form for the second. Letting

$$dv = \sin x\,dx \qquad u = e^x$$
$$v = -\cos x \qquad du = e^x\,dx$$

and integrating by parts, we have

$$\int e^x \sin x\,dx = e^x(-\cos x) - \int (-\cos x)e^x\,dx$$

(2)
$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx.$$

If we now use equation (2) to substitute on the right side of equation (1), we obtain

$$\int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\,dx\right],$$

or
$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

Adding $\int e^x \cos x\,dx$ to both sides of the last equation gives us

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x).$$

Finally, dividing both sides by 2 and adding the constant of integration yields

$$\int e^x \cos x\, dx = \tfrac{1}{2}e^x (\sin x + \cos x) + C.$$

We could have evaluated the given integral by using $dv = e^x\, dx$ for both the first and second applications of the integration by parts formula.

We must choose substitutions carefully when evaluating an integral of the type given in Example 5. To illustrate, suppose that in the evaluation of the integral on the right in equation (1) of the solution we had used

$$dv = e^x\, dx \qquad u = \sin x$$
$$v = e^x \qquad du = \cos x\, dx.$$

Integration by parts then leads to

$$\begin{aligned}\int e^x \sin x\, dx &= (\sin x)\, e^x - \int e^x \cos x\, dx \\ &= e^x \sin x - \int e^x \cos x\, dx.\end{aligned}$$

If we now substitute in (1), we obtain

$$\int e^x \cos x\, dx = e^x \sin x - \left[e^x \sin x - \int e^x \cos x\, dx\right],$$

which reduces to

$$\int e^x \cos x\, dx = \int e^x \cos x\, dx.$$

Although this is a true statement, it is not an evaluation of the given integral.

EXAMPLE 6 Evaluate $\int \sec^3 x\, dx$.

SOLUTION The possible choices for dv are

$$dx, \quad \sec x\, dx, \quad \sec^2 x\, dx, \quad \sec^3 x\, dx.$$

The most complicated of these expressions that can be readily integrated is $\sec^2 x\, dx$. Thus, we let

$$dv = \sec^2 x\, dx \qquad u = \sec x$$
$$v = \tan x \qquad du = \sec x \tan x\, dx$$

and integrate by parts as follows:

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$$

Instead of applying another integration by parts, let us change the form of the integral on the right by using the identity $1 + \tan^2 x = \sec^2 x$. This gives us

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x\, (\sec^2 x - 1)\, dx,$$

or $$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$$

Adding $\int \sec^3 x\,dx$ to both sides of the last equation gives us

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx.$$

If we now evaluate $\int \sec x\,dx$ and divide both sides of the resulting equation by 2 (and then add the constant of integration), we obtain

$$\int \sec^3 x\,dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C.$$

Integration by parts may sometimes be employed to obtain **reduction formulas** for integrals. We can use such formulas to write an integral involving powers of an expression in terms of integrals that involve lower powers of the expression.

EXAMPLE 7 Find a reduction formula for $\int \sin^n x\,dx$.

SOLUTION Let

$$dv = \sin x\,dx \qquad u = \sin^{n-1} x$$
$$v = -\cos x \qquad du = (n-1)\sin^{n-2} x \cos x\,dx$$

and integrate by parts as follows:

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we may write

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.$$

Consequently,

$$\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx.$$

The left side of the last equation reduces to $n\int \sin^n x\,dx$. Dividing both sides by n, we obtain

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

EXAMPLE 8 Use the reduction formula in Example 7 to evaluate $\int \sin^4 x\,dx$.

SOLUTION Using the formula with $n = 4$ gives us

$$\int \sin^4 x\,dx = -\tfrac{1}{4}\cos x \sin^3 x + \tfrac{3}{4}\int \sin^2 x\,dx.$$

Applying the reduction formula, with $n = 2$, to the integral on the right, we have

$$\int \sin^2 x\, dx = -\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx$$
$$= -\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x + C.$$

Consequently,

$$\int \sin^4 x\, dx = -\tfrac{1}{4}\cos x \sin^3 x - \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + D$$

with $D = \tfrac{3}{4}C$.

It should be evident that by repeated applications of the formula in Example 7 we can find $\int \sin^n x\, dx$ for any positive integer n, because these reductions end with either $\int \sin x\, dx$ or $\int dx$, and each of these can be evaluated easily.

EXERCISES 9.1

Exer. 1–38: Evaluate the integral.

1 $\int xe^{-x}\, dx$

2 $\int x \sin x\, dx$

3 $\int x^2 e^{3x}\, dx$

4 $\int x^2 \sin 4x\, dx$

5 $\int x \cos 5x\, dx$

6 $\int xe^{-2x}\, dx$

7 $\int x \sec x \tan x\, dx$

8 $\int x \csc^2 3x\, dx$

9 $\int x^2 \cos x\, dx$

10 $\int x^3 e^{-x}\, dx$

11 $\int \tan^{-1} x\, dx$

12 $\int \sin^{-1} x\, dx$

13 $\int \sqrt{x} \ln x\, dx$

14 $\int x^2 \ln x\, dx$

15 $\int x \csc^2 x\, dx$

16 $\int x \tan^{-1} x\, dx$

17 $\int e^{-x} \sin x\, dx$

18 $\int e^{3x} \cos 2x\, dx$

19 $\int \sin x \ln \cos x\, dx$

20 $\int_0^1 x^3 e^{-x^2}\, dx$

21 $\int \csc^3 x\, dx$

22 $\int \sec^5 x\, dx$

23 $\int_0^1 \dfrac{x^3}{\sqrt{x^2+1}}\, dx$

24 $\int \sin \ln x\, dx$

25 $\int_0^{\pi/2} x \sin 2x\, dx$

26 $\int x \sec^2 5x\, dx$

27 $\int x(2x+3)^{99}\, dx$

28 $\int \dfrac{x^5}{\sqrt{1-x^3}}\, dx$

29 $\int e^{4x} \sin 5x\, dx$

30 $\int x^3 \cos (x^2)\, dx$

31 $\int (\ln x)^2\, dx$

32 $\int x\, 2^x\, dx$

33 $\int x^3 \sinh x\, dx$

34 $\int (x+4) \cosh 4x\, dx$

35 $\int \cos \sqrt{x}\, dx$

36 $\int \tan^{-1} 3x\, dx$

37 $\int \cos^{-1} x\, dx$

38 $\int (x+1)^{10}(x+2)\, dx$

Exer. 39–42: Use integration by parts to derive the reduction formula.

39 $\int x^m e^x\, dx = x^m e^x - m \int x^{m-1} e^x\, dx$

40 $\int x^m \sin x\, dx = -x^m \cos x + m \int x^{m-1} \cos x\, dx$

41 $\int (\ln x)^m\, dx = x(\ln x)^m - m \int (\ln x)^{m-1}\, dx$

42 $\int \sec^m x\, dx = \dfrac{\sec^{m-2} x \tan x}{m-1} + \dfrac{m-2}{m-1} \int \sec^{m-2} x\, dx$ for $m \neq 1$.

43 Use Exercise 39 to evaluate $\int x^5 e^x\, dx$.

44 Use Exercise 41 to evaluate $\int (\ln x)^4\, dx$.

45 If $f(x) = \sin \sqrt{x}$, find the area of the region under the graph of f from $x = 0$ to $x = \pi^2$.

46 The region between the graph of $y = x\sqrt{\sin x}$ and the x-axis from $x = 0$ to $x = \pi/2$ is revolved about the x-axis. Find the volume of the resulting solid.

47 The region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$ is revolved about the y-axis. Find the volume of the resulting solid.

48 Suppose the force $f(x)$ acting at the point with coordinate x on a coordinate line l is given by $f(x) = x^5\sqrt{x^3+1}$. Find the work done in moving an object from $x = 0$ to $x = 1$.

49 Find the centroid of the region bounded by the graphs of the equations $y = e^x$, $y = 0$, $x = 0$, and $x = \ln 3$.

50 The velocity (at time t) of a point moving along a coordinate line is t/e^{2t} ft/sec. If the point is at the origin at $t = 0$, find its position at time t.

51 When applying the integration by parts formula (9.1), show that if, after choosing dv, we use $v + C$ in place of v, the same result is obtained.

52 In Section 6.3 the discussion of finding volumes by means of cylindrical shells was incomplete because we did not show that the same result is obtained if the disk method is also applicable. Use integration by parts to prove that if f is differentiable and either $f'(x) > 0$ on $[a, b]$ or $f'(x) < 0$ on $[a, b]$, and if V is the volume of the solid obtained by revolving the region bounded by the graphs of f, $x = a$, and $x = b$ about the x-axis, then the same value of V is obtained using either the disk method or the shell method. (*Hint:* Let g be the inverse function of f, and use integration by parts on $\int_a^b \pi[f(x)]^2\,dx$.)

53 Discuss the following use of Formula (9.1): Given $\int (1/x)\,dx$, let $dv = dx$ and $u = 1/x$ so that $v = x$ and $du = (-1/x^2)\,dx$. Hence

$$\int \frac{1}{x}\,dx = \left(\frac{1}{x}\right)x - \int x\left(-\frac{1}{x^2}\right)dx,$$

or

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

Consequently, $0 = 1$.

54 If $u = f(x)$ and $v = g(x)$, prove that the analogue of Formula (9.1) for definite integrals is

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du$$

for values a and b of x.

9.2 TRIGONOMETRIC INTEGRALS

In Example 7 of Section 9.1 we obtained a reduction formula for $\int \sin^n x\,dx$. Integrals of this type may also be found without using integration by parts. If n is an odd positive integer, we begin by writing

$$\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx.$$

Since the integer $n - 1$ is even, we may then use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ to obtain a form that is easy to integrate, as illustrated in the following example.

EXAMPLE 1 Evaluate $\int \sin^5 x\,dx$.

SOLUTION As in the preceding discussion, we write

$$\begin{aligned}\int \sin^5 x\,dx &= \int \sin^4 x \sin x\,dx\\ &= \int (\sin^2 x)^2 \sin x\,dx\\ &= \int (1 - \cos^2 x)^2 \sin x\,dx\\ &= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx.\end{aligned}$$

If we substitute

$$u = \cos x, \quad du = -\sin x\, dx,$$

we obtain

$$\begin{aligned}\int \sin^5 x\, dx &= -\int (1 - 2\cos^2 x + \cos^4 x)(-\sin x)\, dx \\ &= -\int (1 - 2u^2 + u^4)\, du \\ &= -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + C \\ &= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.\end{aligned}$$

Similarly, for odd powers of $\cos x$ we write

$$\int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$$

and use the fact that $\cos^2 x = 1 - \sin^2 x$ to obtain an integrable form.

If the integrand is $\sin^n x$ or $\cos^n x$ and n is *even*, then the half-angle formula

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{or} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

may be used to simplify the integrand.

EXAMPLE 2 Evaluate $\int \cos^2 x\, dx$.

SOLUTION Using a half-angle formula, we have

$$\begin{aligned}\int \cos^2 x\, dx &= \tfrac{1}{2}\int (1 + \cos 2x)\, dx \\ &= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C.\end{aligned}$$

EXAMPLE 3 Evaluate $\int \sin^4 x\, dx$.

SOLUTION

$$\begin{aligned}\int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\ &= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx \\ &= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx\end{aligned}$$

We appy a half-angle formula again and write

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x) = \tfrac{1}{2} + \tfrac{1}{2}\cos 4x.$$

Substituting in the last integral and simplifying gives us

$$\int \sin^4 x\, dx = \tfrac{1}{4}\int\left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx$$
$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

Integrals involving only products of $\sin x$ and $\cos x$ may be evaluated using the following guidelines.

Guidelines for evaluating $\int \sin^m x \cos^n x\, dx$ **(9.2)**

1 **If m is an odd integer:** Write the integral as

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cos^n x \sin x\, dx$$

and express $\sin^{m-1} x$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. Make the substitution

$$u = \cos x, \quad du = -\sin x\, dx$$

and evaluate the resulting integral.

2 **If n is an odd integer:** Write the integral as

$$\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos^{n-1} x \cos x\, dx$$

and express $\cos^{n-1} x$ in terms of $\sin x$ by using the trigonometric identity $\cos^2 x = 1 - \sin^2 x$. Make the substitution

$$u = \sin x, \quad du = \cos x\, dx$$

and evaluate the resulting integral.

3 **If m and n are even:** Use half-angle formulas for $\sin^2 x$ and $\cos^2 x$ to reduce the exponents by one-half.

EXAMPLE 4 Evaluate $\int \cos^3 x \sin^4 x\, dx$.

SOLUTION By guideline 2 of (9.2),

$$\int \cos^3 x \sin^4 x\, dx = \int \cos^2 x \sin^4 x \cos x\, dx$$
$$= \int (1 - \sin^2 x) \sin^4 x \cos x\, dx.$$

If we let $u = \sin x$, then $du = \cos x\, dx$, and the integral may be written

$$\int \cos^3 x \sin^4 x\, dx = \int (1 - u^2)u^4\, du = \int (u^4 - u^6)\, du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{7}u^7 + C$$
$$= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C.$$

The following guidelines are analogous to those in (9.2) for integrands of the form $\tan^m x \sec^n x$.

Guidelines for evaluating $\int \tan^m x \sec^n x\, dx$ **(9.3)**

1. **If m is an odd integer:** Write the integral as
$$\int \tan^m x \sec^n x\, dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x\, dx$$
and express $\tan^{m-1} x$ in terms of $\sec x$ by using the trigonometric identity $\tan^2 x = \sec^2 x - 1$. Make the substitution
$$u = \sec x, \quad du = \sec x \tan x\, dx$$
and evaluate the resulting integral.
2. **If n is an even integer:** Write the integral as
$$\int \tan^m x \sec^n x\, dx = \int \tan^m x \sec^{n-2} x \sec^2 x\, dx$$
and express $\sec^{n-2} x$ in terms of $\tan x$ by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$. Make the substitution
$$u = \tan x, \quad du = \sec^2 x\, dx$$
and evaluate the resulting integral.
3. **If m is even and n is odd:** There is no standard method of evaluation. Possibly use integration by parts.

EXAMPLE 5 Evaluate $\int \tan^3 x \sec^5 x\, dx$.

SOLUTION By guideline 1 of (9.3),
$$\begin{aligned}\int \tan^3 x \sec^5 x\, dx &= \int \tan^2 x \sec^4 x\, (\sec x \tan x)\, dx\\ &= \int (\sec^2 x - 1) \sec^4 x\, (\sec x \tan x)\, dx.\end{aligned}$$
Substituting $u = \sec x$ and $du = \sec x \tan x\, dx$, we obtain
$$\begin{aligned}\int \tan^3 x \sec^5 x\, dx &= \int (u^2 - 1)u^4\, du\\ &= \int (u^6 - u^4)\, du\\ &= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C\\ &= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C.\end{aligned}$$

EXAMPLE 6 Evaluate $\int \tan^2 x \sec^4 x\, dx$.

SOLUTION By guideline 2 of (9.3),
$$\begin{aligned}\int \tan^2 x \sec^4 x\, dx &= \int \tan^2 x \sec^2 x \sec^2 x\, dx\\ &= \int \tan^2 x\, (\tan^2 x + 1) \sec^2 x\, dx.\end{aligned}$$
If we let $u = \tan x$, then $du = \sec^2 x\, dx$, and
$$\begin{aligned}\int \tan^2 x \sec^4 x\, dx &= \int u^2(u^2 + 1)\, du\\ &= \int (u^4 + u^2)\, du\\ &= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C\\ &= \tfrac{1}{5}\tan^2 x + \tfrac{1}{3}\tan^3 x + C.\end{aligned}$$

Integrals of the form $\int \cot^m x \csc^n x\, dx$ may be evaluated in similar fashion.

Finally, if an integrand has one of the forms $\cos mx \cos nx$, $\sin mx \sin nx$, or $\sin mx \cos nx$, we use a product-to-sum formula to help evaluate the integral, as illustrated in the next example.

EXAMPLE 7 Evaluate $\int \cos 5x \cos 3x\, dx$.

SOLUTION Using the product-to-sum formula for $\cos u \cos v$, we obtain

$$\begin{aligned}\int \cos 5x \cos 3x\, dx &= \int \tfrac{1}{2}(\cos 8x + \cos 2x)\, dx \\ &= \tfrac{1}{16} \sin 8x + \tfrac{1}{4} \sin 2x + C.\end{aligned}$$

EXERCISES 9.2

Exer. 1–30: Evaluate the integral.

1 $\int \cos^3 x\, dx$

2 $\int \sin^2 2x\, dx$

3 $\int \sin^2 x \cos^2 x\, dx$

4 $\int \cos^7 x\, dx$

5 $\int \sin^3 x \cos^2 x\, dx$

6 $\int \sin^5 x \cos^3 x\, dx$

7 $\int \sin^6 x\, dx$

8 $\int \sin^4 x \cos^2 x\, dx$

9 $\int \tan^3 x \sec^4 x\, dx$

10 $\int \sec^6 x\, dx$

11 $\int \tan^3 x \sec^3 x\, dx$

12 $\int \tan^5 x \sec x\, dx$

13 $\int \tan^6 x\, dx$

14 $\int \cot^4 x\, dx$

15 $\int \sqrt{\sin x} \cos^3 x\, dx$

16 $\int \dfrac{\cos^3 x}{\sqrt{\sin x}}\, dx$

17 $\int (\tan x + \cot x)^2\, dx$

18 $\int \cot^3 x \csc^3 x\, dx$

19 $\int_0^{\pi/4} \sin^3 x\, dx$

20 $\int_0^1 \tan^2 (\frac{1}{4}\pi x)\, dx$

21 $\int \sin 5x \sin 3x\, dx$

22 $\int_0^{\pi/4} \cos x \cos 5x\, dx$

23 $\int_0^{\pi/2} \sin 3x \cos 2x\, dx$

24 $\int \sin 4x \cos 3x\, dx$

25 $\int \csc^4 x \cot^4 x\, dx$

26 $\int (1 + \sqrt{\cos x})^2 \sin x\, dx$

27 $\int \dfrac{\cos x}{2 - \sin x}\, dx$

28 $\int \dfrac{\tan^2 x - 1}{\sec^2 x}\, dx$

29 $\int \dfrac{\sec^2 x}{(1 + \tan x)^2}\, dx$

30 $\int \dfrac{\sec x}{\cot^5 x}\, dx$

31 The region bounded by the x-axis and the graph of $y = \cos^2 x$ from $x = 0$ to $x = 2\pi$ is revolved about the x-axis. Find the volume of the resulting solid.

32 The region between the graphs of $y = \tan^2 x$ and $y = 0$ from $x = 0$ to $x = \pi/4$ is revolved about the x-axis. Find the volume of the resulting solid.

33 The velocity (at time t) of a point moving on a coordinate line is $\cos^2 \pi t$ ft/sec. How far does the point travel in 5 seconds?

34 The acceleration (at time t) of a point moving on a coordinate line is $\sin^2 t \cos t$ ft/sec^2. At $t = 0$ the point is at the origin and its velocity is 10 ft/sec. Find its position at time t.

35 (a) Prove that if m and n are positive integers,

$$\int \sin mx \sin nx\, dx = \begin{cases} \dfrac{\sin (m-n)x}{2(m-n)} - \dfrac{\sin (m+n)x}{2(m+n)} + C & \text{if } m \neq n \\ \dfrac{x}{2} - \dfrac{\sin 2mx}{4m} + C & \text{if } m = n \end{cases}$$

(b) Obtain formulas similar to that in part (a) for

$$\int \sin mx \cos nx\, dx$$

and

$$\int \cos mx \cos nx\, dx.$$

36 (a) Use part (a) of Exercise 35 to prove that

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

(b) Find

(i) $\int_{-\pi}^{\pi} \sin mx \cos nx\, dx$

(ii) $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx$

9.3 TRIGONOMETRIC SUBSTITUTIONS

In Example 1 of Section 1.3 we showed how to change the expression $\sqrt{a^2 - x^2}$, with $a > 0$, into a trigonometric expression without radicals, by using the *trigonometric substitution* $x = a \sin \theta$. We can use a similar procedure for $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$. This technique is useful for eliminating radicals from certain types of integrands. The substitutions are listed in the following table.

