

Eikonal Approximation

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This is a study report on eikonal approximations which was undertaken under the guidance of Kimball Milton. This report was submitted to partially fulfill the requirements for the general examination.

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I. INTRODUCTION

The etymology of the word eikon traces it back to the word *eikenai* which is the transliteration of the word $\epsilon\iota\kappa\epsilon\upsilon\alpha\iota$ [1] in the Greek language meaning 'to resemble' [2]. In the Greek language it evolved into the word *eikōn* which is the transliteration of the word $\epsilon\iota\kappa\omicron\nu$ [1] meaning 'image' [2]. It was borrowed into the Latin language and later into the English language as eikon. In the English language it transformed into the words icon and ikon which are the variants of the word eikon. Notice that throughout its evolution it has meant 'image.'

The main goal of this report is to understand the eikonal approximation in quantum mechanics and quantum field theories. Approximations play a very important role in the understanding of processes that cannot be solved

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exactly. The Born approximation in quantum mechanics is an example of an approximation that has been extensively used for studying low energy processes. In quantum field theories involving coupling constants smaller than one we use the standard weak-coupling perturbation series which is parallel in its approach to the Born approximation in quantum mechanics. In the 1950's and 1960's when high energy physics was ascending towards its peak, it was realized among the high energy physicists of those times that the Born approximation is not a valid approximation for studying processes involving high energies. This period in fact was the golden age in the development of the eikonal approximation in quantum mechanics and quantum field theories. The descendents of this era took the theory of eikonal approximation for granted in quantum mechanics and quantum field theories. There was prolific activity in the application of eikonal approximation in high energy physics, especially in QCD.

The eikonal approximation was not born in the study of quantum mechanics. It originated far back in optics. Light we know obeys Maxwell's equations. In terms of Maxwell's equations light is understood as a wave obeying a wave equation. But two hundred years before Maxwell wrote down his equations scientists understood reflection and refraction of light which was extensively studied under the branch of science called ray optics. Today in elementary school we learn ray optics without ever introducing Maxwell's equations. In ray optics we assume that light travels in a straight line. This assumption works fine as long as the size of the obstacle is large compared to the wavelength of light. This is called the eikonal approximation in optics. Here we make contact with the meaning of the word eikon in the sense that eikon meaning 'image' is formed by light only in the straight line approximation. In processes involving diffraction we encounter the limit of the validity of the eikonal approximation in optics. We quickly switch over to Maxwell's equations to study diffraction.

Optics is described by Maxwell's equations which can be written as a wave equation with the dispersion relation given by $\omega = kc$. Similarly, quantum mechanics is described in terms of Schrodinger's equation which is a diffusion equation (in imaginary time) with a dispersion relation given by $\omega = \frac{\hbar k^2}{2m}$. With this correspondence we ask the question, can we not have a corresponding eikonal approximation in quantum mechanics? Yes we can. The eikonal approximation in quantum mechanics works for processes involving the scattering of particles with large incoming momentum and when the scattering angle is very small.

In the language of differential equations, the main advantage the eikonal approximation offers is that the equations reduce to a differential equation in a single variable. This reduction into a single variable is the result of the straight line approximation or the eikonal approximation which allows us to choose the straight line as a special direction. The early steps involved in the eikonal approximation in quantum mechanics are very closely related to the WKB approximation in quantum mechanics. The WKB approximation involves an expansion in terms of Planck's constant \hbar . The WKB approximation also reduces the equations into a differential equation in a single variable. But the complexity involved in the WKB approximation is that this variable is described by the trajectory of the particle which in general is complicated. The advantage of the eikonal approximation is in the classical trajectory being a straight line. Thus in this manner the eikonal approximation is a very stringent semi-classical limit.

A very comprehensive collection of work on scattering theory in general with a very extensive bibliography which covers scattering theory in both electromagnetism and quantum mechanics is the book by Roger G. Newton [3]. A couple of textbooks among many which I have used are [6] and [7].

An extensive list of literature on the theory of eikonal approximation in quantum mechanics and quantum field theories developed between the years 1950 and 1970 is available in [4]. It is unanimously accepted by everyone that R. J. Glauber's lecture notes on eikonal approximation [5] is the best available work on the subject. Glauber's lecture notes addresses the question of the conditions for the validity of the eikonal approximations in quantum mechanics. In the section on eikonal approximations in this article starting out from his ideas we derive the conditions for the validity of the eikonal approximations more concretely. We verify our conclusions by evaluating the total scattering cross section for the simplest potential (an inverted finite spherical well) using the eikonal approximation and by comparing it with the result obtained by the partial wave expansion.

After our study of eikonal approximation in quantum mechanics we concentrate on quantum electrodynamics. To study the eikonal approximation in quantum electrodynamics we need to first write the field theoretical equations in a suitable form. Schwinger's formalism for quantum electrodynamics is the most suitable for this purpose. We develop Schwinger's formalism in detail based on [15-18] in section V.

In section VI we concentrate on electron-electron scattering. We introduce the quenched approximation. We outline the derivation of the electron-electron scattering in the eikonal approximation based on [19,20]. We point out the similarity of the result of electron-electron scattering obtained in the eikonal approximation to the non-relativistic scattering amplitude due to the Coulomb potential.

The terminology used in this study report is the following

$$f(\theta, \phi) \equiv \text{Scattering amplitude}$$

$$\sigma_{\text{scatt.}} \equiv \text{Scattering cross section}$$

$\sigma_{\text{abs.}} \equiv$ Absorption cross section

$\sigma_{\text{tot.}} \equiv \sigma_{\text{scatt.}} + \sigma_{\text{abs.}}$

II. SCATTERING IN QUANTUM MECHANICS

A scattering process in quantum mechanics is described by the solution of the Schroedinger equation

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}, t) \right] \psi'(\vec{r}, t) = -i\hbar \frac{\partial}{\partial t} \psi'(\vec{r}, t) \quad (1)$$

with the boundary conditions dictated by the requirement that the wave function $\psi'(\vec{r}, t)$ must have a component that involves an incident plane wave with energy $E = \frac{\hbar^2 k^2}{2m}$ moving in the positive z direction and another component that involves a spherical outgoing wave. It should be emphasized that in scattering problems we do not require $\psi'(\vec{r}, t)$ to go to zero at $\vec{r} \rightarrow \infty$. In fact the scattering amplitude which is the quantity of interest is contained in the $\vec{r} \rightarrow \infty$ (asymptotic) part of $\psi'(\vec{r}, t)$.

A. Formulation of the scattering problem

For the case when the initial beam can be described by a state of definite energy we can write

$$\psi'(\vec{r}, t) = e^{-\frac{i}{\hbar}Et} \psi(\vec{r}) \quad (2)$$

where $\psi(\vec{r})$ satisfies the differential equation

$$[\nabla^2 + k^2] \psi(\vec{r}) = U(\vec{r})\psi(\vec{r}) \quad (3)$$

with the boundary conditions on $\psi(\vec{r})$ dictated by the boundary conditions on $\psi'(\vec{r}, t)$. We have used the notation $k^2 = \frac{2mE}{\hbar^2}$ and $U(\vec{r}) = \frac{2mV(\vec{r})}{\hbar^2}$.

The differential equation for $\psi(\vec{r})$ in eqn. (3) can be rewritten as an integral equation [8] given by

$$\psi(\vec{r}) = \phi(\vec{r}) + \int d^3r' G_0(\vec{r}, \vec{r}') U(\vec{r}') \psi(\vec{r}') \quad (4)$$

where $\phi(\vec{r})$ satisfies the potential free equation

$$[\nabla^2 + k^2] \phi(\vec{r}) = 0 \quad (5)$$

and the Green's function $G_0(\vec{r}, \vec{r}')$ is the solution to

$$[\nabla^2 + k^2] G_0(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}'). \quad (6)$$

The boundary conditions on $\phi(\vec{r})$ and $G_0(\vec{r}, \vec{r}')$ are prescribed by the boundary conditions on $\psi(\vec{r})$. The integral equation in eqn. (4) is called the Lippmann-Schwinger equation [9]¹. The solutions to $\phi(\vec{r})$ and $G_0(\vec{r}, \vec{r}')$ are

$$\begin{aligned} \phi(\vec{r}) &= A_0 e^{i\vec{k}\cdot\vec{r}} + B_0 e^{-i\vec{k}\cdot\vec{r}} \quad (7) \\ G_0(\vec{r}, \vec{r}') &= -\frac{1}{4\pi} \left[A \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} + B \frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right] \quad (8) \end{aligned}$$

¹Schwinger's view of the paper was [10]: '... I thought the importance of the paper was the variational principle ... the so-called Lippmann-Schwinger scattering equation is to me conventional scattering theory written in operator notation. Nothing new. But that is what everybody paid attention to.'

with the constraint $A + B = 1$. Using eqn. (7) and eqn. (8) in eqn. (4) the Lippmann-Schwinger equation takes the form

$$\psi(\vec{r}) = A_0 e^{i\vec{k}\cdot\vec{r}} + B_0 e^{-i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi} \int d^3r' \left[A \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} + B \frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right] U(\vec{r}')\psi(\vec{r}'). \quad (9)$$

Imposing the boundary conditions that the wave function consists of a component that is a plane wave moving in the positive z direction and another that is an outgoing spherical wave we get

$$\psi(\vec{r}) = A_0 e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi} \int d^3r' \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} U(\vec{r}')\psi(\vec{r}'). \quad (10)$$

As we emphasized before the information related to the scattering amplitude is contained in the asymptotic region of the wave function. In most of the problems of interest the potential $V(\vec{r})$ is confined to a finite volume in space and the detectors are very far from the region containing the potential. For these cases we can safely conclude $r' \ll r$ and thus approximate

$$|\vec{r}-\vec{r}'| = r - \frac{\vec{r}\cdot\vec{r}'}{r} + O\left[\left(\frac{r'}{r}\right)^2\right]. \quad (11)$$

We can thus write

$$\psi_{r\rightarrow\infty}(\vec{r}) = A_0 e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi} \int d^3r' \frac{1}{r} e^{ik(r-\frac{\vec{r}\cdot\vec{r}'}{r})} U(\vec{r}')\psi(\vec{r}') \quad (12)$$

$$= e^{i\vec{k}\cdot\vec{r}} + f(\theta, \phi) \frac{e^{ikr}}{r} \quad (13)$$

where we have set $A_0 = 1$ so that

$$f(\theta, \phi) = -\frac{1}{4\pi} \int d^3r' e^{-i\vec{k}'\cdot\vec{r}'} U(\vec{r}')\psi(\vec{r}') \quad (14)$$

can be interpreted as the scattering amplitude, where $\vec{k}' = k\frac{\vec{r}}{r}$. The illustration of the variables involved is shown in fig. 1.

