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On the Shoulders of Hipparchus

A Reappraisal of Ancient Greek Combinatorics

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1. Introduction

To write about combinatorics in ancient Greek mathematics is to write about an empty subject. The surviving evidence is so scanty that even the possibility that "the Greeks took [any] interest in these matters" has been denied by some historians of mathematics.[1] *Tranchant* judgments of this sort reveal, if not a cursory scrutiny of the extant sources, at least a defective acquaintance with the strong selectivity of the process of textual transmission the ancient mathematical corpus has been exposed to – a fact that should induce, about a sparingly attested field of research, a cautious attitude rather than *a priori* negative assessments.[2] (Should not the *onus probandi* be required also when the existence in the past of a certain domain of research is being denied?) I suspect that, behind such a strongly negative historiographic position, two different motives could have conspired: the prejudice of "ancient Greek mathematicians" as geometrically-minded and the attempt to revalue certain aspects of non-western mathematics by tendentiously maintaining that such aspects could not have been even conceived by "the Greeks". Combinatorics is the ideal field in this respect, since many interesting *instances* of combinatorial calculations come from other cultures,[3] whereas combinatorial examples in the ancient Greek mathematical corpus have been considered, as we have seen, worse than disappointing, and in any event such as to justify a very negative attitude. The situation was somewhat complicated (and the obscurity of the reference was taken as index of its unreliability) by the fact that the most relevant piece of evidence is the following, astonishing passage in PLUTARCH's *De Stoicorum repugnantiis*:

ἀλλὰ μὴν αὐτὸς τὰς διὰ δέκα ἀξιωμάτων συμπλοκὰς πλήθει φησὶν ὑπερβάλλειν ἑκατὸν μυριάδας οὔτε δι' αὑτοῦ ζητήσας ἐπιμελῶς οὔτε διὰ τῶν ἐμπείρων τἀληθὲς ἱστορήσας. [...] Χρύσιππον δὲ πάντες ἐλέγχουσιν οἱ ἀριθμετικοί, ὧν καὶ Ἵππαρχός ἐστιν ἀποδεικνύων τὸ διάπτωμα τοῦ λογισ-

[1] BIGGS 1979, p. 114.

[2] In this respect the position of ROME, who says that an attentive "dépouillement des 'mathematici graeci minores'" is needed (ROME 1930, p. 104), appears more reasonable, even if his study clearly displays a strong skepticism about the ancients' achievements in the field.

[3] See e.g. BIGGS 1979 on this.

μοῦ παμμέγεθες αὐτῷ γεγονός, εἴγε τὸ μὲν καταφατικὸν ποιεῖ συμπεπλεγμένων ἀξιωμάτων μυριάδας δέκα καὶ πρὸς ταύταις τρισχίλια τεσσαράκοντα ἐννέα τὸ δ' ἀποφατικὸν ἐνακόσια πεντήκοντα δύο πρὸς τριάκοντα καὶ μιᾷ μυριάσι. (1047C-E)

But now he [Chrysippus] says himself that the number of conjunctions produced by means of ten assertibles exceeds a million, though he had neither investigated the matter carefully by himself nor sought out the truth with the help of experts. [...] Chrysippus is refuted by all the arithmeticians, among them Hipparchus himself who proves that his error in calculation is enormous if in fact affirmation gives 103049 conjoined assertibles and negation 310952.[4]

Past attempts at explaining these numbers (which I shall call henceforth HIPPARCHUS' first and second number) have been completely unsuccessful,[5] and historians of science and of logic have generally been bound to a sceptical attitude about the subject.[6] Recently, the numbers have been finally (and fortuitously) identified by experts of enumerative combinatorics: 103049 is the *tenth* Schröder number, 310952 is only very slightly different from half the sum of the tenth and the eleventh such number.[7] The effects of this observation are disruptive: it is absolutely plain that the whole issue of ancient Greek combinatorics *must* be reconsidered from an entirely different perspective, taking also into account that, as we shall see, comparatively refined techniques are required to compute the above numbers. The fact is that the problem of identifying HIPPARCHUS' numbers on the sole basis of the Plutarchean passage was simply *too difficult* to be given a solution which were more than fortuitous, considering also the widespread belief that, if combinatorics in "Greek mathematics" had existed, it could not have gone beyond utterly trivial results. But this was not the case, and we are forced to conclude that the vagaries of textual tradition have (almost) annihilated the field. The historian of ancient thought is now compelled to face two distinct problems:

(i) To explain the numbers in terms of Stoic logic.
(ii) To reconstruct the calculations performed by HIPPARCHUS; moreover, to try to outline the composition of the combinatorial *humus* in which such calculations *must* have grown out, possibly relating it to some extra-mathematical field of research.

4 PLUTARCH 1976, p. 527. The translation is slightly reworked, insofar as the logical terms are rendered in accordance with BOBZIEN 1999 (elsewhere, unless otherwise specified, all translations are mine). The standard references on Stoic logic are MATES 1953 and FREDE 1974. A very good recent survey is BOBZIEN 1999, and see also BOBZIEN 1996. The extant fragments and testimonies on Stoic dialectics are conveniently collected in HÜLSER 1987–1988, which has superseded the corresponding portions of VON ARNIM's *Stoicorum Veterum Fragmenta*.

5 BIERMANN-MAU 1958.

6 See e.g. HEATH 1921, vol. 2, p. 256; NEUGEBAUER 1975, p. 338; KNEALE-KNEALE 1971, p. 162; BIGGS 1979, pp. 113–114; ROME 1930, p. 101; TOOMER 1978, pp. 223–224.

7 STANLEY 1997, HABSIEGER *et al.* 1998.

My interest here is to reconstruct the way HIPPARCHUS' numbers were arrived at, trying *en passant* to convince the reader that time has come for a reappraisal of ancient Greek combinatorics. I shall provide only a *resumé* of the striking (to modern eyes) interpretation of the numbers in terms of Stoic logic, leaving a wider discussion of the logico-philosophical background for a separate study.

The present work is organized as follows. The rationale beyond the identification of 103049 as the tenth Schröder number is briefly summarized in Sect. 2. Section 3 presents a short discussion of the Plutarchean passage, focussing on the technical lexicon employed, and an outline of the interpretation in terms of Stoic logic. Section 4 contains a discussion of combinatorial results in ancient Greek sources, with additional evidence with respect to the one usually adduced, the data being organized in such a way as to follow HIPPARCHUS' probable calculations. Section 5 offers some provisional conclusions.

2. Hipparchus, Schröder, and Stanley

Schröder numbers were first introduced in SCHRÖDER 1870 in order to solve a series of "bracketing problems". Suppose a string of n letters to be given: it is requested to find all possible ways to put the letters between brackets. The bracketing of a single letter is always omitted, as well as overall brackets enclosing the whole string of letters and brackets. A bracketing of a string of ten letters looks as follows:

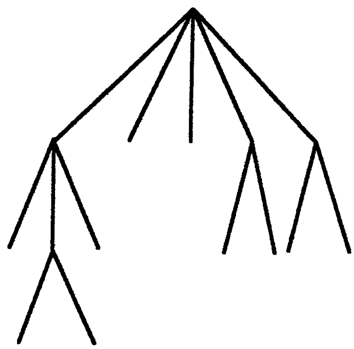
$$(x(xx)x)xx(xx)(xx).$$

If $s(n)$ denotes the number of possible bracketings of a string of n letters, then we have[8]

$$\begin{aligned} &s(1), s(2), \ldots, s(11), \ldots \\ &\quad = 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots . \end{aligned}$$

HIPPARCHUS' first number coincides thus with the tenth Schröder number $s(10)$. This striking observation has been published for the first time in STANLEY 1997 (the discovery is due to D. HOUGH). The Schröder numbers are of course the common solution of a series of combinatorial problems. For instance, $s(n)$ counts in how many ways non-intersecting diagonals can be drawn inside a convex polygon with $n+1$ vertices, or how many trees there are with one single root and n endpoints (with the condition that no vertex has one single further branch). The representation of the process of bracketing as a plane tree takes into account the various levels of parentheses in a natural way, clearly showing the recursive character of the operation. For instance, the plane tree which corresponds to the above bracketing is

[8] $s(1)$ is in the list just as a matter of convention, very much in the same way as 0! is given a sense and set equal to 1.



An identification of HIPPARCHUS' second number has been proposed, too. Shortly after STANLEY's paper, HABSIEGER, KAZARIAN, and LANDO pointed out that $(s(10) + s(11))/2 = 310954$, and proposed the following explanation: consider the "number of bracketings on the string $NO\ x_1\ x_2 \dots x_{10}$, with the following convention: take the negation of all the simple propositions included in the first brackets that include *NO*. Since the bracketings $[NO[P_1] \dots [P_k]]$ and $[NO[[P_1] \dots [P_k]]]$ give the same result, most of the negative compound propositions will be obtained in two different ways; the only case which is obtained in an unique way is when one only takes the negation of x_1".[9] They also briefly commented on the discrepancy between their result and the second number reported by Plutarch, suggesting a "misprint", a mistake in the calculation, or some reason connected with Stoic logic.

3. Stoic logic and Hipparchus' numbers

3.1. Lexical features

The above Plutarchean passage is repeated *almost* verbatim in *Quaestiones Conviviales* VIII 9, 732F, the main difference between the two texts being that the manuscript tradition of the latter reports 101049 as first number (χίλια instead of τρισχίλια). The number has been restored to 103049 by HUBERT on the *sole* basis of the *De Stoicorum repugnantiis* passage.[10] Presumably, HUBERT regarded the falling of the prefix τρισ during a transcription as more likely, in that its undue insertion is a highly implausible mistake for a copyist: the former kind of mistake is made more likely by the fact that, in the main manuscripts of the Plutarchean treatises, the numbers are not written with numerals. (But this does not entail that during the whole tradition the numbers had never been written with numerals. It is plausible instead that the opposite actually occurred,

[9] HABSIEGER *et al.* 1998. The authors provide the terms x with unnecessary subscript indices. *NO* denotes the negation, and is also the first term in the string of *eleven* terms $NO\ x_1\ x_2 \dots x_{10}$. The P_i's denote strings of terms and (possibly) brackets.

[10] In the apparatus of the Teubner edition of the *Quaestiones Conviviales* one finds "add. Hu. ex 1047 d" (PLUTARCH 1938, p. 296).

since a scribal error is the most likely source of the discrepancy in the last digit between HIPPARCHUS' second number as attested and as reconstructed by HABSIEGER *et al.*)

As we shall presently see, the relevant terms in the Plutarchean passage conform very precisely to the lexicon of Stoic logic; since it is well known that (following the ancient practice of quoting from memory) PLUTARCH often quotes inaccurately,[11] this suggests that he is actually consulting a source. It is convenient to stick to such an assumption if we want to accept the received numbers as the correct ones, i.e. those originally calculated by HIPPARCHUS (even if, as we have seen, the last digit in 310952 is very likely wrong). The identification of the first number as the tenth Schröder number strongly supports such a working hypothesis as well as HUBERT's emendation, to the extent that 101049 cannot be given a reasonable combinatorial explanation.

3.2. *Stoic logic, Hipparchus' numbers*

The Stoic "conjunction" (συμπεπλεγμένον ἀξίωμα,[12] or, interchangeably, συμπλοκή) is *defined* as a non-simple assertible formed by means of the connective "and": "A conjunction is an assertible which is conjoined by certain conjunctive connectives, like 'And it is day and it is light'"[13]. The truth values of a conjunction depend on the truth values of the conjuncts: "In every conjunction, if one [among the conjuncts] is false, even if the others are true, the whole is said to be false".[14] Referring to the conjunction, CHERNISS observes that "[t]hese technical Stoic definitions should have been assumed by Hipparchus if his calculations were supposed to refute Chrysippus (though the latter may himself have used συμπλοκή non-technically in the context criticized)".[15] The bracketed part of CHERNISS' suggestion should more properly be applied to the linguistic distinction PLUTARCH himself introduces (that is to the problem whether he was able

[11] Cfr. ZIEGLER 1951, especially cc. 915.15 ff. and 928.10 ff..

[12] "Assertible" (ἀξίωμα) is a technical term: it is defined (see e.g. SEXTUS EMPIRICUS, *Pyrr. Hyp.* ii.104) as a "self-complete sayable that can be stated as far as itself is concerned". Moreover, the assertibles are the carriers of truth-values: "an assertible is that which is true or false" (DIOGENES LAERTIUS, *Vitae Phil.* vii.65). It is crucial for our interpretation to keep in mind that the former is *the* definition of ἀξίωμα, the latter its main property.

[13] καὶ ἡμέρα ἔστι καὶ φῶς ἔστι (DIOGENES LAERTIUS, *Vitae Phil.* vii.72). I have chosen to translate the prefixed καὶ in such an unidiomatic and uncommon way ("both... [and]" would have been the standard choice) for two reasons. First, the prefixed καὶ is unidiomatic in Greek as well; as we shall see below, prefixing particles was a peculiarity of "Stoic syntax" in *strictly technical* contexts. Second, using "both... and" entails introducing a lexical differentiation between the first conjunctive particle and the others in a conjunction, a feature not shared by the Greek original expression. (My choice of translation has been induced by some very pertinent criticisms by HENK BOS).

