If the mean colors or color excesses without regard to magnitude are taken over successive values of csc $|\beta|$ for northern and southern latitudes separately, they show linear relations with csc $|\beta|$. The difference in slope must, in view of the present results, be attributed to the greater mean z distances of the stars in northern latitudes, but the difference between the intercepts is not compatible with uniform reddening. With the present more inclusive observational material, this anomaly still persists. Taking for the intrinsic colors of the B2 and earlier stars $-.24$, the median color is found, for csc $|\beta| < 12.0$, as

$$C_m = -.24 + .018 \csc |\beta| \quad \beta > 0,$$
$$C_m = -.24 + .013 \csc |\beta| \quad \beta < 0. \qquad (5)$$

If this is broken up, the percentages having colors $> C_m$ are:

csc $\|\beta\|$	$\beta > 0$	$\beta < 0$
1– 6	41	50
6–12	57	51
1–12	50	50

In order to equalize the percentages for $\beta > 0$, the constant in the first line of (5) should be decreased to $-.26$ (which raises the coefficient to $+.021$), and this reaffirms the very same difference in the intercepts found before. The only way out seems that it should be ascribed to irregular absorption which would be expected to be more noticeable at the higher latitudes, i.e., shorter distances than at lower latitudes, because whereas the effect of distance dispersion increases with csc $|\beta|$, the effect of irregular absorption increases only with $(\csc |\beta|)^{\frac{1}{2}}$.

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THE EARTH'S EQUATORIAL RADIUS AND THE DISTANCE OF THE MOON

By JOHN A. O'KEEFE and J. PAMELIA ANDERSON

Introduction. B. Lindblad has remarked that geodetic measurements made by the eclipse method are sensitive to the assumed value of the lunar parallax.[1] Hence, an accurate measurement of the lunar parallax appeared to be useful, now that the eclipse technique is coming into wider use.

The instruments for this program are to be described elsewhere by D. D. Mears. It is sufficient here to say that, with the kind help of A. E. Whitford of the University of Wisconsin, Mr. Mears constructed a photoelectric system for recording the instants of occultation. The system was attached to a portable 12-inch Cassegrain telescope having an equatorial mounting. The output of the photocell was fed to a pen oscillograph constructed by the Brush Recorder Co.

It has been thought best to present the theory before the observations, in order that the program may be more easily understood, since it contains an unfamiliar combination of familiar ideas.

Theory. The theory of the method is a modification of that usual in the calculation of occultations. Let

$$\sigma^2 = (\xi - x)^2 + (\eta - y)^2, \qquad (1)$$

where x, y, ξ, η have the meanings given to them in the *Nautical Almanac*, except that they are here supposed to be calculated in meters. Specifically

$$x = p \sin (\alpha_☾ - \alpha_*) \cos \delta_☾,$$
$$y = p[\sin \delta_☾ \cos \delta_* - \cos \delta_☾ \sin \delta_* \cos (\alpha_☾ - \alpha_*)], \qquad (2)$$

where $\alpha_☾$, $\delta_☾$ refer to the moon; α_*, δ_* to the star; and p is the moon's distance in meters. Instead of the expressions in the *Nautical Almanac* for ξ and η, in terms of h, ϕ' and ρ, let us employ a system of rectangular coordinates u, v, and w, fixed in the earth. The w axis coincides with the earth's mean axis of rotation; the u axis is perpendicular to the w axis and is parallel to the plane of the mean astronomical meridian at Greenwich; the v axis is perpendicular to the other two, and positive toward India in order to yield a right-handed coordinate system. For the same reason, longitudes, λ, are counted eastward from Greenwich. If ϕ' is the geocentric latitude, and ρ the geocentric radius in meters,

$$u = \rho \cos \phi' \cos \lambda,$$
$$v = \rho \cos \phi' \sin \lambda,$$
$$w = \rho \sin \phi. \qquad (3)$$

It is considerably more practical, however, to make use of the radius of curvature ν of the ellipsoid perpendicular to the meridian, called the radius of curvature in the prime vertical, or, sometimes, the great normal. In terms of ν

$$u = (\nu + h) \cos \phi \cos \lambda,$$
$$v = (\nu + h) \cos \phi \sin \lambda,$$
$$w = [(1 - e^2)\nu + h] \sin \phi, \qquad (4)$$

where h is the height in meters above the ellipsoid, and ϕ is the geodetic latitude. The advantages of employing ν are:

1. 10-figure tables of ν are available.[2]

2. The direction of the great normal coincides with the vertical, whereas the geocentric radius is inclined to the vertical at an angle of about 11′ in middle latitudes. It follows that the effect of elevation can be taken into account by adding the elevation to the length of the great normal, leaving the value of ϕ unchanged. The value of ϕ', on the other hand, varies by a significant amount with a change in the elevation.

