Matter and Radiation

Most of the information we gather regarding astronomical bodies is through the radiation received from them. It is therefore essential for us to learn both the production of radiation as well as the interaction of radiation with matter.

The form of radiation that we are most sensitive to at present is electromagnetic, and almost the entire electromagnetic spectrum has been used by astronomers, to the technologically feasible level of sensitivity. Astronomy at some bands, notably ultraviolet, X-rays and low-energy gamma rays, cannot be carried out from the ground because our atmosphere absorbs radiation at these wavelengths (see fig. 2). One resorts to space platforms for observations at these wavelengths.

In India, strong groups and excellent instruments exist for carrying out ground-based observations at radio wavelengths (e.g. GMRT). At optical wavelengths, we have a number of telescopes of up to 2-m diameter. Small, but effective space payloads for astronomy at X-ray and Gamma-ray wavelengths have been flown by ISRO in the past, and the country is now gearing up to launch, in the year 2006, a major space mission ASTROSAT, which will carry four X-ray payloads and one ultraviolet/optical payload. New facilities have also just been commissioned for ground-based study of gamma rays at very high energy, above $\sim 10^{12}$ eV.

Figure 1: Giant Metrewave Radio Telescope (GMRT) near Pune, a radio interferometer consisting of 30 dishes of 45 m diameter each, spread over distances spanning $\sim$ 30 km, is the world's largest radio telescope operating at metre wavelengths

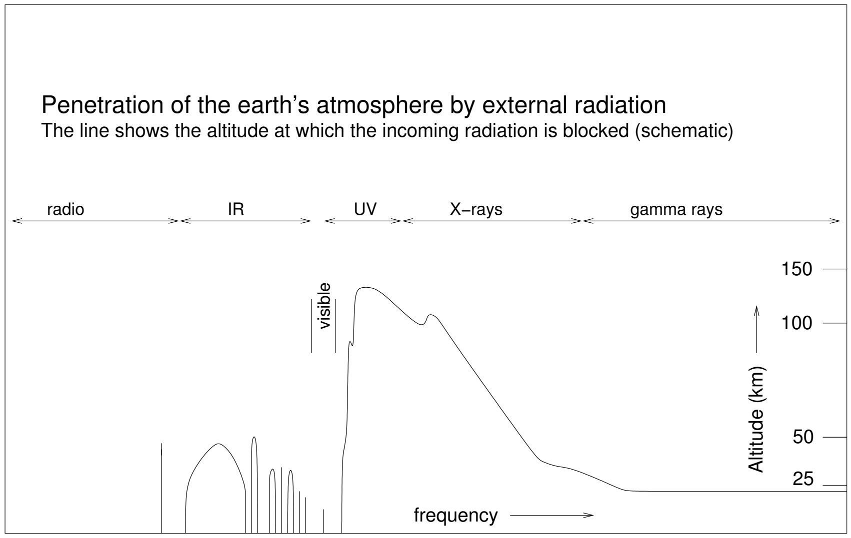

Figure 2: Height above which astronomy is possible at different wavelengths

Interaction of matter and radiation

Radiation passing through matter can be modified due to several reasons:

- Radiation could be absorbed, the energy going into excitation of discrete atomic or molecular energy levels (bound-bound transitions), or ionisation/dissociation (bound-free transitions), or even in free-free transitions between continuum levels which increase the kinetic energy of particles (heating).
- Radiation could cause stimulated emission, which is a process inverse to the absorption mentioned above and exists for all transitions – bound-bound,

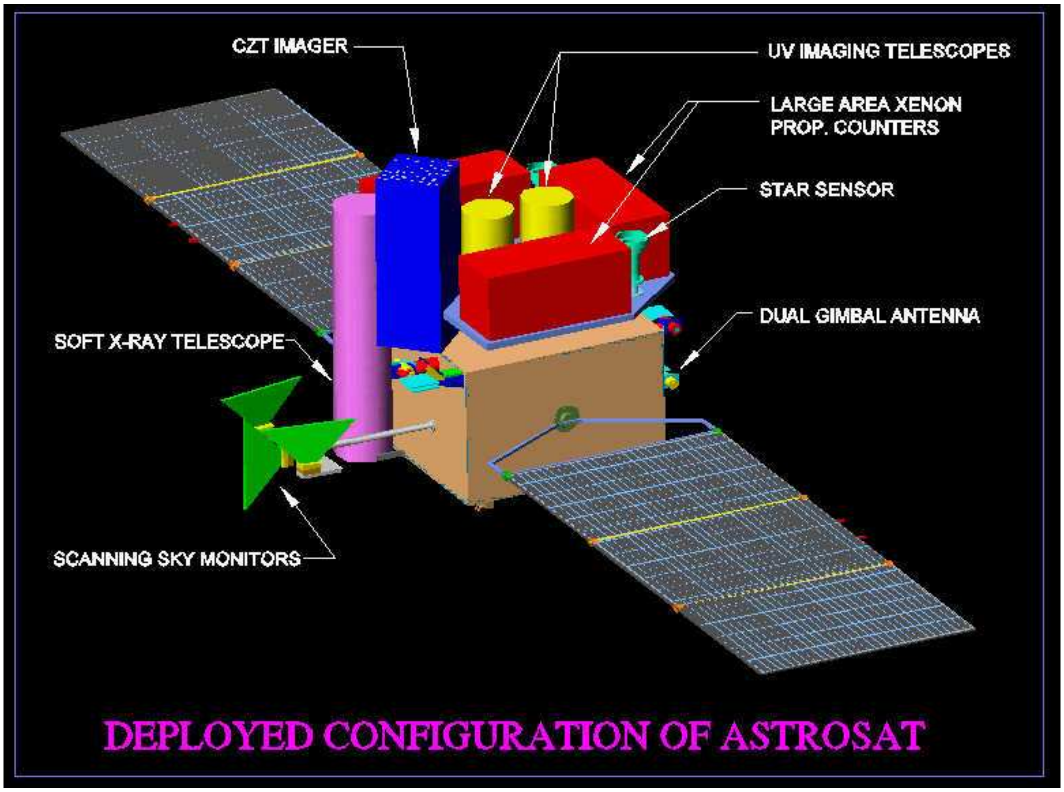

Figure 3: A schematic representation of the deployed configuration of the Indian multi-wavelength astronomy satellite ASTROSAT planned for launch in the year 2006

free-bound and free-free. This will add extra energy to the radiation at the expense of the excitation or kinetic energy of matter.

- Spontaneous emission in all three categories may also take place and add to the passing radiation.
- Radiation may be scattered by matter, either imparting kinetic energy to the matter present, or gaining energy at the expense of kinetic energy.
- If the radiation is sufficiently energetic, it could produce particle-antiparticle pairs (e.g. $e^{\pm}$ at photon energies > 1 MeV) and be destroyed in the process. The inverse process, if free pairs are available, would add photons of the corresponding energy.