[14] "In omni autem conjuncto si unum est mendacium, etiamsi cetera vera sunt, totum esse mendacium dicitur" (AULUS GELLIUS, *Noctes Atticae* xvi.8.11). Cfr. also SEXTUS EMPIRICUS, *Adv. Math.* viii.125. Notice that the one referring to the truth values is not the *definition* of "conjunction": the definition is purely syntactical. Other Stoic connectives, e.g. disjunction or several forms of implication, are not even truth-functional.

[15] PLUTARCH 1976, p. 527 note *c*.

to appreciate the pregnancy of the technical terms in his source or not): when referring to CHRYSIPPUS he speaks of συμπλοκάς, while HIPPARCHUS is said to have calculated the total number of συμπεπλεγμένων.[16] At any rate, all that we are interested in are HIPPARCHUS' calculations, so that ascribing to him the use of the main technical term for "conjunctive assertible" is highly significant.

Two main features characterize the Stoic conjunction:

(i) It is an *n*-place connective, as clearly results from e.g. the second negated conjunction in the following Chrysippean statement reported by PLUTARCH: "[...] it is not the case that the day is a body [and] it is not the case that and the first day of the month [is] a body and the tenth and the fifteenth and the thirtieth and the month and the summer and the autumn and the year".[17]

(ii) As the preceding examples show, one of the connectives was prefixed to the first constituent assertible: i.e., the standard form of a conjunction was "and... and... and..." ("καί... καί... καί..." – "et... et... et..." in Latin sources).

A similar rule held for disjunction. In an analogous way a negation (ἀποφατικόν) was always formed by *prefixing* to a whole assertible the particle 'not'.[18] It is clear that the above rules were intended to avoid the ambiguities which could arise in the case of (incorrectly formed) assertibles such as "the first and the second or the third", or "not the first and the second".[19] The case of negation excepted, no *explicit* discussions of these prescriptions are attested, but, in technical contexts and whenever points are touched on in which ambiguities could arise, the prescription is adhered to with remarkable consistency: assertibles are often encountered where a negated conjunction is expressed in its complete form: "It is not the case that and it is day and it is night (οὐχὶ καὶ ἡμέρα ἔστι καὶ νὺξ ἔστι)".[20] Moreover, even if no original Stoic work on logic has come to us,

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[16] ARISTOTLE employed both terms in the broader sense of "compound expression". A good sample of their use is already provided by the *Categoriae* and *De interpretatione*, but see also BONITZ 1870, p. 718 *sub vocibus* (cfr. also Sect. 4.2.3 below). The restriction of συμπλοκή to a technical term for "conjunction" only was regarded as a distinguishing feature of Stoic logic: "We say that those are followers of the Stoics who have reserved the name συμπλοκή to the utterance containing the conjunctive connective only; Aristotle, who came before them, followed the custom of the ancients, who called συμπλοκή the assemblage of several parts of speech" (DEXIPPUS, *In Arist. Cat.*, pp. 22.18 ff. (Busse)). PLUTARCH very likely intended συμπλοκή in its broader sense: in *Quaestiones Conviviales* 733B both CHRYSIPPUS' claim and HIPPARCHUS' correction are referred to in order to support the statement that, given all kinds of aliments and drinks the body assimilates, and given the various metabolic processes within the latter, "the συμπλοκαί of all these can sometimes produce new and unfamiliar diseases".

[17] *De communibus notitiis adversus Stoicos* 1084D. Translation from PLUTARCH 1976, p. 859, slightly reworked. Cfr. also AULUS GELLIUS, *Noctes Atticae* xvi.8.10.

[18] οὐκ, οὐχί,...; see e.g. SEXTUS EMPIRICUS, *Adv. Math.* viii.89.

[19] The Stoics used such a way of writing (non-simple) assertibles in the representation of a generic argument called its *mode*. A modern transcription of the given examples could be $a \wedge b \vee c$ and $\neg a \wedge b$ – where a, b, c are propositional variables –, which are not well-formed as well, unless further rules for handling connectives in a parentheses-free notation are spelled out.

[20] See e.g. SEXTUS EMPIRICUS, *Adv. Math.* viii.226. There is a very important passage in ALEXANDER (*In An. pr.*, pp. 401.16–405.16 (Wallies)) attesting to the care with which the Stoics

"formalistic" issues such as forming unambiguous expressions are of the kind the Stoics were criticized for in antiquity.[21] The works devoted to such arguments were thus the most suited to be left out in the process of textual transmission, whereas the arguments themselves were the less suited to be resumed in philosophical debates outside the ancient Stoa. Among CHRYSIPPUS' writings, as listed by DIOGENES LAERTIUS, we find a treatise in two books "On the conjunction",[22] as well as seven treatises (in seventeen books) dedicated to various forms of ambiguity (ἀμφιβολία).[23] On the other hand, as we have seen and shall see just below, our sources attest to a careful effort to avoid ill-formed expressions, and the possibility that some technical treatises had contained sharper prescriptions cannot be ruled out. For what concerns us, the care in avoiding ambiguity in the use of the connectives means that both the lexicon by means of which the connectives were expressed, and the relative position in non-simple assertibles of both the connectives and the constituent assertibles, were strictly fixed. In this way, the logical properties of non-simple assertibles were determinable through a syntactical analysis.

Representing a sequence of assertibles as a string of letters, a correspondence between bracketed strings of letters and conjoined assertibles immediately suggests itself. On the affirmative side, it is enough to conjoin all the assertibles corresponding to the letters contained in each bracket, considering every nested bracket as a non-simple assertible to be conjoined with the rest of the collection of assertibles/brackets lying at the same level. Let us stop and consider such a prescription from the point of view of first-order propositional logic: the connective "and" is associative (the fact of being two- or n-placed is immaterial exactly for this reason), so that every conjunction/bracketing actually collapses and gives rise to the same proposition: *the* logical product of the constituent propositions. In general, the criterion by which different molecular propositions are distinguished is grounded on their truth-table: since all possible conjunctions of the

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discussed the various ambiguous (διττόν) expressions arising from not placing the negation prefixed to the whole sentence to be negated. A paradox generated by playing with ambiguity in the scope of the negative particle οὐχί is reported in SEXTUS EMPIRICUS, *Pyrr. Hyp.* ii.241 (cfr. also *Pyrr. Hyp.* ii.231).

[21] See for instance GALENUS, *Inst. Log.* iv.6 (Kalbfleisch): "The followers of Chrysippus, fixing their attention more to the manner of speech than to the things spoken about, use the term 'conjunction' for all propositions compounded by means of the conjunctive connectives, whether they are consequents of one another or incompatibles" (translation from MATES 1953) and cfr. LUKASIEWICZ 1957, pp. 18–19; FREDE 1974, pp. 198–201; BOBZIEN 1999, pp. 103–104. See also GALENUS, *Inst. Log.* iii.5 (Kalbfleisch) and ALEXANDER, *In An. pr.*, p. 283.28–30 (Wallies). Recall moreover that the Stoics introduced the notion of subsyllogistic argument in order to classify those arguments not sharing the exact linguistic format of the corresponding syllogisms, but equivalent to them. As an example, "If the first, the second. But the first. Therefore the second" (a mode of a first indemonstrable) is a syllogism, whereas "The second follows from the first. But the first. Therefore the second" is subsyllogistic (See ALEXANDER, *In An. pr.*, p. 373.31–35 (Wallies) and GALENUS, *Inst. Log.* xix.6 (Kalbfleisch)).

[22] Περὶ τοῦ συμπεπλεγμένου πρὸς Ἀθηνάδην α′ β′ (DIOGENES LAERTIUS, *Vitae Phil.* vii.190).

[23] DIOGENES LAERTIUS, *Vitae Phil.* vii.193.

same terms have the same truth-table, irrespective of where the brackets have been put, they *are* the same proposition.[24] The correct number in CHRYSIPPUS' statement would then be 1. As we have seen, Stoic logic provides instead for a way out from such a collapsing: a καὶ must be *prefixed* in a conjunction. Hence, "and and the first and and the second and the third and the fourth" and "and the first and the second and and the third and the fourth", to which the bracketed strings $(x(xx))x$ and $xx(xx)$ unambiguously correspond,[25] count as *different* conjunctions, insofar as they are syntactically distinguishable.[26] It is easy to see that *almost* every bracketing gives rise to one and only one conjoined assertible: there is in fact a residual ambiguity left. Take for instance the strings $(xx)xx$ and $(xxx)x$: they would admit the same expression: "and and the first and the second and the third and the fourth". Clearly, the problem lies in the fact that the rule of prefixing the connective determines where to open the bracket, but not where to close it. Hence, whenever in a bracketed string an arrangement occurs like this: $\ldots x)xx \ldots$, the conjoined assertible corresponding to it could have been generated as well by the slightly different string $\ldots xx)x \ldots$: the mapping from bracketed sequences to prose statements as above is surjective but not injective.

Could the Stoics have detected the problem? Does it undermine the interpretation just sketched? Two answers can be envisaged. First of all, HIPPARCHUS' results could be viewed as an indirect evidence that at least in his *milieu* the ambiguity had been resolved.[27] In fact, we are told by PLUTARCH that "Chrysippus is refuted by all the arithmeticians", i.e. by people able to recognize whether a problem is well-formulated or not. Hence, even supposing CHRYSIPPUS had employed συμπλοκή in its broader meaning, a fact that could explain his claim that the conjunctions "exceed a million", the "arithmeticians" had to stick to a sharply determined meaning in order to set up their calculations; moreover, without removing the ambiguity pointed out above no calculations could have been carried out, and HIPPARCHUS' numbers attest to the fact that at least *he* (and very likely the arithmeticians together with him) did not use the broader meaning and attacked an ambiguity-free problem. Whether such moves were accomplished either by the arithmeticians or by CHRYSIPPUS himself or by his followers we are not told nor are we entitled to guess.[28]

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[24] Such a criterion underlies the modern result (originally proved by SCHRÖDER) that the number of distinct molecular propositions that can be formed from n given elementary propositions is equal to 2^{2^n} (for a proof and a discussion see e.g. HILBERT-ACKERMANN 1950, pp. 18–19).

[25] To be precise, the various x should have been differentiated, for instance using subscript indices, in the same way as the several assertibles in the modes to which the bracketed strings correspond are represented by a suitable numeral. But counting bracketings does not require at all that the assertibles be differentiated, so I prefer not to introduce unnecessary notational complications. See also the remarks in Sect. 4.3 below.

[26] See also BOBZIEN 1999, p. 105 on this point.

[27] Recall that CHRYSIPPUS lived between 280–276 and 208–204 B.C., whereas HIPPARCHUS' astronomical observations reported in the *Almagest* range from 147 to 127 B.C., and that the field of logic was extensively developed by the Stoics well beyond CHRYSIPPUS' times (cfr. e.g. BOBZIEN 1999).

[28] But strict adherence to the syntactic approach cannot be reasonably ascribed to dialecticians much later than CHRYSIPPUS, even less to any arithmetician. The latter, or even HIPPARCHUS

Alternatively, suppose the ambiguity had not been detected by the dialecticians nor by the arithmeticians. If we allow for the possibility that a sort of *symbolic* translation had underlain the real calculations (and this could simply have consisted in an abstract mathematical representation; see also the discussion under point 4.3 below), it is enough that the rules of such a translation had been established for a proper subset of the less subtle cases, namely the unambiguous ones, even if the "translator" was completely unaware of the fact that the prescription does not set up a one-to-one correspondence in all instances. Once the problem is represented in mathematical language, the computational procedures can independently produce a well-defined answer. The arithmeticians could hence have unintentionally worked out a *different* problem with respect to the intended one – provided we can speak of "intended problem" even if the problem is not properly posed. Mathematics is a language that requires definiteness, and sometimes forces it in problems that naturally arise as "unformalized" in character, or, as in our case, "formalized" but under another conception of "formalization". Paradoxically, then, it may have happened that neither the Stoics nor the arithmeticians had perceived the original ambiguity in the Stoic prescription for conjunction.

To complete the correspondence sketched above, notice that the character of the constituent assertibles is left undetermined, so that no permutation of them is required. Moreover, the conjunction being at least a binary connective, the bracketing of a single letter is forbidden, and it is understood that all the constituent assertibles have to be conjoined. Hence counting conjunctions coincides with counting bracketings of strings of letters, and the number of conjoined assertibles produced by means of ten assertibles equals what is now called the tenth Schröder number, i.e. 103049.