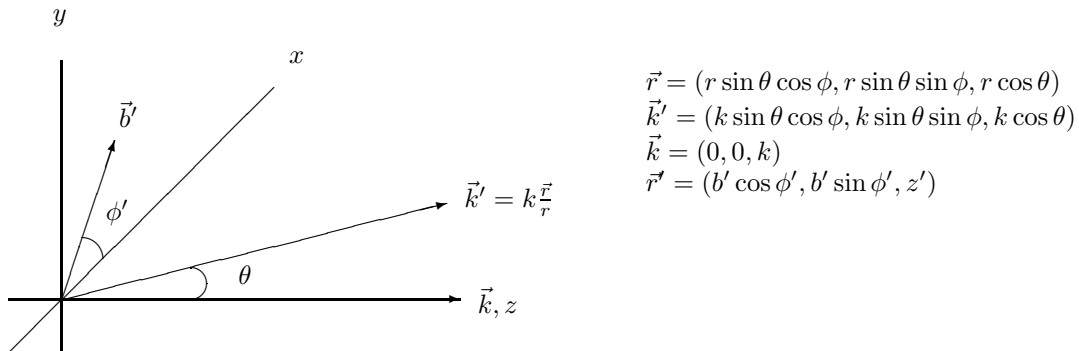


FIG. 1. Illustration of the various variables used in the calculation. Note that \vec{r} , \vec{k}' , and \vec{k} are in spherical polar coordinates and \vec{r}' is in cylindrical polar coordinates.

Often it is more useful to denote $f(\theta, \phi)$ as a function of \vec{k} and \vec{k}' and thus write $f(\theta, \phi) = f(\vec{k}', \vec{k})$. Observe that even though the information related to $f(\theta, \phi)$ is contained in the asymptotic region of $\psi(\vec{r})$ the only contributions to $f(\theta, \phi)$ in eqn. (14) comes from regions where the potential is not zero. To complete the formulation of the scattering problem we define the scattering cross section as

$$\sigma_{\text{scatt.}} = \int d\Omega |f(\theta, \phi)|^2 \quad (15)$$

and the total cross section as

$$\sigma_{\text{tot.}} = \sigma_{\text{scatt.}} + \sigma_{\text{abs.}} \quad (16)$$

where $\sigma_{\text{abs.}}$ is the absorption cross section which will be explicitly defined in eqn. (27).

To summarize this section on formulation of the scattering problem, we have concluded that a prescription to evaluate the scattering amplitude is to use

$$f(\theta, \phi) = -\frac{1}{4\pi} \int d^3r' e^{-i\vec{k}' \cdot \vec{r}'} U(\vec{r}') \psi(\vec{r}') \quad (17)$$

where $\psi(\vec{r})$ is determined by solving

$$[\nabla^2 + k^2] \psi(\vec{r}) = U(\vec{r}) \psi(\vec{r}) \quad (18)$$

under the boundary conditions described after eqn. (1).

B. Identities satisfied by the scattering amplitude

Starting from the Schroedinger equation we shall derive two identities satisfied by the scattering amplitude. The first, the dynamical reversibility theorem was first derived by R. J. Glauber and V. Schomaker [11] in 1953. The second, the optical theorem, is a statement of probability conservation in quantum mechanics written as a continuity equation in reference to the scattering amplitude. The optical theorem was first derived by E. Feenberg [12] in 1932.

1. Dynamical reversibility theorem

In the Schroedinger equation if we write the wave function as

$$\psi'(\vec{r}, t; \vec{k}) = e^{-\frac{i}{\hbar}Et} \psi(\vec{r}; \vec{k}) \quad (19)$$

where $2mE = \hbar^2 k^2$, we can quickly derive the following identity

$$\psi(\vec{r}; -\vec{k}_2) \nabla^2 \psi(\vec{r}; \vec{k}_1) - \psi(\vec{r}; \vec{k}_1) \nabla^2 \psi(\vec{r}; -\vec{k}_2) = (k_1^2 - k_2^2) \psi(\vec{r}; \vec{k}_1) \psi(\vec{r}; \vec{k}_2). \quad (20)$$

Evaluating the above expression on a sphere of radius $r \rightarrow \infty$ for the case $|\vec{k}_1| = |\vec{k}_2|$ we have

$$\oint_{r \rightarrow \infty} d\vec{S} \cdot \left[\psi_{r \rightarrow \infty}(\vec{r}; -\vec{k}_2) \nabla^2 \psi_{r \rightarrow \infty}(\vec{r}; \vec{k}_1) - \psi_{r \rightarrow \infty}(\vec{r}; \vec{k}_1) \nabla^2 \psi_{r \rightarrow \infty}(\vec{r}; -\vec{k}_2) \right] = 0. \quad (21)$$

Using eqn. (13) in the above expression and evaluating the surface integrals after taking the limit $r \rightarrow \infty$ we get [5]

$$f(\vec{k}_2, \vec{k}_1) = f(-\vec{k}_1, -\vec{k}_2) \quad (22)$$

where we have used the notation we defined for $f(\theta, \phi)$ in the paragraph after eqn. (14). Thus the scattering amplitude for a process going from \vec{k}_1 to \vec{k}_2 is identical to that of a process going from $-\vec{k}_2$ to $-\vec{k}_1$. This is dynamical reversibility.

2. Optical theorem

In a very similar manner as we did in the earlier section (in this case we take the complex conjugate of the wave function) we arrive at the following continuity equation

$$\frac{\partial \rho(\vec{r})}{\partial t} + \vec{\nabla} \cdot \vec{j}(\vec{r}) = s(\vec{r}) \quad (23)$$

where

$$\begin{aligned}
\rho(\vec{r}) &= |\psi(\vec{r})|^2 \\
\vec{j}(\vec{r}) &= \frac{\hbar}{2im} \left[\psi^*(\vec{r}) \vec{\nabla} \psi(\vec{r}) - \psi(\vec{r}) \vec{\nabla} \psi^*(\vec{r}) \right] \\
s(\vec{r}) &= \frac{2}{\hbar} [\text{Im } V(\vec{r})] |\psi(\vec{r})|^2.
\end{aligned} \tag{24}$$

Integrating the continuity equation over a sphere of radius $r \rightarrow \infty$ and noting that the derivative of the ρ term does not contribute because both the states have the same energy we get

$$\oint_{r \rightarrow \infty} d\vec{S} \cdot [\psi_{r \rightarrow \infty}^*(\vec{r}) \nabla^2 \psi_{r \rightarrow \infty}(\vec{r}) - \psi_{r \rightarrow \infty}(\vec{r}) \nabla^2 \psi_{r \rightarrow \infty}^*(\vec{r})] = \lim_{r \rightarrow \infty} \int d^3r s(\vec{r}). \tag{25}$$

Again using eqn. (13) in the above expression and evaluating the surface integrals after taking the limit $r \rightarrow \infty$ we get [5]

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta |f(\theta, \phi)|^2 - \lim_{r \rightarrow \infty} \frac{2m}{\hbar^2 k} \int d^3r [\text{Im } V(\vec{r})] |\psi(\vec{r})|^2 = \frac{4\pi}{k} \text{Im} f(\theta = 0). \tag{26}$$

Using eqn. (15) and defining the absorption scattering cross section as

$$\sigma_{\text{abs.}} = - \lim_{r \rightarrow \infty} \frac{2m}{\hbar^2 k} \int d^3r [\text{Im } V(\vec{r})] |\psi(\vec{r})|^2 \tag{27}$$

we have the optical theorem

$$\sigma_{\text{scatt.}} + \sigma_{\text{abs.}} = \frac{4\pi}{k} \text{Im} f(\theta = 0). \tag{28}$$

III. PARTIAL WAVE EXPANSION

In this article we will be concerned with approximation methods for evaluating the scattering amplitude. In particular we will be interested in the eikonal approximation to the scattering amplitude. After we have made an approximation we would like to be aware of how much we have deviated from the exact result. The closest we can have to an exact result in a scattering problem is the result got by the method of partial wave expansion. We shall thus find it very useful to use the results obtained from the method of partial wave expansion as a benchmark.

The method of partial wave expansion breaks down the initial wavefunction into an infinite sum over angular momentum components labeled by the quantum number l . The contribution to the scattering amplitude from each l is calculated separately and called the partial wave scattering amplitude. The complete scattering amplitude is then obtained by summing over all the partial wave scattering amplitudes. We say it is the closest we can have to an exact expression because the solution cannot in general be written in a closed form. Nevertheless for an incoming beam of a given energy characterized by the parameter k we can always define its angular momentum as ka where a is the scattering length defined in section III B in terms of the partial wave scattering amplitude corresponding to $l = 0$. It turns out that for most of the potentials the contributions from l terms very much higher than ka converge very rapidly and thus can be neglected. Physically this is understood as, only those beams that have sufficient initial energy and those that pass through the scattering length of the potential are deflected.