So if we follow the passage of a ray through a medium, there will be processes which will remove photons from the ray path by absorption, destruction or scattering, and there will be processes which will add photons to it such as spontaneous or stimulated emission or pair annihilation. In astrophysics, we lump together these effects in two quantities which characterize the medium: the absorption coefficient α_ν and the emission coefficient j_ν. Both are functions of frequency ν. j_ν is defined as the amount of energy created by spontaneous emission per unit volume per unit time per unit frequency interval per unit solid angle, at the frequency ν. α_ν is the amount of energy removed from the beam per unit volume per unit time per unit frequency interval per unit solid angle per unit incident specific intensity at the frequency ν. Since stimulated emission is an inverse process to absorption and is proportional to the incident intensity like absorption is, its effects are also taken into account through α_ν. If stimulated emission dominates over loss of intensity due to absorption and scattering, α_ν becomes negative, as is the situation in lasers. Astronomers sometimes use a related quantity called *opacity* $\kappa_\nu = \alpha_\nu/\rho$, which is the absorption coefficient per unit mass.

Accounting for the radiative energy removed from and added to the propagating beam one can write the Radiative Transfer equation:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

which can be rewritten as

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

where $S_\nu = j_\nu/\alpha_\nu$ is called the *source function* of the material and

$$\tau_\nu = \int_0^s \alpha_\nu(s')ds'$$

is called the *optical depth*.

The solution of the transfer equation yields

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

where I_ν^0 is the incident intensity on the material and I_ν is the emergent intensity after passage through the material. The first term gives the exponential decay of background radiation due to absorption effects and the second term corresponds to the amount of radiation added by the material, including effects of self-absorption.

If $\tau_\nu \ll 1$, the medium is called "optically thin", while if $\tau_\nu \geq 1$ it is called "optically thick". As τ_ν becomes large, the incident background radiation is practically fully extinguished, and the intensity observed from the source approaches S_ν.

According to thermodynamic principles, any material immersed in a blackbody radiation bath at a temperature T, at equilibrium, must attain the same temperature T. This imposes a strict condition on the source function for a thermal body:

$$S_\nu = B_\nu$$

where

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

is the *Planck function*. So the radiation emitted by a thermal body approaches that from a blackbody at the same temperature as the optical depth τ_ν becomes large, and this is the maximum radiation this body can emit at that frequency. If the optical depth is smaller, the radiation emitted by the body is also smaller, but of course any background radiation can then shine through the body more easily. Note that τ_ν depends on frequency, so a body could be optically thick at some frequencies and optically thin at others at the same time. For example a body that produces spectral lines has higher optical depth at line frequencies and lower optical depth away from line frequencies. A true blackbody spectrum will be generated only if the body is optically thick at *all frequencies*.

In this context, it is often convenient to express the actual intensity I_ν at a given frequency in terms of an equivalent blackbody temperature $T_{b,\nu}$, called the *brightness temperature*. This is the temperature that a blackbody needs to have to produce the intensity I_ν at the frequency ν. If the radiation is a true blackbody then T_b becomes independent of frequency and equal to the actual temperature of the blackbody. From the transfer equation one can then easily conclude that at any given frequency ν the passage of radiation through a medium results in a net absorption (i.e. emergent intensity I_ν less than incident intensity I_ν^0) if the temperature of the medium T is less than the brightness temperature of the background $T_{b,\nu}^0(I_\nu^0)$ and in a net emission (i.e. $I_\nu > I_\nu^0$) if $T > T_{b,\nu}^0$.

The thermodynamic principle stated above also results in an equivalent relation between the emission and absorption probabilities described by quantum mechanics. If we consider two energy levels 1 and 2, at energies E_1 and $E_2 = E_1 + h\nu$, then the transition rates between these levels are described by three Einstein coefficients:

A_{21}	=	transition (2→1) probability per unit time for spontaneous emission
B_{12}	=	transition probability (1→2) per unit time for absorption, per unit incident intensity at frequency ν
B_{21}	=	transition probability (2→1) per unit time for stimulated emission, per unit incident intensity at frequency ν

The thermodynamic relations between these coefficients are

$$g_1 B_{12} = g_2 B_{21}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

where g_1 and g_2 are the statistical weights of the two levels respectively. These are called the Einstein relations.

In terms of these coefficients, the emission and absorption coefficients defined before can be written as

$$j_\nu = \frac{h\nu}{4\pi} n_2 A_{21}$$

and

$$\alpha_\nu = \frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})$$

where n_1 and n_2 represent the population at the lower and the upper levels respectively. In reality a spectral transition is not infinitely sharp but will have a certain width depending on the lifetime of the levels, as well as the velocity distribution of the atoms. In order to account for this a normalised profile function $\phi(\nu)$ ($\int_0^\infty \phi(\nu) d\nu = 1$) should be multiplied to the right hand side of both the above relations.

One clearly sees from the above that if the level populations of the atom are such that $n_2 > n_1(g_2/g_1)$, then α_ν becomes negative due to the dominance of stimulated emission. The corresponding τ_ν becomes negative, and the Radiative Transfer equation predicts an exponential amplification of the incident background radiation at that frequency. This is the so-called "Amplification by Stimulated Emission of Radiation", namely LASER or MASER effect. Given a thermal distribution of level populations this can never be achieved:

$$n_2 = \frac{g_2}{g_1} n_1 e^{-h\nu/kT}; \quad \frac{n_2}{n_1} \to \frac{g_2}{g_1} \text{ as } T \to \infty$$

The population at the upper level must exceed this for amplification to occur. Such a situation is called "population inversion" and is manifestly "non-thermal".

It is not uncommon in Astronomy to encounter situations where such population inversion is naturally established. In ionised regions where strong ionising photon flux from, say, a nearby hot star is present, hydrogen atoms are first ionised, and then many electrons recombine at levels with very high principal quantum number. They then undergo gradual de-excitation, level by level, by spontaneous emission. But since the radiative lifetime of upper levels are longer than those of the lower levels, in a steady state the population in the upper levels exceed those at the lower levels, for a certain range of quantum numbers. Population inversion is also encountered in the cool envelopes of giant stars, where maser emission from molecules such as Silicon Monoxide are seen. Dense molecular clouds where young stars are being formed display maser transitions of Water, Methanol, Ammonia, OH radical etc.

Production of radiation

Let us now look at microscopic electromagnetic processes that lead to the production of radiation by matter, as well as scattering of radiation by single particles.

From classical electromagnetic theory one obtains the radiation field from an accelerated charge:

$$\mathbf{E}_{\rm rad}(\mathbf{r}, t) = \frac{q}{c}\left[\frac{\mathbf{n}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\right]$$

$$\mathbf{B}_{\rm rad}(\mathbf{r}, t) = [\mathbf{n} \times \mathbf{E}_{\rm rad}]$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, $\mathbf{v}$ being the velocity of the particle. $\mathbf{n}$ is the unit vector pointing from the particle to the observer and R is the distance between the particle and the observer. The expressions in square brackets are evaluated at the retarted time $(t - R/c)$. q is the charge of the particle. These give the instantaneous electric and magnetic fields experienced by the observer. The plane containing the direction of the electric field and the propagation vector $\mathbf{n}$ defines the plane of polarization of the radiation.