Concerning the negative case, the very definition of ἀποφατικόν entails that the particle "not" must be prefixed to the assertible it negates, and that precisely this is the feature making it an ἀποφατικόν, regardless of the affirmative or negative (or non-simple) character of the original assertible.[29] Supposing HIPPARCHUS had calculated the number of negative conjunctions via the same techniques he employed in the affirmative

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himself, could have adopted the pragmatic attitude of considering, by stipulation and on the analogy of the unambiguous cases, also the seemingly – i.e. syntactically – ambiguous ones well defined (I owe this remark to S. BOBZIEN). At any rate, to remove the ambiguity an enclitic τε could have been placed after each nested conjoined assertible as in the following mixed Greek-English example: καὶ καὶ the first καὶ the second τε καὶ the third καὶ the fourth, which is ungrammatical, insofar as redundant, but is unambiguous – it corresponds to the bracketing $(xx)xx$.

[29] The best testimony in this respect is in SEXTUS EMPIRICUS, *Adv. Math.* viii.89–90, where we are told that "[the Stoics] say in fact that 'contradictory (ἀντικείμενα) [assertibles] are those which exceed (πλεονάζει) each other by a negation', e.g. 'it is day' – 'it is not the case that it is day'"; SEXTUS proposes then an objection – a conjunction in which the negation encompasses the second conjunct only, being hence not contradictory of the original conjunction but nevertheless exceeding it by a negation – and reports the reply of the Stoics: "Yes – they say – but they are contradictory with this [added condition], that the negation is prefixed (προτετάχθαι) to one or the other: for then it has scope over (κυριεύει) the whole assertible". The rule of prefixing the "not" is thus strictly functional to the requirement that the negation transforms one assertible into its contradictory. Thus, if καταφατικόν and ἀποφατικόν in the Plutarchean passage are to be taken in their technical meaning, they are assertibles which are one the contradictory of the other: they are respectively an assertible without or with the prefix "not".

case, had he allowed for the insertion of further negative particles inside the string of conjoined assertibles and had he taken into account all possible cases, a number which is greater than the received one by several orders of magnitude would have been produced.[30] HIPPARCHUS must hence have followed the Stoic prescription of prefixing *one* "not". The latter cannot simply encompass the *whole* subsequent series of conjoined assertibles: the resulting number of negative conjunctions would of course be equal to 103049. One must therefore allow for the "not" to act also upon an initial segment of the string (in other words, the negation οὐχί must be allowed to step over some among the καί particles the conjoined assertible begins by). This way an unambiguous rule is provided about which (sub)set of conjoined assertibles the negation actually has scope over. Translating it into the language of bracketings, we are given a prescription similar to that by HABSIEGER *et al.* quoted above: consider the number of bracketings on the string $\neg xx \ldots x$, with the following convention: take the negation of the conjoined assertible corresponding to the first bracket that immediately follows $\neg$, otherwise take the negation of the first assertible in the string. Since the bracketings $\neg(\ldots)$ and $(\neg(\ldots))$ give the same result, many of the negative assertibles will be obtained in two different ways, the only ones which are obtained in an unique way being those in which the sign $\neg$ encompasses the whole string of letters and brackets.

It is not difficult to see that with the above convention one gets the same result as HABSIEGER *et al.*, i.e. that the number of negative conjunctions of n assertibles is $(s(n) + s(n+1))/2$. In fact, the string $\neg xx \ldots x$ has $n+1$ symbols, the related number of bracketings being thus $s(n+1)$. As explained above, since the negation maps simple assertibles onto simple assertibles, the same negative assertible corresponds in many cases to a pair of bracketings, the exception being constituted by all configurations of the form $\neg(\ldots)$, where the bracket contains all n symbols x, possibly further bracketed. There are exactly $s(n)$ such configurations: adding them again to the whole collection of bracketings of $\neg xx \ldots x$, i.e. summing $s(n)$ to $s(n+1)$, *every* negative compound will correspond to one (and only one) *pair* of bracketings, that is

$$s(n+1) + s(n) = \text{twice the number of negative conjunctions}$$

from which the above result follows at once. As an example, I list the strings relative to the seven negative conjunctions of 3 assertibles. The right column contains, in the case of assertibles obtained in two different ways, the corresponding duplicates.

[30] Recall, moreover, that every Stoic description of logical entities built up from assertibles is formulated in a way that is independent of the affirmative or negative character of the constituent assertibles (see on this e.g. BOBZIEN 1996, p. 137). To call a conjoined assertible an ἀποφατικόν would then have been justified only whenever the negative particle had been intended to act upon a previously formed conjunction.

$$(\neg x)xx \quad \neg xxx$$

$$(\neg x)(xx) \quad \neg x(xx)$$

$$((\neg x)x)x \quad (\neg xx)x$$

$$(\neg(xx))x \quad \neg(xx)x$$

$$\neg(xxx)$$

$$\neg(x(xx))$$

$$\neg((xx)x)$$

As $s(11) = 518859$, we are given this way 310954 as the total number of negative conjunctions of ten assertibles and this leaves us with the discrepancy with the attested 310952. As far as I know, Stoic logic does not allow for the elimination of two cases, and it seems very difficult to accommodate for the second number reported by PLUTARCH without radically changing the interpretation. The hypothesis of a miscalculation is not plausible either: the correctness of the result for $s(10)$ entails that every $s(n)$ with $n \leq 10$ had to have been correctly calculated (see the beginning of the next Section for the details). Moreover, even if HIPPARCHUS had also to compute $s(11)$, a discrepancy of two units only is suspect insofar as it is too low: by the very nature of the calculations involved, an error at a certain stage is amplified exponentially by the subsequent steps. It follows that a would-be mistake could have occurred only during the very final, trivial steps in the calculation of $s(11)$, computations whose exactness is very easy to check. The safest attitude is perhaps, as suggested by Reviel NETZ, to ascribe the discrepancy to a mistake of a copyist, who read a numeral Δ (i.e. 4) as a shorthand for δύο. After all, it is really surprising that such abstruse numbers, deprived of any supporting calculation, have come to us affected by two scribal errors only (counting also the variant reading in *Quaestiones Conviviales* for the thousands in the first number).[31]

4. Hipparchus' calculations

My aim is now to inquire what kind of calculations HIPPARCHUS performed. One of the main reasons of interest of the whole issue lies in the mathematical techniques he employed in order to reach a result which is impossible to obtain by direct inspection of all possible cases. In other words, the sheer possibility of the calculations *presupposes* acquaintance with some basic (in a modern perspective) facts of combinatorial analysis. STANLEY suggests,[32] in order to compute the number of bracketings of a string of n letters, an algorithm which is sufficiently effective and which could be regarded in some sense as "natural", since it explicitly takes into account the recursive character of the process. It can be expressed by the following formula:

[31] For comparison's sake, see the wealth of variant readings in the extant (and highly corrupt) Greek text of ARCHIMEDES' *Dimensio circuli*.

[32] STANLEY 1997, p. 349.

$$s(n) = \sum_{i_1+\cdots+i_k=n} s(i_1)\ldots s(i_k), \quad n \geq 2 \tag{$*$}$$

where the sum is over all ordered partitions of n into $k \geq 2$ positive addenda. In words, one starts by fixing the first level of brackets, i.e. the more external one (here the representation in terms of trees is useful). The building-blocks at this level are single letters or brackets. To calculate the total number of possible bracketings given a specific configuration of first-level brackets one has to take the product of the numbers of possible bracketings associated with each building-block. The only non-trivial contributions to the product arise when the building-block is a bracketed string made of i $(2 < i < n)$ letters and possibly further brackets, so that its contribution amounts to $s(i)$. As an example, take the bracketing of 10 letters in Sect. 2 above. There are five first-level building-blocks: $(x(xx)x)$, x, x, (xx), (xx), which can be associated with the following string of digits, each corresponding to the number of x's in the relative building-block: 4, 1, 1, 2, 2. With this partition of 10 fixed, the corresponding contribution to the sum (*) is $s(4)s(1)s(1)s(2)s(2) = s(4)$ since $s(1) = 1 = s(2)$. Fixing the first level of brackets corresponds to picking up one specific ordered partition of n, i.e. one specific addend in the above sum. Summing over all possible partitions (i.e. all ways of fixing the first level of brackets) we get the result. One is thus enabled to determine the numbers $s(n)$ in succession starting from the obvious $s(2) = 1$.

These calculations (and the combinatorial shortcuts which are necessary in order to make the problem workable) could, I think, be a good approximation of what HIPPARCHUS actually did. Around them I shall organize the following discussion of some direct evidence of combinatorial results in ancient Greco-Roman sources.

4.1. Recursive arguments and proofs

The recursive character of the procedure described above is so patent that it could not have escaped HIPPARCHUS. A comprehensive survey of recursive methods in the ancient Greek mathematical corpus would fall outside the aim of the present study. Yet, a selected series of examples should suffice, I hope, to show that what we would nowadays call recursive mathematical arguments were a matter of course over the whole range of the corpus, though no systematic "metamathematical" thinking concerning such a kind of arguments can be shown to have occurred (and very likely did not occur). In particular, recognizing the widespread use of iterative proofs is a key element of the present reconstruction, since such proofs constitute one of the main mathematical tools founding any sort of combinatorial reasoning, especially whenever general rules and results are to be found and subsequently to be expressed in compact form. I arrange the examples by increasing degree of linguistic explicitness; the reader may note the recurrent presence of standard words and phrases, which will eventually find a synthesis in the verbal construct described under item *g*). The existence of a circumscribed and stable linguistic wording for the iteration of a well-defined proof-step shows that the entire process of iteration was recognised as an autonomous, meaningful unit in a proof, ready to be transferred to other mathematical fields.

a) Many instances of recursive reasoning in philosophical contexts can be adduced. It is enough to recall, e.g., the Aristotelian definition of the continuum as "divisible into further (αἰεί) divisibles",[33] or the argument of ARCHYTAS on the infinity of the Cosmos (reach the boundary and then stretch out a hand),[34] or paradoxical arguments such as the "Sorites". A form of the latter reads as follows: "It cannot be that if two is few, three is not so likewise, nor that if two or three are few, four is not so; and so on up to (καὶ οὕτω μέχρι) ten. But two is few, therefore so also is ten".[35] The allusion to the paradox made in SEXTUS EMPIRICUS, *Pyrr. Hyp.* iii.80 contains the adverb ἀεί to express the iteration.

b) Concerning the much debated problem of the existence of proofs by complete induction in the ancient mathematical corpus, I have argued elsewhere for the survival of one such proof in PLATO, *Parmenides* 149A–C, and for its being in relation with the elaboration of soritical arguments. It suffices to our purposes to recall that PLATO's text displays a series of phrases, adverbs, and syntactical constructs which enable him to word in a very refined way the explicitly iterative character of the proof.[36] Yet, such an impressive apparatus is distributed among the various steps of the proof, which lacks therefore a formulaic expression able to summarize the entire process.

c) A problem very likely involving recursive calculations, and similar to that of finding the partitions of a number into ordered addenda, is already hinted at by PLATO in *Leges* 737E-738A, when he says that "[5040] could not be divided into more than 60 minus 1 divisions, in succession from 1 to 10". The reference here is to the fact that 5040 ($= 2^4 \cdot 3^2 \cdot 5 \cdot 7$) admits as factors all integers from 1 to 10, and that the total number of its factors (the number itself excepted) is equal to 59. As already observed by BECKER,[37] PLATO's curious wording of 59 as "60 minus 1" seems to reflect a general computational prescription, according to the fact that the total number of factors of a number of the form $p^a \cdot q^b \cdot r^c \cdot s^d$ is $(a+1)(b+1)(c+1)(d+1)$, from which one unit must be subtracted if the number itself is not to be counted as a factor. The calculations are well in the range of *psêphoi*-arithmetic, when the prime factors are at most three, via the device of plane and solid representations of numbers.[38] The general result could be reached by applying a sort of reasoning by recurrence, observing that every composite number is also some (though not univocally determined) plane or solid number, repeating the remark for the sides of the latter and so on, so that one has only to determine the number of factors of plane numbers (or, if one likes, of solid numbers with three different prime factors). As we shall see repeatedly in what follows, the construct of plane and solid numbers, unfavourably looked upon in a modern perspective insofar as not providing an univocal classification of numbers in terms of their parts (factors), has the surprising virtue, *not* shared by the modern approach to factorization, of being very well suited to enter into iterative procedures.

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[33] See e.g. *Physica* Z 1, 231b15–16.

[34] SIMPLICIUS, *In Phys.*, p. 467.26 ff. (Diels).

[35] DIOGENES LAERTIUS, *Vitae Phil.* vii.82. For a full discussion of the extant sources see BARNES 1982.

[36] An analysis of the Platonic text is in ACERBI 2000.

[37] BECKER 1936, p. 553.