The absolute values of u, v, and w are uncertain by several hundred meters, chiefly because of uncertainties in the dimensions of the earth. On the other hand, differences of u, v, w between points in a single triangulation scheme, probably have errors of the order of 10 meters. The error can be considered as amounting to 5 meters in each of the two horizontal directions, due to errors in the triangulation,[3] and 5 meters in the vertical direction, due to errors in the determination of the form of the sea-level surface as extrapolated under the land. This surface is known as the geoid; its form is determined by the process known as astronomical leveling.[4] The measurement of heights above the geoid by the process of spirit-leveling may be regarded as error-free.

In considering a section of the geoid such as that which underlies the United States, it is customary to say that although its form is well-known, nevertheless the actual orientation may differ from the measured orientation by a tilt, either in the north-south direction or in the east-west direction. The tilts so described are with reference to the center of the earth; the orientation of the given region of the geoid with respect to a stellar frame of reference is well-established by astronomical observations. It is better to say that the direction of the center of the earth from the given region of the geoid is not known accurately.

The orientation of the known region of the geoid around a vertical axis is fixed by astronomic measurements of azimuth, A. Since the Laplace equation relating astronomic measurements (subscript A) to geodetic measurements (subscript G),

$$\sin \phi(\lambda_A - \lambda_G) = A_A - A_G \qquad (5)$$

is used in the adjustment of the U. S. net, it can be shown that the orientation of the net is likewise accurate as referred to the stars, within the accuracy of measurement. We conclude that for all points on 1927 North American datum for which geoidal heights are known as well as ordinary elevations the corrections Δu, Δv, Δw to the u, v, w coordinates are approximately constant.

In addition, these three corrections are independent. This would not be true of $\Delta\lambda$, $\Delta\phi'$, $\Delta\rho$, for example; a change of the assumed value of $\Delta\phi'$, keeping the measured lengths the same, implies changes of $\Delta\lambda$ which vary from point to point, owing to the convergence of the meridians.

By substituting equations (3) in the *Nautical Almanac* equations for ξ and η, it can easily be shown that:

$$\xi = u \sin \mu_* + v \cos \mu_*,$$
$$\eta = w \cos \delta_* - u \sin \delta_* \cos \mu_* + v \sin \delta_* \sin \mu_*, \qquad (6)$$

where μ_* is the star's Greenwich hour angle.

Referring to Eq. (1), we set

$$\Delta\sigma = \sigma - k, \qquad (7)$$

where k is the moon's radius in meters.

We shall now seek an equation of the form:

$$\Delta\sigma = b_1\Delta u + b_2\Delta v + b_3\Delta w + b_4\Delta\alpha_☾ + b_5\Delta\delta_☾ + b_6\Delta p + b_7\Delta k, \qquad (8)$$

in which the b's are coefficients whose expressions are found below. In equations of this type for the discussion of occultations it is customary to let the quantities Δu, Δv, etc., represent corrections to the computed quantities. On the other hand, the quantity $\Delta\sigma$ represents the distance measured outward from the calculated position of the moon's limb to the calculated position of the star. It has the nature of an error, rather than a correction. Thus if we form the coefficients b_1, b_2, b_3, etc. from the equation

$$d\sigma = \frac{\partial\sigma}{\partial u} du + \frac{\partial\sigma}{\partial v} dv + \cdots, \qquad (9)$$

we must notice the minus signs in the equation

$$b_1 = -\frac{\partial\sigma}{\partial u},\ b_2 = -\frac{\partial\sigma}{\partial v},\ \cdots \text{ etc.} \qquad (10)$$

and similar equations for $\Delta\xi$, Δx, etc. We differentiate (7), keeping (1) in mind, and find, using the above fact about the difference of sign:

$$\Delta\sigma = -\frac{\xi - x}{\sigma}(\Delta\xi - \Delta x) - \frac{\eta - y}{\sigma}(\Delta\eta - \Delta y) + \Delta k. \qquad (11)$$

Setting

$$\frac{\xi - x}{\sigma} = \sin\chi, \quad \frac{\eta - y}{\sigma} = \cos\chi, \qquad (12)$$

and rearranging

$$\Delta\sigma = (\Delta x - \Delta\xi)\sin\chi + (\Delta y - \Delta\eta)\cos\chi + \Delta k. \qquad (13)$$