The power radiated per unit solid angle at a direction Θ with respect to the acceleration $\dot{\mathbf{v}}$ can be obtained from the above:

$$\frac{dW}{dtd\Omega} = \frac{q^2\dot{\mathrm{v}}^2}{4\pi c^3}\sin^2\Theta$$

for $\beta \ll 1$. Integrating over all angles, one obtains the expression for total power radiated (Larmor's formula):

$$\frac{dW}{dt} = -\frac{dE}{dt} = \frac{2q^2\dot{\mathrm{v}}^2}{3c^3}$$

For a relativistic particle with Lorentz factor γ this assumes the form

$$\frac{dW}{dt} = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2)$$

where $a_\perp$ and $a_\parallel$ are the components of acceleration perpendicular and parallel to the velocity respectively.

For a collection of charges, this can be written, if the particle velocities $\mathrm{v} \ll c$, as

$$\frac{dW}{dtd\Omega} = \frac{\ddot{D}^2}{4\pi c^3}\sin^2\Theta$$

and

$$\frac{dW}{dt} = \frac{2\ddot{D}^2}{3c^3}$$

where D is the dipole moment of the system of charges. This is called the Dipole approximation.

In general the magnitude or the direction or both of the radiation field $\mathbf{E}_{\mathrm{rad}}$ at the observer changes with time. Any arbitrary time dependent function such as this can be described as being composed of sinusoidal oscillations of different frequencies, each with a different amplitude and phase. This is called Fourier decomposition. The complex amplitude of a sinusoidal component with frequency ν is given by

$$\hat{E}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t) e^{i2\pi\nu t} dt$$

the power at that frequency component being $P(\nu) = |\hat{E}(\nu)|^2$. This $P(\nu)$ vs ν is what we normally call the "spectrum" of the radiation. There are some simple properties of the Fourier transform which allow us to guess the nature of the spectrum:

1. If $E(t)$ has a characteristic duration T then $P(\nu)$ is significant for frequencies $\nu \leq 1/T$. The sharper the spike in $E(t)$, the higher is the frequency content.

2. If $E(t)$ is periodic with a period T_0 then $P(\nu)$ only contains contributions at the frequency $\nu_0 = 1/T_0$ and its harmonics. If each pulse of the periodic signal has a width $\Delta T < T_0$ then $N_{\rm h} \sim T_0/\Delta T$ harmonics will be present in the spectrum.

Common radiation processes in astrophysics

Thermal Bremsstrahlung This relates to the radiation emitted by free electrons in an ionised plasma due to encounter with ions. This is one of the most common continuum radiation processes in astrophysics.

We can deduce the character of this radiation from the following simple considerations. Let b be the impact parameter in one such encounter between an electron of charge $-e$ and an ion of charge $+Ze$. We can approximate the encounter as an acceleration

$$a \approx \frac{Ze^2}{m_e b^2}$$

for a duration

$$\Delta t \approx \frac{2b}{\mathrm{v}}$$

around the closest approach, and negligible acceleration outside this time window. This gives the power emitted

$$P \approx \frac{2e^2}{3c^3}a^2 = \frac{2}{3}\frac{Z^2e^6}{m_e^2c^2}\frac{1}{b^4}$$

and the total energy emitted in the encounter

$$P\Delta t \approx \frac{4}{3}\frac{Z^2e^6}{m_e^2c^2}\frac{1}{b^3\mathrm{v}}$$

Given the duration of the acceleration, this power is emitted over the frequency range 0 to $\mathrm{v}/2b$, and hence the spectral power per unit frequency is

$$P(\nu; b, \mathrm{v}) \approx \frac{8}{3}\frac{Z^2e^6}{m_e^2c^2}\frac{1}{b^2\mathrm{v}^2}$$

per encounter.

Noticing, now, that the number of encounters per unit time per unit volume with impact parameter between b and $b + db$ with electrons having speed between v and v+dv is

$$2\pi b db n_i n_e f(\mathrm{v}) d\mathrm{v}$$

where n_i and n_e are the number densities of the electrons and ions respectively and $f(\mathrm{v})$ is the normalised speed distribution of the electrons, we can write the emission coefficient due to bremsstrahlung (free-free emission) as

$$j_\nu^{ff} = \frac{1}{4\pi}\int_{\mathrm{v}_{\min}}^{\infty} d\mathrm{v} \int_{b_{\min}}^{b_{\max}} db 2\pi b n_i n_e f(\mathrm{v}) P(\nu; b, \mathrm{v})$$

Here $b_{\min}$ is decided by the fact that after the encounter and the corresponding loss of energy in radiation the electron still remains free, and $b_{\max}$ from the condition that $\mathrm{v}/(2b_{\max}) = \nu$. The minimum cutoff speed $\mathrm{v}_{\min}$ is decided by the condition

$$\frac{1}{2}m_e\mathrm{v}_{\min}^2 = h\nu$$

i.e., the electron must have enough kinetic energy to produce a photon of frequency ν. Here we are considering a non-relativistic situation. For a thermal distribution of electrons, $f(\mathrm{v};T)$ is the Maxwellian distribution of speeds at a temperature T. The above integral yields the result

$$j_\nu^{ff} = \frac{8e^6}{3m_e c^3}\left(\frac{2\pi}{3m_e k}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}(\nu, T)$$

where the factor $\bar{g}_{ff}(\nu, T)$, called the *velocity averaged Gaunt factor*, arises from the limits on the impact parameter and is a very slowly varying function of frequency: $\bar{g}_{ff}(\nu, T) \approx 1$ for $h\nu \leq kT$. (In the above expression for j_ν^{ff} numerical coefficients that result from the exact treatment of the problem have been included). From Kirchhoff's law one can then obtain the corresponding free-free absorption coefficient

$$\begin{aligned} \alpha_\nu^{ff} &= j_\nu^{ff}/B_\nu \\ &= \frac{4e^6}{3m_e hc}\left(\frac{2\pi}{3m_e k}\right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3}(1 - e^{-h\nu/kT})\bar{g}_{ff} \end{aligned}$$

Numerically,

$$j_\nu^{ff} = 5.4 \times 10^{-40} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff} \quad \mathrm{W\ m^{-3}\ Hz^{-1}}$$

$$\alpha_\nu^{ff} = 3.7 \times 10^{10} T^{-1/2} Z^2 n_e n_i \nu^{-3}(1 - e^{-h\nu/kT})\bar{g}_{ff} \quad \mathrm{m^{-1}}$$

The bremsstrahlung emission can cool a hot gas significantly, the net cooling rate per unit volume being given by

$$\epsilon^{ff} = \int_0^\infty 4\pi j_\nu^{ff} d\nu = 1.4 \times 10^{-28} T^{1/2} n_e n_i Z^2 \bar{g}_B \quad \mathrm{W\ m^{-3}}$$

where the frequency averaged Gaunt factor $\bar{g}_B \approx 1.2$. The above numerical coefficients have been evaluated for number densities n_e and n_i in units of $\mathrm{cm^{-3}}$.