[38] From this could come a first perception of what combinations with repetitions are.

d) From the perspective of the present work, the Archimedean notation for large numbers in the *Arenarius* is very relevant even though slightly disappointing. Briefly put, the first step is to call "first numbers" those up to the myriad of myriads, to take the latter as unit of the "second numbers" and to start counting these second numbers till their myriad of myriads is reached. One continues in this way, the end being arrived at with the myriad of myriads of the myriad-myriad-th numbers. The construction goes further: "Let us call in fact numbers of the first period those just named, and let the last number of the first period be called unit of the first numbers of the second period [...] and so on (καὶ ἀεὶ οὕτως)" up to the end of the myriad-miriad-th period. The final number among those named is exceedingly larger than the number of grains of sand which could fill the entire universe (ARCHIMEDES finds for them an upper bound of 10^{63}). The relevant point in the Archimedean notation is that whole numbers are taken as units of the successive level,[39] and that such a procedure can be repeated indefinitely, even if ARCHIMEDES does not expressly state this.[40]

e) More explicit in wording the recursive character of the method employed are examples such as the rule for approximating the square root of a number given in HERON, *Metrica* I.8.[41] The first step in the approximation is described in detail on the grounds of an example: an approximate square root of 720 is found by picking first the nearest square 729, whose side is 27. Divide then 720 by 27, add the result to 27 and halve what is obtained: it gets 26 1/2 1/3, whose square is 720 1/36. The prescription terminates thus: "If we want that the difference be less than 1/36, in place of 729 we put 720 1/36 just found, and doing the same things (ταὐτὰ ποιήσαντες) we shall find that the difference is by far less than 1/36". The peculiar structure of the final expression, a subordinate participial clause followed by a principal clause in which the fulfilment is declared of what is sought, is here employed to mark one single step in the iteration, even if it is of the standard sort we shall encounter under item *g*) below, where the entire series of steps is encompassed by the same kind of construct. As HERON's treatises collect a congeries of techniques summarizing a whole tradition, one is inclined to suspect that the original source had contained more than the first step of the iteration, with explicit indication that arbitrarily precise approximations of the square root could be obtained.

The *Elements* contain several proofs in quasi-inductive format: as a paradigm instance take IX.8, "If any numbers are in continuous proportion from unit, the third from the unit and every other one will be square, the fourth and every third one cube [...]". If the numbers are A, B, Γ, Δ, E, Z, the proof directly shows first that B is square. Since then B, Γ, Δ are in continuous proportion and B is square, also Δ is square by VII.22. "By

[39] A similar idea is Pythagorean in origin: cfr. the names δευτερωδουμένη, τριωδουμένη... μονάς for 10, 100, etc. (the notion is expressly ascribed to the Pythagoreans e.g. in IAMBLICHUS, *In Nichom. arithm. introd.*, pp. 88.21 ff., 103.16 ff. (Pistelli)).

[40] But recall that the notation was fully discussed in an entire book (now lost) addressed to ZEUXIPPUS, while in the *Arenarius* only what is strictly functional to ARCHIMEDES' calculations is reminded. In this respect, the useless introduction of the second period of numbers could be viewed as an indication towards the indefinite repeatability of the process. Recall also that APOLLONIUS returned on the subject in a lost work of unknown title (PAPPUS 1876–78, pp. 18.23–24.20).

[41] HERON 1903, pp. 18.12–20.5.

the very same arguments (διὰ τὰ αὐτὰ δή) also Z is square. We shall prove similarly (ὁμοίως δὲ δείξομεν) that also every other one is square".[42] The actual repeatability of the proof is here invoked (it is very significant that the general level is reached in two steps), a very standard move which is by no means typical of iterative proofs: here the procedure has not a self-contained status. The fact is confirmed by the presence of the two canonical clauses (in Greek above) marking the repetition of the proof: they belong to second-order discourse, indicating that reflection *on* the preceding proof is necessary in order to express its recursive character: the iterative step is not embodied in the proof as a self-contained unit.

f) The quadratures of some (rectilineal and) non-rectilineal figures (e.g. in book XII of the *Elements* and in several Archimedean treatises) are based on the bisection principle, an explicitly recursive procedure. As is well known, ARCHIMEDES pushed the technique up to the very refined iterative proofs in *Quadratura parabolae* 20–24 and in *Dimensio circuli* 3 (cfr. also the determination of the center of gravity of the segment of parabola in *Planorum aequilibria* II).

The proof in *Quadratura parabolae* differs in a decisive respect, insofar as the surface content of the (inscribed) polygons can be exactly determined at each step, from other, similar demonstrations: in the latter all that one can say is that the difference between the original figure and the approximating polygons can be made as small as one likes, and this is enough to let the proof by *reductio* work. In *Quadratura parabolae*, instead, the reference to *Elements* X.1 (even if not explicit) is supported by a direct control of the process of successively taking away more than half of the residual figure: in this case one is given a segment of parabola, from which a suitable triangle is subtracted; two segments of parabola are left, from each of which a triangle constructed in the same way as the one in the preceding step is subtracted, and so on. The key point lies in the fact that the Archimedean method fournishes an exact estimate of the error committed in taking a partial sum of the succession of inscribed triangles, and this is made possible by the explicitly recursive character of the latter operation, which generates a (readily summable) geometrical series of ratio 1/4. That the proof ends with the canonical *reductio* is of no real significance, for summing the complete series is meaningless without having proved that it converges, and this is done nowadays by an indirect argument of the same sort we encounter in the ancient approach.

In *Dimensio circuli* 3, an estimate of the ratio between circumference and diameter is provided by approximating the circle by means of inscribed and circumscribed regular polygons. Starting from the hexagon, successive bisections lead to the 96-gon. It is absolutely clear that the procedure can be iterated at will, providing better and better approximations of the sought for ratio (the text of the *Dimensio circuli* is too corrupt to allow us any conjecture about whether ARCHIMEDES could have *explicitly* made this point or not). Better approximations had in fact been calculated by ARCHIMEDES himself and by APOLLONIUS,[43] but it is worthwhile to remark the Archimedean care in matching the requirement of a good approximation with the one of having it using the lowest

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[42] EUCLIDES, vol. II, p. 195.5–7.

[43] Cfr. HERON, *Metrica* I.<26> (HERON 1903, p. 66.13–17) and EUTOCIUS, *In dim. circ.* 3, p. 258.16–20 (Heiberg) respectively.

possible denominators in the numerical ratios involved in the upper and in the lower bound.

The common feature of the above sketched proofs, and in general of all quadratures, consists in subtracting in sequence portions of the figure at issue until a certain condition is met (typically that the residual figure has become less than a preassigned one). The process of subtraction is recursive, since the same kind of construction (bisection of an arc, etc.) is performed at each step on the very figure resulting from the preceding step. Now, the last move in the proof is *not* the *end* of the iterative procedure, in which case the latter had not been truly recursive, but the fulfilment of a condition external to it, a sort of parameter of control which, and this is the important point, is not *a priori* fixed, but is functional to the development of the proof by *reductio*, and hence can be set arbitrarily far away from the first step of the iteration. Such a parameter of control (which can also be construed as a preassigned bound to the precision required in an explicit calculation) does not undermine at all the length of the iteration, which is then indefinitely extendable in its very conception. In a well-defined sense there are infinitely many steps in the process, and the very interesting point lies in the fact that the latter is worded in such a way as to constitute one *single* step in a proof, as we shall see presently.

g) A peculiar, compact linguistic tool was in fact developed at some comparatively early stage in order to express the recursive procedure of successive bisections on which the quadratures – or the successive subtractions in an ἀνθυφαίρεσις (*Elements* VII.1, 2; X.1–3) – are grounded. There are two basic forms of expression,[44] usually compounded of two clauses, one subordinate to the other. When the first clause is subordinate to the second, the former employs participial forms (as in the interpolation lemma to THEODOSIUS' *Sphaerica*, III.9: "Cutting DE in two and the half of it in two and doing this continually (ἀεί), we leave (λείπομεν) a certain magnitude less than AZ"[45]), or a genitive absolute construction; conversely, ἕως+verbal form is used in the second when the first is a principal clause (e.g. in XII.5). The presence of the adverb ἀεί in the first clause is the rule, with the notable exception of ARCHIMEDES' *De sphaera et cylindro* I.11, where ἑξῆς appears. Forms of the verb λείπω in the second clause are the standard format, the sole exceptions being represented by *Elements* VII.31 and by the version of book XII contained in the Bologna manuscript, where λαμβάνω is regularly employed.

In our perspective, it is interesting to remark the presence of the same verbal format (in the genitive absolute + principal clause form) in the just mentioned *Elements* VII.31, where no bisection principle is at issue. It is to be proved that "every composite number is measured by some prime number". But every composite number A is by definition measured by some number: if the latter is prime, end of proof. If not, it is measured by some number, which measures also the one originally set out. If the last number found is prime, end of proof; if not, "[t]hen, such a procedure being done, some prime number will be taken, which will measure the one before itself, and which will measure also

[44] In *Quadratura parabolae* 24 the analogous step is expressed by means of two principal clauses, correlated by δή; otherwise the manner of wording is identical.

[45] HEIBERG 1927, pp. 193.20–194.9.

A".[46] In fact, if this is not the case, "infinitely many numbers will measure A", which is impossible "in numbers". This example shows that the verbal format we are discussing was not intended as specifically associated with the bisection principle, but was conceived as the standard wording for those iterative procedures of which the bisection principle had to become a widely used, and hence paradigmatic, instance. (Remark, in the above proof, also the explicit recognition that the outcome of such a *single* proof-step *could* be infinitely many objects.) Thus, also the linguistic side displays a remarkable degree of standardization, a fact that confirms the full extent to which the procedure had been recognised as an independent proof-technique: noticeably, and unlike the case of *Elements* IX.8 above, there is no second-order (the use of the first person in the verb "we leave" can obviously be dispensed with) discourse in the proof.

4.2. Combinatorics

For $n = 10$, the sum in formula (*) has 511 terms; in general, the number of ordered partitions of a positive integer n is $2^{n-1} - 1$. [47] The calculation is easy: "write" n as a sequence of n conveniently spaced *psêphoi*: you can generate an ordered partition of n by inserting a mark in some of the spaces between the *psêphoi*. There are $n - 1$ spaces between n *psêphoi*, and you can decide for each of them whether to insert the mark or not. Hence you can partition the n *psêphoi* into $2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2$ ($n - 1$ factors), i.e. 2^{n-1}, ways. You have to subtract 1 since the case in which no mark at all is inserted gives rise to no partition. 511 terms are not an unreasonable amount, but the sum (*) can be further shortened if one realizes that several ordered partitions correspond to one and the same unordered partition, that every such ordered partition gives the same contribution to the sum, and that a general and elementary combinatorial calculation provides for the number of ordered partitions corresponding to a single unordered one. It is not difficult to write down all unordered partitions of an integer n, and the list can be effectively (i.e. recursively) and quickly ($n = 10$ has 41 different unordered partitions) computed starting from $n = 2$.[48] Given an unordered partition $n = n_1 + n_2 + \cdots + n_k$, containing $l \leq k$ different addenda, the number of ordered partitions corresponding to it is $k!/k_1!k_2!\ldots k_l!$, where $k_1, \ldots, k_l$ are the occurrences of the different addenda (so that $k_1 + k_2 + \cdots + k_l = k$). I find it incredible that HIPPARCHUS had performed his calculations without noticing the possibility that several different conjunctions actually gave the same contribution to the sum, and that they differed only by a rearrangement of

[46] EUCLIDES, vol. II, p. 138.14–15. I adopt the wording of the Theonine manuscripts, which add the last two clauses. The reading is confirmed by the mediaeval Latin translations from Arabic: see BUSARD 1983, p. 217.509–510 and BUSARD 1984, c. 182.17–20. Cfr. also HEIBERG's perplexities about the text carried by the manuscript **P** in EUCLIDES, vol. II, p. 138 *in app.*

[47] By the way, $2^{n-1} - 1$ is the sum of the first $n - 2$ terms of the geometric progression of ratio 2; the sum is calculated for a generic ratio in *Elements* IX.35.

[48] An introduction to the problem of partitions can be found in HARDY-WRIGHT 1979, chap. XIX. There is no closed formula providing the number of unordered partitions $p(n)$ for generic n; however, a generating function can easily be written. The values of $p(n)$ from $n = 2$ to $n = 10$ are 1, 2, 4, 6, 10, 14, 21, 29, 41.

the first-level conjunctions. Their common contribution to the sum is always a product of $s(i)$'s with $i < n$, operation which is of course commutative (*Elements* VII.16), and the calculation of the number of equivalent first-level ordered conjunctions is a problem of the kind a mathematician cannot but feel himself compelled to solve. The technical tool needed is not, of course, the expression written above in its symbolic form, but simply a clear understanding of how to use multiplication and division in order to calculate the possible combinations (in a generic, extended sense) of objects out of a given set. This raises a general point: the fact is that our symbolic representation of calculations tends to obscure the meaning of the operation actually carried on, making us lose contact with the complex concatenation of arguments the symbols stand for. The meaning is transferred to the symbolic representation itself, a fact which should be forbidden by the very definition of "symbolic representation", and this has the natural consequence of generating the belief that whoever does not possesses the symbolism cannot thereby be able to perform the corresponding calculations, which should obviously be too complicated or even incomprehensible to him.