Trigonometric substitutions **(9.4)**

EXPRESSION IN INTEGRAND	TRIGONOMETRIC SUBSTITUTION
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$

When making a trigonometric substitution we shall assume that θ is in the range of the corresponding inverse trigonometric function. Thus, for the substitution $x = a \sin \theta$, we have $-\pi/2 \le \theta \le \pi/2$. In this case, $\cos \theta \ge 0$ and

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta}\\ &= \sqrt{a^2(1 - \sin^2 \theta)}\\ &= \sqrt{a^2 \cos^2 \theta}\\ &= a \cos \theta.\end{aligned}$$

If $\sqrt{a^2 - x^2}$ occurs in a denominator, we add the restriction $|x| \ne a$, or, equivalently, $-\pi/2 < \theta < \pi/2$.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx.$

SOLUTION The integrand contains $\sqrt{16 - x^2}$, which is of the form $\sqrt{a^2 - x^2}$ with $a = 4$. Hence, by (9.4), we let

$$x = 4 \sin \theta \quad \text{for} \quad -\pi/2 < \theta < \pi/2.$$

It follows that

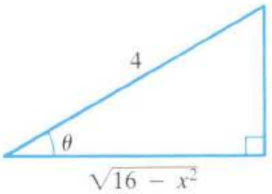
$$\sqrt{16 - x^2} = \sqrt{16 - 16 \sin^2 \theta} = 4\sqrt{1 - \sin^2 \theta} = 4\sqrt{\cos^2 \theta} = 4 \cos \theta.$$

Since $x = 4 \sin \theta$, we have $dx = 4 \cos \theta\, d\theta$. Substituting in the given integral yields

$$\begin{aligned}\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx &= \int \frac{1}{(16 \sin^2 \theta)4 \cos \theta}\, 4 \cos \theta\, d\theta\\ &= \frac{1}{16}\int \frac{1}{\sin^2 \theta}\, d\theta\\ &= \tfrac{1}{16}\int \csc^2 \theta\, d\theta\\ &= -\tfrac{1}{16} \cot \theta + C.\end{aligned}$$

FIGURE 9.1

$\sin \theta = \dfrac{x}{4}$



We must now return to the original variable of integration, x. Since $\theta = \arcsin(x/4)$, we could write $-\frac{1}{16}\cot\theta$ as $-\frac{1}{16}\cot\arcsin(x/4)$, but this is a cumbersome expression. Since the integrand contains $\sqrt{16 - x^2}$, it is preferable that the evaluated form also contain this radical. There is a simple geometric method for ensuring that it does. If $0 < \theta < \pi/2$ and $\sin\theta = x/4$, we may interpret θ as an acute angle of a right triangle having opposite side and hypotenuse of lengths x and 4, respectively (see Figure 9.1). By the Pythagorean theorem, the length of the adjacent side is $\sqrt{16 - x^2}$. Referring to the triangle, we find

$$\cot\theta = \frac{\sqrt{16 - x^2}}{x}.$$

It can be shown that the last formula is also true if $-\pi/2 < \theta < 0$. Thus, Figure 9.1 may be used if θ is either positive or negative.

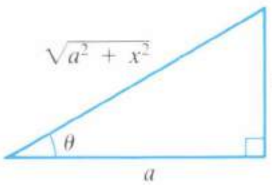
Substituting $\sqrt{16 - x^2}/x$ for $\cot\theta$ in our integral evaluation gives us

$$\begin{aligned}\int \frac{1}{x^2\sqrt{16 - x^2}}\,dx &= -\frac{1}{16}\cdot\frac{\sqrt{16 - x^2}}{x} + C \\ &= -\frac{\sqrt{16 - x^2}}{16x} + C.\end{aligned}$$

If an integrand contains $\sqrt{a^2 + x^2}$ for $a > 0$, then, by (9.4), we use the substitution $x = a\tan\theta$ to eliminate the radical. When using this substitution we assume that θ is in the range of the inverse tangent function; that is, $-\pi/2 < \theta < \pi/2$. In this case, $\sec\theta > 0$ and

FIGURE 9.2

$\tan\theta = \dfrac{x}{a}$



$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2\tan^2\theta} \\ &= \sqrt{a^2(1 + \tan^2\theta)} \\ &= \sqrt{a^2\sec^2\theta} \\ &= a\sec\theta.\end{aligned}$$

After substituting and evaluating the resulting trigonometric integral, it is necessary to return to the variable x. We can do this by using the formula $\tan\theta = x/a$ and referring to the right triangle in Figure 9.2.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{1}{\sqrt{4 + x^2}}\,dx$.

SOLUTION The denominator of the integrand has the form $\sqrt{a^2 + x^2}$ with $a = 2$. Hence, by (9.4), we make the substitution

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\,d\theta.$$

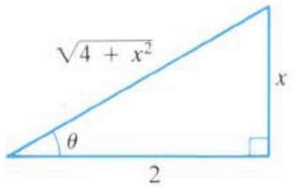
Consequently

$$\sqrt{4 + x^2} = \sqrt{4 + 4\tan^2\theta} = 2\sqrt{1 + \tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta$$

and
$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{1}{2\sec\theta}\,2\sec^2\theta\,d\theta$$
$$= \int \sec\theta\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C.$$

FIGURE 9.3

$\tan\theta = \dfrac{x}{2}$



Using $\tan\theta = x/2$, we sketch the triangle in Figure 9.3, from which we obtain

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Hence

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

The expression on the right may be written

$$\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + C = \ln|\sqrt{4+x^2}+x| - \ln 2 + C.$$

Since $\sqrt{4+x^2} + x > 0$ for every x, the absolute value sign is unnecessary. If we also let $D = -\ln 2 + C$, then

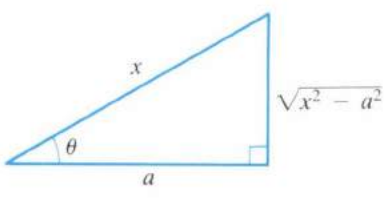
$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln(\sqrt{4+x^2}+x) + D.$$

If an integrand contains $\sqrt{x^2-a^2}$, then using (9.4) we substitute $x = a\sec\theta$, where θ is chosen in the range of the inverse secant function; that is, either $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$. In this case, $\tan\theta \ge 0$ and

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$$
$$= \sqrt{a^2(\sec^2\theta - 1)}$$
$$= \sqrt{a^2\tan^2\theta}$$
$$= a\tan\theta.$$

FIGURE 9.4

$\sec\theta = \dfrac{x}{a}$



Since

$$\sec\theta = \frac{x}{a},$$

we may refer to the triangle in Figure 9.4 when changing from the variable θ to the variable x.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$.

SOLUTION The integrand contains $\sqrt{x^2-9}$, which is of the form $\sqrt{x^2-a^2}$ with $a = 3$. Referring to (9.4), we substitute as follows:

$$x = 3\sec\theta, \quad dx = 3\sec\theta\tan\theta\,d\theta$$

Consequently

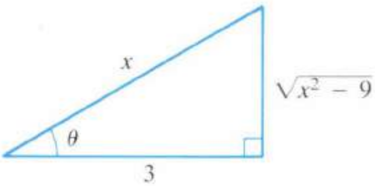
$$\sqrt{x^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = 3\sqrt{\sec^2 \theta - 1} = 3\sqrt{\tan^2 \theta} = 3 \tan \theta$$

and

$$\begin{aligned}\int \frac{\sqrt{x^2 - 9}}{x}\, dx &= \int \frac{3 \tan \theta}{3 \sec \theta}\, 3 \sec \theta \tan \theta \, d\theta \\ &= 3 \int \tan^2 \theta \, d\theta \\ &= 3 \int (\sec^2 \theta - 1)\, d\theta = 3 \int \sec^2 \theta \, d\theta - 3 \int d\theta \\ &= 3 \tan \theta - 3\theta + C.\end{aligned}$$

FIGURE 9.5

$\sec \theta = \dfrac{x}{3}$



Since $\sec \theta = x/3$, we may refer to the right triangle in Figure 9.5. Using $\tan \theta = \sqrt{x^2 - 9}/3$ and $\theta = \sec^{-1}(x/3)$, we obtain

$$\begin{aligned}\int \frac{\sqrt{x^2 - 9}}{x}\, dx &= 3\, \frac{\sqrt{x^2 - 9}}{3} - 3 \sec^{-1}\left(\frac{x}{3}\right) + C \\ &= \sqrt{x^2 - 9} - 3 \sec^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$

As shown in the next example, we can use trigonometric substitutions to evaluate certain integrals that involve $(a^2 - x^2)^n$, $(a^2 + x^2)^n$, or $(x^2 - a^2)^n$, in cases other than $n = \frac{1}{2}$.

EXAMPLE 4 Evaluate $\displaystyle\int \frac{(1 - x^2)^{3/2}}{x^6}\, dx$.

SOLUTION The integrand contains the expression $1 - x^2$, which is of the form $a^2 - x^2$ with $a = 1$. Using (9.4), we substitute

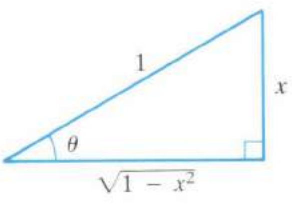
$$x = \sin \theta, \quad dx = \cos \theta \, d\theta.$$

Thus, $1 - x^2 = 1 - \sin^2 \theta = \cos^2 \theta$, and

$$\begin{aligned}\int \frac{(1 - x^2)^{3/2}}{x^6}\, dx &= \int \frac{(\cos^2 \theta)^{3/2}}{\sin^6 \theta} \cos \theta \, d\theta \\ &= \int \frac{\cos^4 \theta}{\sin^6 \theta}\, d\theta = \int \frac{\cos^4 \theta}{\sin^4 \theta} \cdot \frac{1}{\sin^2 \theta}\, d\theta \\ &= \int \cot^4 \theta \csc^2 \theta \, d\theta \\ &= -\tfrac{1}{5} \cot^5 \theta + C.\end{aligned}$$

FIGURE 9.6

$\sin \theta = x$



To return to the variable x, we note that $\sin \theta = x = x/1$ and refer to the right triangle in Figure 9.6, obtaining $\cot \theta = \sqrt{1 - x^2}/x$. Hence

$$\begin{aligned}\int \frac{(1 - x^2)^{3/2}}{x^6}\, dx &= -\frac{1}{5}\left(\frac{\sqrt{1 - x^2}}{x}\right)^5 + C \\ &= -\frac{(1 - x^2)^{5/2}}{5x^5} + C.\end{aligned}$$

Although we now have additional integration techniques available, it is a good idea to keep earlier methods in mind. For example, the integral $\int (x/\sqrt{9+x^2})\,dx$ could be evaluated by means of the trigonometric substitution $x = 3\tan\theta$. However, it is simpler to use the algebraic substitution $u = 9 + x^2$ and $du = 2x\,dx$, for in this event the integral takes on the form $\frac{1}{2}\int u^{-1/2}\,du$, which is readily integrated by means of the power rule. The following exercises include integrals that can be evaluated using simpler techniques than trigonometric substitutions.

EXERCISES 9.3

Exer. 1–22: Evaluate the integral.

1 $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx$

2 $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

3 $\displaystyle\int \frac{1}{x\sqrt{9+x^2}}\,dx$

4 $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$

5 $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

6 $\displaystyle\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$

7 $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

8 $\displaystyle\int \frac{x}{x^2+9}\,dx$

9 $\displaystyle\int \frac{1}{(x^2-1)^{3/2}}\,dx$

10 $\displaystyle\int \frac{1}{\sqrt{4x^2-25}}\,dx$

11 $\displaystyle\int \frac{1}{(36+x^2)^2}\,dx$

12 $\displaystyle\int \frac{1}{(16-x^2)^{5/2}}\,dx$

13 $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

14 $\displaystyle\int \frac{1}{49+x^2}\,dx$

15 $\displaystyle\int \frac{x}{(16-x^2)^2}\,dx$

16 $\displaystyle\int x\sqrt{x^2-9}\,dx$

17 $\displaystyle\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$

18 $\displaystyle\int \frac{1}{x\sqrt{25x^2+16}}\,dx$

19 $\displaystyle\int \frac{1}{x^4\sqrt{x^2-3}}\,dx$

20 $\displaystyle\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx$

21 $\displaystyle\int \frac{(4+x^2)^2}{x^3}\,dx$

22 $\displaystyle\int \frac{3x-5}{\sqrt{1-x^2}}\,dx$

23 The region bounded by the graphs of $y = x(x^2+25)^{-1/2}$, $y = 0$, and $x = 5$ is revolved about the y-axis. Find the volume of the resulting solid.

24 Find the area of the region bounded by the graph of $y = x^3(10-x^2)^{-1/2}$, the x-axis, and the line $x = 1$.

Exer. 25–26: Solve the differential equation subject to the given initial condition.

25 $x\,dy = \sqrt{x^2-16}\,dx$; $\quad y = 0$ if $x = 4$

26 $\sqrt{1-x^2}\,dy = x^3\,dx$; $\quad y = 0$ if $x = 0$

Exer. 27–32: Use a trigonometric substitution to derive the formula. (See Formulas 21, 27, 31, 36, 41, and 44 in Appendix IV.)

27 $\displaystyle\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right| + C$

28 $\displaystyle\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

29 $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

30 $\displaystyle\int \frac{1}{u^2\sqrt{a^2-u^2}}\,du = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

31 $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{u} + C$

32 $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

9.4 INTEGRALS OF RATIONAL FUNCTIONS

Recall that if q is a rational function, then $q(x) = f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials. In this section we shall state rules for evaluating $\int q(x)\,dx$.

Let us consider the specific case $q(x) = 2/(x^2 - 1)$. It is easy to verify that

$$\frac{1}{x-1} + \frac{-1}{x+1} = \frac{2}{x^2-1}.$$

The expression on the left side of the equation is called the *partial fraction decomposition* of $2/(x^2 - 1)$. To find $\int q(x)\,dx$, we integrate each of the fractions that make up the decomposition, obtaining

$$\begin{aligned}\int \frac{2}{x^2-1}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{-1}{x+1}\,dx \\ &= \ln|x-1| - \ln|x+1| + C \\ &= \ln\left|\frac{x-1}{x+1}\right| + C.\end{aligned}$$

It is theoretically possible to write *any* rational expression $f(x)/g(x)$ as a sum of rational expressions whose denominators involve powers of polynomials of degree not greater than two. Specifically, if $f(x)$ and $g(x)$ are polynomials *and the degree of $f(x)$ is less than the degree of $g(x)$*, then it can be proved that

$$\frac{f(x)}{g(x)} = F_1 + F_2 + \cdots + F_r$$

such that each term F_k of the sum has one of the forms

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

for real numbers A and B and a nonnegative integer n, where $ax^2 + bx + c$ is **irreducible** in the sense that this quadratic polynomial has no real zeros (that is, $b^2 - 4ac < 0$). In this case, $ax^2 + bx + c$ cannot be expressed as a product of two first-degree polynomials with real coefficients.

The sum $F_1 + F_2 + \cdots + F_r$ is the **partial fraction decomposition** of $f(x)/g(x)$, and each F_k is a **partial fraction**. We shall not prove this algebraic result but shall, instead, state guidelines for obtaining the decomposition.

The guidelines for finding the partial fraction decomposition of $f(x)/g(x)$ should be used only if $f(x)$ has lower degree than $g(x)$. If this is not the case, then we may use long division to arrive at the proper form. For example, given

$$\frac{x^3 - 6x^2 + 5x - 3}{x^2 - 1},$$

we obtain, by long division,

$$\frac{x^3 - 6x^2 + 5x - 3}{x^2 - 1} = x - 6 + \frac{6x - 9}{x^2 - 1}.$$

We then find the partial fraction decomposition for $(6x - 9)/(x^2 - 1)$.

Guidelines for partial fraction decompositions of $f(x)/g(x)$ **(9.5)**

1. If the degree of $f(x)$ is not lower than the degree of $g(x)$, use long division to obtain the proper form.
2. Express $g(x)$ as a product of linear factors $ax + b$ or irreducible quadratic factors $ax^2 + bx + c$, and collect repeated factors so that $g(x)$ is a product of *different* factors of the form $(ax + b)^n$ or $(ax^2 + bx + c)^n$ for a nonnegative integer n.
3. Apply the following rules.

Rule a For each factor $(ax + b)^n$ with $n \geq 1$, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n},$$

where each numerator A_k is a real number.

Rule b For each factor $(ax^2 + bx + c)^n$ with $n \geq 1$ and with $ax^2 + bx + c$ irreducible, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n},$$

where each A_k and B_k is a real number.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x}\,dx$.

SOLUTION We may factor the denominator of the integrand as follows:

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) = x(x+3)(x-1)$$

Each factor has the form stated in Rule a of (9.5), with $m = 1$. Thus, to the factor x there corresponds a partial fraction of the form A/x. Similarly, to the factors $x + 3$ and $x - 1$ there correspond partial fractions $B/(x + 3)$ and $C/(x - 1)$, respectively. Therefore the partial fraction decomposition has the form

$$\frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}.$$

Multiplying by the lowest common denominator gives us

$$(*) \qquad 4x^2 + 13x - 9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3).$$

In a case such as this, in which the factors are all linear and nonrepeated, the values for A, B, and C can be found by substituting values for x that make the various factors zero. If we let $x = 0$ in $(*)$, then

$$-9 = -3A, \quad \text{or} \quad A = 3.$$

Letting $x = 1$ in $(*)$ gives us

$$8 = 4C, \quad \text{or} \quad C = 2.$$

Finally, if $x = -3$ in $(*)$, then

$$-12 = 12B, \quad \text{or} \quad B = -1.$$

The partial fraction decomposition is, therefore,

$$\frac{4x^2 + 13x - 9}{x(x + 3)(x - 1)} = \frac{3}{x} + \frac{-1}{x + 3} + \frac{2}{x - 1}.$$

Integrating and letting K denote the sum of the constants of integration, we have

$$\begin{aligned}
\int \frac{4x^2 + 13x - 9}{x(x + 3)(x - 1)}\,dx &= \int \frac{3}{x}\,dx + \int \frac{-1}{x + 3}\,dx + \int \frac{2}{x - 1}\,dx \\
&= 3 \ln |x| - \ln |x + 3| + 2 \ln |x - 1| + K \\
&= \ln |x^3| - \ln |x + 3| + \ln |x - 1|^2 + K \\
&= \ln \left| \frac{x^3(x - 1)^2}{x + 3} \right| + K.
\end{aligned}$$

Another technique for finding A, B, and C is to expand the right-hand side of (*) and collect like powers of x as follows:

$$4x^2 + 13x - 9 = (A + B + C)x^2 + (2A - B + 3C)x - 3A$$

We now use the fact that if two polynomials are equal, then coefficients of like powers of x are the same. It is convenient to arrange our work in the following way, which we call **comparing coefficients of x**.

$$\begin{aligned}
&\textit{coefficients of } x^2\text{:} && A + B + C = 4 \\
&\textit{coefficients of } x\text{:} && 2A - B + 3C = 13 \\
&\textit{constant terms:} && -3A = -9
\end{aligned}$$

We may show that the solution of this system of equations is $A = 3$, $B = -1$, and $C = 2$.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{3x^3 - 18x^2 + 29x - 4}{(x + 1)(x - 2)^3}\,dx$.

SOLUTION By Rule a of (9.5), there is a partial fraction of the form $A/(x + 1)$ corresponding to the factor $x + 1$ in the denominator of the integrand. For the factor $(x - 2)^3$ we apply Rule a (with $m = 3$), obtaining a sum of three partial fractions $B/(x - 2)$, $C/(x - 2)^2$, and $D/(x - 2)^3$. Consequently, the partial fraction decomposition has the form

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x + 1)(x - 2)^3} = \frac{A}{x + 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2} + \frac{D}{(x - 2)^3}.$$

Multiplying both sides by $(x + 1)(x - 2)^3$ gives us

$$(*) \qquad \begin{aligned} 3x^3 - 18x^2 + 29x - 4 &= A(x - 2)^3 + B(x + 1)(x - 2)^2 \\ &\quad + C(x + 1)(x - 2) + D(x + 1). \end{aligned}$$

Two of the unknown constants may be determined easily. If we let $x = 2$ in (*), we obtain

$$6 = 3D, \quad \text{or} \quad D = 2.$$

Similarly, letting $x = -1$ in $(*)$ yields

$$-54 = -27A, \quad \text{or} \quad A = 2.$$

The remaining constants may be found by comparing coefficients. Examining the right-hand side of $(*)$, we see that the coefficient of x^3 is $A + B$. This must equal the coefficient of x^3 on the left. Thus, by comparison,

$$\textit{coefficients of } x^3\text{:} \quad 3 = A + B.$$

Since $A = 2$, it follows that $B = 1$.