Synchrotron Emission Synchrotron emission arises from the motion of relativistic charged particles in a magnetic field $\mathbf{B}$. Let us consider the example of an electron. The Lorentz force on it, $e\mathbf{v} \times \mathbf{B}/c$, is perpendicular to $\mathbf{v}$, causing the electron to gyrate around the magnetic field. The gyration frequency is $\omega_c = eB/m_e c$, which in the case of a relativistic electron with Lorentz factor γ becomes $\omega_B = eB/\gamma m_e c$. When the electron is extreme non relativistic, the electric field at the observer oscillates sinusoidally, and only the frequency $\omega_c/2\pi$ is present in the spectrum. As the speed of the electron is increased, Lorentz transformation causes a front-back asymmetry in the emission, introducing non-sinusoidal character to the periodic oscillations of the electric field, hence generating higher harmonics in the radiation spectrum. As the electron becomes highly relativistic ($\gamma \gg 1$), the radiation gets beamed into a forward cone of half-width $1/\gamma$. The observer experiences the electric field only when this cone crosses the line of sight. The typical duration of the electric field in each pass, after correcting for doppler effect, turns out to be $1/\gamma^3\omega_B \sin\alpha$, where α, called the "pitch angle", is the angle between the magnetic field and the velocity vector of the electron. Thus the synchrotron radiation peaks at a frequency

$$\nu_{\rm sy} \approx \frac{\gamma^3 \omega_B \sin\alpha}{2\pi} \propto BE^2$$

with harmonics present every $\omega_B/2\pi$. In a collection of electrons with different pitch angles and energies the harmonic structure of the spectrum is washed out, and one is left with a continuum. The power emitted by a single electron is given by

$$P = \frac{2}{3}\frac{e^4}{m_e^2 c^3}\gamma^2\beta_\perp^2 B^2$$

which, on averaging over an uniform pitch angle distribution yields

$$\begin{aligned} P &= \frac{4}{9} r_0^2 c \gamma^2 \beta^2 B^2 \quad (\propto B^2 E^2) \\ &= \frac{4}{3}\sigma_T c \gamma^2 \beta^2 u_B \end{aligned}$$

where $r_0 = e^2/m_e c^2$ is the classical electron radius, $\sigma_T = 8\pi r_0^2/3$ is the Thomson scattering cross section (see discussion later) and $u_B = B^2/8\pi$ is the magnetic energy density. This also expresses the average energy loss rate of a single electron in a tangled magnetic field often encountered in astrophysical situations.

Often the relativistic electrons that produce the synchrotron emission have a non-thermal power-law energy spectrum

$$N(E) \propto E^{-{\rm p}}$$

which results in a broadband, power-law synchrotron emissivity of the form

$$j_\nu \propto \nu^{-({\rm p}-1)/2}$$

Just like any other radiation process this too has its inverse, commonly referred to as *Synchrotron self absorption*. For a thermal distribution of relativistic electrons, the source function would be the Planck function B_ν. In most of the situations we will deal with, the photons produced by Synchrotron emission have a much lower energy than the particles themselves. The source function in this Rayleigh-Jeans regime is proportional to $\nu^2 kT$, where kT is the typical energy of electrons in the distribution. Once the electron distribution is non-thermal, the source function will depart from this. The particular case of power-law distribution mentioned above has the special property that it is *scale-free*, i.e. no *typical* energy can be defined for the distribution. The typical energy of electrons responsible for synchrotron emission at a frequency ν, according to the expression above, is $E \propto \nu^{1/2}$. In this case of scale-free distribution, this energy replaces kT in the expression for the source function, which now goes as $S_\nu \propto \nu^{5/2}$. The absorption coefficient is therefore

$$\alpha_\nu = \frac{j_\nu}{S_\nu} \propto \nu^{-(\mathrm{p}+4)/2}$$

Compton Scattering Compton scattering causes momentum exchange between a photon and a charged scatterer such as an electron. In a reference frame where the electron is initially at rest, the change in wavelength of the scattered radiation w.r.t. the incident is given by

$$\lambda_{\mathrm{scattered}} - \lambda_{\mathrm{incident}} = \lambda_c (1 - \cos\theta)$$

where θ is the scattering angle and $\lambda_c \equiv h/m_e c$ is the "Compton wavelength" of the electron. Clearly, in this frame the incident radiation loses some energy which is picked up by the electron recoil. The cross section for Compton scattering is given by the Klein-Nishina formula, which reduces in the non-relativistic regime ($x \equiv h\nu/m_e c^2 \ll 1$) to

$$\sigma \approx \sigma_T \left(1 - 2x + \frac{26x^2}{5} + \cdots\right)$$

and for $x \gg 1$ to

$$\sigma = \frac{3}{8}\sigma_T x^{-1}\left(\ln 2x + \frac{1}{2}\right)$$

For low x, the cross section approaches σ_T and the change in energy of the photon is also very small. This limit is called the Thomson scattering. As x increases, the energy transfer becomes larger, and the cross section drops.

If the electron is originally in motion then the problem is analysed simply by making a Lorentz transformation to the rest frame of the electron before scattering, and transforming back to the lab frame after scattering. If the electrons are relativistic, this shows that initially low energy photons ($h\nu/m_e c^2 \ll \gamma^2 - 1$) gain energy by a factor γ^2 in the lab frame, at the expense of the kinetic energy of the electron. This is often called the "Inverse Compton Effect". The inverse compton power for a single electron works out to be

$$P_{\mathrm{comp}} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 u_{\mathrm{ph}}$$

where u_{ph} is the photon energy density. Note the similarity to P_{sy}.

A photon of energy ϵ passing through a thermal distribution of electrons at a temperature T gains, on an average, an amount of energy

$$\Delta\epsilon = \frac{\epsilon}{m_e c^2}(4kT - \epsilon)$$

per scattering if $kT \ll m_e c^2$. This shows that photons of energy higher than $4kT$ lose energy and those of lower energy gain energy from the electron distribution. In the relativistic limit, however, passing photons can gain a substantial amount of energy:

$$\Delta\epsilon = 16\epsilon\left(\frac{kT}{m_e c^2}\right)^2$$

One ought to note that the scattering process preserves the number of photons.

Bound-free transitions

We will now discuss processes involving bound states. We begin with free-bound/bound-free transitions, the most important such processes in astrophysical context are photoionisation and recombination. One encounters this in hot, thermal regions where a large fraction of the gas is ionised.

