Students in my classes respond with interest and enthusiasm to the game of J-decompositions. While playing, they develop an appreciation for, and a hint of understanding behind, the process of partial fraction decomposition. Since the proof outlined above involves an interesting application of the division algorithm, it could provide an entertaining and instructive supplement to an abstract algebra or number theory course.

References

1. Nathan Jacobson, *Basic Algebra I*, 2nd ed., W. H. Freeman, 1985.
2. Roland E. Larson, et. al., *Calculus with Analytic Geometry*, 6th ed., Houghton Mifflin, 1998.

Euler's Theorem for Generalized Quadrilaterals

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In [1], J. B. Dence and T. P. Dence gave a proof of a theorem of Euler on convex quadrilaterals $ABCD$ (see Figure 1).

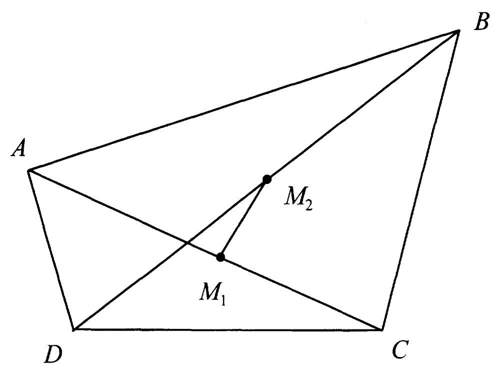

Figure 1.

Theorem. Let M_1 and M_2 denote the midpoints of AC and BD, respectively. Then

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{M_1M_2}^2.$$

(In other words. the sum of the squares of the sides is equal to the sum of the squares of the diagonals, increased by four times the square of the segment joining the midpoints of the diagonals.)

Actually, Euler's theorem is valid for a much broader class of quadrilaterals, which I refer to as generalized quadrilaterals. A *generalized quadrilateral* $ABCD$ in R^n is the figure that has A, B, C, and D (any points in R^n) as *vertices*, AB, BC, CD, DA, as *sides*, and AC and BD as *diagonals*. The vertices A, B, C, D need not be coplanar and need not be distinct. In Figure 2 are two examples of generalized quadrilaterals that are not quadrilaterals in the usual sense of the word. The sides are represented by solid lines, the diagonals by dashed lines.

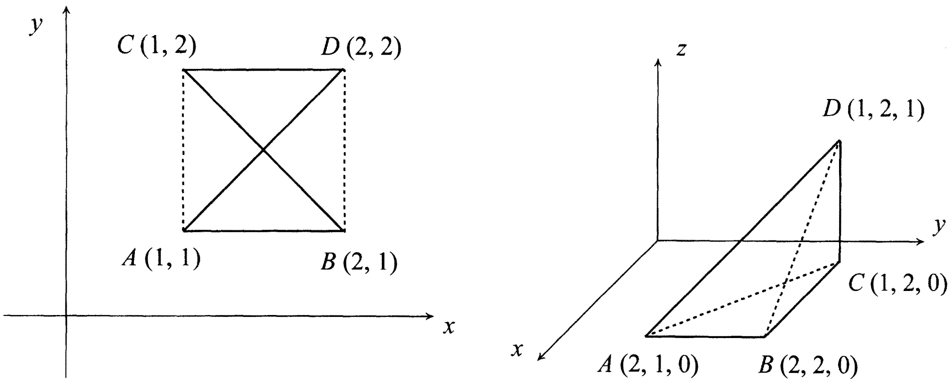

Figure 2.

The proof of Euler's theorem is most easily effected by using vectors. We identify a point X in R^n with the vector $\overrightarrow{OX}$ (O is the origin), and for convenience we write $X^2 \equiv X \cdot X = |X|^2$. This notation is felicitous inasmuch as $(X \pm Y)^2 = X^2 \pm 2\,X \cdot Y + Y^2$. Note that $\overline{XY}^2 = (Y - X)^2$.

Proof of Euler's theorem.. Suppose $ABCD$ is any generalized quadrilateral in R^n, and M_1 and M_2 are the respective midpoints of the diagonals AC and BD. Then

$$M_1 = \tfrac{1}{2}(A + C), \quad M_2 = \tfrac{1}{2}(B + D), \text{ so } M_2 - M_1 = \tfrac{1}{2}((B - A) + (D - C)).$$

We have to show that

$$(B - A)^2 + (C - B)^2 + (D - C)^2 + (A - D)^2 = (C - A)^2 + (D - B)^2 + 4(M_1 - M_2)^2,$$

which reduces to

$$(C - B)^2 + (A - D)^2 = (C - A)^2 + (D - B)^2 + 2(B - A) \cdot (D - C).$$

It is a simple matter to verify that each side of this equation is equal to

$$A^2 + B^2 + C^2 + D^2 - 2\,A \cdot D - 2\,B \cdot C. \qquad \blacksquare$$

Reference

1. Joseph B. Dence and Thomas P. Dence, A property of quadrilaterals, *College Mathematics Journal* **32** (2001) No. 4, 292–294.

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