If the method of partial wave expansion gives us the exact solution why do we need approximation methods? Firstly because for almost all potentials (even for the simple Gaussian potential) we need to depend on numerical computational methods to solve the radial differential equation. Secondly for high energy scattering problems we need to sum over a large range of l .

For potentials involving a boundary beyond which the potential is zero it sometimes becomes possible to evade the need for computational methods to a significant extent. For the simplest potential of this class, that of an inverted finite square well, the radial differential equation mentioned above can be solved exactly. Thus we shall solve the inverted finite square well in detail and use it for comparison with the results obtained from other approximation methods.

A. Method of partial wave expansion

For a spherically symmetric potential of the form

$$V(\vec{r}) = V(r) \quad (29)$$

we can write the solution to eqn. (18) as

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} (2l+1) i^l R_l(r) P_l(\cos \theta) \quad (30)$$

where where $R_l(r)$ is the solution to the radial differential equation

$$\frac{d^2 R_l(r)}{dr^2} + \frac{2}{r} \frac{dR_l(r)}{dr} + \left[k^2 - \frac{l(l+1)}{r^2} - U(r) \right] R_l(r) = 0. \quad (31)$$

For potentials that die off faster than $\frac{1}{r^2}$ we can write the solution to eqn. (31) in the $r \rightarrow \infty$ region to be

$$\lim_{r \rightarrow \infty} R_l(r) = C_l \left[\cos \delta_l \left(\lim_{r \rightarrow \infty} j_l(kr) \right) - \sin \delta_l \left(\lim_{r \rightarrow \infty} \eta_l(kr) \right) \right]. \quad (32)$$

Substituting eqn. (32) in eqn. (30) and using the identity

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta) \quad (33)$$

in eqn. (14) we have

$$f(\theta, \phi) \frac{e^{ikr}}{r} = \sum_{l=0}^{\infty} (2l+1) i^l P_l(\cos \theta) \left[(C_l \cos \delta_l - 1) \left(\lim_{r \rightarrow \infty} j_l(kr) \right) - C_l \sin \delta_l \left(\lim_{r \rightarrow \infty} \eta_l(kr) \right) \right]. \quad (34)$$

Using

$$\begin{aligned} \lim_{r \rightarrow \infty} j_l(kr) &= \frac{1}{kr} \cos \left[kr - (l+1) \frac{\pi}{2} \right] \\ &= \frac{1}{2kr} \left[i^{-(l+1)} e^{ikr} + i^{+(l+1)} e^{-ikr} \right] \end{aligned} \quad (35)$$

$$\begin{aligned} \lim_{r \rightarrow \infty} \eta_l(kr) &= \frac{1}{kr} \sin \left[kr - (l+1) \frac{\pi}{2} \right] \\ &= \frac{1}{2kr} \left[i^{-(l+2)} e^{ikr} - i^l e^{-ikr} \right] \end{aligned} \quad (36)$$

and equating the coefficients of e^{ikr} and e^{-ikr} in eqn. (34) we get

$$f(\theta) = \frac{1}{2k} \sum_{l=0}^{\infty} (2l+1) i^l P_l(\cos \theta) i^{-(l+1)} [C_l e^{+i\delta_l} - 1] \quad (37)$$

$$0 = \frac{1}{2k} \sum_{l=0}^{\infty} (2l+1) i^l P_l(\cos \theta) i^{+(l+1)} [C_l e^{-i\delta_l} - 1]. \quad (38)$$

The second equation above further reduces to $C_l = e^{i\delta_l}$ which when used in eqn. (37) to eliminate C_l gives

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) [e^{2i\delta_l} - 1]. \quad (39)$$

This is the standard expression for the scattering amplitude given as a sum of partial waves. δ_l 's are defined as the phase shifts in Bessel functions (which are trigonometric functions in the $r \rightarrow \infty$ limit) as introduced in eqn. (32).

Observe that the phase shifts δ_l 's in eqn. (39) are in principle determined by solving the radial differential equation in the presence of $U(r)$ which can be solved analytically only for special cases. In practice the phase shifts are evaluated by solving the radial equation numerically. For problems involving high energies a further complication arises because we need to sum over a sufficiently high number of partial waves.

For the case of a inverted square well potential it is possible to solve the radial equation exactly. In the subsequent section we shall write down an expression for δ_l for the inverted square well potential. We shall find it useful to compare the results later when we are doing eikonal approximations.

B. Scattering length of a potential

The scattering cross section is obtained by squaring eqn. (39) and integrating over all directions. Using the orthogonality of Legendre polynomials it turns out to be

$$\sigma_{\text{scatt.}} = \frac{4\pi}{k^2} \sum_0^{\infty} (2l+1) \sin^2 \delta_l. \quad (40)$$

The contribution to the scattering cross section from the s-wave ($l=0$) is

$$\{\sigma_{\text{scatt.}}\}_{l=0} = \frac{4\pi}{k^2} \sin^2 \delta_0. \quad (41)$$

The scattering length ‘ a ’ is defined as the radius of the sphere that will give a contribution in a classical scattering problem equal to the s-wave contribution above. Correspondingly we write

$$\{\sigma_{\text{scatt.}}\}_{l=0} = 4\pi a^2. \quad (42)$$

Thus we get

$$a = \frac{1}{k} \sin \delta_0. \quad (43)$$

Observe that $\sin \delta_0$ will require the solution to the differential equation in eqn. (31) for $l=0$ which normally can only be solved numerically.

C. Inverted finite spherical well potential

For a spherically symmetric potential of the form

$$V(\vec{r}) = \begin{cases} V(r) & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} \quad (44)$$

we can write

$$R_l(r) = \begin{cases} R_{l<}(r) & \text{for } r < a \\ C_l [\cos \delta_l j_l(kr) - \sin \delta_l \eta_l(kr)] & \text{for } r > a \end{cases} \quad (45)$$

where $R_{l<}(r)$ is the solution to eqn. (31) for $r < a$. Requiring the wave function and its derivative to be continuous and taking the ratio of the equations got by requiring the continuity we get

$$\frac{a \left. \frac{dR_{l<}(r)}{dr} \right|_{r=a}}{R_{l<}(a)} = \frac{ka \left[\frac{d}{dka} j_l(ka) - \tan \delta_l \frac{d}{dka} \eta_l(ka) \right]}{[j_l(ka) - \tan \delta_l \eta_l(ka)]}. \quad (46)$$

The above expression can be solved for $\tan \delta_l$ if $R_{l<}(r)$ is known at $r = a$.

For an inverted square well potential defined by

$$V(\vec{r}) = \begin{cases} V & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} \quad (47)$$

we further have

$$R_{l<}(r) = B_l j_l(\alpha r) \quad (48)$$

where $\alpha = k \sqrt{1 - \frac{V}{E}}$ with $\frac{V}{E} < 1$. For this case we can write the explicit expression for the phase shifts to be

$$\sin^2 \delta_l = \frac{N^2(ka, \alpha a)}{N^2(ka, \alpha a) + D^2(ka, \alpha a)} \quad (49)$$

where

$$\begin{aligned} N(ka, \alpha a) &= \alpha a j_{l-1}(\alpha a) j_l(ka) - ka j_l(\alpha a) j_{l-1}(ka) \\ D(ka, \alpha a) &= \alpha a j_{l-1}(\alpha a) \eta_l(ka) - ka j_l(\alpha a) \eta_{l-1}(ka). \end{aligned} \quad (50)$$

In fig. 2 we plot $\sigma_{\text{tot.}}$ versus $ka \frac{V}{E}$ for the inverted spherical well potential for $ka = 500$.

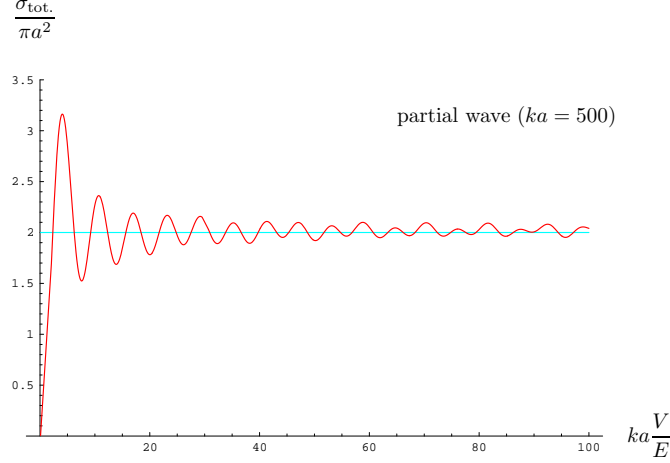


FIG. 2. $\sigma_{\text{tot.}}$ versus $ka\frac{V}{E}$ for $ka = 500$ using partial wave method.

IV. EIKONAL APPROXIMATION IN QUANTUM MECHANICS

In this section we shall convince ourselves that the eikonal approximation is valid for processes involving small angle scattering and very large incoming momentum. More rigorously the conditions are $\frac{V}{E} \ll 1$ and $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$. In this stringent parameter zone the expression for scattering amplitude takes the form

$$f(\theta, \phi) = \frac{k}{2\pi i} \int d^2b' e^{-i\vec{k}' \cdot \vec{b}'} \left[e^{i\chi(\vec{b}')} - 1 \right] \quad (51)$$

where

$$\chi(\vec{b}') = -\frac{1}{2k} \frac{2m}{\hbar^2} \int_{-\infty}^{+\infty} dz' V(\vec{b}', z'). \quad (52)$$

The scattering cross section takes the form

$$\sigma_{\text{scatt.}} = 8\pi a^2 \int_0^\infty dt t \sin^2 \left[\frac{k}{E} \int_{-\infty}^{+\infty} dz' V(\vec{b}', z') \right]. \quad (53)$$

We shall derive the above formulas in this section.