A photoionisation/recombination process is represented by

$$(Z + 1)e^+(n') + e^-(p) \rightleftharpoons Ze^+(n) + \gamma(h\nu = \chi_{Z,n} - \chi_{Z+1,n'} + p^2/2m_e)$$

where n denotes the level of excitation of the lower ionisation species, and $\chi_{Z,n}$ is its ionisation potential from this level. n' and $\chi_{Z+1,n'}$ are the corresponding quantities for the higher ionisation species. In thermal equilibrium, a relation can be obtained for the population ratio of the different species.

Let $N_{Z+1}(n')$, $N_Z(n)$ and N_e be the number densities of the species in question. Let the temperature of the gas be T. The number density of electrons with momentum p, per unit momentum range, is then

$$N_e(p) = \frac{N_e}{(2\pi m_e kT)^{3/2}} e^{-p^2/2m_e kT} \cdot 4\pi p^2$$

In the language of Einstein coefficients, the above process can be thought to be composed of three distinct parts:

- Spontaneous Recombination, equivalent to spontaneous emission:

  $$(Z+1)e^+(n') + e^-(p) \to Ze^+(n) + \gamma(h\nu = \chi_{Z,n} - \chi_{Z+1,n'} + p^2/2m_e)$$

  the rate for which is

  $$R_S = N_{Z+1}(n')N_e(p)A_{21}(n, n', p)$$

- Induced Recombination, equivalent to stimulated emission:

  $$(Z+1)e^+(n') + e^-(p) + \gamma(h\nu = \chi_{Z,n} - \chi_{Z+1,n'} + p^2/2m_e) \to Ze^+(n)$$

  for which the rate is

  $$R_I = N_{Z+1}(n')N_e(p)I_\nu B_{21}(n, n', p)$$

- Ionisation, equivalent to absorption:

  $$Ze^+(n) + \gamma(h\nu = \chi_{Z,n} - \chi_{Z+1,n'} + p^2/2m_e) \to (Z+1)e^+(n') + e^-(p)$$

  at a rate

  $$I = N_Z(n)I_\nu B_{12}(n, n', p)$$

In thermal equilibrium, detailed balance requires that the total radiative transitions downwards must equal the absorbing transitions upward, namely

$$I = R_S + R_I$$

i.e.

$$N_Z(n) I_\nu B_{12}(n, n', p) = N_{Z+1}(n') N_e(p) [A_{21}(n, n', p) + I_\nu B_{21}(n, n', p)]$$

Hence

$$\frac{N_{Z+1}(n') N_e(p)}{N_Z(n)} = \frac{B_{12}(n, n', p)/B_{21}(n, n', p)}{1 + A_{21}(n, n', p)/I_\nu B_{21}(n, n', p)}$$

Now

$$B_{12}(n, n', p)/B_{21}(n, n', p) = g_2/g_1$$

the ratio of statistical weights. The statistical weight of state 2 is

$$g_2 = g_{Z+1,n'} g_e \frac{4\pi p^2}{h^3}$$

per unit volume per unit momentum range. The spin degeneracy factor g_e is 2 and $g_{Z+1,n'}$ is the statistical weight of level n' of the higher ionisation species. The statistical weight of state 1 is $g_1 = g_{Z,n}$. Thus, per unit momentum range,

$$B_{12}(n, n', p)/B_{21}(n, n', p) = \frac{g_{Z+1,n'} g_e}{g_{Z,n}} \frac{4\pi p^2}{h^3}$$

Noting also that

$$A_{21}(n, n', p) = \frac{2h\nu^3}{c^2} B_{21}(n, n', p)$$

and

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

we find

$$\frac{N_{Z+1}(n') N_e(p)}{N_Z(n)} = \frac{g_{Z+1,n'} g_e}{g_{Z,n}} \frac{4\pi p^2}{h^3} e^{-(\chi_{Z,n} - \chi_{Z+1,n'} + p^2/2m_e)/kT}$$

which in terms of the total electron density can be written as

$$\frac{N_{Z+1}(n') N_e}{N_Z(n)} = \frac{g_{Z+1,n'} g_e}{g_{Z,n}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-(\chi_{Z,n} - \chi_{Z+1,n'})/kT}$$

Noting that $\chi_{Z,n} = \chi_{Z,1} - E_{Z,n}$ where $E_{Z,n}$ is the excitation energy of level n from the ground state of the species Z, and likewise for the upper ionisation state, a sum can be performed over all states n' of the higher ionisation state and the states n of the lower ionisation state:

$$\frac{N_{Z+1}N_e}{N_Z} = \frac{g_{Z+1}g_e}{g_Z}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-(\chi_Z - \chi_{Z+1})/kT}$$

where we have dropped the suffix 1 for the ground state ionisation potential. The quantities g_Z and g_{Z+1} now denote the partition functions of the corresponding species. This expression, first derived by Meghnad Saha, goes by the name *Saha ionisation equation* and is vital for the determination of the abundance of different ionised species in a hot gas.

The ionisation and recombination probabilities are often expressed in terms of the respective cross sections: $\sigma_{n',n,Z}(\mathrm{v})$ is the cross section for the capture of an electron of speed v by an ion of charge $Z + 1$ at the level n', resulting in an ion of charge Z at a level n. This cross section is defined per unit velocity range.

a_ν is the corresponding photoionisation cross section, for a photon of energy $h\nu = \chi_{Z,n} - \chi_{Z+1,n'} + m_e\mathrm{v}^2/2$ to be absorbed by an ion of charge Z at a level n and be ionised to charge $Z + 1$ at a level n' as well as produce a photoelectron travelling with a speed v. This cross section is defined per unit frequency range. We note that since $h\nu$ obeys the above relation, an unit velocity range corresponds to a frequency range

$$\frac{d\nu}{d\mathrm{v}} = \frac{m_e\mathrm{v}}{h}$$

The cross sections $\sigma_{n',n,Z}(\mathrm{v})$ and a_ν are related to the A and B coefficients defined above, and are therefore related among themselves. Let us slightly modify the foregoing discussion to consider unit velocity range in place of unit momentum range. A_{21}, defined for unit velocity range, will now equal $\sigma_{n',n,Z}(\mathrm{v})\mathrm{v}$. The relation between a_ν and B_{12} can be worked out as follows. Since in an isotropic radiation field a radiation intensity I_ν corresponds to a photon flux $4\pi I_\nu/h\nu$, and an unit velocity range corresponds to a frequency range $m_e\mathrm{v}/h$, we have

$$I_\nu B_{12} = \frac{4\pi I_\nu}{h\nu}\frac{m_e\mathrm{v}}{h}a_\nu$$

or

$$B_{12} = \frac{4\pi m_e \mathrm{v}}{h^2 \nu} a_\nu$$

Noting that the statistical weight of the upper level

$$g_2 = g_{Z+1,n'} g_e \frac{4\pi m_e^3 \mathrm{v}^2}{h^3}$$

per unit volume per unit *velocity* range, and the statistical weight of the lower state

$$g_1 = g_{Z,n}$$

we get

$$B_{21} = \frac{g_1}{g_2} B_{12} = \frac{g_{Z,n}}{g_{Z+1,n'} g_e} \frac{h^3}{4\pi m_e^3 \mathrm{v}^2} \frac{4\pi m_e \mathrm{v}}{h^2 \nu} a_\nu$$

and then using

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

one finds

$$\sigma_{n',n,Z}(\mathrm{v}) = \frac{2g_{Z,n}}{g_{Z+1,n'} g_e} \frac{h^2 \nu^2}{c^2 m_e^2 \mathrm{v}^2} a_\nu$$

This is called the *Milne relation*. Like in the case of Saha equation, the levels n and n' of the ion species can be summed over, replacing the respective statistical weights by the corresponding partition functions.