The plan is now to evaluate Δx, Δy, $\Delta\xi$, and $\Delta\eta$ in terms of Δu, Δv, Δw, $\Delta\alpha_☾$, $\Delta\delta_☾$, and Δp. In this process, we shall ignore the effect of the errors $\Delta\alpha_*$, $\Delta\delta_*$, and $\Delta\mu_*$. These errors are probably of the order of 0".1, or 5×10^{-7} radians according to the errors in the Zodiacal Catalog. It will be seen at once from Eq. (6) in which u, v, and w are of the order of 5×10^6 meters that the net effect on ξ and η is of the order of a few meters. Hence, we may write

$$\Delta\xi = \sin\mu_*\Delta u + \cos\mu_*\Delta v,$$
$$\Delta\eta = \cos\delta_*\Delta w - \sin\delta_*\cos\mu_*\Delta u + \sin\delta_*\sin\mu_*\Delta v. \qquad (14)$$

In the Eqs. (2), let us first differentiate, including $\Delta\alpha_*$ and $\Delta\delta_*$. We find

$$\Delta x = p\cos\delta_☾(\Delta\alpha_☾ - \Delta\alpha_*) + \frac{x}{p}\Delta p,$$
$$\Delta y = p(\Delta\delta_☾ - \Delta\delta_*) + \frac{y}{p}\Delta p. \qquad (15)$$

Here, we can get rid of the $\Delta\alpha_*$ and $\Delta\delta_*$ by defining $\Delta\alpha_☾$ and $\Delta\delta_☾$ as measured relative to the star which is to be occulted. This yields

$$\Delta x = p\cos\delta_☾\Delta\alpha_☾ + \frac{x}{p}\Delta p,$$
$$\Delta y = p\Delta\delta_☾ + \frac{y}{p}\Delta p. \qquad (16)$$

Substituting (14) and (16) into (13), we obtain

$$\Delta\sigma = +(-\sin\mu_*\sin\chi + \sin\delta_*\cos\mu_*\cos\chi)\Delta u + (-\cos\mu_*\sin\chi - \sin\delta_*\sin\mu_*\cos\chi)\Delta v + (-\cos\delta_*\cos\chi)\Delta w + (p\cos\delta_☾\sin\chi)\Delta\alpha_☾ + (p\cos\chi)\Delta\delta_☾ + \frac{x\sin\chi + y\cos\chi}{p}\Delta p + \Delta k. \qquad (17)$$

Equation (17) is of the form (8), as sought; the b's may be identified as follows:

$$\begin{aligned} b_1 &= -\sin\mu_*\sin\chi + \sin\delta_*\cos\mu_*\cos\chi, \\ b_2 &= -\cos\mu_*\sin\chi - \sin\delta_*\sin\mu_*\cos\chi, \\ b_3 &= -\cos\delta_*\cos\chi, \\ b_4 &= p\cos\delta_☾\sin\chi, \\ b_5 &= p\cos\chi, \\ b_6 &= \frac{x\sin\chi + y\cos\chi}{p}, \\ b_7 &= +1. \end{aligned} \qquad (18)$$

For a single occultation of a single star observed at several places, the quantities μ_*, x, and y vary markedly; the quantities p, $\delta_☾$, δ_* vary only slightly. Hence, along a line of constant χ, the quantity

$$U = b_3\Delta w + b_4\Delta\alpha_☾ + b_5\Delta\delta_☾ + b_7\Delta k \qquad (19)$$

is a constant, provided that the Δw, $\Delta\alpha_☾$, $\Delta\delta_☾$, and Δk can be considered constant. It has been noted above that this is a reasonable assumption for Δw. It is obvious for $\Delta\alpha_☾$ and $\Delta\delta_☾$, provided that the lunar tables are sufficiently accurate. For Δk, this condition implies that the lunar radius is constant at the point where the occultation takes place. If the moon did not librate, this could easily be insured by making contact at the same position angle at all stations which observe a given occultation. This can be done within a certain degree of accuracy; for the moment let us take Δk also as constant. We thus find, for each station at which a lunar occultation is observed, an equation of the form:

$$\Delta\sigma = b_1\Delta u + b_2\Delta v + b_6\Delta p + U. \qquad (20)$$

Equation (20) can be understood if we consider the calculated position of the edge of the moon's shadow, as projected on the fundamental plane, at the instant of occultation. See Figure 1. This line bears a certain resemblance to a line of position, as it is used in navigational astronomy. From the observation of the occultation at the first site, we find the correction to the position line, in the direction perpendicular to the edge of the shadow. We do not care about the correction in a direction parallel to the edge of the shadow, since the other measurements will also be made at the position angle χ. Thus what might

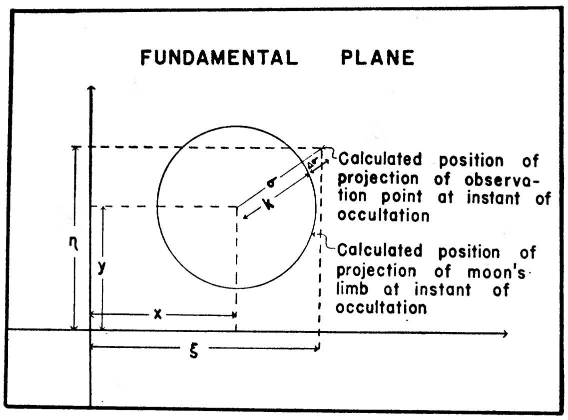

Figure 1. Relations in the fundamental plane.

appear to be a problem in 2 unknowns is reduced to one.