Finally, we compare the constant terms in $(*)$ by letting $x = 0$. This gives us the following:

$$\textit{constant terms}\text{:} \quad -4 = -8A + 4B - 2C + D$$

Substituting the values we have found for A, B, and D into the preceding equation yields

$$-4 = -16 + 4 - 2C + 2,$$

which has the solution $C = -3$. The partial fraction decomposition is, therefore,

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x + 1)(x - 2)^3} = \frac{2}{x + 1} + \frac{1}{x - 2} + \frac{-3}{(x - 2)^2} + \frac{2}{(x - 2)^3}.$$

To find the given integral, we integrate each of the partial fractions on the right side of the last equation, obtaining

$$2 \ln |x + 1| + \ln |x - 2| + \frac{3}{x - 2} - \frac{1}{(x - 2)^2} + K$$

with K the sum of the four constants of integration. This may be written in the form

$$\ln [(x + 1)^2 |x - 2|] + \frac{3}{x - 2} - \frac{1}{(x - 2)^2} + K.$$

EXAMPLE 3 Evaluate $\displaystyle\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4}\, dx$.

SOLUTION The denominator may be factored by grouping as follows:

$$2x^3 - x^2 + 8x - 4 = x^2(2x - 1) + 4(2x - 1) = (x^2 + 4)(2x - 1)$$

Applying Rule b of (9.5) to the irreducible quadratic factor $x^2 + 4$, we see that one of the partial fractions has the form $(Ax + B)/(x^2 + 4)$. By Rule a, there is also a partial fraction $C/(2x - 1)$ corresponding to the factor $2x - 1$. Consequently,

$$\frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{Ax + B}{x^2 + 4} + \frac{C}{2x - 1}.$$

As in previous examples, this leads to

$$(*) \qquad x^2 - x - 21 = (Ax + B)(2x - 1) + C(x^2 + 4).$$

We can find one constant easily. Substituting $x = \frac{1}{2}$ in $(*)$ gives us

$$-\tfrac{85}{4} = \tfrac{17}{4}C, \quad \text{or} \quad C = -5.$$

The remaining constants may be found by comparing coefficients of x in $(*)$:

$$\begin{aligned} \textit{coefficients of } x^2\text{:} &\quad 1 = 2A + C \\ \textit{coefficients of } x\text{:} &\quad -1 = -A + 2B \\ \textit{constant terms:} &\quad -21 = -B + 4C \end{aligned}$$

Since $C = -5$, it follows from $1 = 2A + C$ that $A = 3$. Similarly, using the coefficients of x with $A = 3$ gives us $-1 = -3 + 2B$, or $B = 1$. Thus the partial fraction decomposition of the integrand is

$$\begin{aligned} \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} &= \frac{3x + 1}{x^2 + 4} + \frac{-5}{2x - 1} \\ &= \frac{3x}{x^2 + 4} + \frac{1}{x^2 + 4} - \frac{5}{2x - 1}. \end{aligned}$$

The given integral may now be found by integrating the right side of the last equation. This gives us

$$\frac{3}{2}\ln(x^2 + 4) + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{5}{2}\ln|2x - 1| + K.$$

EXAMPLE 4 Evaluate $\displaystyle\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}\,dx$.

SOLUTION Applying Rule b of (9.5), with $n = 2$, yields

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

Multiplying by the lcd $(x^2 + 1)^2$ gives us

$$\begin{aligned} 5x^3 - 3x^2 + 7x - 3 &= (Ax + B)(x^2 + 1) + Cx + D \\ 5x^3 - 3x^2 + 7x - 3 &= Ax^3 + Bx^2 + (A + C)x + (B + D). \end{aligned}$$

We next compare coefficients as follows:

$$\begin{aligned} \textit{coefficients of } x^3\text{:} &\quad 5 = A \\ \textit{coefficients of } x^2\text{:} &\quad -3 = B \\ \textit{coefficients of } x\text{:} &\quad 7 = A + C \\ \textit{constant terms:} &\quad -3 = B + D \end{aligned}$$

This gives us $A = 5$, $B = -3$, $C = 7 - A = 2$, and $D = -3 - B = 0$. Therefore

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}$$
$$= \frac{5x}{x^2+1} - \frac{3}{x^2+1} + \frac{2x}{(x^2+1)^2}.$$

Integrating yields

$$\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2}\,dx = \frac{5}{2}\ln(x^2+1) - 3\tan^{-1} x - \frac{1}{x^2+1} + K.$$

EXERCISES 9.4

Exer. 1–32: Evaluate the integral.

1 $\int \frac{5x-12}{x(x-4)}\,dx$

2 $\int \frac{x+34}{(x-6)(x+2)}\,dx$

3 $\int \frac{37-11x}{(x+1)(x-2)(x-3)}\,dx$

4 $\int \frac{4x^2+54x+134}{(x-1)(x+5)(x+3)}\,dx$

5 $\int \frac{6x-11}{(x-1)^2}\,dx$

6 $\int \frac{-19x^2+50x-25}{x^2(3x-5)}\,dx$

7 $\int \frac{x+16}{x^2+2x-8}\,dx$

8 $\int \frac{11x+2}{2x^2-5x-3}\,dx$

9 $\int \frac{5x^2-10x-8}{x^3-4x}\,dx$

10 $\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\,dx$

11 $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx$

12 $\int \frac{2x^2-12x+4}{x^3-4x^2}\,dx$

13 $\int \frac{9x^4+17x^3+3x^2-8x+3}{x^5+3x^4}\,dx$

14 $\int \frac{5x^2+30x+43}{(x+3)^3}\,dx$

15 $\int \frac{x^3+6x^2+3x+16}{x^3+4x}\,dx$

16 $\int \frac{2x^2+7x}{x^2+6x+9}\,dx$

17 $\int \frac{5x^2+11x+17}{x^3+5x^2+4x+20}\,dx$

18 $\int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,dx$

19 $\int \frac{x^2+3x+1}{x^4+5x^2+4}\,dx$

20 $\int \frac{4x}{(x^2+1)^3}\,dx$

21 $\int \frac{2x^3+10x}{(x^2+1)^2}\,dx$

22 $\int \frac{x^4+2x^2+4x+1}{(x^2+1)^3}\,dx$

23 $\int \frac{x^3+3x-2}{x^2-x}\,dx$

24 $\int \frac{x^4+2x^2+3}{x^3-4x}\,dx$

25 $\int \frac{x^6-x^3+1}{x^4+9x^2}\,dx$

26 $\int \frac{x^5}{(x^2+4)^2}\,dx$

27 $\int \frac{2x^3-5x^2+46x+98}{(x^2+x-12)^2}\,dx$

28 $\int \frac{-2x^4-3x^3-3x^2+3x+1}{x^2(x+1)^3}\,dx$

29 $\int \frac{4x^3+2x^2-5x-18}{(x-4)(x+1)^3}\,dx$

30 $\int \frac{10x^2+9x+1}{2x^3+3x^2+x}\,dx$

31 $\int \frac{x^3+3x^2+3x+63}{(x^2-9)^2}\,dx$

32 $\int \frac{x^5-x^4-2x^3+4x^2-15x+5}{(x^2+1)^2(x^2+4)}\,dx$

Exer. 33–36: Use partial fractions to evaluate the integral (see Formulas 19, 49, 50, and 52 of the table of integrals in Appendix IV).

33 $\int \frac{1}{a^2-u^2}\,du$

34 $\int \frac{1}{u(a+bu)}\,du$

35 $\int \frac{1}{u^2(a+bu)}\,du$

36 $\int \frac{1}{u(a+bu)^2}\,du$

37 If $f(x) = x/(x^2 - 2x - 3)$, find the area of the region under the graph of f from $x = 0$ to $x = 2$.

38 The region bounded by the graphs of $y = 1/(x-1)(4-x)$, $y = 0$, $x = 2$, and $x = 3$ is revolved about the y-axis. Find the volume of the resulting solid.

39 If the region described in Exercise 38 is revolved about the x-axis, find the volume of the resulting solid.

40 In the *law of logistic growth*, it is assumed that at time t, the rate of growth $f'(t)$ of a quantity $f(t)$ is given by $f'(t) = Af(t)[B - f(t)]$, where A and B are constants. If $f(0) = C$, show that

$$f(t) = \frac{BC}{C + (B - C)e^{-ABt}}.$$

41 As an alternative to partial fractions, show that an integral of the form

$$\int \frac{1}{ax^2 + bx}\,dx$$

may be evaluated by writing it as

$$\int \frac{(1/x^2)}{a + (b/x)}\,dx$$

and using the substitution $u = a + (b/x)$.

42 Generalize Exercise 41 to integrals of the form

$$\int \frac{1}{ax^n + bx}\,dx.$$

43 Suppose $g(x) = (x - c_1)(x - c_2)\cdots(x - c_n)$ for a positive integer n and distinct real numbers $c_1, c_2, \ldots, c_n$. If $f(x)$ is a polynomial of degree less than n, show that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - c_1} + \frac{A_2}{x - c_2} + \cdots + \frac{A_n}{x - c_n}$$

with $A_k = f(c_k)/g'(c_k)$ for $k = 1, 2, \ldots, n$. (This is a method for finding the partial fraction decomposition if the denominator can be factored into distinct linear factors.)

44 Use Exercise 43 to find the partial fraction decomposition of

$$\frac{2x^4 - x^3 - 3x^2 + 5x + 7}{x^5 - 5x^3 + 4x}.$$

9.5 INTEGRALS INVOLVING QUADRATIC EXPRESSIONS

Partial fraction decompositions may lead to integrands containing an irreducible quadratic expression $ax^2 + bx + c$. If $b \neq 0$, it is sometimes necessary to complete the square as follows:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

The substitution $u = x + b/(2a)$ may then lead to an integrable form.

EXAMPLE 1 Evaluate $\int \frac{2x - 1}{x^2 - 6x + 13}\,dx.$

SOLUTION Note that the quadratic expression $x^2 - 6x + 13$ is irreducible, since $b^2 - 4ac = -16 < 0$. We complete the square as follows:

$$\begin{aligned} x^2 - 6x + 13 &= (x^2 - 6x \qquad) + 13 \\ &= (x^2 - 6x + 9) + 13 - 9 = (x - 3)^2 + 4 \end{aligned}$$

Thus,

$$\int \frac{2x - 1}{x^2 - 6x + 13}\,dx = \int \frac{2x - 1}{(x - 3)^2 + 4}\,dx.$$

We now make the substitution

$$u = x - 3, \quad x = u + 3, \quad dx = du.$$

Thus,

$$\begin{aligned}\int \frac{2x - 1}{x^2 - 6x + 13}\,dx &= \int \frac{2(u + 3) - 1}{u^2 + 4}\,du \\ &= \int \frac{2u + 5}{u^2 + 4}\,du \\ &= \int \frac{2u}{u^2 + 4}\,du + 5\int \frac{1}{u^2 + 4}\,du \\ &= \ln(u^2 + 4) + \frac{5}{2}\tan^{-1}\frac{u}{2} + C \\ &= \ln(x^2 - 6x + 13) + \frac{5}{2}\tan^{-1}\frac{x - 3}{2} + C.\end{aligned}$$

We may also employ the technique of completing the square if a quadratic expression appears under a radical sign.

EXAMPLE 2 Evaluate $\int \frac{1}{\sqrt{8 + 2x - x^2}}\,dx$.

SOLUTION We complete the square for the quadratic expression $8 + 2x - x^2$ as follows:

$$\begin{aligned}8 + 2x - x^2 = 8 - (x^2 - 2x) &= 8 + 1 - (x^2 - 2x + 1) \\ &= 9 - (x - 1)^2\end{aligned}$$

Thus,

$$\int \frac{1}{\sqrt{8 + 2x - x^2}}\,dx = \int \frac{1}{\sqrt{9 - (x - 1)^2}}\,dx.$$

Using the substitution

$$u = x - 1, \quad du = dx$$

yields

$$\begin{aligned}\int \frac{1}{\sqrt{8 + 2x - x^2}}\,dx &= \int \frac{1}{\sqrt{9 - (x - 1)^2}}\,dx \\ &= \int \frac{1}{\sqrt{9 - u^2}}\,du \\ &= \sin^{-1}\frac{u}{3} + C \\ &= \sin^{-1}\frac{x - 1}{3} + C.\end{aligned}$$

In the next example we make a trigonometric substitution after completing the square.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+8x+25}}\,dx$.

SOLUTION We complete the square for the quadratic expression as follows:

$$\begin{aligned} x^2+8x+25 &= (x^2+8x \qquad)+25 \\ &= (x^2+8x+16)+25-16 \\ &= (x+4)^2+9 \end{aligned}$$

Thus,

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx.$$

If we make the trigonometric substitution

$$x+4 = 3\tan\theta, \quad dx = 3\sec^2\theta\,d\theta,$$

then

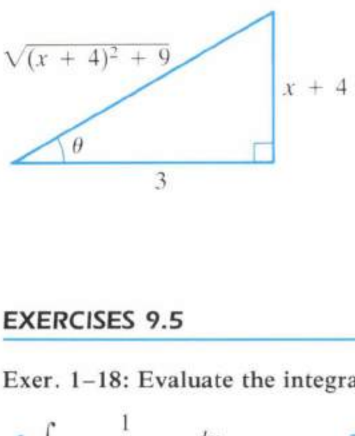
$$\sqrt{(x+4)^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1} = 3\sec\theta$$

and

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+8x+25}}\,dx &= \int \frac{1}{3\sec\theta}\,3\sec^2\theta\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta|+C. \end{aligned}$$

FIGURE 9.7

$\tan\theta = \dfrac{x+4}{3}$



To return to the variable x, we use the triangle in Figure 9.7, obtaining

$$\begin{aligned} \int \frac{1}{\sqrt{x^2+8x+25}}\,dx &= \ln\left|\frac{\sqrt{x^2+8x+25}}{3}+\frac{x+4}{3}\right|+C \\ &= \ln\left|\sqrt{x^2+8x+25}+x+4\right|-\ln|3|+C \\ &= \ln\left|\sqrt{x^2+8x+25}+x+4\right|+K \end{aligned}$$

with $K = C-\ln 3$.

EXERCISES 9.5

Exer. 1–18: Evaluate the integral.

1 $\displaystyle\int \frac{1}{(x+1)^2+4}\,dx$

2 $\displaystyle\int \frac{1}{\sqrt{16-(x-3)^2}}\,dx$

3 $\displaystyle\int \frac{1}{x^2-4x+8}\,dx$

4 $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$

5 $\displaystyle\int \frac{1}{\sqrt{4x-x^2}}\,dx$

6 $\displaystyle\int \frac{1}{\sqrt{7+6x-x^2}}\,dx$

7 $\displaystyle\int \frac{2x+3}{\sqrt{9-8x-x^2}}\,dx$

8 $\displaystyle\int \frac{x+5}{9x^2+6x+17}\,dx$

9 $\displaystyle\int \frac{1}{(x^2+4x+5)^2}\,dx$

10 $\displaystyle\int \frac{1}{(x^2-6x+34)^{3/2}}\,dx$

11 $\displaystyle\int \frac{1}{(x^2+6x+13)^{3/2}}\,dx$

12 $\displaystyle\int \sqrt{x(6-x)}\,dx$

13 $\displaystyle\int \frac{1}{2x^2 - 3x + 9}\,dx$

14 $\displaystyle\int \frac{2x}{(x^2 + 2x + 5)^2}\,dx$

15 $\displaystyle\int \frac{e^x}{e^{2x} + 3e^x + 2}\,dx$

16 $\displaystyle\int \sqrt{x^2 + 10x}\,dx$

17 $\displaystyle\int_2^3 \frac{x^2 - 4x + 6}{x^2 - 4x + 5}\,dx$

18 $\displaystyle\int_0^1 \frac{x - 1}{x^2 + x + 1}\,dx$

19 Find the area of the region bounded by the graphs of $y = 1/(x^2 + 4x + 29)$, $y = 0$, $x = -2$, and $x = 3$.

20 The region bounded by the graph of $y = 1/(x^2 + 2x + 10)$, the coordinate axes, and the line $x = 2$ is revolved about the x-axis. Find the volume of the resulting solid.

9.6 MISCELLANEOUS SUBSTITUTIONS

In this section we shall consider substitutions that are useful for evaluating certain types of integrals. The first example illustrates that if an integral contains an expression of the form $\sqrt[n]{f(x)}$, then one of the substitutions $u = \sqrt[n]{f(x)}$ or $u = f(x)$ may simplify the evaluation.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2 + 4}}\,dx$.

SOLUTION 1 The substitution $u = \sqrt[3]{x^2 + 4}$ leads to the following equivalent equations:

$$u = \sqrt[3]{x^2 + 4}, \quad u^3 = x^2 + 4, \quad x^2 = u^3 - 4$$

Taking the differential of each side of the last equation, we obtain

$$2x\,dx = 3u^2\,du, \quad \text{or} \quad x\,dx = \tfrac{3}{2}u^2\,du.$$

We now substitute as follows:

$$\begin{aligned}
\int \frac{x^3}{\sqrt[3]{x^2 + 4}}\,dx &= \int \frac{x^2}{\sqrt[3]{x^2 + 4}} \cdot x\,dx \\
&= \int \frac{u^3 - 4}{u} \cdot \frac{3}{2}u^2\,du = \frac{3}{2}\int (u^4 - 4u)\,du \\
&= \tfrac{3}{2}(\tfrac{1}{5}u^5 - 2u^2) + C = \tfrac{3}{10}u^2(u^3 - 10) + C \\
&= \tfrac{3}{10}(x^2 + 4)^{2/3}(x^2 - 6) + C
\end{aligned}$$

SOLUTION 2 If we substitute u for the expression *underneath* the radical, then

$$u = x^2 + 4, \quad \text{or} \quad x^2 = u - 4$$

and

$$2x\,dx = du, \quad \text{or} \quad x\,dx = \tfrac{1}{2}\,du.$$

In this case we may write

$$\begin{aligned}
\int \frac{x^3}{\sqrt[3]{x^2 + 4}}\,dx &= \int \frac{x^2}{\sqrt[3]{x^2 + 4}} \cdot x\,dx \\
&= \int \frac{u - 4}{u^{1/3}} \cdot \frac{1}{2}\,du = \frac{1}{2}\int (u^{2/3} - 4u^{-1/3})\,du \\
&= \tfrac{1}{2}(\tfrac{3}{5}u^{5/3} - 6u^{2/3}) + C = \tfrac{3}{10}u^{2/3}(u - 10) + C \\
&= \tfrac{3}{10}(x^2 + 4)^{2/3}(x^2 - 6) + C.
\end{aligned}$$

EXAMPLE 2 Evaluate $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$.

SOLUTION To obtain a substitution that will eliminate the two radicals $\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x} = x^{1/3}$, we use $u = x^{1/n}$, where n is the least common denominator of $\frac{1}{2}$ and $\frac{1}{3}$. Thus, we let

$$u = x^{1/6}, \quad \text{or, equivalently,} \quad x = u^6.$$

Hence

$$dx = 6u^5\, du, \quad x^{1/2} = (u^6)^{1/2} = u^3, \quad x^{1/3} = (u^6)^{1/3} = u^2$$

and, therefore,

$$\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx = \int \frac{1}{u^3 + u^2}\, 6u^5\, du = 6 \int \frac{u^3}{u+1}\, du.$$

By long division,

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Consequently,

$$\begin{aligned}\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx &= 6 \int \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 6\left(\tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 + u - \ln |u+1|\right) + C \\ &= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln\left(\sqrt[6]{x} + 1\right) + C.\end{aligned}$$

If an integrand is a rational expression in $\sin x$ and $\cos x$, then the substitution

$$u = \tan \frac{x}{2} \quad \text{for} \quad -\pi < x < \pi$$

will transform the integrand into a rational (algebraic) expression in u. To prove this, first note that

$$\cos \frac{x}{2} = \frac{1}{\sec (x/2)} = \frac{1}{\sqrt{1 + \tan^2 (x/2)}} = \frac{1}{\sqrt{1+u^2}}$$

$$\sin \frac{x}{2} = \tan \frac{x}{2} \cos \frac{x}{2} = u \frac{1}{\sqrt{1+u^2}}.$$

Consequently,

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2u}{1+u^2}$$

$$\cos x = 1 - 2 \sin^2 \frac{x}{2} = 1 - \frac{2u^2}{1+u^2} = \frac{1-u^2}{1+u^2}.$$

Moreover, since $x/2 = \tan^{-1} u$, we have $x = 2 \tan^{-1} u$ and, therefore,

$$dx = \frac{2}{1+u^2}\, du.$$

The following theorem summarizes this discussion.