In astronomy, the ionisation state of an atom is designated by a roman numeral starting with I for the neutral state. Neutral Hydrogen is thus referred to as HI and ionised hydrogen as HII. Neutral Helium is HeI, singly ionised Helium is HeII and doubly ionised Helium is HeIII. The photoionisation processes, where important, converts the region into a predominantly ionised one, and such ionised diffuse gas is usually referred to as an HII region.

The free-bound processes in an HII region produces a continuum radiation, called the recombination radiation. The HI free-bound radiation at a frequency ν will result from recombinations of free electrons with speed v to levels with principal quantum number $n \geq n_1$, where

$$h\nu = \frac{1}{2} m_e \mathrm{v}^2 + \chi_n$$

and

$$h\nu \geq \chi_{n_1}$$

the emission coefficient then being given by

$$j_\nu = \frac{1}{4\pi} n_p n_e \sum_{n=n_1}^{\infty} \sum_{l=0}^{n-1} \mathrm{v}\sigma_{nl}(\mathrm{HI}, \mathrm{v}) f(\mathrm{v}) h\nu \frac{d\mathrm{v}}{d\nu}$$

The recombination cross section σ can be calculated from the corresponding a_ν using the Milne relation. The nature of a_ν is such that at the ionisation threshold from each n the cross section jumps to a large value and then falls roughly as ν^{-3} at frequencies higher than the threshold. These thresholds thus make their appearance in j_ν too, and due to the Milne relation it decays roughly linearly with wavelength beyond each threshold. This gives the emission coefficient of recombination radiation a "sawtooth" appearance.

In most situations, the total emission by the free-free process far exceeds the recombination radiation, but the recombination radiation can introduce characteristic spectral features at ionisation thresholds in the continuum radiation. For Hydrogen, the highest ionisation threshold, called the Lyman Limit, corresponds to an energy of 13.6 eV or an wavelength of 912Å. Recombination radiation from Hydrogen at wavelengths shorter than this is called the "Lyman continuum". Similarly, the recombination radiation shortward of 3646Å, the Balmer Limit, is called the "Balmer continuum" and so on. It is to be noted that radiation at wavelengths shorter than Lyman limit have a great difficulty escaping from regions of diffuse gas, since such a photon is almost immediately absorbed by a nearby Hydrogen atom. Observed light from gas-rich galaxies abruptly cuts off above the Lyman limit and this feature, called the "Lyman break", has been used to infer the redshift of very distant galaxies. At wavelengths much shorter than Lyman limit, however, the absorption cross section dies away as $\sim \lambda^3$, and the absorption by Hydrogen is no longer significant. However heavier elements with higher ionisation energies continue to provide significant opacity to photons of energy until about 1 keV, above which the continuum absorption by diffuse gas is no longer significant.

Photoionisation is a very important heating process for diffuse gas. Given a source of ionising photons, this process converts the energy of the ionising radiation into the kinetic energy of photoelectrons, thereby heating the gas. HII regions around hot stars can be heated quickly to high temperatures by this process. If there were

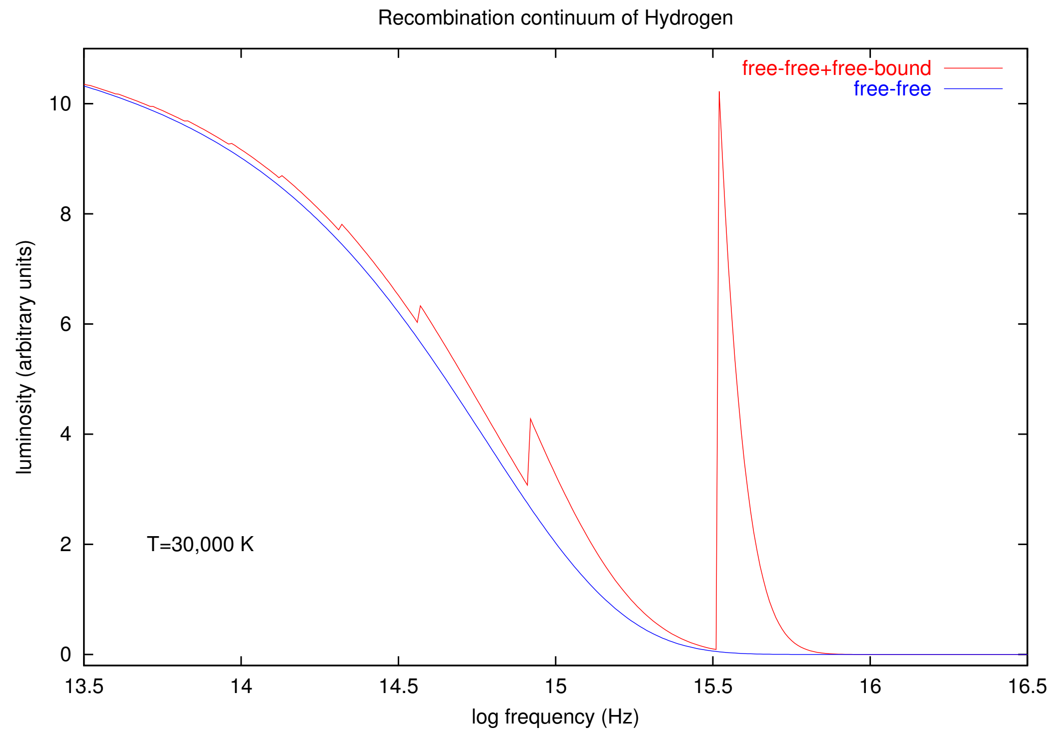

Figure 4: Recombination radiation from a pure hydrogen nebula compared to free-free emission. The red line includes both free-free and recombination radiation, the blue line free-free radiation alone

no cooling processes available such a region would eventually attain the temperature of the star itself. Free-free emission, however, provides a substantial amount of cooling, and recombination continuum a smaller amount. When metals are present, certain bound-bound transitions in them may become the most important cooling agents. In practice, one finds that the temperatures of HII regions lie in the range $\sim$ 6000 to 15000 K.

Spectral lines

We will now discuss bound-bound transitions. There are many important spectral lines used in Astronomical studies of various phenomena, and it is not possible to discuss here all of them. We will mention some of the most commonly used ones, and the physical parameters they can help us derive.