A. Formalism for the eikonal approximation

In eqns. (17) and (18) we found the prescription for calculating the scattering amplitude to be

$$f(\theta, \phi) = -\frac{1}{4\pi} \int d^3r' e^{-i\vec{k}' \cdot \vec{r}'} U(\vec{r}') \psi(\vec{r}') \quad (54)$$

where $\psi(\vec{r}')$ is determined by solving

$$[\nabla^2 + k^2] \psi(\vec{r}') = U(\vec{r}') \psi(\vec{r}'). \quad (55)$$

To solve for $\psi(\vec{r}')$ let us write

$$\psi(\vec{r}') = e^{i\vec{k} \cdot \vec{r}'} \phi(\vec{r}'). \quad (56)$$

With the above substitution and with the choice \vec{k} along the z direction eqn. (55) takes the form

$$\left[2ik \frac{\partial}{\partial z} - U(\vec{b}, z) \right] \phi(\vec{b}, z) = -\nabla^2 \phi(\vec{b}, z) \quad (57)$$

where we have used the notation $\vec{r} \equiv (\vec{b}, z)$. We can write the formal solution to eqn. (57) as

$$\phi(\vec{b}, z) = \eta(\vec{b}, z) - \int d^2b' \int_{-\infty}^{\infty} dz' G_e(\vec{b}, z, \vec{b}', z') \nabla'^2 \phi(\vec{b}', z') \quad (58)$$

where $\eta(\vec{b}, z)$ satisfies

$$\left[2ik \frac{\partial}{\partial z} - U(\vec{b}, z) \right] \eta(\vec{b}, z) = 0 \quad (59)$$

and $G_e(\vec{b}, z, \vec{b}', z')$ satisfies

$$\left[2ik \frac{\partial}{\partial z} - U(\vec{b}, z) \right] G_e(\vec{b}, z, \vec{b}', z') = \delta^{(2)}(\vec{b} - \vec{b}') \delta(z - z') \quad (60)$$

with boundary conditions on $\eta(\vec{b}, z)$ and $G_e(\vec{b}, z, \vec{b}', z')$ prescribed by the boundary conditions on $\psi(\vec{r})$. The solutions to eqns. (59) and (60) are

$$\eta(\vec{b}, z) = e^{\frac{1}{2ik} \int_{-\infty}^z du U(\vec{b}, u)} \quad (61)$$

where we imposed the boundary condition $\eta_0(\vec{b}) = \eta(\vec{b}, z = -\infty) = 1$, and

$$G_e(\vec{b}, z, \vec{b}', z') = \frac{1}{2ik} \delta^{(2)}(\vec{b} - \vec{b}') \theta(z - z') e^{\frac{1}{2ik} \int_z^{z'} du U(\vec{b}, u)} \quad (62)$$

$$= \frac{1}{2ik} \delta^{(2)}(\vec{b} - \vec{b}') \theta(z - z') \eta(\vec{b}, z) \eta^{-1}(\vec{b}, z'). \quad (63)$$

Using eqns. (61) and (63) in eqn. (58) we have

$$\phi(\vec{b}, z) = \eta(\vec{b}, z) \left[1 - \frac{1}{2ik} \int_{-\infty}^z dz' \eta^{-1}(\vec{b}, z') \left(\nabla_b^2 + \frac{\partial^2}{\partial z'^2} \right) \phi(\vec{b}, z') \right] \quad (64)$$

which after iteration takes the form

$$\phi(\vec{b}, z) = \eta(\vec{b}, z) \left[1 + \int_{-\infty}^z dz' K(\vec{b}, z', \vec{\nabla}_b, \frac{\partial}{\partial z'}) + \int_{-\infty}^z dz' K(\vec{b}, z', \vec{\nabla}_b, \frac{\partial}{\partial z'}) \int_{-\infty}^{z'} dz'' K(\vec{b}, z'', \vec{\nabla}_b, \frac{\partial}{\partial z''}) + \dots \right] \quad (65)$$

where the expression for $K(\vec{b}, z, \vec{\nabla}_b, \frac{\partial}{\partial z})$ acting on an arbitrary function $g(z)$ is given by

$$K(\vec{b}, z, \vec{\nabla}_b, \frac{\partial}{\partial z}) g(z) = \frac{-1}{2ik} \eta^{-1}(\vec{b}, z) \left(\nabla_b^2 + \frac{\partial^2}{\partial z^2} \right) \eta(\vec{b}, z) g(z). \quad (66)$$

Using the above series expansion for $\phi(\vec{b}, z)$ in eqn. (56) and plugging it in eqn. (54) we can write the scattering amplitude as

$$f(\theta, \phi) = f^{(0)}(\theta, \phi) + f^{(1)}(\theta, \phi) + f^{(2)}(\theta, \phi) + \dots \quad (67)$$

where

$$\begin{aligned} f^{(0)}(\theta, \phi) &= -\frac{1}{4\pi} \int d^2b' \int_{-\infty}^{+\infty} dz' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} U(\vec{b}', z') \eta(\vec{b}', z') \\ f^{(1)}(\theta, \phi) &= -\frac{1}{4\pi} \int d^2b' \int_{-\infty}^{+\infty} dz' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} U(\vec{b}', z') \eta(\vec{b}', z') \int_{-\infty}^{z'} dz'' K(\vec{b}', z'') \\ f^{(2)}(\theta, \phi) &= -\frac{1}{4\pi} \int d^2b' \int_{-\infty}^{+\infty} dz' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} U(\vec{b}', z') \eta(\vec{b}', z') \int_{-\infty}^{z'} dz'' K(\vec{b}', z'') \int_{-\infty}^{z''} dz''' K(\vec{b}', z'''). \end{aligned} \quad (68)$$

We have used $K(\vec{b}, z, \vec{\nabla}_b, \frac{\partial}{\partial z}) \equiv K(\vec{b}, z)$ for compactness. The power in the exponent $i(\vec{k} - \vec{k}') \cdot \vec{r}'$ in the above expressions can be explicitly written as

$$i(\vec{k} - \vec{k}') \cdot \vec{r}' = -ikb' \sin \theta \cos(\phi - \phi') + ikz' 2 \sin^2 \frac{\theta}{2}. \quad (69)$$

The coordinates used are pictorially described in fig. 1.

B. Validity of the eikonal approximation

Starting from the expression for $f^{(0)}(\theta, \phi)$ in eqn. (68) and switching to dimensionless variables $z \rightarrow at$, $\vec{b} \rightarrow a\vec{w}$ and $V(\vec{b}, z) \rightarrow Vv(\vec{w}, z)$ where 'a' is the scattering length, we can write

$$f(\theta, \phi) = -\frac{a}{4\pi}(ka)^2 \frac{V}{E} \left[g^{(0)}(ka) + \frac{V}{E} g_A^{(1)}(ka) + ka \left(\frac{V}{E} \right)^2 g_B^{(1)}(ka) + \dots \right] \quad (70)$$

where we have suppressed the θ and ϕ dependence in $g^{(n)}$'s. The explicit expressions for the $g^{(0)}$, $g_A^{(1)}$ and $g_B^{(1)}$ are

$$\begin{aligned} g^{(0)} &= \int d^2w' \int_{-\infty}^{+\infty} dt' e^{-ikaw' \sin \theta \cos(\phi-\phi') + ikat' 2 \sin^2 \frac{\theta}{2} v(\vec{w}', t')} \eta(\vec{w}', t') \\ |g^{(0)}| &\leq \int d^2w' \int_{-\infty}^{+\infty} dt' |v(\vec{w}', t')| \\ g_A^{(1)} &= -\frac{1}{4} \int d^2w' \int_{-\infty}^{+\infty} dt' e^{-ikaw' \sin \theta \cos(\phi-\phi') + ikat' 2 \sin^2 \frac{\theta}{2} v(\vec{w}', t')^2} \eta(\vec{w}', t') \\ &\quad -\frac{1}{4} \int d^2w' \int_{-\infty}^{+\infty} dt' e^{-ikaw' \sin \theta \cos(\phi-\phi') + ikat' 2 \sin^2 \frac{\theta}{2} v(\vec{w}', t')} \eta(\vec{w}', t') \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' |\nabla_{w'}^2 v(\vec{w}', t''')| \\ |g_A^{(1)}| &\leq \frac{1}{4} \int d^2w' \int_{-\infty}^{+\infty} dt' |v(\vec{w}', t')|^2 + \frac{1}{4} \int d^2w' \int_{-\infty}^{+\infty} dt' |v(\vec{w}', t')| \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' |\nabla_{w'}^2 v(\vec{w}', t''')| \\ g_B^{(1)} &= -\frac{i}{8} \int d^2w' \int_{-\infty}^{+\infty} dt' e^{-ikaw' \sin \theta \cos(\phi-\phi') + ikat' 2 \sin^2 \frac{\theta}{2} v(\vec{w}', t')} \eta(\vec{w}', t') \int_{-\infty}^{t'} dt'' v(\vec{w}', t'')^2 \\ &\quad -\frac{i}{8} \int d^2w' \int_{-\infty}^{+\infty} dt' e^{-ikaw' \sin \theta \cos(\phi-\phi') + ikat' 2 \sin^2 \frac{\theta}{2} v(\vec{w}', t')} \eta(\vec{w}', t') \int_{-\infty}^{t'} dt'' \left[\int_{-\infty}^{t''} dt''' \nabla_{w'}^2 v(\vec{w}', t''') \right]^2 \\ |g_B^{(1)}| &\leq \frac{1}{8} \int d^2w' \int_{-\infty}^{+\infty} dt' |v(\vec{w}', t')| \int_{-\infty}^{t'} dt'' |v(\vec{w}', t'')|^2 \\ &\quad + \frac{1}{8} \int d^2w' \int_{-\infty}^{+\infty} dt' |v(\vec{w}', t')| \int_{-\infty}^{t'} dt'' \left| \int_{-\infty}^{t''} dt''' \nabla_{w'}^2 v(\vec{w}', t''') \right|^2. \end{aligned} \quad (72)$$

It is easy to verify from the above expressions that for potentials that fall away faster than $\frac{1}{t\sqrt{w}}$ the contributions to $g^{(n)}$'s are finite. For these potentials it can thus be concluded that major contribution to the scattering amplitude comes from the zeroth order for $\frac{V}{E} \ll 1$ and $ka \left(\frac{V}{E} \right)^2 \ll 1$.