Theorem (9.6)

If an integrand is a rational expression in $\sin x$ and $\cos x$, the following substitutions will produce a rational expression in u:

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\, du,$$

where $u = \tan \dfrac{x}{2}$.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{1}{4\sin x - 3\cos x}\, dx$.

SOLUTION Applying Theorem (9.6) and simplifying the integrand yields

$$\begin{aligned}
\int \frac{1}{4\sin x - 3\cos x}\, dx &= \int \frac{1}{4\left(\dfrac{2u}{1+u^2}\right) - 3\left(\dfrac{1-u^2}{1+u^2}\right)} \cdot \frac{2}{1+u^2}\, du \\
&= \int \frac{2}{8u - 3(1-u^2)}\, du \\
&= 2\int \frac{1}{3u^2 + 8u - 3}\, du.
\end{aligned}$$

Using partial fractions, we have

$$\frac{1}{3u^2+8u-3} = \frac{1}{10}\left(\frac{3}{3u-1} - \frac{1}{u+3}\right)$$

and hence

$$\begin{aligned}
\int \frac{1}{4\sin x - 3\cos x}\, dx &= \frac{1}{5}\int \left(\frac{3}{3u-1} - \frac{1}{u+3}\right) du \\
&= \frac{1}{5}(\ln|3u-1| - \ln|u+3|) + C \\
&= \frac{1}{5}\ln\left|\frac{3u-1}{u+3}\right| + C \\
&= \frac{1}{5}\ln\left|\frac{3\tan(x/2) - 1}{\tan(x/2) + 3}\right| + C.
\end{aligned}$$

Theorem (9.6) may be used for any integrand that is a rational expression in $\sin x$ and $\cos x$. However, it is important to also consider simpler substitutions, as illustrated in the next example.

EXAMPLE 4 Evaluate $\displaystyle\int \frac{\cos x}{1 + \sin^2 x}\, dx$.

SOLUTION We could use the formulas in Theorem (9.6) to change the integrand into a rational expression in u. The following substitution is simpler:

$$u = \sin x, \quad du = \cos x\, dx$$

Thus,

$$\begin{aligned}\int \frac{\cos x}{1+\sin^2 x}\, dx &= \int \frac{1}{1+u^2}\, du \\ &= \arctan u + C \\ &= \arctan \sin x + C.\end{aligned}$$

EXERCISES 9.6

Exer. 1–26: Evaluate the integral.

1 $\int x\sqrt[3]{x+9}\, dx$

2 $\int x^2\sqrt{2x+1}\, dx$

3 $\int \frac{x}{\sqrt[5]{3x+2}}\, dx$

4 $\int \frac{5x}{(x+3)^{2/3}}\, dx$

5 $\int_4^9 \frac{1}{\sqrt{x}+4}\, dx$

6 $\int_0^{25} \frac{1}{\sqrt{4+\sqrt{x}}}\, dx$

7 $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\, dx$

8 $\int \frac{1}{\sqrt[4]{x}+\sqrt[3]{x}}\, dx$

9 $\int \frac{1}{(x+1)\sqrt{x-2}}\, dx$

10 $\int_0^4 \frac{2x+3}{\sqrt{1+2x}}\, dx$

11 $\int \frac{x+1}{(x+4)^{1/3}}\, dx$

12 $\int \frac{x^{1/3}+1}{x^{1/3}-1}\, dx$

13 $\int e^{3x}\sqrt{1+e^x}\, dx$

14 $\int \frac{e^{2x}}{\sqrt[3]{1+e^x}}\, dx$

15 $\int \frac{e^{2x}}{e^x+4}\, dx$

16 $\int \frac{\sin 2x}{\sqrt{1+\sin x}}\, dx$

17 $\int \sin\sqrt{x+4}\, dx$

18 $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

19 $\int_2^3 \frac{x}{(x-1)^6}\, dx$

20 $\int \frac{x^2}{(3x+4)^{10}}\, dx$

21 $\int \frac{\sin x}{\cos x(\cos x - 1)}\, dx$

22 $\int \frac{\cos x}{\sin^2 x - \sin x - 2}\, dx$

23 $\int \frac{e^x}{e^{2x}-1}\, dx$

24 $\int \frac{1}{e^x+e^{-x}}\, dx$

25 $\int \frac{\sin 2x}{\sin^2 x - 2\sin x - 8}\, dx$

26 $\int \frac{\sin x}{5\cos x + \cos^2 x}\, dx$

Exer. 27–32: Use Theorem (9.6) to evaluate the integral.

27 $\int \frac{1}{2+\sin x}\, dx$

28 $\int \frac{1}{3+2\cos x}\, dx$

29 $\int \frac{1}{1+\sin x+\cos x}\, dx$

30 $\int \frac{1}{\tan x + \sin x}\, dx$

31 $\int \frac{\sec x}{4-3\tan x}\, dx$

32 $\int \frac{1}{\sin x - \sqrt{3}\cos x}\, dx$

Exer. 33–34: Use Theorem (9.6) to derive the formula.

33 $\int \sec x\, dx = \ln\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right| + C$

34 $\int \csc x\, dx = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C$

9.7 TABLES OF INTEGRALS

Mathematicians and scientists who use integrals in their work sometimes refer to tables of integrals. Many of the formulas contained in these tables may be obtained by methods we have studied. In general, tables of integrals should be used only after gaining experience with standard methods of integration. For complicated integrals it is often necessary to make substitutions or to use partial fractions, integration by parts, or other techniques to obtain integrands to which the table is applicable.

The following examples illustrate the use of several formulas stated in the brief table of integrals in Appendix IV. To guard against errors introduced when using the table, you should always check answers by differentiation.

EXAMPLE 1 Evaluate $\int x^3 \cos x\, dx$.

SOLUTION We first use reduction Formula 85 in the table of integrals with $n = 3$ and $u = x$, obtaining

$$\int x^3 \cos x\, dx = x^3 \sin x - 3 \int x^2 \sin x\, dx.$$

Next we apply Formula 84 with $n = 2$, and then Formula 83, obtaining

$$\begin{aligned}\int x^2 \sin x\, dx &= -x^2 \cos x + 2 \int x \cos x\, dx \\ &= -x^2 \cos x + 2(\cos x + x \sin x) + C.\end{aligned}$$

Substitution in the first expression gives us

$$\int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x - 6 \cos x - 6x \sin x + C.$$

EXAMPLE 2 Evaluate $\int \frac{1}{x^2\sqrt{3 + 5x^2}}\, dx$ for $x > 0$.

SOLUTION The integrand suggests that we use that part of the table dealing with the form $\sqrt{a^2 + u^2}$. Specifically, Formula 28 states that

$$\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C.$$

(In tables, the differential du is placed in the numerator instead of to the right of the integrand.) To use this formula, we must adjust the given integral so that it matches *exactly* with the formula. If we let

$$a^2 = 3 \quad \text{and} \quad u^2 = 5x^2,$$

then the expression underneath the radical is taken care of; however, we also need

(i) u^2 to the left of the radical

(ii) du in the numerator

We can obtain (i) by writing the integral as

$$5 \int \frac{1}{5x^2\sqrt{3 + 5x^2}}\, dx.$$

For (ii) we note that

$$u = \sqrt{5}x \quad \text{and} \quad du = \sqrt{5}\, dx$$

and write the preceding integral as

$$5 \cdot \frac{1}{\sqrt{5}} \int \frac{1}{5x^2\sqrt{3 + 5x^2}} \sqrt{5}\, dx.$$

The last integral matches exactly with that in Formula 28, and hence

$$\begin{aligned}\int \frac{1}{x^2\sqrt{3 + 5x^2}}\, dx &= \sqrt{5}\left[-\frac{\sqrt{3 + 5x^2}}{3(\sqrt{5}x)}\right] + C \\ &= -\frac{\sqrt{3 + 5x^2}}{3x} + C.\end{aligned}$$

As illustrated in the next example, it may be necessary to make a substitution of some type before a table can be used to help evaluate an integral.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{\sin 2x}{\sqrt{3 - 5\cos x}}\, dx.$

SOLUTION Let us begin by rewriting the integral:

$$\int \frac{\sin 2x}{\sqrt{3 - 5\cos x}}\, dx = \int \frac{2\sin x \cos x}{\sqrt{3 - 5\cos x}}\, dx$$

Since no formulas in the table have this form, we consider making the substitution $u = \cos x$. In this case $du = -\sin x\, dx$ and the integral may be written

$$\begin{aligned}2\int \frac{\sin x \cos x}{\sqrt{3 - 5\cos x}}\, dx &= -2\int \frac{\cos x}{\sqrt{3 - 5\cos x}}(-\sin x)\, dx \\ &= -2\int \frac{u}{\sqrt{3 - 5u}}\, du.\end{aligned}$$

Referring to the table of integrals, we see that Formula 55 is

$$\int \frac{u\, du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu}.$$

Using this result with $a = 3$ and $b = -5$ gives us

$$-2\int \frac{u}{\sqrt{3 - 5u}}\, du = -2\left(\frac{2}{75}\right)(-5u - 6)\sqrt{3 - 5u} + C.$$

Finally, since $u = \cos x$, we obtain

$$\int \frac{\sin 2x}{\sqrt{3 - 5\cos x}}\, dx = \frac{4}{75}(5\cos x + 6)\sqrt{3 - 5\cos x} + C.$$

We have discussed various methods for evaluating indefinite integrals; however, the types of integrals we have considered constitute only a small percentage of those that occur in applications. The following are examples of indefinite integrals for which antiderivatives of the integrands cannot be expressed in terms of a finite number of algebraic or transcendental functions:

$$\int \sqrt[3]{x^2 + 4x - 1}\, dx, \quad \int \sqrt{3 \cos^2 x + 1}\, dx, \quad \int e^{-x^2}\, dx$$

In Chapter 11 we shall consider methods involving *infinite* sums that are sometimes useful in evaluating such integrals.

EXERCISES 9.7

Exer. 1–30: Use the table of integrals in Appendix IV to evaluate the integral.

1 $\int \frac{\sqrt{4 + 9x^2}}{x}\, dx$

2 $\int \frac{1}{x\sqrt{2 + 3x^2}}\, dx$

3 $\int (16 - x^2)^{3/2}\, dx$

4 $\int x^2\sqrt{4x^2 - 16}\, dx$

5 $\int x\sqrt{2 - 3x}\, dx$

6 $\int x^2\sqrt{5 + 2x}\, dx$

7 $\int \sin^6 3x\, dx$

8 $\int x \cos^5 (x^2)\, dx$

9 $\int \csc^4 x\, dx$

10 $\int \sin 5x \cos 3x\, dx$

11 $\int x \sin^{-1} x\, dx$

12 $\int x^2 \tan^{-1} x\, dx$

13 $\int e^{-3x} \sin 2x\, dx$

14 $\int x^5 \ln x\, dx$

15 $\int \frac{\sqrt{5x - 9x^2}}{x}\, dx$

16 $\int \frac{1}{x\sqrt{3x - 2x^2}}\, dx$

17 $\int \frac{x}{5x^4 - 3}\, dx$

18 $\int \cos x \sqrt{\sin^2 x - \frac{1}{4}}\, dx$

19 $\int e^{2x} \cos^{-1} e^x\, dx$

20 $\int \sin^2 x \cos^3 x\, dx$

21 $\int x^3\sqrt{2 + x}\, dx$

22 $\int \frac{7x^3}{\sqrt{2 - x}}\, dx$

23 $\int \frac{\sin 2x}{4 + 9 \sin x}\, dx$

24 $\int \frac{\tan x}{\sqrt{4 + 3 \sec x}}\, dx$

25 $\int \frac{\sqrt{9 + 2x}}{x}\, dx$

26 $\int \sqrt{8x^3 - 3x^2}\, dx$

27 $\int \frac{1}{x(4 + \sqrt[3]{x})}\, dx$

28 $\int \frac{1}{2x^{3/2} + 5x^2}\, dx$

29 $\int \sqrt{16 - \sec^2 x} \tan x\, dx$

30 $\int \frac{\cot x}{\sqrt{4 - \csc^2 x}}\, dx$

9.8 REVIEW EXERCISES

Exer. 1–100: Evaluate the integral.

1 $\int x \sin^{-1} x\, dx$

2 $\int \sec^3 (3x)\, dx$

3 $\int_0^1 \ln (1 + x)\, dx$

4 $\int_0^1 e^{\sqrt{x}}\, dx$

5 $\int \cos^3 2x \sin^2 2x\, dx$

6 $\int \cos^4 x\, dx$

7 $\int \tan x \sec^5 x\, dx$

8 $\int \tan x \sec^6 x\, dx$

9 $\int \frac{1}{(x^2 + 25)^{3/2}}\, dx$

10 $\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx$

11 $\int \frac{\sqrt{4 - x^2}}{x}\, dx$

12 $\int \frac{x}{(x^2 + 1)^2}\, dx$

13 $\int \frac{x^3 + 1}{x(x - 1)^3}\, dx$

14 $\int \frac{1}{x + x^3}\, dx$

15 $\int \frac{x^3 - 20x^2 - 63x - 198}{x^4 - 81}\, dx$

16 $\int \frac{x - 1}{(x + 2)^5}\, dx$

17 $\int \frac{x}{\sqrt{4 + 4x - x^2}}\, dx$

18 $\int \frac{x}{x^2 + 6x + 13}\, dx$

19 $\int \frac{\sqrt[3]{x + 8}}{x}\, dx$

20 $\int \frac{\sin x}{2 \cos x + 3}\, dx$

21 $\int e^{2x} \sin 3x\, dx$

22 $\int \cos (\ln x)\, dx$

23 $\int \sin^3 x \cos^3 x\, dx$

24 $\int \cot^2 3x\, dx$

25 $\int \frac{x}{\sqrt{4 - x^2}}\, dx$

26 $\int \frac{1}{x\sqrt{9x^2 + 4}}\, dx$

27 $\int \frac{x^5 - x^3 + 1}{x^3 + 2x^2}\,dx$

28 $\int \frac{x^3}{x^3 - 3x^2 + 9x - 27}\,dx$

29 $\int \frac{1}{x^{3/2} + x^{1/2}}\,dx$

30 $\int \frac{2x+1}{(x+5)^{100}}\,dx$

31 $\int e^x \sec e^x\,dx$

32 $\int x \tan x^2\,dx$

33 $\int x^2 \sin 5x\,dx$

34 $\int \sin 2x \cos x\,dx$

35 $\int \sin^3 x \cos^{1/2} x\,dx$

36 $\int \sin 3x \cot 3x\,dx$

37 $\int e^x \sqrt{1 + e^x}\,dx$

38 $\int x(4x^2 + 25)^{-1/2}\,dx$

39 $\int \frac{x^2}{\sqrt{4x^2 + 25}}\,dx$

40 $\int \frac{3x + 2}{x^2 + 8x + 25}\,dx$

41 $\int \sec^2 x \tan^2 x\,dx$

42 $\int \sin^2 x \cos^5 x\,dx$

43 $\int x \cot x \csc x\,dx$

44 $\int (1 + \csc 2x)^2\,dx$

45 $\int x^2(8 - x^3)^{1/3}\,dx$

46 $\int x (\ln x)^2\,dx$

47 $\int \sqrt{x} \sin \sqrt{x}\,dx$

48 $\int x\sqrt{5 - 3x}\,dx$

49 $\int \frac{e^{3x}}{1 + e^x}\,dx$

50 $\int \frac{e^{2x}}{4 + e^{4x}}\,dx$

51 $\int \frac{x^2 - 4x + 3}{\sqrt{x}}\,dx$

52 $\int \frac{\cos^3 x}{\sqrt{1 + \sin x}}\,dx$

53 $\int \frac{x^3}{\sqrt{16 - x^2}}\,dx$

54 $\int \frac{x}{25 - 9x^2}\,dx$

55 $\int \frac{1 - 2x}{x^2 + 12x + 35}\,dx$

56 $\int \frac{7}{x^2 - 6x + 18}\,dx$

57 $\int \tan^{-1} 5x\,dx$

58 $\int \sin^4 3x\,dx$

59 $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

60 $\int \frac{x}{\csc 5x^2}\,dx$

61 $\int \frac{1}{\sqrt{7 + 5x^2}}\,dx$

62 $\int \frac{2x + 3}{x^2 + 4}\,dx$

63 $\int \cot^6 x\,dx$

64 $\int \cot^5 x \csc x\,dx$

65 $\int x^3\sqrt{x^2 - 25}\,dx$

66 $\int (\sin x)10^{\cos x}\,dx$

67 $\int (x^2 - \operatorname{sech}^2 4x)\,dx$

68 $\int x \cosh x\,dx$

69 $\int x^2 e^{-4x}\,dx$

70 $\int x^5\sqrt{x^3 + 1}\,dx$

71 $\int \frac{3}{\sqrt{11 - 10x - x^2}}\,dx$

72 $\int \frac{12x^3 + 7x}{x^4}\,dx$

73 $\int \tan 7x \cos 7x\,dx$

74 $\int e^{1 + \ln 5x}\,dx$

75 $\int \frac{4x^2 - 12x - 10}{(x-2)(x^2 - 4x + 3)}\,dx$

76 $\int \frac{1}{x^4\sqrt{16 - x^2}}\,dx$

77 $\int (x^3 + 1)\cos x\,dx$

78 $\int (x-3)^2(x+1)\,dx$

79 $\int \frac{\sqrt{9 - 4x^2}}{x^2}\,dx$

80 $\int \frac{4x^3 - 15x^2 - 6x + 81}{x^4 - 18x^2 + 81}\,dx$

81 $\int (5 - \cot 3x)^2\,dx$

82 $\int x(x^2 + 5)^{3/4}\,dx$

83 $\int \frac{1}{x(\sqrt{x} + \sqrt[4]{x})}\,dx$

84 $\int \frac{x}{\cos^2 4x}\,dx$

85 $\int \frac{\sin x}{\sqrt{1 + \cos x}}\,dx$

86 $\int \frac{4x^2 - 6x + 4}{(x^2 + 4)(x - 2)}\,dx$

87 $\int \frac{x^2}{(25 + x^2)^2}\,dx$

88 $\int \sin^4 x \cos^3 x\,dx$

89 $\int \tan^3 x \sec x\,dx$

90 $\int \frac{x}{\sqrt{4 + 9x^2}}\,dx$

91 $\int \frac{2x^3 + 4x^2 + 10x + 13}{x^4 + 9x^2 + 20}\,dx$

92 $\int \frac{\sin x}{(1 + \cos x)^3}\,dx$

93 $\int \frac{(x^2 - 2)^2}{x}\,dx$

94 $\int \cot^2 x \csc x\,dx$

95 $\int x^{3/2} \ln x\,dx$

96 $\int \frac{x}{\sqrt[3]{x} - 1}\,dx$

97 $\int \frac{x^2}{\sqrt[3]{2x + 3}}\,dx$

98 $\int \frac{1 - \sin x}{\cot x}\,dx$

99 $\int x^3 e^{(x^2)}\,dx$

100 $\int (x + 2)^2(x + 1)^{10}\,dx$

CHAPTER

10

INDETERMINATE FORMS AND IMPROPER INTEGRALS

INTRODUCTION

The first important limit we considered in Chapter 3 was the derivative formula

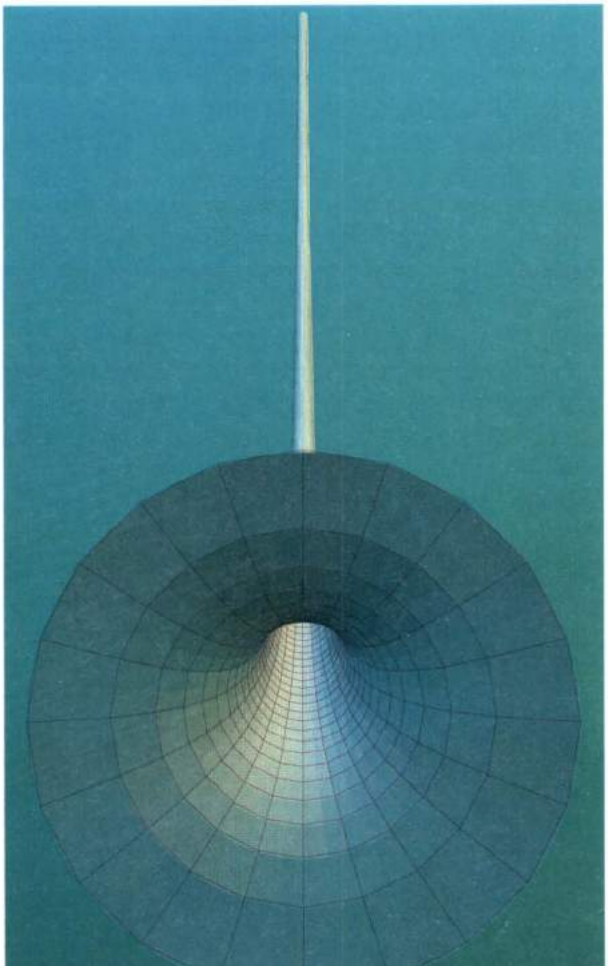
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

If f is continuous at $x = a$, then taking the limit of the numerator and denominator separately gives us

$$f'(a) = \frac{f(a) - f(a)}{a - a} = \frac{0}{0},$$

an undefined expression. However, we know that derivatives are not always undefined. You may recall that to arrive at each rule for finding derivatives we used an algebraic or trigonometric simplification, which was sometimes accompanied by an ingenious manipulation or geometric argument. In this chapter we introduce techniques that allow us to proceed in a more direct manner when considering similar problems about limits. The most important result we shall discuss is *L'Hôpital's rule*, used for investigating limits of quotients in which both numerator and denominator approach 0 or both approach ∞ or $-\infty$. Other so-called *indeterminate forms* are considered in Section 10.2. In the last two sections we study definite integrals that have discontinuous integrands or infinite limits of integration.