Equation (20) expresses rather precisely the real situation; we are uncertain about Δu, Δv, and Δp by significant quantities. It is unfortunately not possible from the present material, consisting of only 9 observations on 4 different occultations to solve for 7 unknowns, namely, Δu, Δv, Δp, and the 4 values of U.

It may be shown, however, that each of the three physically significant unknowns, namely, Δu, Δv, and Δp is uncertain largely because of the uncertainty in the earth's equatorial radius, or more exactly the uncertainty in the position of the center of the International Ellipsoid. This ellipsoid is taken tangent to the geoid at Meade's Ranch, but its relation to the earth's center is unknown. In the first place, since, according to Eq. (4)

$$u = (\nu + h)\cos\phi\cos\lambda,$$

we find by differentiation

$$\Delta u = \Delta(\nu + h)\cos\phi\cos\lambda - (\nu + h)\sin\phi\cos\lambda\Delta\phi - (\nu + h)\cos\phi\sin\lambda\Delta\lambda. \quad (21)$$

In this equation, in which $\Delta\phi$ and $\Delta\lambda$ are understood to be in radians, we can reasonably estimate their uncertainties as 1/200 000. The trigonometric functions are each of the order of unity. Hence, each of the second terms is likely to be less than 30 meters. On the other hand, the uncertainty of the earth's radius, $\Delta(\nu + h)$ which appears in the first term, is 300 meters, as judged by the difference between the International and the Jeffreys ellipsoids; and again the coefficient is of the order of unity.

The difference $\Delta(\nu + h)$ in turn is largely due to the uncertainty in Δa. Since

$$\nu = a(1 - e^2\sin^2\phi)^{-\frac{1}{2}} \quad (22)$$

by the usual geodetic formula, in which a is the length in meters of the earth's equatorial radius, and e is the eccentricity, or, omitting terms of the fourth order and higher in e,

$$\nu = a(1 + \tfrac{1}{2}e^2\sin^2\phi), \quad (23)$$

we have

$$\Delta(\nu + h) = \Delta a(1 + \tfrac{1}{2}e^2\sin^2\phi) + \tfrac{1}{2}a\sin^2\phi\Delta e^2 + ae^2\sin\phi\cos\phi\Delta\phi + \Delta h. \quad (24)$$

In this equation, the error of Δh, including the geoidal height, may be estimated as 5 meters. The error Δe^2 may be estimated as about 1 part in 600 of the value of e^2, or about 1×10^{-5}. Hence, the value of $\frac{1}{2}a\sin^2\phi\Delta e^2$ will be about 15 meters. The term $ae^2\sin\phi\cos\phi\Delta\phi$ will have a value of a few tenths of a meter only. Thus the last 3 terms may be ignored, and we may set

$$\Delta(\nu + h) = \Delta a\frac{\nu}{a}. \quad (25)$$

We find, substituting in Eq. (21), and neglecting the difference between $(\nu + h)/a$ and ν/a,

$$\Delta u = u\frac{\Delta a}{a} - v\Delta\lambda - u\tan\phi\Delta\phi. \quad (26)$$

By exactly similar steps, we may establish that

$$\Delta v = v\frac{\Delta a}{a} + u\Delta\lambda - v\tan\phi\Delta\phi. \quad (27)$$

As pointed out above, it is reasonable to believe that the values of Δu and Δv are constant for the whole U. S. triangulation. Hence, they may be evaluated at any point. It is convenient to evaluate them at the triangulation station Meade's Ranch, the datum-point for the triangulation of North America. At this point

$$\phi = 39°\,13'\,26''.686\text{ N.}, \quad \lambda = -98°\,32'\,30''.506.$$

The height may conveniently be taken as zero, even though the station itself is at a considerable elevation. We thus find

$$u_0 = -734\,910.3 \text{ meters},$$
$$v_0 = -4\,892\,974.0 \text{ meters}.$$

From Rice's study of the deflections of the vertical in the United States,[5] we find the values:

$$\Delta\phi'' = -1''.2, \quad \Delta\lambda'' = -0''.5.$$

The value of $-0''.5$ for $\Delta\lambda$ is not explicitly given by Rice; it is obtained from his value of $-0''.3$ for $\Delta\eta$ after allowing for the cosine of the latitude. Before substituting these values into Eqs. (26) and (27) they must, of course, be reduced to radians. The resulting numerical values are:

$$\begin{aligned} \Delta u &= -0.1151\,\Delta a - 14.94 \text{ meters}, \\ \Delta v &= -0.767\,\Delta a - 21.45 \text{ meters}. \end{aligned} \tag{28}$$

