DE LA VALLÉE POUSSIN ON APPROXIMATIONS

Leçons sur l'Approximation des Fonctions d'une Variable Réelle. By C. de la Vallée Poussin. Paris, Gauthier-Villars, 1919. vi + 150 pp.

In 1908, M. de la Vallée Poussin was engaged in a study of certain questions in the theory of approximate representation by means of polynomials and finite trigonometric sums. In discussing the representation of a function whose graph is a broken line, he was led by the limitations of the formulas which he employed to make the following observation:*

"Il serait très intéressant de savoir s'il est impossible de représenter l'ordonnée d'une ligne polygonale avec une approximation d'ordre supérieur à $1 : n$ par un polynôme de degré n."

This sentence, occurring incidentally, with no particular emphasis, in a footnote attached to some supplementary pages following a memoir on a different phase of the subject, has been the direct or indirect occasion of about thirty published articles and memoirs which I call to mind at the moment,† and probably of numerous others which would be disclosed by a thorough search of the literature. The resulting theory is concerned not only with inner limits of approximation, as contemplated in the passage just quoted, but also with outer limits for the approximation attainable by various means, the degree of convergence of Fourier's and other series, and a variety of related topics. While this theory also has other origins and beginnings,‡ it is a fact that the numerous papers referred to can in each case be traced back in the personal experience and associations of the authors at least partly to de la Vallée Poussin's formulation of the problem.

The book under review, a monograph in the Borel series, is a summary, not of the entire literature of the subject, for citations are few and informal, but of its principal results, in systematic and often novel presentation. It is particularly appropriate that the man to whom the theory chiefly owes its inception, a man who has made essential contributions to it at various stages of its development, should now have performed the service of setting it before the general reader in its most attractive form.

Of the contents of the book in detail I shall speak more briefly than would otherwise be desirable, for the reason that I had occasion to refer to it extensively, as well as to the other literature, in an expository paper recently published in this BULLETIN.§

* de la Vallée Poussin, *Note sur l'approximation par un polynôme d'une fonction dont la dérivée est à variation bornée*, BULLETINS DE L'ACADÉMIE DE BELGIQUE, CLASSE DES SCIENCES, 1908, pp. 403–410; p. 403, footnote.

† It will be a convenience to the reviewer in the present connection if he may be allowed the occasional use of the pronoun in the first person.

‡ See particularly Lebesgue, *Sur la représentation approchée des fonctions*, RENDICONTI DI PALERMO, vol. 26 (1908), pp. 325–328.

§ D. Jackson, *The general theory of approximation by polynomials and trigonometric sums. Chicago Symposium paper.* March 25, 1921. This BULLETIN, vol. 27, Nos. 9–10, June–July, 1921, pp. 415–431.

An Introduction deals with Weierstrass's theorems on the uniform approximation of continuous functions by means of polynomials and trigonometric sums, and with simple properties of the "modulus of continuity" of a function.* From certain references (later in the book) to passages in which I have made use of the latter, the reader might obtain the impression that it was, in the words of the White Knight, "my own invention." As I have said on occasion elsewhere, however, it came to me from the work of Lebesgue,† who used it in essentially the same way. I do not know what its earlier history may have been, if any.

Chapters I–III are concerned with the approximate representation of a given function by the partial sums of its Fourier series, the Fejér means of the Fourier series, and special trigonometric sums designed to give more rapid convergence in certain cases. In the main, the discussion is aimed at the determination of outer limits for the error of the approximation; an important part of the second chapter, however, is devoted to the theorems of S. Bernstein on the inner limit of the approximation of $|x|$ by a polynomial and on the derivative of a trigonometric sum.‡ Among the notable features of the treatment may be mentioned, in the case of the Fourier series, the simultaneous consideration of the original series and its conjugate, and in the case of the Fejér mean, the representation of the mean by the formula

$$\sigma_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(x + \frac{2t}{n}\right)\frac{\sin^2 t}{t^2}\,dt.$$

A generalization of the same formula is used for the purposes of the third chapter. In consideration of the acknowledgment made in the preceding paragraph, I may perhaps be allowed to say, in connection with certain comments on my thesis (e.g., pp. 43, 52), that its methods were considerably improved and its results extended in papers of mine published subsequently in the TRANSACTIONS.§

The author's methods are particularly convenient for the determination of the numerical constants contained in some of the error formulas. Take, for example, the theorem that if a function satisfies a Lipschitz condition with coefficient λ it can be represented by a trigonometric sum of order n with a maximum error not exceeding

$$\frac{K\lambda}{n},$$

* By the *modulus of continuity*, or *modulus of oscillation*, $\omega(\delta)$, of a function $f(x)$, is meant the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$.

† Lebesgue, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 38 (1910), pp. 184–210; see, e.g., p. 202. Also, Lebesgue, *Sur les intégrales singulières*, ANNALES DE TOULOUSE, (3), vol. 1 (1909), pp. 25–117; p. 114 and elsewhere.

‡ Cf. the Symposium paper cited above.

§ *On approximation by trigonometric sums and polynomials*, TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), pp. 491–515; *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS, vol. 14 (1913), pp. 343–364.

where K is an absolute constant. In one of the papers just referred to, it was shown that K can have the value 2.90, but certainly not a value less than $\pi/2$. Gronwall* later justified the use of the value $K = 2.76$. The author's calculation on page 45, applied to the case of a simple Lipschitz condition, and supplemented by the use of the relation†

$$\int_0^\infty \frac{\sin^4 t}{t^3}\,dt = \log 2,$$

gives at once

$$K = \frac{12 \log 2}{\pi} = 2.65.$$

Dr. Gronwall has suggested to me in correspondence a method by which I believe it would be possible to reduce the value considerably further, but I have not carried through the computation. From page 46 of the text it can be deduced readily that if $f(x)$ has a first derivative satisfying a Lipschitz condition with coefficient λ, it is possible to obtain an approximation with an error not exceeding $6\lambda/n^2$. The constant 6 is an appreciable improvement over the value 9 (more closely, $2.90^2 = 8.41$, or, by inference from Gronwall's work, $2.76^2 = 7.62$) previously published.

The remaining chapters are no less significant, but will be dismissed very briefly. Chapter IV presents general theorems on the inner limit of approximation by trigonometric sums; the fifth chapter deals with outer and inner limits for polynomial approximation; the next two are devoted to the Tchebychef theory of approximation by polynomials and trigonometric sums, with some of its recent extensions. The last three chapters contain a study of the approximate representation of analytic functions of a complex variable.

A number of misprints have been noted, particularly in the early pages. For the most part they are not such as to cause any serious inconvenience. In the footnote on page 3, however, the reference should be to the BULLETIN DES SCIENCES MATHÉMATIQUES.

The book needs no higher praise than a statement that it is worthy of its author. It is qualified to give pleasant hours to any student of mathematics who turns its leaves, and will be of inestimable value to anyone who has occasion seriously to study the subject with which it deals. If many a mathematical treatise seems to place itself under the motto, quoted in a different connection not long ago by a reviewer in these pages, "All hope abandon ye who enter here," it would be appropriate to inscribe on the fly-leaf of a text by M. de la Vallée Poussin:

"The crooked shall be made straight, and the rough places plain."

DUNHAM JACKSON.

* *On approximation by trigonometric sums*, this BULLETIN, vol. 21 (1914–15), pp. 9–14.

† I do not remember seeing a proof of this relation in print; I am personally indebted for various demonstrations of it to Messrs. Gronwall, Landau, M. Riesz, and I. Schur. See, however, Gronwall, loc. cit., p. 14.