The topics discussed in this chapter have many mathematical and physical applications. Our most important uses for them will occur in the next chapter, when we discuss infinite series.



10.1 THE INDETERMINATE FORMS 0/0 AND ∞/∞

In Chapter 2 we considered limits of quotients such as

$$\lim_{x\to 3} \frac{x^2-9}{x-3} \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x}.$$

In each case, taking the limits of the numerator and denominator gives us the undefined expression 0/0. We say that the indicated quotients have the **indeterminate form 0/0** at $x = 3$ and at $x = 0$, respectively. We previously used algebraic, geometric, and trigonometric methods to calculate such limits. In this section we develop another technique that employs the derivatives of the numerator and denominator of the quotient. We also consider the **indeterminate form ∞/∞**, where both the numerator and the denominator approach ∞ or $-\infty$. The following table displays general definitions of the forms we shall discuss.

Indeterminate form	Limit form: $\lim_{x\to c} \frac{f(x)}{g(x)}$
$\frac{0}{0}$	$\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$
$\frac{\infty}{\infty}$	$\lim_{x\to c} f(x) = \infty$ or $-\infty$ and $\lim_{x\to c} g(x) = \infty$ or $-\infty$

The main tool for investigating these indeterminate forms is *L'Hôpital's rule*. The proof of this rule makes use of the following formula, which bears the name of the French mathematician Augustin Cauchy (1789–1857).

Cauchy's formula **(10.1)**

If f and g are continuous on $[a, b]$ and differentiable on (a, b) and if $g'(x) \neq 0$ for every x in (a, b), then there is a number w in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(w)}{g'(w)}.$$

PROOF We first note that $g(b) - g(a) \neq 0$, because otherwise $g(a) = g(b)$ and, by Rolle's theorem (4.10), there is a number c in (a, b) such that $g'(c) = 0$, contrary to our assumption about g'.

Let us introduce a new function h as follows:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

for every x in $[a, b]$. It follows that h is continuous on $[a, b]$ and differentiable on (a, b) and that $h(a) = h(b)$. By Rolle's theorem, there is a number w in (a, b) such that $h'(w) = 0$; that is,

$$[f(b) - f(a)]g'(w) - [g(b) - g(a)]f'(w) = 0.$$

This is equivalent to Cauchy's formula. ■

Cauchy's formula is a generalization of the mean value theorem (4.12), for if we let $g(x) = x$ in (10.1), we obtain

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(w)}{1} = f'(w).$$

The next result is the main theorem on indeterminate forms.

L'Hôpital's rule* (10.2)

Suppose f and g are differentiable on an open interval (a, b) containing c, except possibly at c itself. If $f(x)/g(x)$ has the indeterminate form 0/0 or ∞/∞ at $x = c$ and if $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided either

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists} \quad \text{or} \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} = \infty.$$

*G. L'Hôpital (1661–1704) was a French nobleman who published the first calculus book. The rule appeared in that book; however, it was actually discovered by his teacher, the Swiss mathematician Johann Bernoulli (1667–1748), who communicated the result to L'Hôpital in 1694.

PROOF Suppose $f(x)/g(x)$ has the indeterminate form 0/0 at $x = c$ and $\lim_{x \to c} [f'(x)/g'(x)] = L$ for some number L. We wish to prove that $\lim_{x \to c} [f(x)/g(x)] = L$. Let us introduce two functions F and G as follows:

$$F(x) = f(x) \quad \text{if } x \neq c \quad \text{and} \quad F(c) = 0$$

$$G(x) = g(x) \quad \text{if } x \neq c \quad \text{and} \quad G(c) = 0$$

Since

$$\lim_{x \to c} F(x) = \lim_{x \to c} f(x) = 0 = F(c),$$

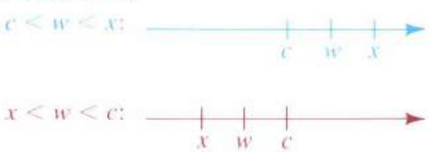
the function F is continuous at c and hence is continuous *throughout* the interval (a, b). Similarly, G is continuous on (a, b). Moreover, at every $x \neq c$ we have $F'(x) = f'(x)$ and $G'(x) = g'(x)$. It follows from Cauchy's formula, applied either to the interval $[c, x]$ or to $[x, c]$, that there is a number w between c and x such that

$$\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(w)}{G'(w)} = \frac{f'(w)}{g'(w)}.$$

Using the fact that $F(x) = f(x)$, $G(x) = g(x)$, and $F(c) = G(c) = 0$ gives us

$$\frac{f(x)}{g(x)} = \frac{f'(w)}{g'(w)}.$$

FIGURE 10.1



Since w is always between c and x (see Figure 10.1), it follows that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(w)}{g'(w)} = \lim_{w \to c} \frac{f'(w)}{g'(w)} = L,$$

which is what we wished to prove.

A similar argument may be given if $\lim_{x \to c} [f'(x)/g'(x)] = \infty$. The proof for the indeterminate form ∞/∞ is more difficult and may be found in texts on advanced calculus.

L'Hôpital's rule is sometimes used incorrectly, by applying the quotient rule to $f(x)/g(x)$. Note that (10.2) states that the derivatives of $f(x)$ and $g(x)$ are taken *separately*, after which the limit of $f'(x)/g'(x)$ is investigated.

EXAMPLE 1 Find $\lim_{x \to 0} \dfrac{\cos x + 2x - 1}{3x}$.

SOLUTION Both the numerator and the denominator have the limit 0 as $x \to 0$. Hence the quotient has the indeterminate form 0/0 at $x = 0$. By L'Hôpital's rule (10.2),

$$\lim_{x \to 0} \frac{\cos x + 2x - 1}{3x} = \lim_{x \to 0} \frac{-\sin x + 2}{3},$$

provided the limit on the right exists or equals ∞. Since

$$\lim_{x \to 0} \frac{-\sin x + 2}{3} = \frac{2}{3},$$

it follows that

$$\lim_{x \to 0} \frac{\cos x + 2x - 1}{3x} = \frac{2}{3}.$$

Sometimes it is necessary to employ L'Hôpital's rule several times in the same problem, as illustrated in the next example.

EXAMPLE 2 Find $\lim_{x \to 0} \dfrac{e^x + e^{-x} - 2}{1 - \cos 2x}$.

SOLUTION The given quotient has the indeterminate form 0/0. By L'Hôpital's rule,

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x} = \lim_{x \to 0} \frac{e^x - e^{-x}}{2 \sin 2x},$$

provided the second limit exists. Because the last quotient has the indeterminate form 0/0, we apply L'Hôpital's rule a second time, obtaining

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2 \sin 2x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{4 \cos 2x} = \frac{2}{4} = \frac{1}{2}.$$

It follows that the given limit exists and equals $\frac{1}{2}$.

L'Hôpital's rule is also valid for one-sided limits, as illustrated in the following example.

EXAMPLE 3 Find $\lim_{x\to(\pi/2)^-} \dfrac{4\tan x}{1+\sec x}$.

SOLUTION The indeterminate form is ∞/∞. By L'Hôpital's rule,

$$\lim_{x\to(\pi/2)^-} \frac{4\tan x}{1+\sec x} = \lim_{x\to(\pi/2)^-} \frac{4\sec^2 x}{\sec x\tan x} = \lim_{x\to(\pi/2)^-} \frac{4\sec x}{\tan x}.$$

The last quotient again has the indeterminate form ∞/∞ at $x = \pi/2$; however, additional applications of L'Hôpital's rule always produce the form ∞/∞ (verify this fact). In this case the limit may be found by using trigonometric identities to change the quotient as follows:

$$\frac{4\sec x}{\tan x} = \frac{4/\cos x}{\sin x/\cos x} = \frac{4}{\sin x}$$

Consequently

$$\lim_{x\to(\pi/2)^-} \frac{4\tan x}{1+\sec x} = \lim_{x\to(\pi/2)^-} \frac{4}{\sin x} = \frac{4}{1} = 4.$$

Another form of L'Hôpital's rule can be proved for $x \to \infty$ or $x \to -\infty$. Let us give a partial proof of this fact. Suppose

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

If we let $u = 1/x$ and apply L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{u\to0^+} \frac{f(1/u)}{g(1/u)} = \lim_{u\to0^+} \frac{D_u f(1/u)}{D_u g(1/u)}.$$

By the chain rule,

$$D_u f(1/u) = f'(1/u)(-1/u^2) \quad \text{and} \quad D_u g(1/u) = g'(1/u)(-1/u^2).$$

Substituting in the last limit and simplifying, we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{u\to0^+} \frac{f'(1/u)}{g'(1/u)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

We shall also refer to this result as L'Hôpital's rule. The next two examples illustrate the application of the rule to the form ∞/∞.

EXAMPLE 4 Find $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.

SOLUTION The indeterminate form is ∞/∞. By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})}.$$

The last expression has the indeterminate form 0/0. However, further applications of L'Hôpital's rule would again lead to 0/0 (verify this fact). If, instead, we simplify the expression algebraically, we can find the limit as follows:

$$\lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

EXAMPLE 5 Find $\lim_{x\to\infty} \dfrac{e^{3x}}{x^2}$, if it exists.

SOLUTION The indeterminate form is ∞/∞. We apply L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{e^{3x}}{x^2} = \lim_{x\to\infty} \frac{3e^{3x}}{2x}$$

The last quotient has the indeterminate form ∞/∞, so we apply L'Hôpital's rule a second time, obtaining

$$\lim_{x\to\infty} \frac{3e^{3x}}{2x} = \lim_{x\to\infty} \frac{9e^{3x}}{2} = \infty.$$

Thus, e^{3x}/x^2 has no limit, increasing without bound as $x \to \infty$.

It is extremely important to verify that a given quotient has the indeterminate form $0/0$ *or* ∞/∞ *before using L'Hôpital's rule.* If we apply the rule to a form that is not indeterminate, we may obtain an incorrect conclusion, as illustrated in the next example.

EXAMPLE 6 Find $\lim_{x\to 0} \dfrac{e^x + e^{-x}}{x^2}$, if it exists.

SOLUTION The quotient does *not* have either of the indeterminate forms, $0/0$ or ∞/∞, at $x = 0$. To investigate the limit, we write

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2} = \lim_{x\to 0} (e^x + e^{-x})\left(\frac{1}{x^2}\right).$$

Since $\qquad \lim_{x\to 0} (e^x + e^{-x}) = 2 \quad \text{and} \quad \lim_{x\to 0} \dfrac{1}{x^2} = \infty,$

it follows that

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2} = \infty.$$

If we had overlooked the fact that the quotient does not have the indeterminate form $0/0$ or ∞/∞ at $x = 0$ and had (incorrectly) applied L'Hôpital's rule, we would have obtained

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x}.$$

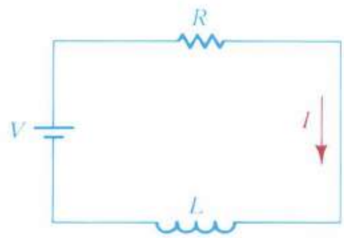
Since the last quotient has the indeterminate form 0/0, we might have applied L'Hôpital's rule, obtaining

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2} = \frac{1+1}{2} = 1.$$

This would have given us the (wrong) conclusion that the given limit exists and equals 1.

The next example illustrates an application of an indeterminate form in the analysis of an electrical circuit.

FIGURE 10.2



EXAMPLE 7 The schematic diagram in Figure 10.2 illustrates an electrical circuit consisting of an electromotive force V, a resistor R, and an inductor L. The current I at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L}).$$

When the voltage is first applied (at $t = 0$), the inductor opposes the rate of increase of current and I is small; however, as t increases, I approaches V/R.

(a) If L is the only independent variable, find $\lim_{L\to 0^+} I$.

(b) If R is the only independent variable, find $\lim_{R\to 0^+} I$.

SOLUTION

(a) If we consider V, R, and t as constants and L as a variable, then the expression for I is not indeterminate at $L = 0$. Using standard limit theorems, we obtain

$$\begin{aligned}\lim_{L\to 0^+} I &= \lim_{L\to 0^+} \frac{V}{R}(1 - e^{-Rt/L}) \\ &= \frac{V}{R}\left(1 - \lim_{L\to 0^+} e^{-Rt/L}\right) \\ &= \frac{V}{R}(1 - 0) = \frac{V}{R}.\end{aligned}$$

Thus, if $L \approx 0$, then the current can be approximated by Ohm's law $I = V/R$.

(b) If V, L, and t are constant and if R is a variable, then I has the indeterminate form 0/0 at $R = 0$. Applying L'Hôpital's rule, we have

$$\begin{aligned}\lim_{R\to 0^+} I &= V \lim_{R\to 0^+} \frac{1 - e^{-Rt/L}}{R} \\ &= V \lim_{R\to 0^+} \frac{0 - e^{-Rt/L}(-t/L)}{1} \\ &= V[0 - (1)(-t/L)] = \frac{V}{L}t.\end{aligned}$$

This may be interpreted as follows. As $R \to 0^+$, the current I is directly proportional to the time t, with the constant of proportionality V/L. Thus, at $t = 1$ the current is V/L, at $t = 2$ it is $(V/L)(2)$, at $t = 3$ it is $(V/L)(3)$, and so on.

EXERCISES 10.1

Exer. 1–52: Find the limit, if it exists.

1 $\lim_{x\to 0} \dfrac{\sin x}{2x}$

2 $\lim_{x\to 0} \dfrac{5x}{\tan x}$

3 $\lim_{x\to 5} \dfrac{\sqrt{x-1}-2}{x^2-25}$

4 $\lim_{x\to 4} \dfrac{x-4}{\sqrt[3]{x+4}-2}$

5 $\lim_{x\to 2} \dfrac{2x^2-5x+2}{5x^2-7x-6}$

6 $\lim_{x\to -3} \dfrac{x^2+2x-3}{2x^2+3x-9}$

7 $\lim_{x\to 1} \dfrac{x^3-3x+2}{x^2-2x-1}$

8 $\lim_{x\to 2} \dfrac{x^2-5x+6}{2x^2-x-7}$

9 $\lim_{x\to 0} \dfrac{\sin x - x}{\tan x - x}$

10 $\lim_{x\to 0} \dfrac{\sin x}{x-\tan x}$

11 $\lim_{x\to 0} \dfrac{x+1-e^x}{x^2}$

12 $\lim_{x\to 0^+} \dfrac{x+1-e^x}{x^3}$

13 $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$

14 $\lim_{x\to \pi/2} \dfrac{1-\sin x}{\cos x}$

15 $\lim_{x\to \pi/2} \dfrac{1+\sin x}{\cos^2 x}$

16 $\lim_{x\to 0^+} \dfrac{\cos x}{x}$

17 $\lim_{x\to (\pi/2)^-} \dfrac{2+\sec x}{3\tan x}$

18 $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$

19 $\lim_{x\to \infty} \dfrac{x^2}{\ln x}$

20 $\lim_{x\to \infty} \dfrac{\ln x}{x^2}$

21 $\lim_{x\to 0^+} \dfrac{\ln \sin x}{\ln \sin 2x}$

22 $\lim_{x\to 0} \dfrac{2x}{\tan^{-1} x}$

23 $\lim_{x\to 0} \dfrac{e^x-e^{-x}-2\sin x}{x\sin x}$

24 $\lim_{x\to 2} \dfrac{\ln(x-1)}{x-2}$

25 $\lim_{x\to 0} \dfrac{x\cos x + e^{-x}}{x^2}$

26 $\lim_{x\to 0} \dfrac{2e^x-3x-e^{-x}}{x^2}$

27 $\lim_{x\to \infty} \dfrac{2x^2+3x+1}{5x^2+x+4}$

28 $\lim_{x\to \infty} \dfrac{x^3+x+1}{3x^3+4}$

29 $\lim_{x\to \infty} \dfrac{x\ln x}{x+\ln x}$

30 $\lim_{x\to \infty} \dfrac{e^{3x}}{\ln x}$

31 $\lim_{x\to \infty} \dfrac{x^n}{e^x}, n > 0$

32 $\lim_{x\to \infty} \dfrac{e^x}{x^n}, n > 0$

33 $\lim_{x\to 2^+} \dfrac{\ln(x-1)}{(x-2)^2}$

34 $\lim_{x\to 0} \dfrac{\sin^2 x + 2\cos x - 2}{\cos^2 x - x\sin x - 1}$

35 $\lim_{x\to 0} \dfrac{\sin^{-1} 2x}{\sin^{-1} x}$

36 $\lim_{x\to \infty} \dfrac{\ln(\ln x)}{\ln x}$

37 $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3 \tan x}$

38 $\lim_{x\to 1} \dfrac{2x^3-5x^2+6x-3}{x^3-2x^2+x-1}$

39 $\lim_{x\to -\infty} \dfrac{3-3^x}{5-5^x}$

40 $\lim_{x\to 0} \dfrac{2-e^x-e^{-x}}{1-\cos^2 x}$

41 $\lim_{x\to 1} \dfrac{x^4-x^3-3x^2+5x-2}{x^4-5x^3+9x^2-7x+2}$

42 $\lim_{x\to 1} \dfrac{x^4+x^3-3x^2-x+2}{x^4-5x^3+9x^2-7x+2}$

43 $\lim_{x\to 0} \dfrac{x-\tan^{-1} x}{x\sin x}$

44 $\lim_{x\to \infty} \dfrac{e^{-x}}{1+e^{-x}}$

45 $\lim_{x\to \infty} \dfrac{x^{3/2}+5x-4}{x\ln x}$

46 $\lim_{x\to 0} \dfrac{x\sin^{-1} x}{x-\sin x}$

47 $\lim_{x\to \infty} \dfrac{\sqrt{x^2+1}}{\tan^{-1} x}$

48 $\lim_{x\to (\pi/2)^-} \dfrac{\tan x}{\cot 2x}$

49 $\lim_{x\to \infty} \dfrac{2e^{3x}+\ln x}{e^{3x}+x^2}$

50 $\lim_{x\to 0^+} \dfrac{e^{-1/x}}{x}$

51 $\lim_{x\to \infty} \dfrac{x-\cos x}{x}$

52 $\lim_{x\to \infty} \dfrac{x+\cosh x}{x^2+1}$

c **Exer. 53–54: Predict the limit after substituting the indicated values of x for $k = 1, 2, 3$, and 4.**

53 $\lim_{x\to 0^+} \dfrac{\ln(\tan x + \cos x)}{\sqrt{\ln(x^2+1)}};\quad x = 10^{-k}$

54 $\lim_{x\to 0} \dfrac{\tan^2(\sin^{-1} x)}{1-\cos[\ln(1+x)]};\quad x = \pm 10^{-k}$

55 An object of mass m is released from a hot-air balloon. If the force of resistance due to air is directly proportional to the velocity $v(t)$ of the object at time t, then it can be shown that

$$v(t) = (mg/k)(1 - e^{-(k/m)t}),$$

where $k > 0$ and g is a gravitational constant. Find $\lim_{k \to 0^+} v(t)$.