The correction to the moon's distance, Δp is a function of the correction Δa to the earth's equatorial radius and the correction $\Delta\pi$ in radians to the moon's parallax, π,

$$p = \frac{a}{\sin \pi}, \tag{29}$$

$$\Delta p = \frac{\Delta a}{\sin \pi} - \frac{a}{\sin \pi} \cot \pi \Delta\pi. \tag{30}$$

To show that the uncertainty of $\Delta\pi$ is chiefly due to the uncertainty in a, we make use of the expressions for the dynamical parallax.[6]

$$\Delta\pi = \frac{\tan \pi}{3a}\Delta a - \frac{\tan \pi}{3g_e}\Delta g_e - \frac{\tan \pi}{3n(n-1)}\Delta n + \frac{\tan \pi}{\frac{3}{\mu}\left(\frac{1}{\mu}+1\right)}\Delta\left(\frac{1}{\mu}\right), \tag{31}$$

where g_e is the mean equatorial acceleration of gravity; n is the reciprocal of the flattening, f, and is related to e by the equation:

$$e^2 = \frac{1}{n^2}(2n-1), \tag{32}$$

and μ is the mass of the moon in units of the earth's mass. Substituting numerical values, and taking $\Delta\pi$ in seconds instead of radians, Δa in km, and Δg_e in cm/sec^2, Lambert[6] found

$$\Delta\pi = 0.179\,\Delta a - 1.17\,\Delta g_e - 0.0130\,\Delta n + 0.170\,\Delta\left(\frac{1}{\mu}\right). \tag{33}$$

Lambert's reference value $\pi = 57'\,02''.682$ corresponds to

$$\begin{aligned} a &= 6\,378\,388 \text{ meters}, \\ g_e &= 978.052 \text{ gals}, \\ \mu &= \frac{1}{81.53}, \\ n &= 297.0. \end{aligned}$$

Applying the well-known correction of approximately 16 milligals, chiefly due to the error of the absolute determination of Potsdam, the value

$$g_e = 978.036 \text{ gals}$$

is accepted. Its uncertainty may be roughly estimated as 2 milligals; the effect of the uncertainty of the new value on the parallax may be estimated at $0''.002$.

With respect to n, the uncertainty mentioned above of 1×10^{-5} in e^2 corresponds to an uncertainty of $\frac{1}{2}$ unit in n, and hence to an uncertainty of $0''.007$ in π. The best value appears to be that quoted above.

With respect to μ, the new value by Spencer Jones[7] is

$$\frac{1}{\mu} = 81.27.$$

The uncertainty is not greater than 0.05, leading to an uncertainty in π of $0''.008$.

The uncertainty of a, on the other hand, is about 0.3 kilometers (the difference between the International Ellipsoid and the ellipsoid of Jeffreys), which leads to an uncertainty in $\Delta\pi$ of $0''.05$. Hence, we are justified in ignoring the other uncertainties, after making use of the best values; and regarding $\Delta\pi$ as a function of a alone, as follows:

$$\Delta\pi = \frac{\tan \pi}{3a \sin 1''}\Delta a - 0.025. \tag{34}$$

The constant term in (34) includes a change of the reference value from that of Lambert's paper, $57'\,6''.682$, to the value employed as the basis for Brown's Tables of the Moon, $57'\,2''.70$. Hence:

$$\Delta p = \frac{2}{3 \sin \pi}\Delta a + 0.7731\frac{\cot \pi}{\sin \pi}. \tag{35}$$

Substituting the values of Δu, Δv, and Δp in Eq. (20), we have

$$\Delta\sigma = (-0.1151\,\Delta a - 14.94)b_1 + (-0.767\,\Delta a - 21.45)b_2 + \left(\frac{2}{3 \sin \pi}\Delta a + 0.7731\frac{\cot \pi}{\sin \pi}\right)b_6 + U. \tag{36}$$

The Eq. (36) will be put in a more comprehensible form if we transpose the quantities

$$-14.94\,b_1, \quad -21.45\,b_2, \quad +0.7731\frac{\cot \pi}{\sin \pi}b_6,$$

to the left-hand side, and regard them as corrections to $\Delta\sigma$. Let us denote the corrected value of $\Delta\sigma$ by $\Delta\sigma'$; then

$$\Delta\sigma' = \Theta\Delta a + U, \tag{37}$$

where

$$\Theta = -0.1151\, b_1 - 0.767\, b_2 + \tfrac{2}{3}\,\frac{x \sin\chi + y\cos\chi}{a}. \tag{38}$$

Equation (37) is the one finally employed in the solution.