We note that the probability of a radiative transition from an initial state $|i\rangle$ to a final state $|f\rangle$ of an atom (or molecule) is proportional to the squared expectation value of the interaction Hamiltonian $e\vec{A}\cdot\vec{p}/m_e c$. For radiation, writing $\vec{A}$ as $A(t)\exp(i\vec{k}\cdot\vec{r})\hat{n}$, where $\hat{n}$ is the unit vector in the direction of propagation of the radiation, one finds that the transition probability is proportional to

$$|\langle f|\exp(i\vec{k}\cdot\vec{r})\hat{n}\cdot\vec{\nabla}|i\rangle|^2$$

This is written for one electron, but can be generalised to a many-electron system by summing the individual momentum operators. The exponential in the above expression can be expanded as

$$\exp(i\vec{k}\cdot\vec{r}) = 1 + i\vec{k}\cdot\vec{r} + \frac{1}{2}(i\vec{k}\cdot\vec{r})^2 + \cdots$$

And the lowest order, where the exponential is set to unity, is called the *dipole* approximation. When the result in this order is zero, then one needs to go to the higher orders, called *electric quadrupole*, *magnetic dipole*, *electric octupole*, *magnetic quadrupole* and so on.

One often refers to the transitions in which the electric dipole order is non-zero, as *allowed* transitions. These need to obey certain *selection rules*, which ensure that there is a change in the net electric dipole moment of the atom (or molecule), and also that the change in the angular momentum compensates for the angular momentum carried away (or brought in) by a photon. In general, there can be no dipole transition between states of the same parity. This condition does not need to be fulfilled for higher order transitions, which are usually called *forbidden* transitions. The forbidden transitions have much lower transition probabilities than allowed transitions, and hence have much smaller optical depths associated with them. This property makes them very valuable as tools to study dense regions or long pathlengths encountered in Astronomy. Forbidden lines also provide some of the most important cooling mechanisms encountered in hot regions.

Since radiative excitation/de-excitation has a low probability for forbidden transitions, transitions between these levels can often be caused by collisions. If we designate two levels participating in this as 1 (lower) and 2 (upper) respectively,

E_{21} being the energy difference between these two levels, then the collisional processes are

$$\text{Atom (state 1)} + e^-(E_{\text{kin}} = \frac{1}{2}m_e v^2) \rightleftharpoons \text{Atom (state 2)} + e^-(E_{\text{kin}} = \frac{1}{2}m_e v^2 - E_{21})$$

The rate of the collisional processes can be written as

$$\begin{aligned} \text{Excitation} &: n_e n_1 q_{12} \\ \text{De-excitation} &: n_e n_2 q_{21} \end{aligned}$$

The collision probabilities q_{12} and q_{21} are the products of collision cross sections and velocities, averaged over the electron speed distribution. They therefore are functions of temperature, and are related by

$$q_{12} = \frac{g_2}{g_1} q_{21} e^{-E_{21}/kT}$$

Levels that collisionally excite and radiatively de-excite can be a major source of cooling for hot gas. A number of metal ions have levels that can participate in the cooling of this form. The statistical equilibrium for such a pair of levels can be written as

$$n_e n_1 q_{12} = n_e n_2 q_{21} + n_2 A_{21}$$

where A_{21} is the spontaneous radiative transition probability. This yields

$$\frac{n_2}{n_1} = \frac{n_e q_{12}}{n_e q_{21} + A_{21}}$$

which provides a cooling rate

$$L_c = n_2 A_{21} E_{21} = n_e n_1 q_{12} E_{21} \left(\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right)$$

This shows that the line cooling increases with increasing n_e until $n_e q_{21}$ reaches values $\sim A_{21}$. Beyond that the cooling rate becomes practically independent of n_e.

Important cooling transitions of this kind are [OII]$\lambda\lambda$3726,3729; [OIII]$\lambda\lambda$4959,5007; [NII]$\lambda\lambda$6548,6583 at temperatures of a few thousand K. The square brackets around the species indicate a forbidden transition, and the wavelengths are in Angstroms. The A_{21} coefficients for these transitions lie in the range 10^{-2} to 10^{-5} s^{-1}. Compare this with the A_{21} coefficient of a typical allowed transition: $\sim 10^8$ s^{-1}. At higher temperatures, the dominant cooling transitions are [NeV]$\lambda\lambda$3345,3425; CIV $\lambda\lambda$1548,1550 etc.

The main Hydrogen lines seen from hot HII regions are the recombination lines, of which Balmer transitions (Hα, Hβ, Hγ) are the most important. electrons captured at higher levels cascade down, eventually producing these photons. Nearly every recombination results in one of these Balmer photons (Hα λ6563, $\sim$ 60%, Hβ λ4861 $\sim$ 20%, Hγ λ4340 $\sim$ 10%) that escape from the region. Lyman series photons that are emitted are almost immediately absorbed in the vicinity, since the vast majority of the HI component resides in the ground state. As a result, Lyman photons are usually unable to escape from the nebula. The Balmer line emissivity of an HII region can then be written as

$$j_\nu^{\mathrm{H}\alpha} \approx n_e n_p \alpha \frac{h\nu^{\mathrm{H}\alpha}}{4\pi}\phi(\nu)$$

where α is the recombination coefficient. This leads to a line intensity

$$I_\nu^{\mathrm{H}\alpha} = \int_0^L j_\nu^{\mathrm{H}\alpha} ds \propto \int_0^L n_e n_p ds = \frac{n_p}{n_e}\int_0^L n_e^2 ds$$

Where L is the total pathlength through the nebula. The quantity $\int_0^L n_e^2 ds$ is called the *Emission Measure* of the nebula, usually expressed in the units cm^{-6} pc. Note that also the free-free emission, the recombination continuum, the free-free optical depth are all proportional to this quantity.

Absorption at Lyman-α wavelength by neutral gas is one of the strongest signatures of Hydrogen observed from even very distant gas clouds. In the spectrum of a distant quasar one may find Lyman-α absorption by a large quantity of Hydrogen gas in the material between us and the quasar. The intergalactic medium is by and large ionised, and neutral gas resides in discrete clouds and filaments, some of which could be galaxies or protogalaxies. Being distributed over a large redshift range, the Lyα absorption from many such clouds in any given line of sight appear to us as a series of distinct lines, often referred to as the "Lyα" forest. As Lyα is absorbed very strongly by HI, the narrow forest lines usually indicate systems where the neutral fraction of Hydrogen is low. If, instead, the amount of HI is relatively high, one gets a deep, saturated absorption with extended wings of the line. These are called "Damped Lyman Alpha" systems or DLAs.