4.2.1. Procedures in which multiplication is needed, and in one case used, to compute simple "combinations" of objects are attested in at least two ancient sources (the Platonic passage under point 4.1, item *c*) above should be added to them).

a) In ARISTOTLE's *Politica* special attention is devoted to arguments in which combinatorial manipulations are decisive. At Δ 4, 1290b25–39 the parts of a state are compared to the parts necessarily an animal must have (mouth, ears, etc.). Every part presents itself in many different forms (διαφοραί). Hence, "the number resulting from their combination (συζεύξεως) will necessarily produce many genera of animals (for it is not possible for the same animal to have several forms of mouth, and similarly for ears), so that, whenever all possible pairings (συνδυασμοί) of these have been taken, they will produce species of animals, and the species of animals will be precisely as many as the combinations (συζεύξεις) of the necessary parts are". συνδυασμός as technical term appears also in Δ 15–16, 1300a31–1301a15,[49] where an extended discussion is presented of all possible systems of government (but the text is here highly corrupt) and judicial elements in a state. However, no reference is made to any general combinatorial prescription, the attested lists of cases being not exhaustive nor entirely consistent (further cases are introduced in the course of the argument).

b) BOETHIUS' *De hypotheticis syllogismis* (composed ca. 515 A.D.) contains detailed lists of all types of hypothetical syllogisms, differentiated on the grounds of the character of the constituent propositions (necessary, contingent, affirmative, negative, etc.). The most interesting passage from the combinatorial point of view is a simple calculation of combinations with repetitions.[50] The problem is similar to the one we find in PLUTARCH: "If someone is inquiring the number of all conditional propositions,

[49] συνδυασμός is employed in the same meaning also, e.g., in *Politica* 1294b2, 1317a3. It is clear that the original meaning of "pairing" (i.e. with reference to only *two* objects coupled) had been already lost in ARISTOTLE's times.

[50] BOETHIUS, *De hypotheticis syllogismis* I,viii,1–7 (BOETHIUS 1969, pp. 244.1–248.55). Since the relevant variable for distinguishing the propositions involved is their character, and not the specific term they contain, real combinations with repetitions are at issue here.

he can find it from categorical [propositions]; and first one must inquire the [conditionals] made up of two simple [...]".[51] The answer runs thus: there are five affirmative categorical propositions and five correlated negative propositions: ten in all. An hypothetical proposition is made of two categorical propositions: one hundred combinations result. Considering also the propositions composed of one categorical and one hypothetical, or of two hypothetical, one obtains one thousand and ten thousand respectively.[52] BOETHIUS also points out that, in the case in which the middle term of the two hypothetical propositions is the same, the last number must be reduced to one thousand. The overall argument is of course trivial, but the idea of taking all possible combinations is expressly stated, and cannot be supported by any direct reckoning or by a diagram: an abstract and general conception of what combinations are is needed. Moreover, BOETHIUS' last remark is followed by the confused statement that "if it is proposed this way: "if a is, b is, and, if b must be, either c is or is not", two conditional propositions, i.e. four predicatives, result. It results that in relation to those that are composed of four predicatives, ten thousands conjunctions are produced".[53] Here it is not clear in what sense the terms in the second and third proposition are to be considered as wholly independent. This could suggest that BOETHIUS is severely abridging an earlier source.

4.2.2. True calculations of combinations without repetitions (in modern, technical sense) have been transmitted in two well-known texts:[54]

a) A thirteenth-century manuscript of EUCLID's *Data* carries a scholium,[55] presenting the odd feature of being unrelated to the main text, in which a diagram corresponding to what is nowadays termed "Pascal's triangle" is written down as far as $n = 10$,

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[51] "Si quis igitur propositionum omnium conditionalium numerum quaerat, ex categoricis poterit invenire; ac primum in conexis ex duabus simplicibus inquirendus est [...]" (*De hyp. syll.* I, vii.7, see BOETHIUS 1969, p. 244.69–71).

[52] I report the most interesting portion of the text: "Sed cum prima propositio secundae propositioni quadam consequentia copuletur, ut una hypothetica fiat, omnes decem affirmativae ac negativae propositiones omnibus decem affirmativis negativisque propositionibus applicabuntur. Itaque complexae centum omnes efficiunt propositiones, haec quae conexae ex simplicibus coniunguntur. Secundum hoc vero modum potest propositionum numerus inveniri etiam in his propositionibus quae ex categorica et hypothetica copulantur vel ex quae duabus conditionalibus fiunt. Nam quae ex categorica et conditionali constant, vel e diverso, haec tribus categoricis iunctae sunt. [...] Quo fit ut tertia propositio cum duabus superioribus, centum inter se modis copulatis atque complexis, iuncta atque commissa, mille omnes faciat complexiones. Centum namque duarum propositionum modi, cum decem modi tertiae propositionis complicati, mille perficiunt. [...]; quod si centum superiorum propositionum categoricarum modi centum posteriorum categoricarum modis complicentur, fient decem milia complexiones" (BOETHIUS, *De hyp. syll.* I,viii.2–5, see BOETHIUS 1969, p. 244.12–246.40).

[53] "si ita proponatur: "si est a, est b, et, si necesse est esse b, est vel non est c", duae propositiones conditionales, id est quatuor praedicativae fiunt. Quo fit ut secundum eas quae ex quatuor praedicativis conectuntur, decem milia faciunt complexiones" (*De hyp. syll.* I,viii.49–51).

[54] Both the scholium and BOETHIUS' remarks were brought to the attention of the scholars by HEIBERG, and are translated and commented on in ROME 1930.

[55] EUCLIDES 1916, p. 290.

supplemented by a series of instructions for computing combinations of terms taken two and three at a time (called δυαδικαὶ συζυγίαι and τριαδικαὶ συζυγίαι respectively; the results are arranged as the third and fourth row of the triangle. See the figure, taken from HEIBERG's text – the empty spot is empty in the manuscript too).

	α	α	α	α	α	α	α	α	α	α
	α	β	γ	δ	ε	ς	ζ	η	ϑ	ι
δυαδικαὶ συζυγίαι		α	γ	ς	ι	ιε	κα	κη	λς	με
τριαδικαὶ συζυγίαι			α	δ		κ	λε	νς	πδ	ρκ
				α	ε	ιε	λε	ο	ρκς	σι
					α	ς	κα	νς	ρκς	σνβ
						α	ζ	κη	πδ	σι
							α	η	λς	ρκ
								α	ϑ	με
									α	ι
										α

Here is the text of the scholium, which is whortwhile to translate in its entirety: "Given any terms whatever[56] to find dyadic combinations (δυαδικὰς συζυγίας). We find them this way: we take of the given 10 terms a number[57] (ἀριθμὸν) a unit less and we multiply it by the one near to it a unit greater and we take the half of the resulting quantity; and we have this as finding of the dyadic combinations of the given terms. The triadic [combinations] in this way: we take the number two units less than the generic quantity (τοῦ ποσοῦ) among the terms given at the beginning and we multiply it by the quantity (τὸν ποσὸν) resulting from the dyadic combinations of the generic quantity (τοῦ ποσοῦ) among the terms given at the beginning and we take the third part of the resulting multiplication; and we have triadic combinations (τριαδικὰς συζυγίας). And similarly in succession (καὶ ἑξῆς ὁμοίως)". A rule for calculating triangular numbers follows, limited to some numerical examples.

The real potentialities of the scholium have been so far neglected.

(i) The scholium is late, but the fact that it reports the rules without any proof could support the assumption of an ancient origin of the rules. A superposition of

[56] The text reads ὁποσαοῦν within a genitive absolute. Maybe a plural genitive should be required.

[57] The text makes here poor sense and is very likely corrupt (for instance, the object the taken number is a unit less than is not specified): a wording such as the one subsequently adopted in the case of triadic combinations should be preferred. The numeral 10 is at all out of place, and I suspect it has been inserted by a late copyist to conform the text to the range covered by the triangle. HEIBERG records a lacuna after "number".

several sources is here very likely: I conjecture that the accompanying diagram could have a late origin, whereas the primary source of the text, subsequently epitomized at several stages, could date back to very early times. It is clear that the written indications in the diagram stop at the triadic combinations *because* the same is done in the text (at least because of the missing denominations of the higher order combinations).

(ii) The instructions are clearly intended to provide a complete description of the procedure for calculating combinations, as the final clause shows. Generality in the prescription shows up at two levels. First, the final clause completely describes the generic step of the procedure *exactly because* the procedure is recursive: use is made of the combinations of the preceding order, the n-th part taken at the end corresponding to the n-adic combination calculated at that stage. Second, it is of some interest to note the presence of the term ποσόν in the particular meaning of "well defined, but otherwise indetermined, quantity" (possibly arising as a result of an operation on well defined, but otherwise indetermined, quantities, as in the second instance above). It is in fact to be intended in this sense, i.e. as the noun corresponding to adjectives such as ὁποσοσοῦν and variations on it (cfr. also the first line of the scholium) we regularly find e.g. in the *Elements*. I have undertaken a survey of ancient (i.e. not later than IAMBLICHUS' treatises) sources in search of similar usages of the noun, with negative results.[58]

(iii) A proof of the rules can easily be outlined fully justifying the way the latter are expressed.[59] First, dyadic combinations can be readily computed through the device of triangular numbers (recall the second part of the scholium). Suppose next that the dyadic combinations of a certain amount of terms (conceive the latter as a row of *psêphoi*) are given, and that the triadic ones are to be computed. Pick one dyadic combination; to have a triadic one, one more term is to be added. The new term can be chosen among a number of terms equal to the original one minus two, since two terms have already been used in the dyadic combination that was fixed. Hence, each dyadic combination gives rise to "total-number-of-terms-minus-two" triadic combinations. Multiplying by the number of dyadic combinations would lead to the result, were it not for the fact that the prescription is redundant. In fact, the new term can be added to the chosen dyadic combination in $2 + 1$ (the first summand comes from the *dyad*ic combinations at the beginning) different places

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58 But cfr. HERON 1912, p. 388.23.

59 ROME proposes that the peculiar enunciation of the rule "mène peut-être plus vite au résultat que le additions successives; et, d'autre part, il est plus maniable pour un Grec que le nôtre, qui comporte une longue multiplication et surtout une division avec un diviseur élevé [. . .]" (ROME 1930, p. 99). But he misses the recursive character of the procedure, and is misled by having reversed the relationship between the text and the triangle, making the content of the former a mere description of the numbers reported in the diagram, i.e. as a set of rules for constructing a particular *Pascal's triangle*, whose main property *should* be (to modern eyes) its being generated by successive additions (cfr. ROME 1930, p. 99: "Il est curieux qu'on n'ait pas observé que le triangle arithmétique pouvait se former par de simples additions"). But in this way the connection between the diagram and combinatorial issues should be proved anew! Such a way of reasoning is a beautiful instance of the way anachronistic viewpoints creep into seemingly harmless arguments.

– in front, in the middle, and after the terms composing the dyad –, but the same triadic combination actually results in each case. The total number of the latter is hence arrived at by dividing by three. And so on. End of proof. Is the above demonstration outside the range of "ancient mathematics"? Compare the passage from ARISTOTLE's *Analytica priora* discussed below.

b) A rule for computing combinations of terms taken two at a time is provided in some ancient commentaries to ARISTOTLE's *Categoriae*. PORPHYRY's *Isagoge* (written ca. 270 A.D.) is an "exposition in few words" of what the five predicables, namely *genus*, *species*, *differentia*, *proprium*, *accidens*, are; in particular, both common features (κοινωνίαι) of and differences (διαφοραί) among them are discussed. How many differences did PORPHYRY have to expound? He answers the question in an excursus: "It has been said in what the genus is differing from the other four, but it happens that each [term] differs from the other four, so that, being five and each one differing from the [other] four, it [should] result four times five, twenty differences in all. But this is not so, rather, [the terms] being reckoned every time in succession, and the [ones which are reckoned as] second lacking one difference (since it has already been considered), the third two, the fourth three, the fifth four, in all ten differences result, four, three, two, one".[60] He then provides the complete list of the ten differences at issue. The argument triggered BOETHIUS' *glossae* in his commentary (written ca. 508 A.D.) to PORPHYRY's *Isagoge*:[61] he first expands and explains with an example PORPHYRY's (intentionally

[60] PORPHYRY, *Isagoge*, p. 17.14–20 (Busse). I report BOETHIUS' faithful Latin translation: "Genus vero quo aliis quattuor differat dictum est. Contingit autem etiam unumquodque aliorum differre ab aliis quattuor, ut cum quinque sint, unumquodque autem ab aliis quattuor differat, quater quinque viginti fiant omnes differentiae; sed semper posterioribus enumeratis et secundis quidem una differentia superatis propterea quoniam iam sumpta est, tertiis vero duabus, quartis vero tribus, quintis vero quattuor, decem omnes fiunt differentiae, quattuor, tres, duae, una" (BOETHIUS, *Porphyrii Isagoge a Boethio translata*, p. 45.10–17 (Busse)).