56 If a steel ball of mass m is released into water and the force of resistance is directly proportional to the square of the velocity, then the distance $s(t)$ that the ball travels in time t is given by

$$s(t) = (m/k) \ln \cosh (\sqrt{gk/m}\, t),$$

where $k > 0$ and g is a gravitational constant. Find $\lim_{k \to 0^+} s(t)$.

57 Refer to Definition (4.22) for simple harmonic motion. The following is an example of the phenomenon of resonance. A weight of mass m is attached to a spring suspended from a support. The weight is set in motion by moving the support up and down according to the formula $h = A \cos \omega t$, where A and ω are positive constants and t is time. If frictional forces are negligible, then the displacement s of the weight from its initial position at time t is given by

$$s = \frac{A\omega^2}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t),$$

with $\omega_0 = \sqrt{k/m}$ for some constant k and with $\omega \neq \omega_0$. Find $\lim_{\omega \to \omega_0} s$, and show that the resulting oscillations increase in magnitude.

58 The logistic model for population growth predicts the size $y(t)$ of a population at time t by means of the formula $y(t) = K/(1 + ce^{-rt})$, where r and K are positive constants and $c = [K - y(0)]/y(0)$. Ecologists call K the *carrying capacity* and interpret it as the maximum number of individuals that the environment can sustain. Find $\lim_{t \to \infty} y(t)$ and $\lim_{K \to \infty} y(t)$, and discuss the graphical significance of these limits.

59 The *sine integral* $Si(x) = \int_0^x [(\sin u)/u]\, du$ is a special function in applied mathematics. Find

(a) $\displaystyle \lim_{x \to 0} \frac{Si(x)}{x}$ (b) $\displaystyle \lim_{x \to 0} \frac{Si(x) - x}{x^3}$

60 The *Fresnel cosine integral* $C(x) = \int_0^x \cos u^2\, du$ is used in the analysis of the diffraction of light. Find

(a) $\displaystyle \lim_{x \to 0} \frac{C(x)}{x}$ (b) $\displaystyle \lim_{x \to 0} \frac{C(x) - x}{x^5}$

c 61 (a) Refer to Exercise 60. Use Simpson's rule, with $n = 4$, to approximate $C(x)$ for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1.

(b) Graph C on $[0, 1]$ using the values found in (a).

c 62 Refer to Exercise 61. Let R be the region under the graph of C from $x = 0$ to $x = 1$, and let V be the volume of the solid obtained by revolving R about the x-axis. Approximate V by using Simpson's rule with $n = 4$.

63 Let $x > 0$. If $n \neq -1$, then $\int_1^x t^n\, dt = [t^{n+1}/(n + 1)]_1^x$. Show that

$$\lim_{n \to -1} \int_1^x t^n\, dt = \int_1^x t^{-1}\, dt.$$

64 Find $\lim_{x \to \infty} f(x)/g(x)$ if

$$f(x) = \int_0^x e^{(t^2)}\, dt \quad \text{and} \quad g(x) = e^{(x^2)}.$$

10.2 OTHER INDETERMINATE FORMS

In the preceding section we discussed limits of quotients that have the indeterminate forms $0/0$ or ∞/∞. Products may lead to the indeterminate form $0 \cdot \infty$, as defined in the following table.

Indeterminate form	Limit form: $\lim_{x \to c} [f(x)\, g(x)]$
$0 \cdot \infty$	$\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = \infty$ or $-\infty$

In exercises we shall also consider the indeterminate form $0 \cdot \infty$ for the case $x \to \infty$ or $x \to -\infty$. The following guidelines may be used.

Guidelines for investigating $\lim_{x \to c} [f(x)\, g(x)]$ for the form $0 \cdot \infty$ (10.3)

1 Write $f(x)\, g(x)$ as

$$\frac{f(x)}{1/g(x)} \quad \text{or} \quad \frac{g(x)}{1/f(x)}.$$

2 Apply L'Hôpital's rule (10.2) to the resulting indeterminate form $0/0$ or ∞/∞.

The choice in guideline 1 is not arbitrary. The following example shows that using $f(x)/[1/g(x)]$ gives us the limit whereas using $g(x)/[1/f(x)]$ leads to a more complicated expression.

EXAMPLE 1 Find $\lim_{x \to 0^+} x^2 \ln x$.

SOLUTION The indeterminate form is $0 \cdot \infty$. Applying guideline 1 of (10.3), we write

$$x^2 \ln x = \frac{\ln x}{1/x^2}.$$

Because the quotient on the right has the indeterminate form ∞/∞ at $x = 0$, we may apply L'Hôpital's rule:

$$\lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x^2} = \lim_{x \to 0^+} \frac{1/x}{-2/x^3}$$

The last quotient has the indeterminate form ∞/∞; however, further applications of L'Hôpital's rule would again lead to ∞/∞. In this case we simplify the quotient algebraically and find the limit as follows:

$$\lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} \frac{x^3}{-2x} = \lim_{x \to 0^+} \frac{x^2}{-2} = 0$$

If, in applying guideline 1, we had rewritten the given expression as

$$x^2 \ln x = \frac{x^2}{1/\ln x} = \frac{x^2}{(\ln x)^{-1}},$$

then the resulting indeterminate form would have been $0/0$. By L'Hôpital's rule,

$$\begin{aligned} \lim_{x \to 0^+} x^2 \ln x &= \lim_{x \to 0^+} \frac{x^2}{(\ln x)^{-1}} \\ &= \lim_{x \to 0^+} \frac{2x}{-(\ln x)^{-2}(1/x)} \\ &= \lim_{x \to 0^+} [-2x^2(\ln x)^2]. \end{aligned}$$

The expression $-2x^2(\ln x)^2$ is more complicated than $x^2 \ln x$, so this choice in guideline 1 does *not* give us the limit.

EXAMPLE 2 Find $\lim_{x \to (\pi/2)^-} (2x - \pi) \sec x$.

SOLUTION The indeterminate form is $0 \cdot \infty$. Using guideline 1 of (10.3), we begin by writing

$$(2x - \pi) \sec x = \frac{2x - \pi}{1/\sec x} = \frac{2x - \pi}{\cos x}.$$

Because the last expression has the indeterminate form 0/0 at $x = \pi/2$, L'Hôpital's rule may be applied as follows:

$$\lim_{x \to (\pi/2)^-} \frac{2x - \pi}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{2}{-\sin x} = \frac{2}{-1} = -2$$

The indeterminate forms defined in the next table may occur in investigating limits involving exponential expressions.

Indeterminate form	Limit form: $\lim_{x \to c} f(x)^{g(x)}$
0^0	$\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$
∞^0	$\lim_{x \to c} f(x) = \infty$ or $-\infty$ and $\lim_{x \to c} g(x) = 0$
1^∞	$\lim_{x \to c} f(x) = 1$ and $\lim_{x \to c} g(x) = \infty$ or $-\infty$

In exercises we shall also consider cases in which $x \to \infty$ or $x \to -\infty$. One method for investigating these forms is to consider

$$y = f(x)^{g(x)}$$

and take the natural logarithm of both sides, obtaining

$$\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x).$$

If the indeterminate form for y is 0^0 or ∞^0, then the indeterminate form for $\ln y$ is $0 \cdot \infty$, which may be handled using earlier methods. Similarly, if y has the form 1^∞, then the indeterminate form for $\ln y$ is $\infty \cdot 0$. It follows that

$$\text{if} \quad \lim_{x \to c} \ln y = \ln\left(\lim_{x \to c} y\right) = L, \quad \text{then} \quad \lim_{x \to c} y = e^L;$$

that is,

$$\lim_{x \to c} f(x)^{g(x)} = e^L.$$

This procedure may be summarized as follows.

Guidelines for investigating $\lim_{x \to c} f(x)^{g(x)}$ for the forms 0^0, 1^∞, and ∞^0 **(10.4)**

1 Let $y = f(x)^{g(x)}$.

2 Take natural logarithms in guideline 1:

$$\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$$

3 Investigate $\lim_{x \to c} \ln y = \lim_{x \to c} [g(x) \ln f(x)]$ and conclude the following:

(a) If $\lim_{x \to c} \ln y = L$, then $\lim_{x \to c} y = e^L$.

(b) If $\lim_{x \to c} \ln y = \infty$, then $\lim_{x \to c} y = \infty$.

(c) If $\lim_{x \to c} \ln y = -\infty$, then $\lim_{x \to c} y = 0$.

A common error is to stop after showing $\lim_{x\to c} \ln y = L$ and conclude that the given expression has the limit L. Remember that *we wish to find the limit of* y. Thus, if $\ln y$ has the limit L, then y has the limit e^L. The guidelines may also be used if $x \to \infty$ or if $x \to -\infty$ or for one-sided limits.

EXAMPLE 3 Find $\lim_{x\to 0^+} (1 + 3x)^{1/(2x)}$.

SOLUTION The indeterminate form is 1^∞. Employing Guidelines (10.4), we proceed as follows:

Guideline 1 $$y = (1 + 3x)^{1/(2x)}$$

Guideline 2 $$\ln y = \frac{1}{2x}\ln(1 + 3x) = \frac{\ln(1 + 3x)}{2x}$$

Guideline 3 The last expression has the indeterminate form 0/0 at $x = 0$, so we apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1 + 3x)}{2x} = \lim_{x\to 0^+} \frac{3/(1 + 3x)}{2} = \frac{3}{2}$$

Consequently we arrive at the following:

$$\lim_{x\to 0^+} (1 + 3x)^{1/(2x)} = \lim_{x\to 0^+} y = e^{3/2}$$

The final indeterminate form we shall consider is defined in the following table.

Indeterminate form	Limit form: $\lim_{x\to c} [f(x) - g(x)]$
$\infty - \infty$	$\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$

When investigating $\infty - \infty$, we try to change the form of $f(x) - g(x)$ to a quotient or product and then apply L'Hôpital's rule or some other method of evaluation, as illustrated in the next example.

EXAMPLE 4 Find $\lim_{x\to 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right)$.

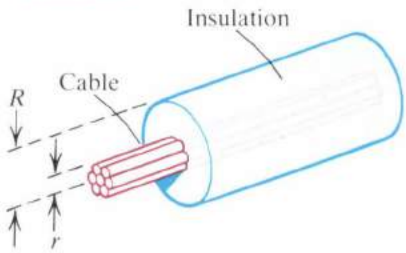
SOLUTION The form is $\infty - \infty$; however, if the difference is written as a single fraction, then

$$\lim_{x\to 0^+} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) = \lim_{x\to 0^+} \frac{x - e^x + 1}{xe^x - x}.$$

This gives us the indeterminate form 0/0. It is necessary to apply L'Hôpital's rule twice, since the first application leads to the indeterminate form 0/0. Thus,

$$\lim_{x\to 0^+} \frac{x - e^x + 1}{xe^x - x} = \lim_{x\to 0^+} \frac{1 - e^x}{xe^x + e^x - 1}$$
$$= \lim_{x\to 0^+} \frac{-e^x}{xe^x + 2e^x} = -\frac{1}{2}.$$

FIGURE 10.3



EXAMPLE 5 The velocity v of an electrical impulse in an insulated cable is given by

$$v = -k\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right),$$

where k is a positive constant, r is the radius of the cable, and R is the distance from the center of the cable to the outside of the insulation, as shown in Figure 10.3. Find

(a) $\lim_{R\to r^+} v$ **(b)** $\lim_{r\to 0^+} v$

SOLUTION

(a) The limit notation implies that r is fixed and R is a variable. In this case the expression for v is not indeterminate, and

$$\lim_{R\to r^+} v = -k \lim_{R\to r^+} \left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right) = -k(1)^2 \ln 1 = -k(0) = 0.$$

(b) If R is fixed and r is a variable, then the expression for v has the indeterminate form $0 \cdot \infty$ at $r = 0$, and we first change the form of the expression algebraically, as follows:

$$\lim_{r\to 0^+} v = -k \lim_{r\to 0^+} \frac{\ln (r/R)}{(r/R)^{-2}} = -kR^2 \lim_{r\to 0^+} \frac{\ln r - \ln R}{r^{-2}}$$

The last quotient has the indeterminate form ∞/∞ at $r = 0$, so we may apply L'Hôpital's rule, obtaining

$$\lim_{r\to 0^+} v = -kR^2 \lim_{r\to 0^+} \frac{(1/r) - 0}{-2r^{-3}}$$
$$= -kR^2 \lim_{r\to 0^+} \left(\frac{r^2}{-2}\right) = -kR^2(0) = 0.$$

EXERCISES 10.2

Exer. 1–42: Find the limit, if it exists.

1 $\lim_{x\to 0^+} x \ln x$

2 $\lim_{x\to (\pi/2)^-} \tan x \ln \sin x$

3 $\lim_{x\to \infty} (x^2 - 1)e^{-x^2}$

4 $\lim_{x\to \infty} x(e^{1/x} - 1)$

5 $\lim_{x\to 0} e^{-x} \sin x$

6 $\lim_{x\to -\infty} x \tan^{-1} x$

7 $\lim_{x\to 0^+} \sin x \ln \sin x$

8 $\lim_{x\to \infty} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$

9 $\lim_{x\to \infty} x \sin \frac{1}{x}$

10 $\lim_{x\to \infty} e^{-x} \ln x$

11 $\lim_{x\to 0} x \sec^2 x$

12 $\lim_{x\to 0} (\cos x)^{x+1}$

13 $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{5x}$

14 $\lim_{x \to 0^+} (e^x + 3x)^{1/x}$

15 $\lim_{x \to 0^+} (e^x - 1)^x$

16 $\lim_{x \to 0^+} x^x$

17 $\lim_{x \to \infty} x^{1/x}$

18 $\lim_{x \to (\pi/2)^-} (\tan x)^{\cos x}$

19 $\lim_{x \to (\pi/2)^-} (\tan x)^x$

20 $\lim_{x \to 2^-} (x - 2)^x$

21 $\lim_{x \to 0^+} (2x + 1)^{\cot x}$

22 $\lim_{x \to 0^+} (1 + 3x)^{\csc x}$

23 $\lim_{x \to \infty} \left(\frac{x^2}{x - 1} - \frac{x^2}{x + 1}\right)$

24 $\lim_{x \to 1^+} \left(\frac{1}{x - 1} - \frac{1}{\ln x}\right)$

25 $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

26 $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$

27 $\lim_{x \to 1^-} (1 - x)^{\ln x}$

28 $\lim_{x \to \infty} (1 + e^x)^{e^{-x}}$

29 $\lim_{x \to 0^+} \left(\frac{1}{\sqrt{x^2 + 1}} - \frac{1}{x}\right)$

30 $\lim_{x \to 0} (\cot^2 x - \csc^2 x)$

31 $\lim_{x \to 0^+} \cot 2x \tan^{-1} x$

32 $\lim_{x \to \infty} x^3 2^{-x}$

33 $\lim_{x \to 0} (\cot^2 x - e^{-x})$

34 $\lim_{x \to \infty} (\sqrt{x^2 + 4} - \tan^{-1} x)$

35 $\lim_{x \to (\pi/2)^-} (1 + \cos x)^{\tan x}$

36 $\lim_{x \to 0^+} (1 + ax)^{b/x}$

37 $\lim_{x \to -3^-} \left(\frac{x}{x^2 + 2x - 3} - \frac{4}{x + 3}\right)$

38 $\lim_{x \to \infty} (\sqrt{x^4 + 5x^2 + 3} - x^2)$

39 $\lim_{x \to 0^+} (x + \cos 2x)^{\csc 3x}$

40 $\lim_{x \to (\pi/2)^-} \sec x \cos 3x$

41 $\lim_{x \to \infty} (\sinh x - x)$

42 $\lim_{x \to \infty} [\ln (4x + 3) - \ln (3x + 4)]$

c **Exer. 43–44: Graph f on the given interval and use the graph to estimate $\lim_{x \to 0} f(x)$.**

43 $f(x) = (x \tan x)^{(x^2)}$; $[-1, 1]$

44 $f(x) = \left[\frac{\ln (x + 1)}{\tan x}\right]^{1/x}$; $[-0.5, 0.5]$

Exer. 45–46: (a) Find the local extrema and discuss the behavior of $f(x)$ near $x = 0$. (b) Find horizontal asymptotes, if they exist. (c) Sketch the graph of f for $x > 0$.

45 $f(x) = x^{1/x}$

46 $f(x) = x^x$

47 The *geometric mean* of two positive real numbers a and b is defined as $\sqrt{ab}$. Use L'Hôpital's rule to prove that

$$\sqrt{ab} = \lim_{x \to \infty} \left(\frac{a^{1/x} + b^{1/x}}{2}\right)^x.$$

48 If a sum of money P is invested at an interest rate of $100r$ percent per year, compounded m times per year, then the principal at the end of t years is given by $P(1 + rm^{-1})^{mt}$. If we regard m as a real number and let m increase without bound, then the interest is said to be *compounded continuously*. Use L'Hôpital's rule to show that in this case the principal after t years is Pe^{rt}.

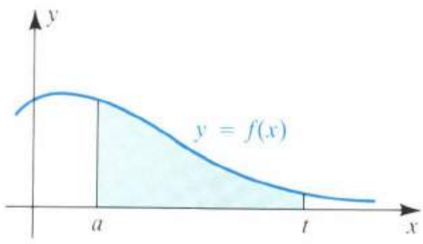
49 Refer to Exercise 55 of Section 10.1. In the velocity formula

$$v(t) = (mg/k)(1 - e^{-(k/m)t}),$$

m represents the mass of the falling object. Find $\lim_{m \to \infty} v(t)$ and conclude that $v(t)$ is approximately proportional to time t if the mass is very large.

10.3 INTEGRALS WITH INFINITE LIMITS OF INTEGRATION

FIGURE 10.4
$\int_a^t f(x)\, dx$



Suppose a function f is continuous and nonnegative on an infinite interval $[a, \infty)$ and $\lim_{x \to \infty} f(x) = 0$. If $t > a$, then the area $A(t)$ under the graph of f from a to t, as illustrated in Figure 10.4, is

$$A(t) = \int_a^t f(x)\, dx.$$

If $\lim_{t \to \infty} A(t)$ exists, then the limit may be interpreted as the area of the region that lies under the graph of f, over the x-axis, and to the right of $x = a$, as illustrated in Figure 10.5. The symbol $\int_a^\infty f(x)\, dx$ is used to denote this number. If $\lim_{t \to \infty} A(t) = \infty$, we cannot assign an area to this (unbounded) region.

Part (i) of the next definition generalizes the preceding remarks to the case where $f(x)$ may be negative for some x in $[a, \infty)$.

Definition **(10.5)**

(i) If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided the limit exists.

(ii) If f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx,$$

provided the limit exists.

FIGURE 10.5
$\int_a^\infty f(x)\,dx$

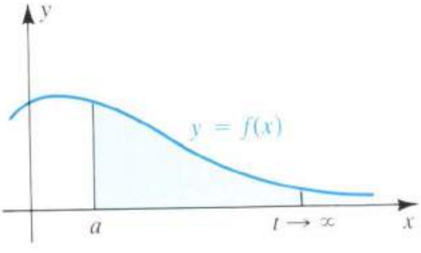
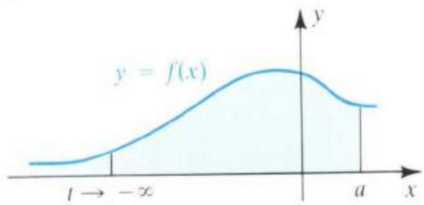


FIGURE 10.6
$\int_{-\infty}^a f(x)\,dx$



If $f(x) \geq 0$ for every x, then the limit in Definition (10.5)(ii) may be regarded as the area under the graph of f, over the x-axis, and to the *left* of $x = a$ (see Figure 10.6).

The expressions in Definition (10.5) are **improper integrals**. They differ from definite integrals in that one of the limits of integration is not a real number. An improper integral is said to **converge** if the limit exists, and the limit is the **value** of the improper integral. If the limit does not exist, the improper integral **diverges**.

Definition (10.5) is useful in many applications. In Example 4 we shall use an improper integral to calculate the work required to project an object from the surface of the earth to a point outside of the earth's gravitational field. Another important application occurs in the investigation of infinite series.

EXAMPLE 1 Determine whether the integral converges or diverges, and if it converges, find its value.

(a) $\displaystyle\int_2^\infty \frac{1}{(x-1)^2}\,dx$ **(b)** $\displaystyle\int_2^\infty \frac{1}{x-1}\,dx$

SOLUTION

(a) By Definition (10.5)(i),

$$\int_2^\infty \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{x-1}\right]_2^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{t-1} + \frac{1}{2-1}\right) = 0 + 1 = 1.$$

Thus, the integral converges and has the value 1.

