SOME IDENTITIES INVOLVING THE EULER AND THE CENTRAL FACTORIAL NUMBERS

Wenpeng Zhang
Department of Mathematics, The University of Georgia, Athens, GA 30602
(Submitted February 1996-Final Revision June 1996)

1. INTRODUCTION AND RESULTS

Let x be a real number with $|x| < \pi/2$. The Euler sequence $E = (E_{2n})$, $n = 1, 2, \ldots$, is defined by the coefficients in the expansion of

$$\sec x = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}.$$

That is, $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50521, \ldots$. These numbers arose in some combinatorial contexts, and were investigated by many authors. For example, see Lehmer [7] and Powell [8]. The main purpose of this paper is to study the calculating problem of the summation involving the Euler numbers, i.e.,

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{E_{2a_1}E_{2a_2}\ldots E_{2a_k}}{(2a_1)!(2a_2)!\ldots(2a_k)!}, \tag{1}$$

where the summation is over all k-dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, and k is any odd number with $k > 1$.

This problem is interesting because it can help us to find some new recurrence properties for (E_{2n}). In this paper we use the differential equation of the generating function of the sequence (E_{2n}) to study the calculating problems of (1), and give an interesting identity for (1) for any fixed odd number $k > 1$. That is, we shall prove the following main conclusion.

Theorem: Let n and m be nonnegative integers and $k = 2m+1$. Then we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{E_{2a_1}E_{2a_2}\ldots E_{2a_k}}{(2a_1)!(2a_2)!\ldots(2a_k)!}$$
$$= \frac{1}{(k-1)!(2n)!}\sum_{i=0}^{m}(-1)^i 4^i t(2m+1, 2m-2i+1)E_{2n+2m-2i},$$

where $t(n, k)$ are central factorial numbers.

From the above theorem, we may immediately deduce the following.

Corollary 1: For any odd prime p, we have the congruence

$$E_{p-1} \equiv \begin{cases} 0 \pmod p, & \text{if } p \equiv 1 \pmod 4, \\ -2 \pmod p, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Corollary 2: For any integer $n > 0$, we have the congruences

(a) $E_{2n+2} + E_{2n} \equiv 0 \pmod 6$,
(b) $E_{2n+4} + 10E_{2n+2} + 9E_{2n} \equiv 0 \pmod{24}$,
(c) $E_{2n+6} + E_{2n} \equiv 0 \pmod{42}$.

2. PROOF OF THE THEOREM

In this section, we shall complete the proof of the theorem. First, we give an elementary lemma which is described as follows.

Lemma: Let $F(x)=1/\cos x$. Then, for any odd number $k=2m+1>1$, $F(x)$ satisfies the differential equation

$$(2m)!F^k(x)=\sum_{i=0}^{m}c_i(m)F^{(2m-2i)}(x),$$

where $F^{(r)}(x)$ denotes the r^{th} derivative of $F(x)$, and the constants $c_i(m)$, $i=0,1,2,\dots,m$, are defined by the coefficients of the polynomial

$$G_m(x)=(x+1^2)(x+3^2)(x+5^2)\cdots(x+(2m-1)^2)=\sum_{i=0}^{m}c_i(m)x^{m-i}.$$

Note: The constants $c_i(m)$ in the Lemma are special cases of the generalized Stirling numbers of the first kind, $s_\xi(n,k)$, introduced by Comtet [2], i.e.,

$$(x-\xi_0)(x-\xi_1)\cdots(x-\xi_{n-1})=\sum_{i=0}^{n}s_\xi(n,i)x^i.$$

Moreover, the constants $c_i(m)$ are, in fact, the central factorial numbers $t(n,k)$ (see Riordan [9]). The inverse and similar numbers are treated in many important papers by Carlitz [3] and [4], and by Carlitz and Riordan [5]. For some generalizations, see Charalambides [6].

Now we prove the Lemma by induction. From the definition of $F(x)$, and differentiating it, we may obtain

$$F'(x)=\frac{\sin x}{\cos^2 x},\quad F''(x)=\frac{\cos^3 x+2\sin^2 x\cos x}{\cos^4 x}=\frac{2}{\cos^3 x}-\frac{1}{\cos x},$$

i.e.,

$$2F^3(x)=F''(x)+F(x). \tag{2}$$

This proves that the Lemma is true for $m=1$. Assume, then, that it is true for a positive integer $m=u$. That is,

$$(2u)!F^{2u+1}(x)=\sum_{i=0}^{u}c_i(u)F^{(2u-2i)}(x). \tag{3}$$

We shall prove it is also true for $m=u+1$. Differentiating (3), we have

$$(2u+1)!F^{2u}(x)F'(x)=\sum_{i=0}^{u}c_i(u)F^{(2u-2i+1)}(x),$$

$$2u(2u+1)!F^{2u-1}(x)(F'(x))^2+(2u+1)!F^{2u}(x)F''(x)=\sum_{i=0}^{u}c_i(u)F^{(2u-2i+2)}(x). \tag{4}$$