We shall later observe that the expression for the scattering amplitude obtained from the zeroth order eikonal approximation satisfies the optical theorem only in the limit $ka \frac{V}{E} \rightarrow \infty$. This requirement can be achieved only if we additionally require $ka \gg \frac{1}{\sqrt{V/E}}$.

Thus to summarize, the conditions under which the eikonal approximation holds are

$$\frac{V}{E} \ll 1 \quad \text{and} \quad \frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}. \quad (73)$$

C. Zeroth order eikonal approximation

Thus for potentials that die out sufficiently fast and for conditions $\frac{V}{E} \ll 1$ and $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$ the zeroth order contribution to $f(\theta, \phi)$ is the most significant. From eqns. (67), (68) and (61) we can write

$$\begin{aligned} f^{(0)}(\theta, \phi) &= -\frac{1}{4\pi} \int d^2b' \int_{-\infty}^{+\infty} dz' e^{i(\vec{k}-\vec{k}') \cdot \vec{r}'} U(\vec{b}', z') \eta(\vec{b}', z') \\ &= -\frac{1}{4\pi} \int d^2b' \int_{-\infty}^{+\infty} dz' e^{-ikb' \sin \theta \cos(\phi-\phi') + ikz' 2 \sin^2 \frac{\theta}{2} U(\vec{b}', z')} e^{\frac{1}{2ik} \int_{-\infty}^{z'} du U(\vec{b}', u)}. \end{aligned} \quad (74)$$

For large incoming momentum and for small scattering angles most of the contribution to the scattering cross section comes from angles less than or of order $\frac{V}{E}$. Thus in the zeroth order eikonal approximation we can approximate

$$-ikb' \sin \theta \cos(\phi - \phi') + ikz' 2 \sin^2 \frac{\theta}{2} \approx -ikb' \theta \cos(\phi - \phi'). \quad (75)$$

Using eqn. (75) in eqn. (74) we have

$$f^{(0)}(\theta, \phi) = -\frac{1}{4\pi} \int_0^\infty b' db' \int_0^{2\pi} d\phi' e^{-ikb' \theta \cos(\phi - \phi')} \int_{-\infty}^{+\infty} dz' U(\vec{b}', z') e^{\frac{1}{2ik} \int_{-\infty}^{z'} du U(\vec{b}', u)}. \quad (76)$$

We notice that the approximation involved in eqn. (75) allows us to carry out the z integral by substituting $\int_{-\infty}^{z'} du U(\vec{b}', u)$ for a new integration variable. After evaluating the z integral we get

$$f^{(0)}(\theta, \phi) = \frac{k}{2\pi i} \int_0^\infty b' db' \int_0^{2\pi} d\phi' e^{-ikb' \theta \cos(\phi - \phi')} \left[e^{i\chi(\vec{b}')} - 1 \right] \quad (77)$$

where

$$\chi(\vec{b}') = -\frac{k}{2} \frac{1}{E} \int_{-\infty}^{+\infty} dz' V(\vec{b}', z'). \quad (78)$$

For the case when the potential is independent of ϕ variable such that the scattering is symmetric about the z -axis we can further carry out the ϕ' integral and thus get

$$f^{(0)}(\theta) = \frac{k}{i} \int_0^\infty b' db' J_0(kb'\theta) \left[e^{i\chi(b')} - 1 \right] \quad (79)$$

where we have used the identity

$$J_0(t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-it \cos \phi} \quad (80)$$

and $J_0(t) = J_0(-t)$. For making the above expressions more illustrative we scale the integral variables with respect to the scattering length of the potential defined in section III B by introducing the dimensionless integral variables u and t defined as $z = au$ and $b = at$, where a is the scattering length of the potential. We also use V to signify the maximum value of the function $V(\vec{r})$. In term of these variables scattering amplitude in eqn. (79) takes the form

$$f(\theta) = a \frac{1}{i} ka \int_0^\infty t dt J_0(t ka\theta) \left[e^{ika \frac{V}{E} \xi(t)} - 1 \right] \quad (81)$$

where

$$\xi(t) = -\frac{1}{2} \frac{1}{V} \int_{-\infty}^{+\infty} du V(at, au). \quad (82)$$

Using eqn. (15) we have the expression for the scattering cross section to be

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 2(ka)^2 \int_0^\infty t dt \int_0^\infty t' dt' \left[e^{+ika \frac{V}{E} \xi(t)} - 1 \right] \left[e^{-ika \frac{V}{E} \xi(t')} - 1 \right] \int_0^\pi \sin \theta d\theta J_0(t ka\theta) J_0(t' ka\theta). \quad (83)$$

We said earlier that $\frac{V}{E}$ is a good estimate for the upper limit of the scattering angle θ . Thus in the limit $\frac{V}{E} \ll 1$ we can write $\sin \theta \approx \theta$ and limit our range of integration in θ from 0 to $\frac{V}{E}$. Further introducing the integration variable $x = ka\theta$ we have

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 2 \int_0^\infty t dt \int_0^\infty t' dt' \left[e^{+ika \frac{V}{E} \xi(t)} - 1 \right] \left[e^{-ika \frac{V}{E} \xi(t')} - 1 \right] \int_0^{ka \frac{V}{E}} x dx J_0(tx) J_0(t'x). \quad (84)$$

Once we have the above expressions for the scattering amplitude in eqn. (81) and scattering cross section in eqn. (84) we would like to check if the optical theorem is satisfied in the eikonal approximation. For potentials that are not complex the optical theorem can be stated as

$$\sigma_{\text{scatt.}} = \frac{4\pi}{k} \text{Im } f(0). \quad (85)$$

Using eqn. (81) in eqn. (85) we have

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 8 \int_0^\infty t dt \sin^2 \left[ka \frac{V}{E} \xi(t) \right]. \quad (86)$$

Comparing eqn. (84) and eqn. (86) we conclude that under the conditions $\frac{V}{E} \ll 1$ and $ka \left(\frac{V}{E}\right)^2 \ll 1$ alone the eikonal approximation does not satisfy the optical theorem. The physical content of the optical theorem is the statement of probability conservation in quantum mechanics. Can we save the eikonal approximation from violating the optical theorem? Yes. We recognize that in eqn. (84) if we take the limit $ka \frac{V}{E} \rightarrow \infty$ and use the identity

$$\int_0^\infty x dx J_0(tx) J_0(t'x) = \frac{\delta(t-t')}{t} \quad (87)$$

we get exactly the expression required for the optical theorem to be satisfied. Thus we observe that even though the expression for the scattering amplitude in the eikonal approximation is derived under the conditions $\frac{V}{E} \ll 1$ and $ka \left(\frac{V}{E}\right)^2 \ll 1$ alone, we still need to put an additional condition $ka \frac{V}{E} \gg 1$ for it to satisfy the optical theorem. Thus we have the eikonal approximation valid under the conditions

$$\frac{V}{E} \ll 1 \quad \text{and} \quad \frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}. \quad (88)$$

For a given ka , the second inequality above puts an upper and lower bound on $\frac{V}{E}$ given by $\frac{1}{ka} \ll \frac{V}{E} \ll \frac{1}{\sqrt{ka}}$. This can be rewritten in the form

$$1 \ll ka \frac{V}{E} \ll \sqrt{ka}. \quad (89)$$

D. Examples

1. Inverted finite spherical well potential

An inverted finite spherical well potential is defined as

$$V(\vec{r}) = \begin{cases} V & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}. \quad (90)$$

Noting that $b^2 + z^2 = r^2$ and thus integrating z from 0 to $\sqrt{r^2 - b^2}$ we have for this potential

$$\begin{aligned} \xi(t) &= -\sqrt{1-t^2} \\ f(\theta) &= a \frac{1}{i} ka \int_0^1 t dt J_0(t ka \theta) \left[e^{-ika \frac{V}{E} \sqrt{1-t^2}} - 1 \right] \\ \frac{\sigma_{\text{scatt.}}}{\pi a^2} &= 8 \int_0^1 t dt \sin^2 \left[ka \frac{V}{E} \sqrt{1-t^2} \right]. \end{aligned} \quad (91)$$

In figs. 3 and 4 we plot $\sigma_{\text{scatt.}}$ versus $ka \frac{V}{E}$ and compare it with the plots got using the partial wave method. Using eqn. (89) for $ka = 50$, we have the region of validity to be $1 \ll ka \frac{V}{E} \ll 7$, and for $ka = 750$, we have the region of validity to be $1 \ll ka \frac{V}{E} \ll 27$. These estimates for the region of validity are in agreement with the region where the curves obtained from the eikonal approximation fit the curve obtained from partial wave method in figs. 3 and 4.