(b) By Definition (10.5)(i),

$$\begin{aligned}\int_2^\infty \frac{1}{x-1}\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x-1}\,dx \\ &= \lim_{t\to\infty} \Big[\ln(x-1)\Big]_2^t \\ &= \lim_{t\to\infty} [\ln(t-1) - \ln(2-1)] \\ &= \lim_{t\to\infty} \ln(t-1) = \infty.\end{aligned}$$

Since the limit does not exist, the improper integral diverges.

The graphs of the two functions given by the integrands in Example 1, together with the (unbounded) regions that lie under the graphs for $x \geq 2$, are sketched in Figures 10.7 and 10.8. Note that although the graphs have the same general shape for $x \geq 2$, we may assign an area to the region under the graph shown in Figure 10.7, but not to that shown in Figure 10.8.

FIGURE 10.7

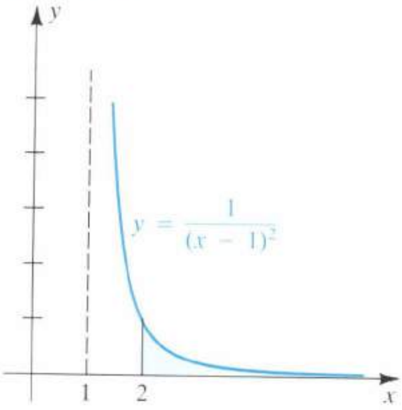


FIGURE 10.8

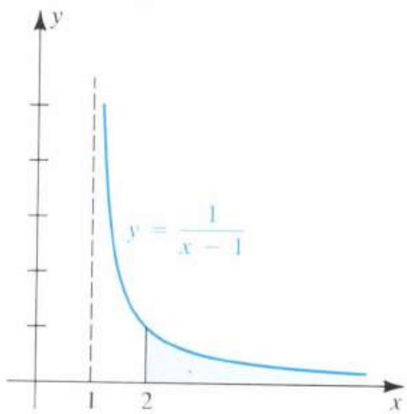
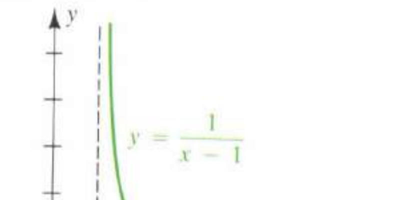


FIGURE 10.9

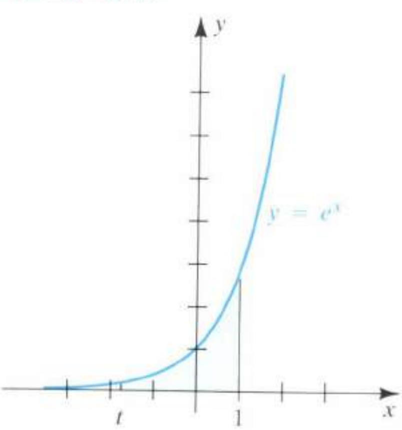


The graph in Figure 10.8 has an interesting property. If the region under the graph of $y = 1/(x-1)$ is revolved about the x-axis, we obtain the (unbounded) solid of revolution shown in Figure 10.9. The improper integral

$$\int_2^\infty \pi \frac{1}{(x-1)^2}\, dx$$

may be regarded as the volume of this solid. By (a) of Example 1, the value of this improper integral is $\pi \cdot 1$, or π. This gives us the curious fact that although we cannot assign an area to the region in Figure 10.8, the volume of the solid of revolution generated by the region (see Figure 10.9) is finite. (A similar situation is described in Exercise 35.)

FIGURE 10.10



EXAMPLE 2 Assign an area to the region that lies under the graph of $y = e^x$, over the x-axis, and to the left of $x = 1$.

SOLUTION The region bounded by the graphs of $y = e^x$, $y = 0$, $x = 1$, and $x = t$, for $t < 1$, is sketched in Figure 10.10. The area of the *unbounded* region to the left of $x = 1$ is

$$\begin{aligned}\int_{-\infty}^{1} e^x\, dx &= \lim_{t \to -\infty} \int_t^1 e^x\, dx = \lim_{t \to -\infty} \Big[e^x\Big]_t^1 \\ &= \lim_{t \to -\infty} (e - e^t) = e - 0 = e.\end{aligned}$$

An improper integral may have *two* infinite limits of integration, as in the following definition.

Definition **(10.6)**

Let f be continuous for every x. If a is any real number, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx,$$

provided both of the improper integrals on the right converge.

If either of the integrals on the right in (10.6) diverges, then $\int_{-\infty}^{\infty} f(x)\,dx$ is said to **diverge**. It can be shown that (10.6) does not depend on the choice of the real number a. It can also be shown that $\int_{-\infty}^{\infty} f(x)\,dx$ is not necessarily the same as $\lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$ (consider $f(x) = x$).

EXAMPLE 3

(a) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

(b) Sketch the graph of $f(x) = \dfrac{1}{1+x^2}$ and interpret the integral in (a) as an area.

SOLUTION **(a)** Using Definition (10.6), with $a = 0$, yields

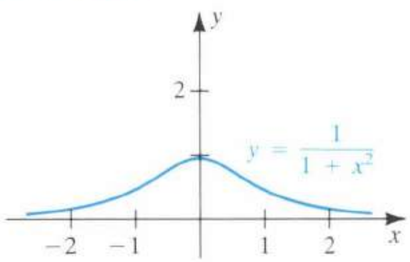
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Next, applying Definition (10.5)(i), we have

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \Big[\arctan x\Big]_0^t$$

$$= \lim_{t\to\infty} (\arctan t - \arctan 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

FIGURE 10.11



Similarly, we may show, by using (10.5)(ii), that

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Consequently the given improper integral converges and has the value $(\pi/2) + (\pi/2) = \pi$.

(b) The graph of $y = 1/(1+x^2)$ is sketched in Figure 10.11. As in our previous discussion, the unbounded region that lies under the graph and above the x-axis may be assigned an area of π square units.

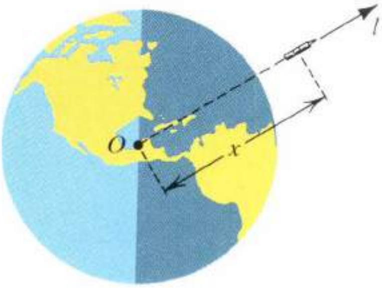
FIGURE 10.12

Let us conclude this section with a physical application of an improper integral. If a and b are the coordinates of two points A and B on a coordinate line l (see Figure 10.12) and if $f(x)$ is the force acting at the point P with coordinate x, then, by Definition (6.21), the work done as P moves from A to B is given by

$$W = \int_a^b f(x)\,dx.$$

In similar fashion, the improper integral $\int_a^{\infty} f(x)\,dx$ may be used to define the work done as P moves indefinitely to the right (in applications, we use the terminology P *moves to infinity*). For example, if $f(x)$ is the force of attraction between a particle fixed at point A and a (movable) particle at P and if $c > a$, then $\int_c^\infty f(x)\,dx$ represents the work required to move P from the point with coordinate c to infinity.

FIGURE 10.13



EXAMPLE 4 Let l be a coordinate line with origin O at the center of the earth, as shown in Figure 10.13. The gravitational force exerted at a point on l that is a distance x from O is given by $f(x) = k/x^2$, for some constant k. Using 4000 miles for the radius of the earth, find the work required to project an object weighing 100 pounds along l, from the surface to a point outside of the earth's gravitational field.

SOLUTION Theoretically, there is *always* a gravitational force $f(x)$ acting on the object; however, we may think of projecting the object from the surface to infinity. From the preceding discussion we wish to find

$$W = \int_{4000}^{\infty} f(x)\,dx.$$

By definition, $f(x) = k/x^2$ is the weight of an object that is a distance x from O, and hence

$$100 = f(4000) = \frac{k}{(4000)^2},$$

or, equivalently,

$$k = 100(4000)^2 = 10^2 \cdot 16 \cdot 10^6 = 16 \cdot 10^8.$$

Thus,

$$f(x) = (16 \cdot 10^8)\frac{1}{x^2}$$

and the required work is

$$\begin{aligned} W &= \int_{4000}^{\infty} (16 \cdot 10^8)\frac{1}{x^2}\,dx = 16 \cdot 10^8 \lim_{t \to \infty} \int_{4000}^{t} \frac{1}{x^2}\,dx \\ &= 16 \cdot 10^8 \lim_{t \to \infty} \left[-\frac{1}{x}\right]_{4000}^{t} = 16 \cdot 10^8 \lim_{t \to \infty} \left(-\frac{1}{t} + \frac{1}{4000}\right) \\ &= \frac{16 \cdot 10^8}{4000} = 4 \cdot 10^5 \text{ mi-lb}. \end{aligned}$$

In terms of foot-pounds,

$$W = 5280 \cdot 4 \cdot 10^5 \approx (2.1)10^9 \text{ ft-lb},$$

or approximately 2 trillion ft-lb.

EXERCISES 10.3

Exer. 1–24: Determine whether the integral converges or diverges, and if it converges, find its value.

1 $\int_1^\infty \frac{1}{x^{4/3}}\,dx$

2 $\int_{-\infty}^{0} \frac{1}{(x-1)^3}\,dx$

3 $\int_1^\infty \frac{1}{x^{3/4}}\,dx$

4 $\int_0^\infty \frac{x}{1+x^2}\,dx$

5 $\int_{-\infty}^{2} \frac{1}{5-2x}\,dx$

6 $\int_{-\infty}^{\infty} \frac{x}{x^4+9}\,dx$

7 $\int_0^\infty e^{-2x}\,dx$

8 $\int_{-\infty}^0 e^x\,dx$

9 $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$

10 $\int_0^\infty \frac{1}{\sqrt[3]{x+1}}\,dx$

11 $\int_{-\infty}^0 \frac{1}{(x-8)^{2/3}}\,dx$

12 $\int_1^\infty \frac{x}{(1+x^2)^2}\,dx$

13 $\int_0^\infty \frac{\cos x}{1+\sin^2 x}\,dx$

14 $\int_{-\infty}^2 \frac{1}{x^2+4}\,dx$

15 $\int_{-\infty}^\infty xe^{-x^2}\,dx$

16 $\int_{-\infty}^\infty \cos^2 x\,dx$

17 $\int_1^\infty \frac{\ln x}{x}\,dx$

18 $\int_3^\infty \frac{1}{x^2-1}\,dx$

19 $\int_0^\infty \cos x\,dx$

20 $\int_{-\infty}^{\pi/2} \sin 2x\,dx$

21 $\int_{-\infty}^\infty \operatorname{sech} x\,dx$

22 $\int_0^\infty xe^{-x}\,dx$

23 $\int_{-\infty}^0 \frac{1}{x^2-3x+2}\,dx$

24 $\int_4^\infty \frac{x+18}{x^2+x-12}\,dx$

Exer. 25–28: If f and g are continuous functions and $0 \le f(x) \le g(x)$ for every x in $[a, \infty)$, then the following *comparison tests for improper integrals* are true:

(I) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

(II) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Determine whether the first integral converges by comparing it with the second integral.

25 $\int_1^\infty \frac{1}{1+x^4}\,dx;\quad \int_1^\infty \frac{1}{x^4}\,dx$

26 $\int_2^\infty \frac{1}{\sqrt[3]{x^2-1}}\,dx;\quad \int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx$

27 $\int_2^\infty \frac{1}{\ln x}\,dx;\quad \int_2^\infty \frac{1}{x}\,dx$

28 $\int_1^\infty e^{-x^2}\,dx;\quad \int_1^\infty e^{-x}\,dx$

Exer. 29–32: Assign, if possible, a value to (a) the area of the region R and (b) the volume of the solid obtained by revolving R about the x-axis.

29 $R = \{(x, y): x \ge 1,\ 0 \le y \le 1/x\}$

30 $R = \{(x, y): x \ge 1,\ 0 \le y \le 1/\sqrt{x}\}$

31 $R = \{(x, y): x \ge 4,\ 0 \le y \le x^{-3/2}\}$

32 $R = \{(x, y): x \ge 8,\ 0 \le y \le x^{-2/3}\}$

33 The unbounded region to the right of the y-axis and between the graphs of $y = e^{-x^2}$ and $y = 0$ is revolved about the y-axis. Show that a volume can be assigned to the resulting unbounded solid, and find the volume.

34 The graph of $y = e^{-x}$ for $x \ge 0$ is revolved about the x-axis. Show that an area can be assigned to the resulting unbounded surface, and find the area.

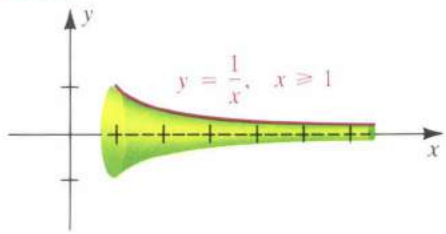
35 The solid of revolution known as *Gabriel's horn* is generated by rotating the region under the graph of $y = 1/x$ for $x \ge 1$ about the x-axis (see figure).

(a) Show that Gabriel's horn has a finite volume of π cubic units.

(b) Is a finite volume obtained if the graph is rotated about the y-axis?

(c) Show that the surface area of Gabriel's horn is given by $\int_1^\infty 2\pi(1/x)\sqrt{1+(1/x^4)}\,dx$. Use a comparison test (see Exercises 25–28) with $f(x) = 2\pi/x$ to establish that this integral diverges. Thus, we cannot assign an area to the surface, even though the volume of the horn is finite.

EXERCISE 35



36 A spacecraft carries a fuel supply of mass m. As a conservation measure, the captain decides to burn fuel at a rate of $R(t) = mke^{-kt}$ g/sec, for some positive constant k.

(a) What does the improper integral $\int_0^\infty R(t)\,dt$ represent?

(b) When will the spacecraft run out of fuel?

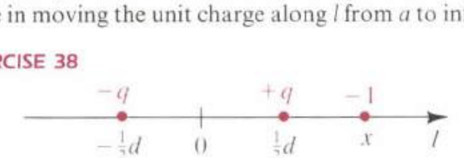
37 The force (in joules) with which two electrons repel one another is inversely proportional to the square of the distance (in meters) between them. If, in Figure 10.12, one electron is fixed at A, find the work done if another electron is repelled along l from a point B, which is 1 meter from A, to infinity.

38 An electric dipole consists of opposite charges separated by a small distance d. Suppose that charges of $+q$ and $-q$ units are located on a coordinate line l at $\frac{1}{2}d$ and $-\frac{1}{2}d$, respectively (see figure). By Coulomb's law, the net force acting on a unit charge of -1 unit at $x > \frac{1}{2}d$ is given by

$$f(x) = \frac{-kq}{(x-\frac{1}{2}d)^2} + \frac{kq}{(x+\frac{1}{2}d)^2}$$

for some positive constant k. If $a > \frac{1}{2}d$, find the work done in moving the unit charge along l from a to infinity.

EXERCISE 38



39 The reliability $R(t)$ of a product is the probability that it will not require repair for at least t years. To design a warranty guarantee, a manufacturer must know the average time of service before first repair of a product. This is given by the improper integral $\int_0^\infty (-t)R'(t)\,dt$.

(a) For many high-quality products, $R(t)$ has the form e^{-kt} for some positive constant k. Find an expression in terms of k for the average time of service before repair.

(b) Is it possible to manufacture a product for which $R(t) = 1/(t+1)$?

40 A sum of money is deposited into an account that pays interest at 8% per year, compounded continuously (see Exercise 48 of Section 10.2). Starting T years from now, money will be withdrawn at the *capital flow rate* of $f(t)$ dollars per year, continuing indefinitely. For future income to be generated at this rate, the minimum amount A that must be deposited, or the *present value of the capital flow*, is given by the improper integral $A = \int_T^\infty f(t)e^{-0.08t}\,dt$. Find A if the income desired 20 years from now is

(a) 12,000 dollars per year

(b) $12{,}000e^{0.04t}$ dollars per year

41 (a) Use integration by parts to establish the formula

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{2a^{3/2}}\int_0^\infty e^{-u^2}\,du.$$

It can be shown that the value of this integral is $\sqrt{\pi}/2$.

(b) The relative number of gas molecules in a container that travel at a speed of v cm/sec can be found by using the *Maxwell-Boltzmann speed distribution* F:

$$F(v) = cv^2 e^{-mv^2/(2kT)},$$

where T is the temperature (in °K), m is the mass of a molecule, and c and k are positive constants. The constant c must be selected so that $\int_0^\infty F(v)\,dv = 1$. Use part (a) to express c in terms of k, T, and m.

42 The *Fourier transform* is useful for solving certain differential equations. The *Fourier cosine transform* of a function f is defined by

$$F_C[f(x)] = \int_0^\infty f(x)\cos sx\,dx$$

for every real number s for which the improper integral converges. Find $F_C[e^{-ax}]$ for $a > 0$.

Exer. 43–48: In the theory of differential equations, if f is a function, then the *Laplace transform* L of $f(x)$ is defined by

$$L[f(x)] = \int_0^\infty e^{-sx}f(x)\,dx$$

for every real number s for which the improper integral converges. Find $L[f(x)]$ if $f(x)$ is the given expression.

43 1 44 x 45 $\cos x$

46 $\sin x$ 47 e^{ax} 48 $\sin ax$

49 The *gamma function* Γ is defined by $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$ for every positive real number n.

(a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Prove that $\Gamma(n+1) = n\Gamma(n)$.

(c) Use mathematical induction to prove that if n is any positive integer, then $\Gamma(n+1) = n!$. (This shows that factorials are special values of the gamma function.)

50 Refer to Exercise 49. Functions given by $f(x) = cx^k e^{-ax}$ with $x > 0$ are called *gamma distributions* and play an important role in probability theory. The constant c must be selected so that $\int_0^\infty f(x)\,dx = 1$. Express c in terms of the positive constants k and a and the gamma function Γ.

c **Exer. 51–52: Approximate the improper integral by making the substitution $u = 1/x$ and then using Simpson's rule with $n = 4$.**

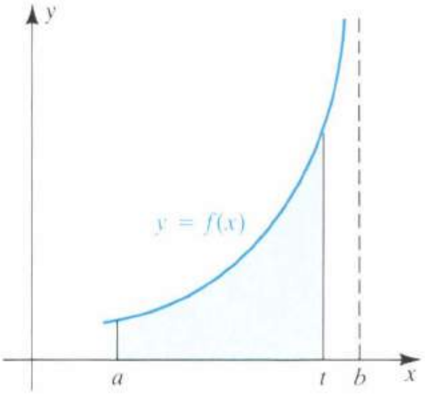
51 $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^4 + x}}\,dx$ 52 $\displaystyle\int_{-\infty}^{-10} \frac{\sqrt{|x|}}{x^3 + 1}\,dx$

10.4 INTEGRALS WITH DISCONTINUOUS INTEGRANDS

If a function f is continuous on a closed interval $[a, b]$, then, by Theorem (5.20), the definite integral $\int_a^b f(x)\,dx$ exists. If f has an infinite discontinuity at some number in the interval, it may still be possible to assign a value to the integral. Suppose, for example, that f is continuous and nonnegative on the half-open interval $[a, b)$ and $\lim_{x\to b^-} f(x) = \infty$. If $a < t < b$, then the area $A(t)$ under the graph of f from a to t (see Figure 10.14, on the next page) is

$$A(t) = \int_a^t f(x)\,dx.$$

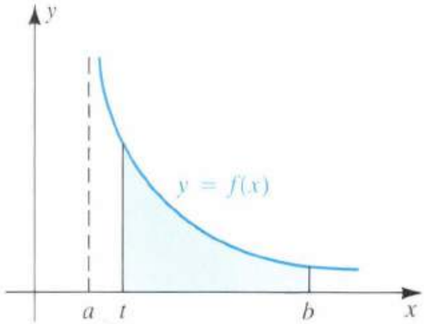
FIGURE 10.14
$\int_a^t f(x)\,dx$



If $\lim_{t \to b^-} A(t)$ exists, then the limit may be interpreted as the area of the unbounded region that lies under the graph of f, over the x-axis, and between $x = a$ and $x = b$. We shall denote this number by $\int_a^b f(x)\,dx$.

For the situation illustrated in Figure 10.15, $\lim_{x \to a^+} f(x) = \infty$, and we define $\int_a^b f(x)\,dx$ as the limit of $\int_t^b f(x)\,dx$ as $t \to a^+$.

FIGURE 10.15
$\int_t^b f(x)\,dx$



These remarks are the motivation for the following definition.