Ephemerides. The ephemerides of the moon required for these calculations must be extremely precise. The error in the moon's position, in feet, is equal to the error in the ground position of its shadow. At the moon's mean distance, an error of 0″.00186 equals 1 meter. Since the Nautical Almanac tabulates the moon's position only to the nearest 0″.1, a request was placed, through the Superintendent of the U. S. Naval Observatory, for the original values. These are calculated at the Greenwich Observatory to the nearest 0″.01 before rounding for publication. The unrounded values, kindly supplied by D. H. Sadler of the Greenwich Observatory, were interpolated by Aiken's method to ten-thousandths of a second of arc, using ten values of the argument. Of course the ten-thousandths were not correct; but the positions so obtained were believed to be consistent with each other within a few thousandths of a second of arc since the longest interval in time between occultations of the same star was about 17 minutes, whereas the tabular interval is 12 hours. As noted in the section on theory, an error of the moon's position which is the same at all stations will lead to no first-order terms in $\Delta\sigma$.

Recently, under the supervision of W. J. Eckert, the Watson Scientific Computing Laboratory has worked out a procedure, using electronic calculators, to obtain a precise lunar ephemeris directly from Brown's theory, bypassing the *Tables*. Dr. Eckert and Dr. E. W. Woolard, of the U. S. Naval Observatory, have discovered that there are sensible discrepancies between Brown's theory and the *Tables*. Hence, it was considered advisable to check the values interpolated from the Greenwich ephemeris by comparing them with the positions from the Watson Computing Laboratory. The latter were first corrected for nutation and the empirical term of Brown. The comparison is shown in Table I. The significant columns, so far as celestial latitudes and longitudes are concerned, are the columns headed $\delta\beta_☾$, $\delta\lambda_☾$, giving the failure of the constancy of error. The largest value, 0″.0026, corresponds to nearly 5 meters; it is thus fully comparable with the errors of the observations themselves.

Librations. The application of the theory of librations to the present problem can be greatly simplified by considering the problem from a point of view situated at the star. From this point of view, the optical and the diurnal, or topocentric librations can be combined as a simple rotation of the moon, with a period of a nodical month, around an axis which, in turn, is inclined to the ecliptic at an angle of 1°.75, and which precesses around the pole of the ecliptic in a period equal to that of one revolution of the moon's nodes. By the Cassini relationship, the descending node of the moon's equator on the ecliptic coincides with the ascending node of the orbit. Instead of considering the observer's selenographic latitude and longitude, we consider the selenographic latitude and longitude of the center

TABLE I. COMPARISON OF EPHEMERIDES

	Celestial Latitude			Celestial Longitude			Parallax		
	Brown's Tables	$\Delta\beta_☾$ unit ″.0001	$\delta\beta_☾$	Brown's Tables	$\Delta\lambda_☾$ unit ″.0001	$\delta\lambda_☾$	Brown's Tables	$\Delta\pi_☾$ unit ″.0001	$\delta\pi_☾$
1.	−4° 47′ 19″.5134	− 839		264° 52′ 38″.7188	−1189		57′ 29″.3403	+69	
2.	−4 47 39.5569	− 832	+ 7	265 2 21.4235	−1215	−26	57 28.8594	+65	−4
3.	+3 50 45.9237	+2123		55 14 3.8855	+ 494		54 20.3305	+10	
4.	+3 51 2.1067	+2123	0	55 18 26.5378	+ 485	− 9	54 20.4192	+15	+5
5.	+2 45 31.4709	+1751		38 46 17.7490	+ 14		54 3.0712	+32	
6.	+2 45 37.1232	+1755	+ 4	38 47 29.3913	+ 15	+ 1	54 3.0668	+32	0
7.	+5 13 56.1080	+1258		96 24 14.2653	− 891		55 1.5878	+46	
8.	+5 13 56.3899	+1265	+ 7	96 28 27.5356	− 873	+18	55 1.7695	+46	0
9.	+5 13 56.5368	+1276	+18	96 30 51.2846	− 866	+25	55 1.8730	+44	−2

1. Antonito, Colo.
2. Clay Center, Kans.
3. El Paso, Tex.
4. Seguin, Tex.
5. Lubbock, Tex.
6. Alvarado, Tex.
7. Hughson, Calif.
8. Desert Center, Calif.
9. Arivoca, Ariz.

of the lunar shadow; or better, the coordinates of the star, which is diametrically opposite the center of the shadow. The star's selenographic latitude varies slowly, owing to the motion of the node; the selenographic longitude increases at the rate of 1 revolution per nodical month.