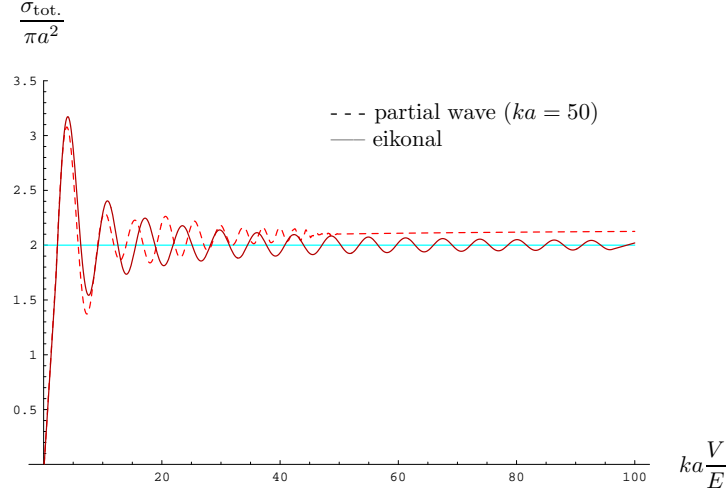


FIG. 3. Comparison of eikonal and partial wave method ($ka = 50$) for the inverted finite spherical well.

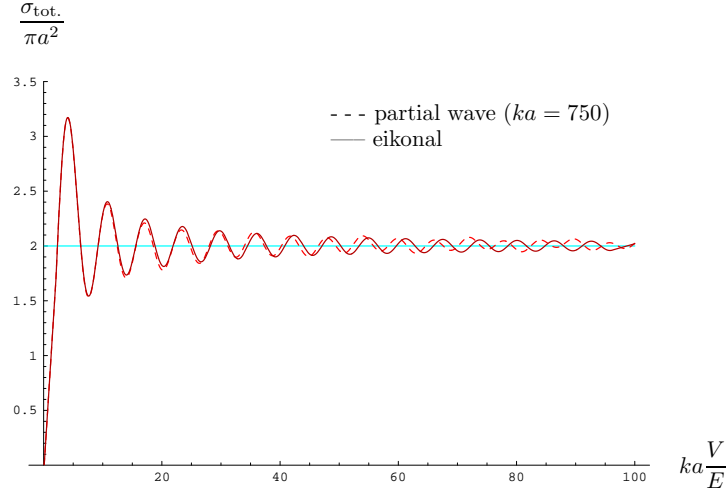


FIG. 4. Comparison of eikonal and partial wave method ($ka = 750$) for the inverted finite spherical well.

2. Yukawa potential

Consider the Yukawa potential

$$V(r) = -V \frac{e^{-r/a}}{r/a} \quad (92)$$

where $V = \frac{e^2}{a}$ and $r^2 = b^2 + z^2$. For this potential we get

$$\xi(t) = \frac{1}{2} \int_{-\infty}^{+\infty} du \frac{e^{-\sqrt{t^2+u^2}}}{\sqrt{t^2+u^2}} = \int_0^\infty d\theta e^{-t \cosh \theta} = K_0(t) \quad (93)$$

where $K_0(t)$ is modified Bessel function of order zero. Using this in eqn. (81) we have

$$f(\theta) = a \frac{1}{i} ka \int_0^\infty t dt J_0(t ka\theta) \left[e^{ika \frac{V}{E} K_0(t)} - 1 \right]. \quad (94)$$

The contribution to the above expression from $J_0(t ka\theta)$ dies off very fast. Thus to a good approximation we can say that we get non-zero contribution to the integral for $t ka\theta \ll 1$. Noting that in the eikonal approximation $\theta \lesssim \frac{V}{E}$ and $1 \ll ka\frac{V}{E} \ll \frac{1}{V/E}$ we conclude that non zero contributions to the integral comes from $t \ll 1$. In the limit $t \ll 1$ the modified Bessel function takes the form

$$K_0(t) \approx \ln \frac{2}{t} - \gamma \quad (95)$$

where $\gamma = 0.577\dots$ is the Euler's constant. Further we observe that one of the terms in eqn. (94) contributes only at $\theta = 0$ and thus is a delta function. Overall after substituting $x = ka\theta$ we get

$$f(\theta) = ika^2\delta(\theta) - ie^{ika\frac{V}{E}(\ln 2 + \ln(ka\theta) - \gamma)} \frac{1}{k\theta^2} \int_0^\infty dx J_0(x) x^{1-ika\frac{V}{E}}. \quad (96)$$

In terms of the gamma functions we have

$$\int_0^\infty dx J_n(x) x^{1+2i\alpha} = 2^{2i\alpha+1} \frac{\Gamma(1 + \frac{n}{2} + i\alpha)}{\Gamma(\frac{n}{2} - i\alpha)}. \quad (97)$$

Thus we can write

$$|f(\theta)|_{\theta \neq 0} = \frac{1}{2k(\frac{\theta}{2})^2} \frac{2}{ka\frac{V}{E}} \frac{\Gamma\left(1 - i\frac{2}{ka\frac{V}{E}}\right)}{\Gamma\left(1 + i\frac{2}{ka\frac{V}{E}}\right)} \quad (98)$$

where $\frac{1}{2}ka\frac{V}{E} = \frac{e^2}{\hbar v}$ using $V = \frac{e^2}{a}$ and $\hbar k = p = mv$. Observe that the above result is identical to the scattering amplitude due to a Coulomb potential for small scattering angles using the approximation $\sin \theta \approx \theta$. It is worth observing the interesting fact that the above result is independent of 'a'.

V. QUANTUM ELECTRODYNAMICS

A. Schwinger's functional differential equations for quantum electrodynamics

We define the vacuum to vacuum persistence amplitude for electrodynamics in the presence of external sources $\mathcal{J}(x) \equiv \{J_\mu(x), \eta(x), \bar{\eta}(x)\}$ to be

$$Z[\mathcal{J}] = \langle 0_+ | 0_- \rangle^{\mathcal{J}}. \quad (99)$$

Using Schwinger's quantum variational principle we can write

$$\delta Z[\mathcal{J}] = i \langle 0_+ | \delta W[\mathcal{A}; \mathcal{J}] | 0_- \rangle^{\mathcal{J}} \quad (100)$$

where $W[\mathcal{A}; \mathcal{J}]$ is the action for quantum electrodynamics given in terms of the fields $\mathcal{A}(x) \equiv \{A_\mu(x), F_{\mu\nu}(x), \psi(x), \bar{\psi}(x)\}$ interacting with external sources is $\mathcal{J}(x)$

$$W[\mathcal{A}; \mathcal{J}] = S[\mathcal{A}] + \int d^4x [J^\mu(x)A_\mu(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] \quad (101)$$

where

$$S[\mathcal{A}] = \int d^4x \left[-\frac{1}{2}F^{\mu\nu}(x)(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) + \frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) + \bar{\psi}(x)[i\gamma^\mu \partial_\mu - m + e\gamma^\mu A_\mu(x)]\psi(x) \right]. \quad (102)$$

Gauge invariance requires us to constrain the external source $J_\mu(x)$ to satisfy the condition $\partial^\mu J_\mu(x) = 0$. Using Schwinger's quantum variational principle to vary $Z[\mathcal{J}]$ with respect to the fields \mathcal{A} and the external sources \mathcal{J} we get

$$\begin{aligned}
\delta Z[\mathcal{J}] &= i \delta \bar{\eta}(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta \bar{\eta}(x)} | 0_- \rangle^{\mathcal{J}} + i \delta \eta(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta \eta(x)} | 0_- \rangle^{\mathcal{J}} + i \delta J^\mu(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta J^\mu(x)} | 0_- \rangle^{\mathcal{J}} \\
&+ i \delta \bar{\psi}(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta \bar{\psi}(x)} | 0_- \rangle^{\mathcal{J}} + i \delta \psi(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta \psi(x)} | 0_- \rangle^{\mathcal{J}} \\
&+ i \delta A^\mu(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta A^\mu(x)} | 0_- \rangle^{\mathcal{J}} + i \delta F^{\mu\nu}(x) \langle 0_+ | \frac{\delta W[\mathcal{A}, \mathcal{J}]}{\delta F^{\mu\nu}(x)} | 0_- \rangle^{\mathcal{J}} \\
&= i \delta \bar{\eta}(x) \langle 0_+ | \psi(x) | 0_- \rangle^{\mathcal{J}} + i \delta \eta(x) \langle 0_+ | \bar{\psi}(x) | 0_- \rangle^{\mathcal{J}} + i \delta J^\mu(x) \langle 0_+ | A_\mu(x) | 0_- \rangle^{\mathcal{J}} \\
&+ i \delta \bar{\psi}(x) [(i\gamma^\mu \partial_\mu - m) \langle 0_+ | \psi(x) | 0_- \rangle^{\mathcal{J}} + e \langle 0_+ | \gamma^\mu A_\mu(x) \psi(x) | 0_- \rangle^{\mathcal{J}} + \eta(x) \langle 0_+ | 0_- \rangle^{\mathcal{J}}] \\
&+ i \left[\langle 0_+ | \bar{\psi}(x) | 0_- \rangle^{\mathcal{J}} (-i\gamma^\mu \overleftarrow{\partial}_\mu - m) + e \langle 0_+ | \bar{\psi}(x) \gamma^\mu A_\mu(x) | 0_- \rangle^{\mathcal{J}} + \bar{\eta}(x) \langle 0_+ | 0_- \rangle^{\mathcal{J}} \right] \delta \psi(x) \\
&+ i \delta A^\mu(x) [-\partial^\mu \langle 0_+ | F_{\mu\nu}(x) | 0_- \rangle^{\mathcal{J}} + e \langle 0_+ | \bar{\psi}(x) \gamma_\mu \psi(x) | 0_- \rangle^{\mathcal{J}} + J_\mu(x) \langle 0_+ | 0_- \rangle^{\mathcal{J}}] \\
&+ i \delta F^{\mu\nu}(x) \frac{1}{2} [\langle 0_+ | F_{\mu\nu}(x) | 0_- \rangle^{\mathcal{J}} - \{\partial_\mu \langle 0_+ | A_\nu(x) | 0_- \rangle^{\mathcal{J}} - \partial_\nu \langle 0_+ | A_\mu(x) | 0_- \rangle^{\mathcal{J}}\}]. \tag{103}
\end{aligned}$$