Definition **(10.7)**

(i) If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx,$$

t^-

provided the limit exists.

(ii) If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx,$$

t^+

provided the limit exists.

As in the preceding section, the integrals defined in (10.7) are referred to as *improper integrals* and they *converge* if the limits exist. The limits are called the *values* of the improper integrals. If the limits do not exist, the improper integrals *diverge*.

Another type of improper integral is defined as follows.

Definition **(10.8)**

If f has a discontinuity at a number c in the open interval (a, b) but is continuous elsewhere on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

provided *both* of the improper integrals on the right converge. If both converge, then the value of the improper integral $\int_a^b f(x)\,dx$ is the sum of the two values.

FIGURE 10.16

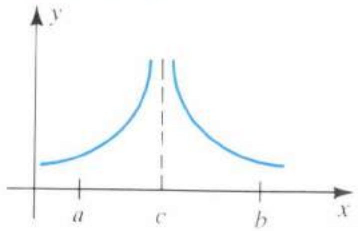
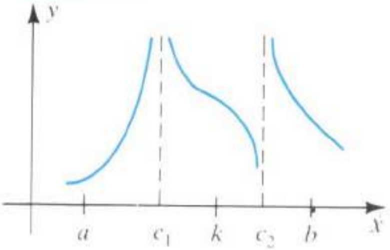


FIGURE 10.17



The graph of a function satisfying the conditions of Definition (10.8) is sketched in Figure 10.16.

A definition similar to (10.8) is used if f has any finite number of discontinuities in (a, b). For example, suppose f has discontinuities at c_1 and c_2, with $c_1 < c_2$, but is continuous elsewhere on $[a, b]$. One possibility is illustrated in Figure 10.17. In this case we choose a number k between c_1 and c_2 and express $\int_a^b f(x)\,dx$ as a sum of four improper integrals over the intervals $[a, c_1]$, $[c_1, k]$, $[k, c_2]$, and $[c_2, b]$, respectively. By definition, $\int_a^b f(x)\,dx$ converges if and only if each of the four improper integrals in the sum converges. We can show that this definition is independent of the number k.

Finally, if f is continuous on (a, b) but has infinite discontinuities at a and b, then we again define $\int_a^b f(x)\,dx$ by means of (10.8).

EXAMPLE 1 Evaluate $\displaystyle\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$.

SOLUTION Since the integrand has an infinite discontinuity at $x = 3$, we apply Definition (10.7)(i) as follows:

$$\begin{aligned}
\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx \\
&= \lim_{t\to 3^-} \Big[-2\sqrt{3-x}\Big]_0^t \\
&= \lim_{t\to 3^-} \left(-2\sqrt{3-t} + 2\sqrt{3}\right) \\
&= 0 + 2\sqrt{3} = 2\sqrt{3}
\end{aligned}$$

EXAMPLE 2 Determine whether the improper integral $\displaystyle\int_0^1 \frac{1}{x}\,dx$ converges or diverges.

SOLUTION The integrand is undefined at $x = 0$. Applying (10.7)(ii) gives us

$$\begin{aligned}
\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx &= \lim_{t\to 0^+} \Big[\ln x\Big]_t^1 \\
&= \lim_{t\to 0^+} (0 - \ln t) = \infty.
\end{aligned}$$

Since the limit does not exist, the improper integral diverges.

EXAMPLE 3 Determine whether the improper integral $\displaystyle\int_0^4 \frac{1}{(x-3)^2}\,dx$ converges or diverges.

SOLUTION The integrand is undefined at $x = 3$. Since this number is in the interval $(0, 4)$, we use Definition (10.8) with $c = 3$:

$$\int_0^4 \frac{1}{(x-3)^2}\,dx = \int_0^3 \frac{1}{(x-3)^2}\,dx + \int_3^4 \frac{1}{(x-3)^2}\,dx$$

For the integral on the left to converge, *both* integrals on the right must converge. Equivalently, the integral on the left diverges if either of the integrals on the right diverges. Applying Definition (10.7)(i) to the first integral on the right gives us

$$\begin{aligned}\int_0^3 \frac{1}{(x-3)^2}\,dx &= \lim_{t\to 3^-} \int_0^t \frac{1}{(x-3)^2}\,dx \\ &= \lim_{t\to 3^-} \left[\frac{-1}{x-3}\right]_0^t \\ &= \lim_{t\to 3^-} \left(\frac{-1}{t-3} - \frac{1}{3}\right) = \infty.\end{aligned}$$

Thus, the given improper integral diverges.

It is important to note that the fundamental theorem of calculus cannot be applied to the integral in Example 3, since the function given by the integrand is not continuous on $[0, 4]$. If we had (incorrectly) applied the fundamental theorem, we would have obtained

$$\left[\frac{-1}{x-3}\right]_0^4 = -1 - \frac{1}{3} = -\frac{4}{3}.$$

This result is obviously incorrect, since the integrand is never negative.

EXAMPLE 4 Evaluate $\displaystyle\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx.$

SOLUTION The integrand is undefined at $x = -1$, which is in the interval $(-2, 7)$. Hence we apply Definition (10.8), with $c = -1$:

$$\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx = \int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx + \int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx$$

We next investigate each of the integrals on the right-hand side of this equation. Using (10.7)(i) with $b = -1$ gives us

$$\begin{aligned}\int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx &= \lim_{t\to -1^-} \int_{-2}^{t} \frac{1}{(x+1)^{2/3}}\,dx \\ &= \lim_{t\to -1^-} \left[3(x+1)^{1/3}\right]_{-2}^{t} \\ &= 3 \lim_{t\to -1^-} \left[(t+1)^{1/3} - (-1)^{1/3}\right] \\ &= 3(0+1) = 3.\end{aligned}$$

Similarly, using (10.7)(ii) with $a = -1$ yields

$$\begin{aligned}\int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx &= \lim_{t \to -1^+} \int_t^7 \frac{1}{(x+1)^{2/3}}\,dx \\ &= \lim_{t \to -1^+} \Big[3(x+1)^{1/3}\Big]_t^7 \\ &= 3 \lim_{t \to -1^+} \left[(8)^{1/3} - (t+1)^{1/3}\right] \\ &= 3(2-0) = 6.\end{aligned}$$

Since both integrals converge, the given integral converges and has the value $3 + 6 = 9$.

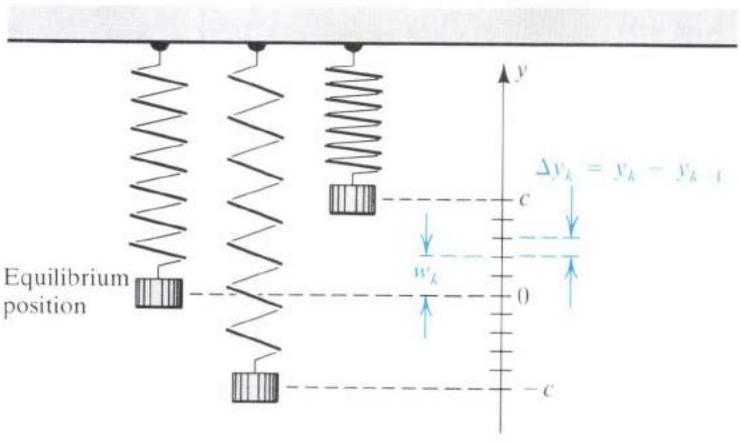
An improper integral may have both a discontinuity in the integrand and an infinite limit of integration. Integrals of this type may be investigated by expressing them as sums of improper integrals, each of which has *one* of the forms previously defined. As an illustration, since the integrand of $\int_0^\infty (1/\sqrt{x})\,dx$ is discontinuous at $x = 0$, we choose any number greater than 0—say 1—and write

$$\int_0^\infty \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{\sqrt{x}}\,dx.$$

We can show that the first integral on the right-hand side of the equation converges and the second diverges. Hence (by definition) the given integral diverges.

Improper integrals of the types considered in this section arise in physical applications. Figure 10.18 is a schematic drawing of a spring with an attached weight that is oscillating between points with coordinates $-c$ and c on a coordinate line y (the y-axis has been positioned at the right for clarity). The **period** T is the time required for one complete oscillation—that is, *twice* the time required for the weight to cover the interval $[-c, c]$. The next example illustrates how an improper integral results when we derive a formula for T.

FIGURE 10.18



EXAMPLE 5 Let $v(y)$ denote the velocity of the weight in Figure 10.18 when it is at the point with coordinate y in $[-c, c]$. Show that the period T is given by

$$T = 2\int_{-c}^{c} \frac{1}{v(y)}\,dy.$$

SOLUTION Let us partition $[-c, c]$ in the usual way, and let $\Delta y_k = y_k - y_{k-1}$ denote the distance the weight travels during the time interval Δt_k. If w_k is any number in the subinterval $[y_{k-1}, y_k]$, then $v(w_k)$ is the velocity of the weight when it is at the point with coordinate w_k. If the norm of the partition is small and if we assume v is a continuous function, then the distance Δy_k may be approximated by the product $v(w_k)\,\Delta t_k$; that is,

$$\Delta y_k \approx v(w_k)\,\Delta t_k.$$

Hence the time required for the weight to cover the distance Δy_k may be approximated by

$$\Delta t_k \approx \frac{1}{v(w_k)}\,\Delta y_k$$

and, therefore,

$$T = 2\sum_k \Delta t_k \approx 2\sum_k \frac{1}{v(w_k)}\,\Delta y_k.$$

By considering the limit of the sums on the right and using the definition of definite integral, we conclude that

$$T = 2\int_{-c}^{c} \frac{1}{v(y)}\,dy.$$

Note that $v(c) = 0$ and $v(-c) = 0$, so the integral is improper.

EXERCISES 10.4

Exer. 1–30: Determine whether the integral converges or diverges, and if it converges, find its value.

1 $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$

2 $\int_0^9 \frac{1}{\sqrt{x}}\,dx$

3 $\int_{-3}^1 \frac{1}{x^2}\,dx$

4 $\int_{-2}^{-1} \frac{1}{(x+2)^{5/4}}\,dx$

5 $\int_0^{\pi/2} \sec^2 x\,dx$

6 $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

7 $\int_0^4 \frac{1}{(4-x)^{3/2}}\,dx$

8 $\int_0^{-1} \frac{1}{\sqrt[3]{x+1}}\,dx$

9 $\int_0^4 \frac{1}{(4-x)^{2/3}}\,dx$

10 $\int_1^2 \frac{x}{x^2-1}\,dx$

11 $\int_{-2}^2 \frac{1}{(x+1)^3}\,dx$

12 $\int_{-1}^1 x^{-4/3}\,dx$

13 $\int_{-2}^0 \frac{1}{\sqrt{4-x^2}}\,dx$

14 $\int_{-2}^0 \frac{x}{\sqrt{4-x^2}}\,dx$

15 $\int_{-1}^2 \frac{1}{x}\,dx$

16 $\int_0^4 \frac{1}{x^2-x-2}\,dx$

17 $\int_0^1 x\ln x\,dx$

18 $\int_0^{\pi/2} \tan^2 x\,dx$

19 $\int_0^{\pi/2} \tan x\,dx$

20 $\int_0^{\pi/2} \frac{1}{1-\cos x}\,dx$

21 $\int_2^4 \frac{x-2}{x^2-5x+4}\,dx$

22 $\int_{1/e}^{e} \frac{1}{x(\ln x)^2}\,dx$

23 $\int_{-1}^2 \frac{1}{x^2}\cos\frac{1}{x}\,dx$

24 $\int_0^{\pi} \sec x\,dx$

25 $\int_0^{\pi} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$

26 $\int_0^{9} \frac{x}{\sqrt[3]{x-1}}\,dx$

27 $\int_0^{4} \frac{1}{x^2-4x+3}\,dx$

28 $\int_{-1}^{3} \frac{x}{\sqrt[3]{x^2-1}}\,dx$

29 $\int_0^{\infty} \frac{1}{(x-4)^2}\,dx$

30 $\int_{-\infty}^{0} \frac{1}{x+2}\,dx$

Exer. 31–34: Suppose that f and g are continuous and $0 \le f(x) \le g(x)$ for every x in $(a, b]$. If f and g are discontinuous at $x = a$, then the following *comparison tests* can be proved:

(i) If $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.

(ii) If $\int_a^b f(x)\,dx$ diverges, then $\int_a^b g(x)\,dx$ diverges.

Analogous tests may be stated for continuity on $[a, b)$ with a discontinuity at $x = b$. Determine whether the first integral converges or diverges by comparing it with the second integral.

31 $\int_0^{\pi} \frac{\sin x}{\sqrt{x}}\,dx;\quad \int_0^{\pi} \frac{1}{\sqrt{x}}\,dx$

32 $\int_0^{\pi/4} \frac{\sec x}{x^3}\,dx;\quad \int_0^{\pi/4} \frac{1}{x^3}\,dx$

33 $\int_0^{2} \frac{\cosh x}{(x-2)^2}\,dx;\quad \int_0^{2} \frac{1}{(x-2)^2}\,dx$

34 $\int_0^{1} \frac{e^{-x}}{x^{2/3}}\,dx;\quad \int_0^{1} \frac{1}{x^{2/3}}\,dx$

Exer. 35–36: Find all real values of n for which the integral converges.

35 $\int_0^1 x^n\,dx$

36 $\int_0^1 x^n \ln x\,dx$

Exer. 37–40: Assign, if possible, a value to (a) the area of the region R and (b) the volume of the solid obtained by revolving R about the x-axis.

37 $R = \{(x, y)\colon 0 \le x \le 1, 0 \le y \le 1/\sqrt{x}\}$

38 $R = \{(x, y)\colon 0 \le x \le 1, 0 \le y \le 1/\sqrt[3]{x}\}$

39 $R = \{(x, y)\colon -4 \le x \le 4, 0 \le y \le 1/(x+4)\}$

40 $R = \{(x, y)\colon 1 \le x \le 2, 0 \le y \le 1/(x-1)\}$

c 41 Approximate $\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$ by making the substitution $u = \sqrt{x}$ and then using the trapezoidal rule with $n = 4$.

c 42 Approximate $\int_0^1 \frac{\sin x}{x}\,dx$ by removing the discontinuity at $x = 0$ and then using Simpson's rule with $n = 4$.

43 Refer to Example 5. If the weight in Figure 10.18 has mass m and if the spring obeys Hooke's law (with spring constant $k > 0$), then, in the absence of frictional forces, the velocity v of the weight is a solution of the differential equation

$$mv\frac{dv}{dy} + ky = 0.$$

(a) Use separation of variables (see Section 7.6) to show that $v^2 = (k/m)(c^2 - y^2)$. (*Hint:* Recall from Example 5 that $v(c) = v(-c) = 0$.)

(b) Find the period T of the oscillation.

44 A simple pendulum consists of a bob of mass m attached to a string of length L (see figure). If we assume that the string is weightless and that no other frictional forces are present, then the angular velocity $v = d\theta/dt$ is a solution of the differential equation

$$v\frac{dv}{d\theta} + \frac{g}{L}\sin\theta = 0,$$

where g is a gravitational constant.

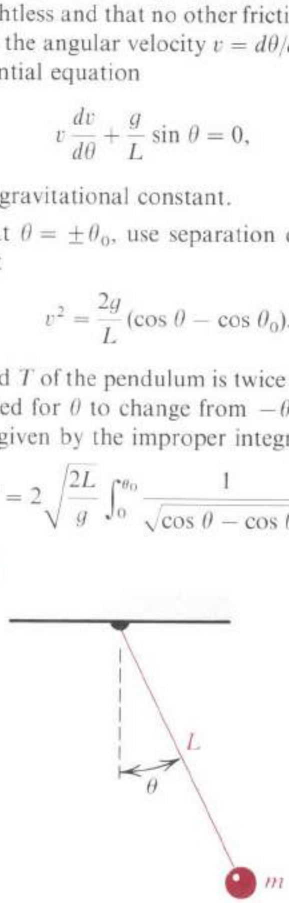
(a) If $v = 0$ at $\theta = \pm\theta_0$, use separation of variables to show that

$$v^2 = \frac{2g}{L}(\cos\theta - \cos\theta_0).$$

(b) The period T of the pendulum is twice the amount of time needed for θ to change from $-\theta_0$ to θ_0. Show that T is given by the improper integral

$$T = 2\sqrt{\frac{2L}{g}}\int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta.$$

EXERCISE 44



45 When a dose of y_0 milligrams of a drug is injected directly into the bloodstream, the average length of time T that a molecule remains in the bloodstream is given by the formula $T = (1/y_0)\int_0^{y_0} t\,dy$ for the time t at which exactly y milligrams is still present.

(a) If $y = y_0e^{-kt}$ for some positive constant k, explain why the integral for T is improper.

(b) If τ is the half-life of the drug in the bloodstream, show that $T = \tau/\ln 2$.

46 In fishery science, the collection of fish that results from one annual reproduction is referred to as a *cohort*. The number N of fish still alive after t years is usually given by an exponential function. For North Sea haddock with initial size of a cohort N_0, $N = N_0e^{-0.2t}$. The average life expectancy T (in years) of a fish in a cohort is given by $T = (1/N_0)\int_0^{N_0} t\,dN$ for the time t when precisely N fish are still alive.

(a) Find the value of T for North Sea haddock.

(b) Is it possible to have a species such that $N = N_0/(1 + kN_0t)$ for some positive constant k? If so, compute T for such a species.

10.5 REVIEW EXERCISES

Exer. 1–16: Find the limit, if it exists.

1 $\lim_{x\to 0} \dfrac{\ln(2-x)}{1+e^{2x}}$

2 $\lim_{x\to 0} \dfrac{\sin 2x - \tan 2x}{x^2}$

3 $\lim_{x\to\infty} \dfrac{x^2+2x+3}{\ln(x+1)}$

4 $\lim_{x\to 0} \dfrac{\tan^{-1} x}{\sin^{-1} x}$

5 $\lim_{x\to 0} \dfrac{e^{2x}-e^{-2x}-4x}{x^3}$

6 $\lim_{x\to(\pi/2)^-} \dfrac{\tan x}{\sec x}$

7 $\lim_{x\to\infty} \dfrac{x^e}{e^x}$

8 $\lim_{x\to(\pi/2)^-} \cos x \ln \cos x$

9 $\lim_{x\to\infty} (1-2e^{1/x})x$

10 $\lim_{x\to 0^+} \tan^{-1} x \csc x$

11 $\lim_{x\to 0} (1+8x^2)^{1/x^2}$

12 $\lim_{x\to 1^+} (\ln x)^{x-1}$

13 $\lim_{x\to\infty} (e^x+1)^{1/x}$

14 $\lim_{x\to 0^+} \left(\dfrac{1}{\tan x} - \dfrac{1}{x}\right)$

15 $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x}$

16 $\lim_{x\to\infty} \dfrac{3^x+2x}{x^3+1}$

Exer. 17–28: Determine whether the integral converges or diverges, and if it converges, find its value.

17 $\int_4^\infty \dfrac{1}{\sqrt{x}}\,dx$

18 $\int_4^\infty \dfrac{1}{x\sqrt{x}}\,dx$

19 $\int_{-\infty}^0 \dfrac{1}{x+2}\,dx$

20 $\int_0^\infty \sin x\,dx$

21 $\int_{-8}^1 \dfrac{1}{\sqrt[3]{x}}\,dx$

22 $\int_{-4}^0 \dfrac{1}{x+4}\,dx$

23 $\int_0^2 \dfrac{x}{(x^2-1)^2}\,dx$

24 $\int_1^2 \dfrac{1}{x\sqrt{x^2-1}}\,dx$

25 $\int_{-\infty}^\infty \dfrac{1}{e^x+e^{-x}}\,dx$

26 $\int_{-\infty}^0 xe^x\,dx$

27 $\int_0^1 \dfrac{\ln x}{x}\,dx$

28 $\int_0^{\pi/2} \csc x\,dx$

c **Exer. 29–30: Approximate the improper integral by making the substitution $u = 1/x$ and then using Simpson's rule with $n = 4$.**

29 $\int_1^\infty e^{-x^2}dx$

30 $\int_1^\infty e^{-x}\sin\sqrt{x}\,dx$

31 Find $\lim_{x\to\infty} f(x)/g(x)$ if $f(x) = \int_1^x (\sin t)^{2/3}\,dt$ and $g(x) = x^2$.

32 Gauss' error integral $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-u^2}\,du$ is used in probability theory. It has the special property $\lim_{x\to\infty} \operatorname{erf}(x) = 1$. Find $\lim_{x\to\infty} e^{(x^2)}[1-\operatorname{erf}(x)]$.