Since the angular libration, as seen from the star, is small in any one series of observations, it is sufficient to consider differential librations. This permits the neglect of the physical librations. It also permits the neglect of the variation of the star's selenographic latitude; and it permits representation of the star's selenographic longitude as a simple increase with time in a period of one sidereal month, instead of a rotation in one nodical month combined with a precession with the period of the nodes.

A further simplification of the problem is possible. S. W. Henrikson has proved that the angular velocity around the center of the moon's face is the same for all points on the moon's limb, and is given by

$$\dot{\chi} = \frac{d\chi}{dt} = \frac{2\pi}{T}\sin i, \qquad (39)$$

where T is the moon's sidereal period and i is the inclination of the moon's axis to the fundamental plane.

We may regard this as a resolution of the moon's angular velocity along an axis parallel to the direction to the star. There is likewise a component whose axis lies in the fundamental plane, and coincides with the projection of the moon's axis on the fundamental plane. Let us call this $\dot{\gamma}$ and take γ as the angle between the physical feature of the moon which cuts off the star and a plane through the moon's center parallel to the fundamental plane. If the moon were smooth and spherical, γ would be zero; the limb of the moon as seen from the star would be a great circle.

It has been found by trial that it is possible and relatively easy to calculate $\dot{\chi}$, and hence to follow the variation of χ from station to station, in such a way that the occultation is always produced by the same feature of the limb. On the other hand, it is not necessary to take account of the variations due to $\dot{\gamma}$. This is because, if the distance from the center of the moon to the given lunar feature be denoted by k_0, and the projection of this distance on the plane of the sky be denoted by k, then

$$k = k_0 \cos \gamma.$$

For a small change $\Delta\gamma$ in γ, the corresponding change in k is

$$\Delta k = k_0 \sin \gamma \Delta\gamma. \qquad (40)$$

For a value of 0.003 radians, or about 11′, for $\Delta\gamma$ and 1 740 000 meters for k_0,

$$\Delta k = 5500 \sin \gamma \text{ meters.}$$

Unless $\sin \gamma$ becomes greater than 0.005, or γ greater than 17′, then the change of k will not exceed 30 meters.

In the original plans for this work, it was assumed that the Hayn[13] charts of the moon's limb could be used to obtain adequate values of γ. Values were determined for peaks which appeared, on the Hayn charts, to be isolated. These were applied; the results were grossly in error. A reexamination of the Hayn charts showed that they were based on too few measurements, spaced too far apart, to give any assurance that the principal crests could be located from them. A marked improvement of the residuals resulted when the Hayn charts were rejected in toto, and a constant value of k of 1 737 987.6 meters was adopted. Since the observation sites had been chosen in order to obtain certain values of γ, not small, the result was totally unexpected. The stars must have missed altogether the peaks which were chosen; and must have been cut off by the relatively level surrounding areas. Whatever the explanation, the procedure of ignoring the variation of k due to changes of γ appears to work satisfactorily.

Considerations governing the choice of time and place. The tape which records an occultation always contains noticeable irregularities in the channel which records the starlight. By custom, such irregularities are called noise; they are to be contrasted with the drop in deflection produced by the occultation, which is called the signal. The ratio of signal to noise is obviously a most important quantity. If this ratio is very low, the signal may be indistinguishable from other tape irregularities; if the drop is barely distinguishable the measured time may be adversely affected. The signal-to-noise ratio is practically unaffected by the amplification which is applied. Since the signal for a given star is usually an approximately fixed quantity, the signal-to-noise ratio is controlled by the noise.

The sources of noise are as follows:

1. Statistical irregularities in the number of electrons leaving the cathode surface due to the star's light.

2. The same for the background light due to moonlight scattered in the air, in the tube, etc.
3. Dark current in the photocell.
4. Scintillation of the star.
5. Noise produced in higher stages of amplification.

Item 1 may be evaluated by reference to Kron.[8] From Kron's figures it can be inferred that an A0 star of magnitude 6.0 gives a current 2.5×10^5 electrons per second. The Fresnel diffraction pattern at the limb of the moon passes over in a few hundredths of a second. In one hundredth of a second there will be 2500 electrons coming from the cathode; hence the natural uncertainty will be $2500^{\frac{1}{2}}$ or 50 electrons, corresponding to an error of 2 per cent. For a 9th magnitude star, the error is 8 per cent. In both cases, it is negligible.