[61] BOETHIUS, *In Porphyrium Commentariorum libri V*, cc. 148A–150A (Migne). I report the relevant texts: in "[s]ed hoc fiet si ad numeri referatur naturam, comparationisque alternationem: nam si ad ipsas differentiarum naturas vigilans lector aspiciat, easdem spe sumptas differentias inveniet. Quo enim genus differt a differentia, eodem differentia distat a genere; et quo differentia distat a specie, eodem species a differentia disgregatur, et in cæteris eodem modo. In hac igitur differentiarum dispositione, quam supra disposui, easdem sæpius annumeravi. At si differentiarum similitudines detrahamus, decem fient omnino differentiæ, quas ad præsentem tractatum velut diversas atque dissimiles oportet assumi" BOETHIUS explains why ten, and not twenty, is the right result, while in "ut tamen has secundum dissimilitudinem differentias non in quinario tantum numero, verum in cæteris quoque notas habere possimus, dabitur regula talis, quæ plenam differentiarum dissimilitudinem in qualibet numeri pluralitate reperiat. Propositarum enim numero rerum si unum dempseris, atque id quod dempto uno relinquitur, in totam summam numeri multiplicaveris, dimidium ejus quod ex multiplicatione factum est, coæqualiter ei pluralitati quam propositarum rerum differentiæ continebant" he gives the rule for calculating combinations. ROME's analysis (ROME 1930, pp. 101–102) of the whole Boethian text is not satisfying. He wrongly interpreted the meaning of *differentia* as "des arrangements ou des combinaisons", and misconstrued BOETHIUS' explanation of the rationale behind the first statement of PORPHYRY, viewing it as a rule (by the way, BOETHIUS gives *no* rule) for computing combinations of terms taken two at a time with repetitions.

wrong) statement that the *differentiae* were twenty, and enunciates then the general rule that the total number of *differentiae* of n objects equals $(n-1)n/2$, providing a numerical example and checking the rule in the case of four objects. He finally promises that "[i]n the exposition of the predicaments, also the reason why this is the case will be explained". In this treatise (written 510 A.D.) we only find an example (namely that there are six *differentiae* of four terms).[62]

As already observed by ROME,[63] the very reasoning of BOETHIUS suggests that the *differentiae* of n objects could easily be calculated as the triangular number of side $n-1$,[64] and it is difficult to imagine that such a result had escaped even the first investigators on the subject (see below for a few clues in ARISTOTLE's *Analytica priora*).

4.2.3. To these texts other passages of combinatorial interest could be added which, albeit often less explicit, are nonetheless interesting from the point of view of both terminology and mathematics involved – and have been written in some cases well before HIPPARCHUS' times.

The first testimony is in PAPPUS' *Collectio* VII.11–12.[65] The author briefly reports there on the content of APOLLONIUS' *Tangencies*, and sets down one of its propositions: "given in position any three points, straight lines, or circles, to draw a circle through each of the given points, if there be given any, and tangent to each of the given (straight or circular) lines. Because of the number of like and unlike givens in the hypothesis, necessarily there are ten propositions differing in part. In fact out of three unlike kinds, ten different unordered groups of three result (ἐκ τριῶν γὰρ ἀνομοίων γενῶν τρίαδες διάφοροι ἄτακτοι γίνονται ῑ)".[66] PAPPUS mentions also another, simpler problem (it is not clear whether it was contained in APOLLONIUS' treatise or not) of the same kind: "given any two points, lines, or circles, to draw [. . .]. Already this contains six problems, since from three different entities one obtains six different unordered pairs (ἐκ τῶν τριῶν γὰρ διαφόρων τινῶν δυάδες ἄτακτοι διάφοροι γίνονται τὸ πλῆθος ϛ̄)".[67] A few remarks are in order.

(i) The very similar sentences I reported also in Greek are paradigm instances of a commentator's remark, and hence in their attested form must be ascribed to PAPPUS, but this does not entail that similar arguments were absent from the *Tangencies*.
(ii) In fact, the sentences themselves are oddly redundant, since the passages are followed by the complete list of all possible triples (resp. pairs). It looks very much as if PAPPUS is *quoting* two cases of a general statement, expressed in a technical lexicon and presenting a formulaic structure, where the spots here occupied by both τρίαδες/δυάδες and the numerals after γίνονται were to be filled with

[62] BOETHIUS, *In Categorias Aristotelis libri IV*, cc. 272C–273A (Migne). The Aristotelian passage commented on is *Categoriae* x, 12b5–16, where no combinatorial reasoning is present.
[63] ROME 1930, p. 103.
[64] I.e. as the sum of all numbers from $n-1$ down to 1, a sum which can obviously be arranged as the triangular number of side $n-1$.
[65] The relevance of the passage for the history of combinatorics had already been pointed out by JONES (PAPPUS 1986, p. 388). The translations are JONES', with one slight modification.
[66] PAPPUS 1986, pp. 91.24–93.2.
[67] PAPPUS 1986, pp. 93.28–95.3.

suitable, corresponding terms. I regard as a further clue in this sense the presence of the expressions ἀνομοίων γενῶν and διάφοροι, which are completely out of context here, and which strongly suggest a connection with investigations into the mathematical structures underlying the genus-species relationship (i.e. the study of categories: recall the *differentiae* in BOETHIUS' passage above). This confirms the feeling that the field of logic was the ideal *milieu* in which combinatorial researches could grow up, so that I could venture to conjecture that PAPPUS was actually drawing on a source of this kind. Even more interesting is the explicit reference to the "unordered" character of the combinations: it presupposes of course an acquaintance with ordered combinations too, and reasons for distinguishing the two kinds (it is at all natural to conjecture some underlying treatment here too).

(iii) It is easy and well within the range of ancient arithmetical techniques to compute the above numbers in the general case: the number of unordered n-tuples of three terms with repetitions is $(n+2)(n+1)/2$, i.e. the triangular number of side $n+1$. The possible n-tuples can in fact be arranged in a triangular array following a rule ensuring that all of them have been taken into account (in other words, this amounts to provide a *proof* of the above formula): starting from each vertex, where each of the three n-tuples containing only one kind of term is placed, every layer of n-tuples situated k steps away from a vertex has n–k terms of the same kind as the vertex, the side opposite to the vertex at issue containing no term of that kind, as follows:

aa	*aaa*	*aaaa*
ab ac	*aab aac*	*aaab aaac*
bb bc cc	*abb abc acc*	*aabb aabc aacc*
	bbb bbc bcc ccc	*abbb abbc abcc accc*
		bbbb bbbc bbcc bccc cccc

(The above prescription is unambiguous and exhaustive since a triangle is a simplex in the plane. The unordered n-tuples of four terms with repetitions can be represented as tetrahedral numbers, so that there are $(n+3)(n+2)(n+1)/6$ of them.) The successive triangles can be generated recursively: one constructs the n+1-th triangle by adding an "*a*" term to the left of all n-tuples present in the n-th triangle and then putting, with an "obvious" formation rule, a further row at the bottom made of "*b*"s and "*c*"s only.

A further passage in *Collectio* VII.16 is interesting. Referring to EUCLID's *Porisms*, PAPPUS claims that a certain result there contained holds in greater generality: "It is not recognised that it is true for every number put forward, if one states it thus: [. . .] 'If any number of lines should intersect each other, not more than two through the same point, and all points on one line be given, the rest being in quantity a triangular number, the side of this having each point touching a line given in position, and no three being at the angles of a triangular area, each remaining point will touch a line given in position"'.[68] I have underscored the clause relevant to us: the problem with it is that, as SIMSON already remarked,[69] the "rest" is *always* in quantity a triangular number, in that, given

[68] JONES' translation. For the text see PAPPUS 1986, p. 99.9–20.
[69] See PAPPUS 1986, p. 393.

n mutually intersecting lines (no more than two of them passing through any point of intersection, so that the number of intersections is $n(n-1)/2$), if the $n-1$ intersections lying on one line are fixed, $(n-1)(n-2)/2$ of them remain, i.e. the triangular number of side $n-2$. I find it reasonable that PAPPUS was actually drawing the more general statement(s) above from some source. He appears in fact to slightly misunderstand the statement he is reporting,[70] to the extent of transforming a side remark about the number of residual intersections into a further hypothesis in the enunciation. He is directly providing an "accidental" feature of the result, very likely connected with the manner of calculation. I am thus inclined to consider the above passage as a beautiful *indirect* evidence of a well defined combinatorial statement, with related calculation, in ancient sources.

In general, the content of some of the treatises PAPPUS is providing lemmas for in book VII of the *Collectio* is very well suited to give rise to combinatorial speculations. Apart from APOLLONIUS' *Tangencies* and EUCLID's *Porisms*, other works such as the *Cutting off of a Ratio* (and *Cutting off of an Area*) and the *Determinate Section* display a subtle cases and subcases structure induced by the several reciprocal positions of points and/or lines.[71] Take the latter treatise; it deals with the following general problem: given four points A, B, Γ, Δ on a straight line, to find a point E on the same straight line such that the rectangle between AE, BE and the one between ΓE, ΔE have a given ratio. Some of the given points are allowed to coincide, some rectangles becoming hence squares, and one of the variable segments is furthermore allowed to be replaced by a fixed one. PAPPUS summarizes the content of the treatise and proves many lemmas useful to it:[72] it is clear that APOLLONIUS dealt with all possible cases, as SIMSON's reconstruction nicely shows. The case structure is generated both by the reciprocal position of the given points and by the possible positions of the point E with respect to them, a little problem whose solution can be worked out by direct enumeration but which, I strongly believe, has raised combinatorial questions in APOLLONIUS or others.

The second set of testimonies comes from some Aristotelian works.

Contrary to what could be expected given the commentaries discussed above, passages of combinatorial interest are absent from the *Categoriae*.

In *Analytica priora* A 25, 42b5–26 a long stretch of text deals with syllogisms ("deductions" in the translation below) with more than two premisses. It is worthwhile to report the entire passage:

> And when the conclusion is reached by means of prior deductions or several continuous middle terms (for instance if premise AB is concluded through terms C and D), then the number of terms will likewise exceed the premises by one (for the term inserted will be put either outside or in the middle; but in both ways it results that the intervals are one fewer than the terms, and the premises are equal to the intervals). However, the premises will not always be even and the terms odd; rather, in alternation, when the premises are

[70] A further indication in this sense is that the general form of the theorem is, as it stands, false. See PAPPUS 1986, p. 393 on this.

[71] See the analyses, containing also details on past reconstructions, in PAPPUS 1986, pp. 510–569.

[72] *Collectio* VII.68–119.

even, the terms will be odd, and when the terms are even, the premises will be odd. (For a single premise is added at the same time as a term, no matter from what side the term may be added, so that since the premises were even and the terms odd, this will necessarily alternate when the same addition has been made.)

But the conclusions will never have the same arrangement either in relation to the terms or in relation to the premises. For when one term is added, conclusions will be added one fewer in number than the terms which were already present: for only in relation to the last term does it fail to produce a conclusion, while it produces one in relation to all the rest. For example, if D is added to A, B, and C, then two conclusions are also added immediately, the one in relation to A and also the one in relation to B (and similarly in the other cases). It will also be the same way if the term is inserted into the middle (for it will only fail to produce a deduction in relation to one term). Consequently, the conclusions will be much greater in number than either the terms or the premises.[73]

Scholars have seen here the best evidence that ARISTOTLE actually envisaged a diagrammatic representation of syllogisms:[74] to be sure, the use of "interval" (διάστημα) as a synonymous of "premiss" is well attested in the *Analytica* and must probably be ascribed to cross-fertilization with other fields,[75] but here it is supplemented by a series of spatial determinations that make one suspect a diagrammatic counterpart is really understood.

Contrary to the belief shared by all modern commentators, it is not so immediate to compute the number of conclusions resulting from ARISTOTLE's remarks. The standard result $n(n-1)/2$ reported in all modern commentaries is in my opinion not well suited to the text of the *Analytica* and must be ascribed to acritical acceptance of a remark by WAITZ: "It is clear that from three propositions [...] three conclusions result, from four six result, from five ten: for from propositions AB, BC, CD, DE, EF the conclusions AC, AD, AE, AF; BD, BE, BF; CE, CF; DF are collected. Thus, the number of conclusions grows according to triangular numbers (Trigonalzahlen). Let the number of propositions be $= n$, the number of terms $= n + 1$, the number of conclusions will be $n(n-1)/2$".[76] WAITZ's calculation is performed supposing that the new term is added on the right of the last term of the given sequence (in WAITZ's instance, F is placed after E of the sequence

[73] Translation from ARISTOTLE 1989, pp. 40–41. Cfr. also the similar, although less precise, claim in *Analytica posteriora* A 32, 88b3–5, where it is stated that the premisses cannot be much fewer than the conclusions. ROSS regards the claim as "a careless remark" and argues that "*A. Pr*. i. 25 must be later than the present chapter" [*viz*. An Post. i. 32] (ARISTOTLE 1949, p. 603). In MENDELL 1998, pp. 201–202 it is argued instead in favour of the plausibility of ARISTOTLE's statement.

[74] See e.g. MENDELL 1998, p. 202 note 78. Cfr. the early remarks in EINARSON 1936, pp. 165–169.

[75] Other *loci* in the *Analytica* are at 35a12,31, 38a4, 82b7, 84a35, 84b14. See EINARSON 1936 on the issue of the borrowing of the term from proportion/musical theory.