In the above equation we have treated the variations in the fields to be c-number variations. Schwinger's quantum action principle states that $Z[\mathcal{J}]$ is stationary with respect to variations in the dynamical parameters (fields $\mathcal{A}(x)$ in the present case). Thus we can write

$$\begin{aligned}
(-i) \frac{\delta Z[\mathcal{J}]}{\delta J^\mu(x)} &= \langle 0_+ | A_\mu(x) | 0_- \rangle^{\mathcal{J}} \\
(-i) \frac{\delta Z[\mathcal{J}]}{\delta \bar{\eta}(x)} &= \langle 0_+ | \psi(x) | 0_- \rangle^{\mathcal{J}} \\
(-i) \frac{\delta Z[\mathcal{J}]}{\delta \eta(x)} &= \langle 0_+ | \bar{\psi}(x) | 0_- \rangle^{\mathcal{J}} \\
0 &= (i\gamma^\mu \partial_\mu - m) \langle 0_+ | \psi(x) | 0_- \rangle^{\mathcal{J}} + e \langle 0_+ | \gamma^\mu A_\mu(x) \psi(x) | 0_- \rangle^{\mathcal{J}} + \eta(x) \langle 0_+ | 0_- \rangle^{\mathcal{J}} \\
0 &= \langle 0_+ | \bar{\psi}(x) | 0_- \rangle^{\mathcal{J}} (-i\gamma^\mu \overleftarrow{\partial}_\mu - m) + e \langle 0_+ | \bar{\psi}(x) \gamma^\mu A_\mu(x) | 0_- \rangle^{\mathcal{J}} + \bar{\eta}(x) \langle 0_+ | 0_- \rangle^{\mathcal{J}} \\
0 &= -\partial^\mu \langle 0_+ | F_{\mu\nu}(x) | 0_- \rangle^{\mathcal{J}} + e \langle 0_+ | \bar{\psi}(x) \gamma_\mu \psi(x) | 0_- \rangle^{\mathcal{J}} + J_\mu(x) \langle 0_+ | 0_- \rangle^{\mathcal{J}} \\
0 &= \langle 0_+ | F_{\mu\nu}(x) | 0_- \rangle^{\mathcal{J}} - \{\partial_\mu \langle 0_+ | A_\nu(x) | 0_- \rangle^{\mathcal{J}} - \partial_\nu \langle 0_+ | A_\mu(x) | 0_- \rangle^{\mathcal{J}}\}. \tag{104}
\end{aligned}$$

Eliminating the fields in the above equations we get

$$-\left[i\gamma^\mu \partial_\mu - m + e\gamma^\mu (-i) \frac{\delta}{\delta J^\mu(x)} \right] (-i) \frac{\delta Z[\mathcal{J}]}{\delta \bar{\eta}(x)} = \eta(x) Z[\mathcal{J}] \tag{105}$$

$$(-i) \frac{\delta Z[\mathcal{J}]}{\delta \eta(x)} \left[-i\gamma^\mu \overleftarrow{\partial}_\mu - m + e\gamma^\mu (-i) \frac{\overleftarrow{\delta}}{\delta J^\mu(x)} \right] = \bar{\eta}(x) Z[\mathcal{J}] \tag{106}$$

$$-\left[g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right] (-i) \frac{\delta Z[\mathcal{J}]}{\delta J^\mu(x)} = J^\mu(x) Z[\mathcal{J}] + e (-i) \frac{\delta}{\delta \eta(x)} \gamma^\mu (-i) \frac{\delta Z[\mathcal{J}]}{\delta \bar{\eta}(x)}. \tag{107}$$

These are the functional differential equations for $Z[\mathcal{J}]$. This was first written down in this form by Julian Schwinger [13,14]. Since $Z[\mathcal{J}]$ is the generating functional for the Green functions it has the information regarding all possible processes in quantum electrodynamics. Conversely a solution to the above functional differential equations solves quantum electrodynamics completely.

B. A formal solution to Schwinger's functional differential equations

Using Schwinger's quantum variation principle in eqn. (100) for the variation of $Z[\mathcal{J}]$ with respect to e we have

$$\begin{aligned}
(-i) \frac{\partial Z[\mathcal{J}]}{\partial e} &= \langle 0_+ | \frac{\partial W[\mathcal{A}, \mathcal{J}]}{\partial e} | 0_- \rangle^{\mathcal{J}} \\
&= \int d^4x \langle 0_+ | \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x) | 0_- \rangle^{\mathcal{J}}. \tag{108}
\end{aligned}$$

Using eqn. (104) to eliminate the fields we get the first order differential equation

$$\frac{\partial Z[\mathcal{J}]}{\partial e} = i \int d^4x (-i) \frac{\delta}{\delta \eta(x)} \gamma^\mu (-i) \frac{\delta}{\delta J^\mu(x)} (-i) \frac{\delta}{\delta \bar{\eta}(x)} Z[\mathcal{J}] \quad (109)$$

Which can be integrated to yield the result

$$Z[\mathcal{J}] = e^{ie} \int d^4x (-i) \frac{\delta}{\delta \eta(x)} \gamma^\mu (-i) \frac{\delta}{\delta J^\mu(x)} (-i) \frac{\delta}{\delta \bar{\eta}(x)} Z_{e=0}[\mathcal{J}] \quad (110)$$

Where $Z_{e=0}[\mathcal{J}]$ is the generating function for the case when the photon field is not coupled to the fermion field. It is fairly straight forward to show [19] that the solution to eqns. (105), (106) and (107) for the $e = 0$ case is

$$Z_{e=0}[\mathcal{J}] = e^{\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_+^{\mu\nu}(x-x') J_\nu(x')} e^i \int d^4x \int d^4x' \bar{\eta}(x) S_+(x-x') \eta(x') \quad (111)$$

where the Green functions $D_+^{\mu\nu}(x-x')$ and $S_+(x-x')$ satisfy the differential equations

$$-(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) D_+^{\mu\nu}(x-x') = g^{\mu\nu} \delta^{(4)}(x-x') \quad (112)$$

$$-(\gamma^\mu \partial_\mu - m) S_+(x-x') = \delta^{(4)}(x-x') \quad (113)$$

with the boundary conditions on the Green functions prescribed by the requirement $|Z|^2 \leq 1$. Using eqn. (111) in eqn. (110) we have

$$Z[\mathcal{J}] = e^{-e} \int d^4x \int d^4x' \frac{\delta}{\delta \eta(x)} \delta^{(4)}(x-x') \gamma^\mu \frac{\delta}{\delta J^\mu(x)} \frac{\delta}{\delta \bar{\eta}(x')} e^i \int d^4x \int d^4x' \bar{\eta}(x) S_+(x-x') \eta(x') e^{\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_+^{\mu\nu}(x-x') J_\nu(x')}. \quad (114)$$

For x_i, y_i being Grassmann variables we have the identity

$$e^{\frac{\partial}{\partial y_m} a_{mn} \frac{\partial}{\partial x_n}} e^{x_i b^{ij} y_j} = e^{\vec{x}^T \cdot \hat{B} \cdot (\hat{1} + \hat{A} \cdot \hat{B})^{-1} \cdot \vec{y}} e^{\text{Tr} \ln(\hat{1} + \hat{A} \cdot \hat{B})} \quad (115)$$

where we have used the symbolic notation, \vec{a} (vector) for a^i and \hat{A} (matrix) for A_{ij} and Tr is the trace over the i, j indices. Generalizing the above result for the case of functionals dependent on Grassmann variables we have

$$e^{\int d^4x \int d^4x' \frac{\delta}{\delta \eta(x)} M(x-x') \frac{\delta}{\delta \bar{\eta}(x')} e^i \int d^4x \int d^4x' \bar{\eta}(x) N(x-x') \eta(x')} = e^{\int d^4x \int d^4x' \bar{\eta}(x) [N(\delta + MN)^{-1}(x,x')] \eta(x)} e^{\text{Tr} \ln(\delta + MN)} \quad (116)$$

where we have used the corresponding symbolic notation to suppress the integrals, Tr is the trace over both the spinor and coordinate index, and δ is the Dirac delta function. Using the above identity, the expression in eqn. (114) simplifies to

$$Z[\mathcal{J}] = e^i \int d^4x \int d^4x' \bar{\eta}(x) G_+[x, x'; (-i) \frac{\delta}{\delta J^\mu}] \eta(x) e^{L[(-i) \frac{\delta}{\delta J^\mu}]} e^{\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_+^{\mu\nu}(x-x') J_\nu(x')} \quad (117)$$

where

$$L[A] = -\text{Tr} \ln G_+[x, x'; A^\mu] + \text{Tr} \ln S_+(x-x') \quad (118)$$

with Tr denoting the trace over both spinor and coordinate index and the Green function $G_+[x, x'; A^\mu]$ satisfies the differential equation

$$[-(i\gamma^\mu \partial_\mu - m) + e\gamma^\mu A_\mu(x)] G_+[x, x'; A^\mu] = \delta^{(4)}(x-x') \quad (119)$$

with the boundary condition prescribed by the initial condition

$$\lim_{e \rightarrow 0} G_+[x, x'; A^\mu] = S_+(x-x'). \quad (120)$$

To get prepared for the next step we write down the following identity involving the Gaussian function and an arbitrary function $F(x)$ in terms of the real variable x

$$F\left(\frac{\partial}{\partial x}\right) e^{\frac{1}{2} x b x} = e^{\frac{1}{2} x b x} e^{\frac{1}{2} \frac{\partial}{\partial b x} b \frac{\partial}{\partial b x}} F(bx). \quad (121)$$