From the equality

$$4^{-n}(4x^2-1^2)(4x^2-3^2)\cdots(4x^2-(2n-1)^2)=\sum_{k=0}^{n}t(2n+1,2k+1)x^{2k},$$

we get

$$c_k(n)=(-1)^k4^kt(2n+1,2n-2k+1). \tag{5}$$

These numbers are tabulated in Riordan [9]. Using this expression and the recursive relation $t(n, k) = t(n-2, k-2) - \frac{1}{4}(n-2)^2 t(n-2, k)$, we have the recurrence relation

$$c_k(n+1) = c_k(n) + (2n+1)^2 c_{k-1}(n), \tag{6}$$

with initial conditions $c_0(n) = 1$, $c_n(n) = 1^2 3^2 \dots (2n-1)^2$. Substituting $(F'(x))^2$ by $F^4(x) - F^2(x)$ and $F''(x)$ by $2F^3(x) - F(x)$ in (4) and applying (3) and (6), we have

$$\begin{aligned}
(2u+2)!\,F^{2u+3}(x) &= (2u)!(2u+1)^2 F^{2u+1}(x) + \sum_{i=0}^{u} c_i(u) F^{(2u+2-2i)}(x) \\
&= (2u+1)^2 \sum_{i=0}^{u} c_i(u) F^{(2u-2i)}(x) + \sum_{i=0}^{u} c_i(u) F^{(2u+2-2i)}(x) \\
&= c_0(u) F^{(2u+2)}(x) + (2u+1)^2 c_u(u) F(x) + \sum_{i=0}^{u-1} (c_{i+1}(u) + (2u+1)^2 c_i(u)) F^{(2u-2i)}(x) \\
&= c_0(u+1) F^{(2u+2)}(x) + c_{u+1}(u+1) F(x) + \sum_{i=1}^{u} c_i(u+1) F^{(2u+2-2i)}(x) \\
&= \sum_{i=0}^{u+1} c_i(u+1) F^{(2u+2-2i)}(x).
\end{aligned}$$

That is, the Lemma is also true for $m = u+1$. This proves the Lemma.

Now we complete the proof of the Theorem. Note that

$$F^{(2i)}(x) = \sum_{n=0}^{\infty} \frac{E_{2n+2i}}{(2n)!} x^{2n}, \quad i = 0, 1, 2, \dots.$$

Comparing the coefficient of x^{2n} on both sides of the Lemma and applying (5), we immediately obtain

$$\begin{aligned}
(2m)! \sum_{a_1+a_2+\dots+a_k=n} \frac{E_{2a_1} E_{2a_2} \dots E_{2a_k}}{(2a_1)!(2a_2)! \dots (2a_k)!} &= \frac{1}{(2n)!} \sum_{i=0}^{m} c_i(m) E_{2n+2m-2i} \\
&= \frac{1}{(2n)!} \sum_{i=0}^{m} (-1)^i 4^i t(2m+1, 2m-2i+1) E_{2n+2m-2i},
\end{aligned}$$

where the constants $c_i(m), i = 0, 1, 2, \dots, m$ are the coefficients of the polynomial

$$G_m(x) = (x+1^2)(x+3^2)(x+5^2) \cdots (x+(2m-1)^2) = \sum_{i=0}^{m} c_i(m) x^{m-i}.$$

This completes the proof of the Theorem.

Proof of the Corollaries: Taking $n = 0$ and $k = p$ in the Theorem, and noting that $E_0 = 1$, $(p-1)! \equiv -1 \pmod p$ (Wilson's theorem, see Apostol [1]), we can get

$$\begin{aligned}
-1 \equiv (p-1)! &= \sum_{i=0}^{\frac{p-1}{2}} c_i\left(\frac{p-1}{2}\right) E_{p-1-2i} \equiv E_{p-1} + c_{\frac{p-1}{2}}\left(\frac{p-1}{2}\right) E_0 \\
&\equiv E_{p-1} + 1^2 3^2 5^2 7^2 \dots (p-2)^2 \equiv E_{p-1} + (-1)^{\frac{p-1}{2}} (p-1)! \equiv E_{p-1} - (-1)^{\frac{p-1}{2}} \pmod p,
\end{aligned}$$

where we have used the congruences

$$c_i\left(\frac{p-1}{2}\right) \equiv 0 \pmod{p}, \quad i = 1, 2, \ldots, \frac{p-3}{2}.$$

Therefore,

$$E_{p-1} \equiv \begin{cases} 0 \pmod{p}, & \text{if } p \equiv 1 \pmod{4}, \\ -2 \pmod{p}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

This completes the proof of Corollary 1.

Taking $m = 1$ and 2 in the Theorem, respectively, we can get

$$E_{2n+4} + E_{2n+2} \equiv E_{2n+2} + E_{2n} \equiv 0 \pmod{2},$$

$$E_{2n+4} + 10E_{2n+2} + 9E_{2n} \equiv 0 \pmod{24}.$$

Thus, $0 \equiv E_{2n+4} + 10E_{2n+2} + 9E_{2n} \equiv E_{2n+4} + E_{2n+2} \equiv 0 \pmod{3}$. Since $(2, 3) = 1$, $E_{2n+4} + E_{2n+2} \equiv 0 \pmod{2}$, we have $E_{2n+4} + E_{2n+2} \equiv 0 \pmod{6}$, that is, $E_{2n+2} + E_{2n} \equiv 0 \pmod{6}$, $n = 1, 2, 3, \ldots$.

Similarly, taking $m = 4$ in the Theorem, we can obtain the congruent equation

$$E_{2n+8} + 84E_{2n+6} + 1974E_{2n+4} + 12916E_{2n+2} + 11025E_{2n} \equiv 0 \pmod{40320}.$$

Thus, $0 \equiv E_{2n+8} + 84E_{2n+6} + 1974E_{2n+4} + 12916E_{2n+2} + 11025E_{2n} \equiv E_{2n+8} + E_{2n+2} \pmod{21}$, that is, $E_{2n+6} + E_{2n} \equiv 0 \pmod{21}$, $n = 1, 2, 3, \ldots$. Noting that $E_{2n+6} + E_{2n} \equiv 0 \pmod{2}$ and $(2, 21) = 1$, we get $E_{2n+6} + E_{2n} \equiv 0 \pmod{42}$. This proves Corollary 2.

ACKNOWLEDGMENTS

The author expresses his gratitude to Professor Andrew Granville and to the anonymous referee for their very helpful and detailed comments.

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AMS Classification Numbers: 11B37, 11B39

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