[76] "Apparet autem ex tribus propositionibus, quae soritem constituant, fieri tres conclusiones, ex quattuor fieri sex, ex quinque decem: colliguntur enim ex propositionibus AB, BC, CD, DE, EF conclusiones AC, AD, AE, AF; BD, BE, BF; CE, CF; DF. Crescit igitur numerus conclusionum secundum numeros triangulos (Trigonalzahlen). Sit numerus propositionum $= n$, numerus terminorum $= n + 1$, erit numerus conclusionum $n(n-1)/2$" (WAITZ 1844–46, vol. I, p. 441, *ad* 42b25).

A, B, C, D, E). But this is only a portion of ARISTOTLE's reasoning: as he explicitly says, one must also take into account the cases in which the new term is added in the middle, or before A.[77] The latter case is actually enough to provide for all new conclusions (FB, FC, FD, FE in the example at issue). As it is easily seen, the $m-1$-th step would contribute thus with $2(m-1)$ conclusions to the total number, and the latter would be equal, if there are $n+1$ terms, to $n(n-1)$, i.e. to the *double* of the triangular number of side $n-1$. But this is true only when the series of premisses to which the new term is added is *one* among the possible series obtained by repeated applications – starting from a single premiss (which in turn can be either AB or BA)[78] – of ARISTOTLE's patently recursive prescription. Taking into account all possible series instead, the final result is equal to all possible ordered arrangements of $n+1$ objects (namely the terms) taken two at a time, i.e. to $n(n+1)$. The same result can be obtained in a recursive way, very much in the style of ARISTOTLE's reasoning: the step in which one passes from $m-1$ premisses to m premisses, i.e. the $m-1$-th step, adds $2m$ conclusions (resulting from combining the new term with the m preceding ones, both on the left and on the right of each of them); summing up to n premisses, i.e. to $n+1$ terms, one gets $n(n+1)$ again. The summation-procedure just described was well known since before ARISTOTLE's times: starting from the dyad, the successive even integers are added as gnomons, generating the so-called *heteromecic* numbers: precisely those of the form $m(m+1)$ for some integer m.[79] The only possible drawback in such an approach is that what is stated as a premiss at a certain stage is obtained as a conclusion at the same stage, although the identical premiss and conclusion never figure in the *same* syllogism, a fact which would render the inference unsyllogistic (in Peripatetic sense).[80] Maybe such complications were responsible for the formulation of the conclusion ARISTOTLE draws from his argument, a conclusion which focusses on the single step of adding one term, without considering the possibility of calculating the total number of conclusions obtained by successively adding terms up to a fixed number of them. In any case, both the kind of reasoning required to prove whichever of the above results and the (conjectural) associated diagrammatic representation are very much like those in BOETHIUS' second passage above. In particular, it is interesting that complex syllogisms could be actually constructed by successive addition of terms, i.e. by a sort of recursive

[77] ALEXANDER (*In An. pr.*, p. 285.18–28 (Wallies)) suggests also the alternative interpretation that ARISTOTLE, when speaking of inserting the new term externally, is actually referring to the second and third syllogistic figures.

[78] Actually, the series of conclusions obtained starting from AB or from BA coincide. The same result obtains if three terms are taken as starting point.

[79] See e.g. NICOMACHUS, *Introductio arithmetica*, p.108.8 (Hoche) and THEON SMYRNAEUS, *Expositio rerum mathematicarum ad legendum Platonem utilium*, p. 26.21–22 (Hiller). KNORR convincingly argued for the Pythagorean (late fifth century B.C.) origin of computation techniques based on figured numbers, in particular *heteromecic* numbers (KNORR 1975, p. 142 ff.). The latter are named in PLATO's *Theaetetus*, 148A–B and in ARISTOTLE's *Metaphysics*, 986a26 (the context is arithmetic; cfr. *Met.* 1093b6); their generation by means of gnomons added to the dyad is usually seen as being hinted at in *Phys.* 203a11–15.

[80] The very definition of syllogism in *An. pr.* 24b19 expressly rules out such a possibility. Cfr. also ALEXANDER, *In An. pr.*, pp. 18.12–19.3 (Wallies), where the author is referring to the fact that in Stoic logic inferences of this sort are not unsyllogistic.

procedure, and that specific attention had been paid to the problem (significant in this respect are the remarks about the irrelevance of the place – outside or in the middle – where the new term is added). Investigations about the structure of complex syllogisms were pursued by later logicians, as an interesting testimony by ALEXANDER confirms us.[81]

In *De generatione et corruptione* B 3, 330a30–b1 the exposition of the ways the "elements of bodies" (hot, cold, dry, and moist) combine begins thus: "Since the elements are four, the combinations (συζεύξεις) of four [terms] are six, and the nature of the contraries does not allow for their pairing (συνδυάζεσθαι) [. . .], it is clear that the combinations of the elements will be four". Just two remarks:

(i) The second sentence is stated in general terms, as in PAPPUS' extract above, and appears to refer to a well-established mathematical fact.
(ii) ARISTOTLE's noun for "combinations" (συζεύξεις) has the same root as that of the term we find in the scholium to the *Data* (συζυγία), but at 332b3, where the sentence is repeated, συζυγία is employed, so that the two terms are in fact synonymous.[82]

Several other Aristotelian passages connected with the above display the latter word with a clear combinatorial meaning.[83] The most important of them, and very likely their common ancestor, is the general discussion in *Topica* B 7 of the ways the contraries combine. The passage begins (112b27–28) by asserting that "[. . .] the contraries conjoin (συμπλέκεται) with one another in six ways, but the conjoined ones (συμπλεκόμενα) give rise to contrariety in four ways"; then, after the six conjunctions have been enumerated, it is stated of some of them which should be object of choice and which of avoidance: the general conclusion that "in each combination (καθ' ἑκάστην συζυγίαν)" one is to be chosen and one to be avoided is finally reached. It is interesting to see which kind of contraries ARISTOTLE is actually considering, and which kind of conjunction is introduced. There are two objects, let us say friends (let us denote the term with F) and enemies (K) and two verbs, to do good (f) and to do harm (k). One can form object-predicate pairs in four ways, and then conjoin pairwise all the resulting combinations; the results sound like "to do good to friends <u>and</u> to do harm to enemies". The latter is the conjunction ARISTOTLE is referring to, being hence intented in the restricted sense. A diagrammatic representation of the above procedure is nearly obvious: put the four pairs as vertices of a square

$$\begin{array}{cc} \mathrm{Ff} & \mathrm{Fk} \\ \mathrm{Kf} & \mathrm{Kk} \end{array}$$

and then draw all possible sides or diagonals. By employing the same metonymy as PAPPUS in *Collectio* VII.16, one could observe that the result of six as the combinations of four objects two at a time can easily be obtained as a particular case of the general problem of finding the triangular number of given side.

[81] *In An. pr.*, pp. 283.3–284.18 (Wallies). Compare also a scholium to AMMONIUS' *In An. pr.* (Wallies, ix–x) concerning investigations about complex syllogisms carried out by GALEN. The scholium is presented and discussed in LUKASIEWICZ 1957, § 14.

[82] LSJ (*sub voce* συζυγία) actually records the two terms as synonymous.

[83] See also *Meteorologica* Δ 1, 378b11, *De sensu et sensato* i, 436a13, *De incessu animalium* ii, 704b19.

The lexical congruences just seen,[84] together with the above mentioned one between PAPPUS' passage and those of PORPHYRY and BOETHIUS, support the contention that the main technical terms were already fixed at early stages of research; the slight variation in the Aristotelian wording is very likely tied to lexical fluctuations typical of initial stages.

4.3. Symbols – conjectures

The last point to be stressed is that HIPPARCHUS could hardly have done without using a diagrammatic or "symbolic" representation. Concerning the way of representing the string of assertibles, it is enough to think of the very old tradition of figured numbers stemming from Pythagorean *psêphoi*-arithmetic.[85] It could also have been useful to represent the couplings among the several representatives of the assertibles. A sort of tree diagrams were used in antiquity in order to represent simple coupling relations among series of terms: they are mainly attested in (and implicit in some passages of) treatises dealing with musical theory, arithmetic ratios theory, or Aristotelian syllogistic (which is not a surprise, since those fields of research were strictly intertwined).[86] The problem is that no text up to much later than HIPPARCHUS' times makes any explicit reference to such representations, and many texts do not seem to presuppose them. Moreover, several occurrences in later commentators provide us with no indications about whether the diagrams were present in the original text or have been introduced at a later stage of the manuscript tradition. Despite the frustrating evidence, one could be reasonably confident that "the arithmeticians" had *reduced* the problem to a form more manageable for them both by operating by way of abstraction and by introducing a series of combinatorial and diagrammatic computing devices. This is the more interesting since the definiteness requirements in the scope of the connectives can be construed, as we have seen above, as being correlated to (and maybe in part a consequence of) the use

[84] Cfr. also SEXTUS EMPIRICUS, *Adv. Math.* viii.175. Recall moreover that συζυγία and the related adjective find their place in the mathematical and astronomical lexicon in a variety of meanings. Cfr. the "conjugate" diameters and sections in APOLLONIUS' *Conica* (e.g. *definitiones primae* 6 and props. I.60, II.17), the "conjugate terms" in HYPSICLES' *Ascensiones* (DE FALCO-KRAUSE 1966, pp. 35.40–46, 36.53, 39.147), the standard sense of "pair", "coupling" in IAMBLICHUS, *In Nichom. arithm. introd.* (see the index in PISTELLI's edition, *sub voce*), and of course the astronomical syzygies.

[85] The primary reference, with analysis of the extant evidence, is BECKER 1936. A more recent discussion, proposing several reconstructed theorems, is in KNORR 1975, chap. V. The diagrams in THEON SMYRNAEUS (II century of our era), *Expositio rerum mathematicarum ad legendum Platonem utilium* (see e.g. pp. 31–33, 39–40, Hiller), represent the figured numbers as arrays of letters.

[86] Cfr. e.g. THEON SMYRNAEUS, *ibidem*, pp. 57, 58, 64, 68, 69, 87 (Hiller). Some acquaintance with hierarchical structures is presupposed in the Porphyrean *scala praedicamentalis*, which codifies the Aristotelian doctrine of subordinations among the various genera and species, even if in PORPHYRY's *Isagoge* no explicit reference is made to diagrammatic structures. A diagram similar to those attested in THEON SMYRNAEUS is attached to BOETHIUS' passage quoted in point 4.2.2, item *b*) above.

of a "symbolic"/diagrammatic representation. "Symbolic" reasoning could in fact have forced some details of the translation of CHRYSIPPUS' claim into mathematical language, and remnants of such a reasoning could be read behind the slightly strained interpretation of the connectives expounded in Sect. 3.2: in particular, the idea of letting the negation step over some of the initial καὶ can present itself as "natural" once a sort of "symbolic" representation is employed.

We could wonder whether we are allowed to skip the quotation marks in the above term "symbolic", and regard HIPPARCHUS' strategy, *as reconstructed above*, as truly representative of a symbolic approach. In fact, his calculations entail a move of abstraction unprecedented in ancient Greek mathematics, a step further than those abstractions underlying e.g. the representation of general magnitudes or of numbers by line segments, or the geometric models employed in mathematical astronomy. HIPPARCHUS' practice of astronomical modelling must nevertheless have played a role in his setting up what we would call an *isomorphic model* of the process of conjoining assertibles. The gist of the model was its strikingly syntactical character: as its sole objects mere *entia rationis* – deprived of any extensional, i.e. geometrical, or linguistic or numerical reference – remained, and the operations on them. HIPPARCHUS' combinatoric calculations were in fact performed on the following "objects":

(a) representatives, very likely letters, of the ἀξιώματα, the basic entities of the problem;
(b) operations on the representatives unambiguously corresponding to the operations on the basic entities, i.e. some device, analogous to our brackets, identifying the several conjunctions among the assertibles (the ambiguities in the Stoic prescriptions pointed out above were already resolved at this stage).

The presence of representatives of the *operations* on the basic representatives is crucial: symbols are best characterized by their being anything that can be acted upon, and by the deplacement of the focus of interest from them to the operations on them. The latter acquire such an "ontological" dignity as to be represented on the same footing as the basic symbols. A further step of abstraction is required to reach the result: to move to a purely arithmetical environment, i.e. to manipulations of numbers, forgetting symbols. After all, a number was sought for. The last step was routine, and has to do with the essence of arithmetic as a science. The step before it is of course the most important in our perspective.