Generalizing the above identity for functionals we have

$$F[(-i)\frac{\delta}{\delta J^\mu}] e^{\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_+^{\mu\nu}(x-x') J_\nu(x')} = e^{\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_+^{\mu\nu}(x-x') J_\nu(x')} \times \left\{ e^{\frac{i}{2} \int d^4x \int d^4x' \frac{\delta}{\delta A^\mu(x)} D_+^{\mu\nu}(x-x') \frac{\delta}{\delta A^\nu(x)} F[A]} \right\}_{A=\int DJ} \quad (122)$$

where $A = \int DJ$ stands for $A^\mu(x) = \int d^4x' D_+^{\mu\nu}(x-x') J_\nu(x')$. Using the above identity in eqn. (117) we have

$$Z[\mathcal{J}] = e^{\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_+^{\mu\nu}(x-x') J_\nu(x')} \times \left\{ e^{\frac{i}{2} \int d^4x \int d^4x' \frac{\delta}{\delta A^\mu(x)} D_+^{\mu\nu}(x-x') \frac{\delta}{\delta A^\nu(x)} e^{i \int d^4x \int d^4x' \bar{\eta}(x) G_+[x,x';A^\mu] \eta(x)} e^{L[A]} \right\}_{A=\int DJ}. \quad (123)$$

We observe that our problem of determining $Z[\mathcal{J}]$ for quantum electrodynamics reduces to the evaluation of the Green function $G_+[x,x';A^\mu]$ for an arbitrary $A^\mu(x)$. Any kind of approximate solution to $G_+[x,x';A^\mu]$ suggests a possible non-perturbative approximate solution to $Z[\mathcal{J}]$.

C. Electron-electron scattering

We can expand $Z[\mathcal{J}]$ around $Z[0]$ and thus write

$$Z[\mathcal{J}] = Z[0] + \int d^4x \bar{\eta}(x) \left\{ \frac{\delta}{\delta \bar{\eta}(x)} Z[\mathcal{J}] \right\}_{\mathcal{J}=0} + \int d^4x \left\{ \frac{\delta}{\delta \eta(x)} Z[\mathcal{J}] \right\}_{\mathcal{J}=0} \eta(x) + \int d^4x J^\mu(x) \left\{ \frac{\delta}{\delta J^\mu(x)} Z[\mathcal{J}] \right\}_{\mathcal{J}=0} + \dots + T + \dots \quad (124)$$

where T is the quantity of interest in the electron-electron scattering process and is given by the expression

$$T = \frac{(-i)^4}{2! 2!} \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 \bar{\eta}_A(x_2) \bar{\eta}_B(x_1) \eta_C(y_2) \eta_D(y_1) G_{ABCD}(x_1, y_1, x_2, y_2) \quad (125)$$

where we have explicitly introduced the Dirac indices for clarity and

$$G_{ABCD}(x_1, y_1, x_2, y_2) = \left\{ (-i)^4 \frac{\delta}{\delta \bar{\eta}_A(x_2)} \frac{\delta}{\delta \bar{\eta}_B(x_1)} \frac{\delta}{\delta \eta_A(y_2)} \frac{\delta}{\delta \eta_B(y_1)} Z[\mathcal{J}] \right\}_{\mathcal{J}=0} = \left\{ e^{M[\frac{\delta}{\delta A}]} \left[G_+[x_1, y_1; A^\mu] G_+[x_2, y_2; A^\mu] e^{L[A]} \right] \right\}_{A=\int DJ, \mathcal{J}=0} \quad (126)$$

where we have introduced the short form for the operator

$$M \left[\frac{\delta}{\delta A} \right] = \frac{i}{2} \int d^4x \int d^4x' \frac{\delta}{\delta A^\mu(x)} D_+^{\mu\nu}(x-x') \frac{\delta}{\delta A^\nu(x)}. \quad (127)$$

T will be interpreted as the probability amplitude for the electron-electron scattering process. Every process in quantum electrodynamics can be generated out of the expansion in eqn. (124). It is due to this reason that $Z[\mathcal{J}]$ is also called the generating function.

For the electron-electron process we shall be interested in the case when the sources of the process $\bar{\eta}(x_1), \bar{\eta}(x_2), \eta(y_1)$ and $\eta(y_2)$ are interaction free states with well defined initial or final momentums. Thus we choose

$$\begin{aligned} \bar{\eta}(x_1) &= e^{-ip_1x_1} \bar{u}(p_1) [\gamma^\mu p_{1\mu} + m] \\ \eta(y_1) &= e^{ip_1'y_1} [\gamma^\mu p'_{1\mu} + m] u(p_1') \\ \bar{\eta}(x_2) &= e^{-ip_2x_2} \bar{u}(p_2) [\gamma^\mu p_{2\mu} + m] \\ \eta(y_2) &= e^{ip_2'y_2} [\gamma^\mu p'_{2\mu} + m] u(p_2'). \end{aligned} \quad (128)$$

Accordingly T will be denoted as $T(p_1, p_1', p_2, p_2')$. In the next section we shall concentrate on approximate solutions to $T(p_1, p_1', p_2, p_2')$.

VI. APPROXIMATION TO ELECTRON-ELECTRON SCATTERING

A. Quenched or ladder approximations

To start with we shall assume that $e^{L[A]}$ in eqn. (126) does not contribute. Further we assume that the operator M in eqn. (127) act in such a way that they connect the two fermion propagators via photon lines. In the diagrammatic language this amounts to omitting all those graphs that involve fermion loops and keeping only those graphs that have photons exchanged between the fermion lines. This is called the quenched or ladder approximation. In this approximation eqn. (126) reads

$$G_{ABCD}(x_1, y_1, x_2, y_2) = \left\{ e^{M_{12}[\frac{\delta}{\delta A}]} [G_+[x_1, y_1; A^\mu] G_+[x_2, y_2; A^\mu]] \right\}_{A=\int DJ, J=0} \quad (129)$$

where we introduced M_{12} in place of M to signify that it connects the two fermion lines. Using the identities

$$e^{M_{12}} = 1 + \int_0^1 d\lambda e^{\lambda M_{12}} M_{12} \quad (130)$$

and

$$\frac{\delta}{\delta A^\mu(z)} G_+[x_1, y_1; A^\mu] = G_+[x_1, z; A^\mu] e\gamma_\mu G_+[z, y_1; A^\mu] \quad (131)$$

we thus get

$$\begin{aligned} T(p_1, p'_1, p_2, p'_2) &= -i \int_0^1 d\lambda e^{\lambda M_{12}} \int d^4 z_1 \int d^4 z_2 D_+^{\mu\nu}(z_1 - z_2) \\ &\times \int d^4 x_1 e^{-ip_1 x_1} \bar{u}(p_1) [\gamma^\mu p_{1\mu} + m] G_+[x_1, z_1; A_1^\mu] e\gamma_\mu \int d^4 y_1 e^{ip'_1 y_1} G_+[z_1, y_1; A_1^\mu] [\gamma^\mu p'_{1\mu} + m] u(p'_1) \\ &\times \int d^4 x_2 e^{-ip_2 x_2} \bar{u}(p_2) [\gamma^\mu p_{2\mu} + m] G_+[x_2, z_2; A_2^\mu] e\gamma_\nu \int d^4 y_2 e^{ip'_2 y_2} G_+[z_2, y_2; A_2^\mu] [\gamma^\mu p'_{2\mu} + m] u(p'_2). \end{aligned} \quad (132)$$

B. Eikonal approximation to $G_+[x, y; A^\mu]$

Next step is to evaluate $G_+[x, y; A^\mu]$ in the eikonal approximation. We defined $G_+[x, y; A^\mu]$ in eqn. (119) to satisfy the differential equation given by

$$[-(i\gamma^\mu \partial_\mu - m) + e\gamma^\mu A_\mu(x)] G_+[x, y; A^\mu] = \delta^{(4)}(x - y). \quad (133)$$

In the eikonal approximation we have [19–22]

$$G_+[x, y; A^\mu] = i \int_0^\infty ds e^{-ims} \delta(x - y - s \frac{p}{m}) e^{ie \int_0^s d\tau \frac{1}{m} p^\mu A_\mu(x - \tau \frac{p}{m})}. \quad (134)$$

Using the above eikonal approximation to $G_+[x, y; A^\mu]$ in eqn. (132) it becomes possible to evaluate $T(p_1, p'_1, p_2, p'_2)$. Here we shall simply state the result without the detailed steps in the derivation. We define

$$T(p_1, p'_1, p_2, p'_2) = (2\pi)^4 \delta^{(4)}(p_1 + p'_1 - p_2 - p'_2) M(p_1, p'_1, p_2, p'_2) \quad (135)$$

and in the eikonal approximation we have the result [19]

$$|M(p_1, p'_1, p_2, p'_2)| = \frac{e^2 (p_1 + p_2)^2 \Gamma(1 - i\frac{e^2}{4\pi})}{2m^2 (p_1 - p'_1)^2 \Gamma(1 + i\frac{e^2}{4\pi})}. \quad (136)$$

We notice the close similarity of the above expression to the non-relativistic scattering amplitude due to the Coulomb potential. This is not surprising because eikonal approximation is a stringent semi-classical limit.

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²Webster's dictionary meaning: a person who champions the less accepted cause for the sake of argument.