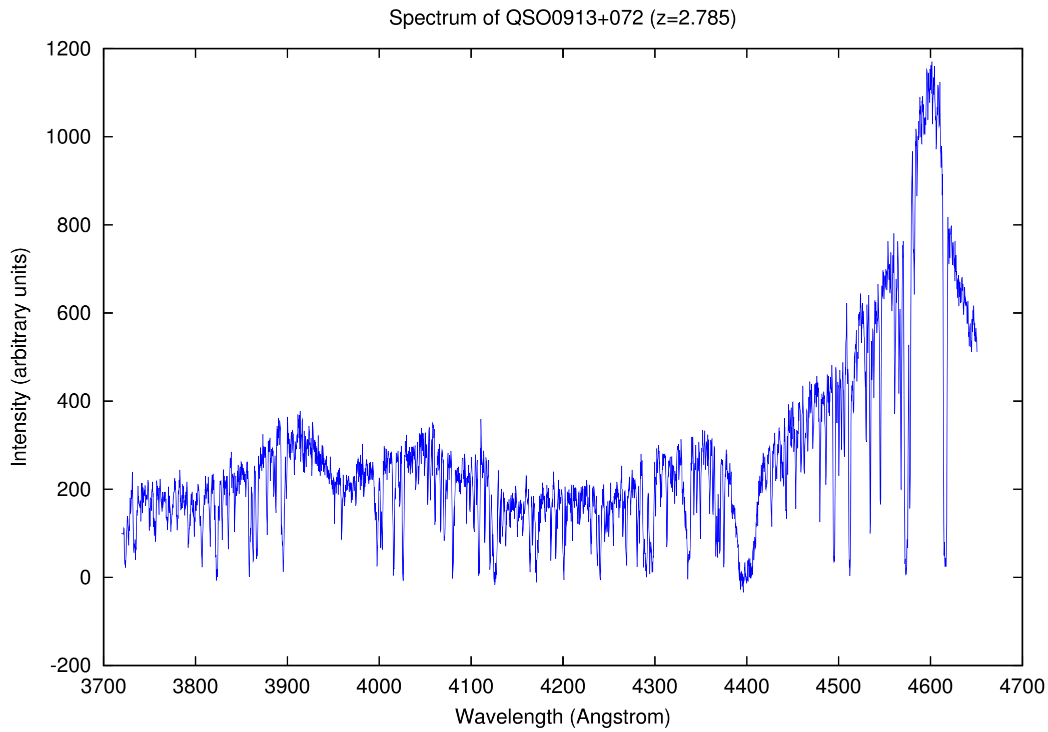

Figure 5: Lyman alpha absorption in the spectrum of the Quasar Q0913+072 (Bechtold 1994 ApJS 91,1). The quasar is at a redshift of 2.785, corresponding to a wavelength of the Lyman alpha line of 4602Å. Absorption at wavelengths shorter than this is caused by intervening medium located at lower redshifts. Note the large number of Lyman alpha absorption lines, called the *Lyman Alpha Forest*. Near 4400Å a *Damped Lyman Alpha* absorption is seen.

The signature of Helium in ionised Helium regions is seen in the strong optical recombination line HeII λ4686, which is a transition from principal quantum number 4 to 3. In very hot gas lines from highly ionised metals provide important diagnostics. For example, absorption lines of OVI at 1032 and 1038 Åare expected from a gas above a few times 10^5 K. At very high temperatures, encountered for example in shock-heated regions or in gas streams accreting onto compact objects, strong X-ray lines of FeXXVI Lyα and Lyβ at 6.7 and 7.8 keV are expected.

We now discuss one of the most important spectral lines in Astronomy that probes neutral atomic Hydrogen gas. This is a hyperfine (spin-flip) transition in the ground state of the Hydrogen atom. In the upper state the nuclear and the electron spins are parallel, and in the lower state they are antiparallel. This falls in the category of forbidden transitions, and has a very low transition probability $A_{21} = 2.869 \times 10^{-15}$ s^{-1}. The frequency of this radiation is 1420.406 MHz, corresponding to a radio wavelength of 21.11 cm. Since the energy difference between the two levels $\Delta E = 5.9 \times 10^{-6}$ eV is very much smaller than kT in any astronomical situation, the populations in these two levels are always in the ratio of their statistical weights, 3:1, namely three-quarters of all neutral hydrogen is in the upper state. The emission coefficient of this line is

$$j_\nu = \frac{h\nu_{21}}{4\pi} n_2 A_{21} \phi(\nu) = \frac{3h\nu_{21}}{16\pi} n_{\rm HI} A_{21} \phi(\nu)$$

where $n_{\rm HI}$ is the neutral hydrogen number density. If the gas is at a temperature T then the absorption coefficient can be obtained from

$$\alpha_\nu = j_\nu / B_\nu(T) = \frac{3c^2 h A_{21}}{32\pi\nu_{21} k} \frac{n_{\rm HI}}{T} \phi(\nu)$$

Using the Rayleigh-Jeans approximation for B_ν. The optical depth in the line is

$$\tau_\nu = \int \alpha_\nu ds = \frac{3c^2 h A_{21}}{32\pi\nu_{21} k} \frac{N_{\rm HI}}{T} \phi(\nu)$$

which gives a line brightness temperature

$$T_{b,\nu} = T(1 - e^{-\tau_\nu})$$

In the above $N_{\rm HI}$ stands for the line-of-sight integral of the number density of neutral hydrogen, called the neutral hydrogen *column density*. From these, one may write

$$N_{\rm HI}\phi(\nu) = \frac{32\pi\nu_{21} k}{3c^2 h A_{21}} \frac{\tau_\nu T_{b,\nu}}{1 - e^{-\tau_\nu}}$$

or

$$N_{\rm HI} = \frac{32\pi\nu_{21} k}{3c^2 h A_{21}} \int d\nu \frac{\tau_\nu T_{b,\nu}}{1 - e^{-\tau_\nu}}$$

Measurement of line emission in a given direction yields $T_{b,\nu}$ and absorption against a strong background radio source yields τ. From these two measurements one can determine the Hydrogen column density and the temperature of neutral hydrogen in the line of sight. The temperature determined this way is commonly referred to as the "spin temperature". Observation of this line allows us to map the hydrogen distribution and study the dynamics of gas in the Galaxy as well as external galaxies.

Nearly half the gas in our galaxy is in molecular form and resides in large *molecular clouds*. A molecule exhibits rotational, vibrational and electronic transitions, the corresponding spectra being, typically, in mm-waves, infrared and ultraviolet respectively. H_2 itself is a homonuclear molecule with no dipole moment, so rotational and vibrational transitions are dipole-forbidden. H_2 molecules are therefore best observed in ultraviolet through its electronic transitions. However a molecular cloud contains other molecular species such as CO which do have permitted rotational-vibrational transitions which are excited by collisions with the H_2 molecules. Study of these transitions allows one to indirectly study the properties of the H_2 gas. The rotational transitions of CO start at a frequency of 115 GHz ($J = 1 \rightarrow 0$), and are almost equidistant in frequency, occurring roughly at an interval of 115 GHz. These transitions of CO have enabled us to study molecular gas in similar amount of detail as the 21-cm line did for the atomic gas.