To appreciate how far are HIPPARCHUS' representatives removed from the data of the problem, consider the steps of abstraction that are necessary as preliminaries to the calculation of the right numbers. First, the Stoics had to extract the general concept of ἀξίωμα, considering it as an independent object of investigation. Recall in fact that for the Stoics the ἀξιώματα, as particular λεκτά, are incorporeal *entities*.[87] Second, the Stoics themselves invented the "modes" as generic representatives of particular arguments. The assertibles within the modes were denoted by ordinals according to their order of apparition in the argument. The same ordinals were used for assertibles appearing more than once in the same argument, and it is clear that the choice of ordinals was dictated

[87] DIOGENES LAERTIUS, *Vitae Phil.* vii.57. See also SEXTUS EMPIRICUS, *Adv. Math.* x.218.

by requirements of univocal denotation, of ordering conservation, of repeatability of the assertibles, and by the need of expressing the arguments in natural language. The stenographic character of the ordinals in the modes is clear from the so-called "mode-arguments", as e.g. "If Plato is alive, Plato breathes. But the first. Therefore the second".[88] HIPPARCHUS was thus provided by his very logical sources with a first step towards using a syntactical representation of the assertibles involved in the problem.[89] Third, from mathematics HIPPARCHUS knew how to use letters to represent geometric entities or monads, and he could have been induced to give up the Stoic ordinals by the simple observation that all ten assertibles in CHRYSIPPUS' problem *are allowed to be represented by the same representative repeated 10 times*, since problems of ordering and of univocal identification of the assertibles are irrelevant there. To be sure, he could have made use of natural language, by taking "the first", "the second", . . . and the conjunction "and" as a sort of abstract representatives of themselves, but a better choice was dictated by the nature of the problem and by a long-standing mathematical practice. The question is: were the representatives intended as true symbols? or better said, could they have been intended as such?[90]

Clearly, the present discussion is strictly related to the one concerning the "algebraic" character of some pieces of ancient mathematics, e.g. the Old Babylonian corpus or DIOPHANTUS' *Arithmetica*. It is useful to resume the terms of the debate, also because of recent, renewed interest in the issue.[91] I take as reference a number of criteria proposed by MAHONEY, and recently resumed and supplemented by HØYRUP, in order to assess the algebraic character of Old Babylonian mathematics. MAHONEY's criteria,[92] with HØYRUP's supplement, for a truly *algebraic approach* are as follows:

(i) the use of "a symbolism for the purpose of abstracting the structure of a problem from its non-essential content"; for instance, the symbolism must be unambiguous;
(ii) the search for "the relationships (usually combinatory operations) that characterize or define that structure or link it to other structures";
(iii) being "totally abstract and free of any ontological commitments";
(iv) moreover, the approach should be analytical (HØYRUP's supplement).[93]

The above discussion should have shown that any syntactical representation HIPPARCHUS had adopted refining Stoic ordinals completely fulfils requirements (i) and (iii); for instance, the representation was clearly free of ambiguities, and the representatives were removed several steps of abstraction away from the "objects" they were intended

88 DIOGENES LAERTIUS, *Vitae Phil.* vii.77. Hence, the ordinals in the modes act both as schematic letters and as mere abbreviations of particular assertibles (see the discussion in BOBZIEN 1999, pp. 129–131).

89 Recall also that the Stoics put syntactical features at the very heart of their systematization of dialectics.

90 This question is of course connected with the use of the so-called "arguments by impossibility" in history of science. Negative, and in my opinion definitive, assessments of arguments of this kind can be found in HØYRUP (forthcoming) and in NETZ (forthcoming).

91 See HØYRUP 2002, Ch. VII.

92 MAHONEY 1971, p. 372.

93 HØYRUP 2002, p. 279.

to represent. As for (iv), it is apparent that the idea behind formula (*) is purely analytical, namely to represent the solution, considered as already done, as a combination of previous, factually well-defined steps. An analytical approach is made easier by the recursive character of the problem, and by the fact that the recursive chain going back from $s(i)$, with $n > 2$ fixed, to $s(2)$ is finite. HIPPARCHUS' own conceptualization of what (*) expresses in modern symbolic form must have been of this kind. The related synthesis consists in performing the steps of calculating the successive $s(i)$, $i \leq n$, in succession. Concerning (ii), observe that the procedure solves at least one *class* of problems, namely those obtained by replacing the 10 assertibles in CHRYSIPPUS' claim with a generic number of assertibles. Moreover, the combinatorial procedures underlying, as we have seen, the calculations are applicable to, and indeed were used to solve, several problems coming from disparate fields, in particular from dialectics.

The crucial point with (i), and this brings us back to our initial question, is the meaning of the term "symbolism". NESSELMANN proposed in 1842 a distinction between rhetorical, syncopated, and symbolic algebra, according to the nature of the abbreviations employed (if any).[94] Rhetorical algebra has no abbreviations, syncopated algebra has them, but they are merely stenograms that can be (and actually must be for a complete understanding) expanded in natural language expressions. Symbolic algebra employs, as modern algebra does, abbreviations enjoying a completely independent ontological status and which are well-defined once the operations to be performed on them are specified – i.e. it deals with objects such as those described under items (a) and (b) above. If we accept this, it is difficult not to conclude that a truly symbolic approach was developed by HIPPARCHUS to solve CHRYSIPPUS' problem. But we cannot suppose that HIPPARCHUS was aware of the extent of his move if such an approach was circumscribed to the solution of one single "combinatorial" problem. It is necessary that such a habit of reasoning had been applied to solve other problems, and possibly transferred to different fields of research (in case creating those fields).

HIPPARCHUS is recorded by the *Fihrist* to have written a mysterious work *On the Subdivision of Numbers* (could this treatise have dealt with partitions of integers?). The same source ascribes to him "*On the art of algebra*, known by the title of the *Rules*".[95] Could these works have contained the calculations leading to the numbers? could they have contained more general combinatorial prescriptions, or applications to other fields? Maybe. I suspect that a general symbolic approach, conflating his high-level refine-

[94] NESSELMANN 1842, p. 302, quoted e.g. in KLEIN 1968, p. 146, HEATH 1921, vol. 2, pp. 455–456, and HØYRUP 2002, p. 298.

[95] IBN AN-NADIM 1871–72, p. 269. Both treatises were commented at the end of the X century by ABU'L-WAFA' AL-BUZAJANI (see IBN AN-NADIM 1871–72, p. 283 and ABU'L-WAFA' 1971, p. 126). The text of the *Fihrist* is in a bad status at that point, and it has been suggested that those treatises should be ascribed to DIOPHANTUS, who follows HIPPARCHUS in the catalogue (cfr. note 97 on pp. 54–55 in SUTER 1892). In view of the existence of the above commentaries, and considering the arguments developed in the present paper, it is in my opinion unreasonable to deny the Hipparchian authorship of the treatises. In SESIANO 1998 it is suggested that some Mesopotamian techniques for solving problems algebraic in character could have been introduced in Greek mathematics by HIPPARCHUS' "algebraic" treatises. HIPPARCHUS' borrowings from Babylonian astronomical data are well known (see e.g. TOOMER 1978).

ments of combinatorial techniques with Mesopotamian "algebraic" procedures, could have been deployed by HIPPARCHUS in those and possibly other treatises. The conceptions and the mathematical techniques needed to achieve the combinatorial calculations explained above seem to be more conducive to the elaboration of a general symbolic perspective than the kind of quasi-algebraic proofs, characterised by suggesting rules by means of numerical examples, typical of Old Babylonian mathematics.[96] On the other hand, the problems constituting the bulk of the Mesopotamian tradition are the ideal field where a symbolic approach to mathematics can find its applications. A crucial point must have been, as we have seen, using symbols to represent also the operations performed on the basic objects of interest, allowing second-order concepts to creep into mathematical practice.

Combinatorics would hence have been only a facet of a general project, though an important facet insofar as providing a founding habit of reasoning for the whole project. That facet was apparently not developed by later mathematicians, contrary to what happened to the cognate field which took its canonized form in DIOPHANTUS' *Arithmetica*.[97] The real conceptual revolution underlying HIPPARCHUS' strategy, namely the truly symbolic character of his representation, was lost in the subsequent developments, leaving as only residues DIOPHANTUS' stenograms.[98] It must be stressed that this is a far cry from asserting that HIPPARCHUS had developed a form of symbolic algebra, that left no traces other than shadows of a conjectured existence. What I think can be reasonably maintained is that, with the general strategy underlying HIPPARCHUS' calculations, the germs of a new approach were all *there*, but that, maybe, he was placed too high, sitting on the shoulders of himself: he was the only one allowed to catch dim glimpses of the new continent.

5. Conclusions

I am inclined to regard as very likely that HIPPARCHUS' calculations have not been merely an episodical performance of a great mathematician. They suggest instead the existence in his times of a reasonably large supply of combinatorial techniques. To be sure, I am not maintaining that specific written expositions had existed devoted to the latter; rather, such techniques are very likely to have been included in the general background a mathematician had to master with complete ease. The evidence presented above supports

[96] Reservations on the use of a true symbolism in Babylonian mathematics are expressed in HØYRUP 2002, pp. 281, 298–299 (cfr. also the very sharp – negative – position on the subject of Old Babylonian "algebra" in MAHONEY 1971).

[97] General principles underlying the process of selection in the transmission of ancient mathematical practices are proposed in NETZ (forthcoming).

[98] But recall that the Arabic text of the *Arithmetica* contains no abbreviations (it is rhetorical algebra), even if this is usually taken to be a translator's choice, and that it is at all unclear the extent of the use of abbreviations in the original redaction of the *Arithmetica*. It is interesting to observe that KLEIN, on the basis of NEUGEBAUER's findings on Old Babylonian mathematics, already conjectured that "[t]he *Arithmetic* of Diophantus may [. . .] itself refer back to a pre- and *non*-Greek, perhaps even a "symbolic", technique of counting" (KLEIN 1968, p. 147).

at least the plausibility of the assumption that HIPPARCHUS had grasped the recursive character of the calculations, and that he was absolutely confident with the "concept" of calculating combinations of objects and with the related computing techniques (possibly employing plane and solid numbers as working tools in the first steps and subsequently letting the machine of recursive proofs turn, as we have seen above). Assuming such a background the numbers reported by PLUTARCH are not difficult to calculate, even in the case in which no combinatorial shortcut is set up in order to reduce the number of terms in the summation (*) from 511 to 41 (but I am strongly convinced that combinatorial factors have actually been used). I calculated with paper and pencil $s(2)$, $s(3)$, ... up to $s(11)$, using combinatorial factors to pass from unordered partitions to ordered ones, but otherwise doing computations in the dullest possible way: it took half an afternoon, checking included. Even without using combinatorial factors, I estimate the calculations could have been worked out in a few days. Of course, the real problem lies in the time needed to understand what one has to do, but my contention is precisely that HIPPARCHUS knew it, since combinatorics had been developed to some extent before him, and that the real point had been to formulate a well-defined problem out of CHRYSIPPUS' claim.

A very interesting picture is thus emerging of the interactions between dialecticians and mathematicians, exponents of fields of research considered widely separated and substantially uncommunicating. Such a picture is confirmed by PLUTARCH's words: "Chrysippus is refuted by all the arithmeticians, among them Hipparchus himself who proves that his error in calculation is enormous [...]". My interpretation of the sentence is that different arithmeticians had given the problem different solutions, in any case grounded on some combinatorial analysis, presumably tied to the several interpretations/extensions CHRYSIPPUS' words can be provided with. Such a quick and plenary answer suggests a protracted interaction between arithmeticians and dialecticians about the subject at issue.[99] In fact, although not mathematically enlightening, the Aristotelian texts presented above attest to the fact that combinatorial reasoning was a matter of course in his times, and suggest that the first steps in the field of combinatorics, stemming from problems which naturally arise in a logical context, can be traced back to his times. After this, the Stoics, with their general conception of logic and, most notably, their attention to syntactical problems and to ambiguities of expression, created the ideal *humus* where combinatorics could spontaneously grow up. Moreover, I conjecture that a distinguished mathematician could have been interested in the subject, thereby giving it a powerful impulse. We have encountered the name of APOLLONIUS at several places and seen how some of his minor works actually attest to their author as being interested in researches in which combinatorial manipulations have a relevant part. I could add to the above items the treatise on *Unordered Irrationals*, in which problems of classification were very likely at issue.[100] HIPPARCHUS could then have constituted a

[99] *Contra* this picture, Reviel NETZ suggested me that the expression "all the arithmeticians" could have been a rethorical expedient by PLUTARCH in order to denote "the science of arithmetics". Such a view could be supported by the fact that the reference to "all the arithmeticians" is absent in the version of the passage contained in *Quaestiones Conviviales*.

[100] See e.g. JUNGE-THOMSON 1930, p. 119. Investigations of a combinatorial kind are advocated by VITRAC (EUCLIDE 1998, see especially pp. 51–62 and 64–67) as underlying the classificatory effort in book X of the *Elements*. The fact that ancient sources report no investigations on this issue,

point of synthesis of two independent mainstreams interested in combinatorial results, the geometrical side stemming from APOLLONIUS' refined researches and the dialectical side, going back to ARISTOTLE's logical works but receiving a decisive spur from the Stoics. After that, rain has begun: the selection in the course of textual tradition has been so strong as to convince some of us that "the Greeks took no interest in these matters".

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as well as on the combinatorial aspects underlying e.g. Menelaus' theorem is a further indication that the interest in such problems had faded very quickly. The residual interest was apparently tied to commentaries on dialectical treatises, as most of our later testimonies attest.

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