Item 2 is the most dangerous. Background illumination in a one millimeter aperture due to the close proximity of the nearly full moon often equals that of a star of magnitude 0.5, or one million electrons in $0^s.01$. The fluctuation in $0^s.01$ amounts to 1000 electrons, which is equal to the total light of a 7th magnitude star.

Item 3 is nearly always negligible compared to Item 1 or 2, as is Item 5. Item 4, the star scintillation, is comparable in amplitude with the signal. It does not have the effect of masking the signal, because in case of violent scintillation the tape before the occultation looks very different from the tape after the occultation. It does, however, undoubtedly distort the form of the drop and thus contributes to the errors of measurement. The amount of this effect has not been analyzed.

A more serious problem than any of the above is the practical problem of retaining the star image in the aperture of the diaphragm up to the moment of occultation. If the sky is very bright the star image will soon become invisible in the finder telescope. This may be delayed by using a relatively high magnification. After the star is lost to view in the finder, it can be followed in the field of the main telescope, usually without serious difficulty. But when it becomes necessary to view the star in a small aperture, other considerations take hold. The light becomes a small, very bright spot on a dark field; it then becomes difficult to distinguish gradations of light and dark in the spot. A ring of light is seen at the edge of the aperture, due to some physiological phenomenon analogous to the Eberhard effect. Under the nervous tension, there is a tendency to a slight blurring of vision. The star is easily lost.

Evidently, both from the practical and the theoretical point of view it is most important to reduce the background illumination. Instrumental changes for this purpose fall outside the scope of this paper; but the computer must evidently do his best to choose occultations in such a way that the best possible ratio of star light to background light is attained. With the equipment used in these observations, it was usually found that occultations were not successful if the difference between moon and star exceeded 17.5 magnitudes. The moon's brightness can be obtained from a table,[9] remembering that the magnitude of the full moon is approximately -12.

Since it is also not possible to work in strong twilight, nor when the moon is less than 15° above the horizon, it follows that we can indicate the area on the earth's surface where an occultation can be observed by drawing two large circles on the globe. The first, with a radius of 75°, is centered on the point opposite the sub-solar point. The other, with the same radius, is centered on the sub-lunar point. The area common to these circles is that in which occultations can be observed, provided the moon is not too bright. Evidently, the breadth of the area is greatest in the tropics. Evidently, also, since only immersions at the dark limb are easily observed, it will be best to work near first quarter. Since, at this time, the moon will have a right ascension about 6^h less than the anti-sun, the most favorable time for observations in the northern hemisphere is when the anti-sun is 3^h following the summer solstice, and the waxing moon is 3^h preceding it; i.e., about the first of February. The worst time is about the first of August. Since the weather in the temperate latitudes in winter is usually cloudy, the argument for the application to tropical latitudes is reinforced.

Calculation of the path. The procedure here described for the calculation of the path is that which was found best after the completion of these observations; the procedure actually used was very much more complicated, but led to approximately the same results.

The first step is to determine approximately for each of the possible days, the moon's path in the sky, as seen from the area proposed, throughout the time from twilight to the setting of the moon below 15° altitude. Stars falling within 15′ of this path should be examined to see whether they can be used. At this stage graphical methods are best. Having found a likely star, a plot is made of the predicted positions of the moon's shadow at several times during the occultation, and the value of the position angle of the occultation is marked off on the trace of the shadow of the limb. A line is chosen along which the position angle is constant. If possible, the northern half of the moon's limb should be employed, since this is smoother than the southern half.

In making the next approximation to the positions, it is possible to use the relation of Henrikson to take care of the change of χ from station to station. Hence one station can be chosen arbitrarily, somewhere in the vicinity of the line determined graphically. It goes without saying that this station should be the one at which it is most difficult to satisfy the other conditions such as accessibility, availability of survey control and the like. For each of the other stations, we begin by estimating the time graphically. For the estimated time, we calculate x and y; next, from relation (39) we calculate the value of χ; and thence the values of ξ and η. The procedure of passing from ξ, η to latitude and longitude is covered in the standard texts.

In order to apply this solution to the practical problem of fixing on a position for the telescope, the field surveyor needs more information. For one thing, the time calculated as above is valid for points at sea-level; but for any elevation more than a few meters, a correction is needed. Again, the point calculated as above may well happen to fall in a lake or a marsh, or some inaccessible part of the mountains. To find a more convenient spot along the chosen path, the surveyor needs the azimuth of the path. Once the location has been changed, the astronomer will wish to know the velocity along the path in order to predict the time accurately.

All of these quantities are easily obtained if we obtain the 9 components of the transformation from coordinates in the fundamental plane to local coordinates near the point. Let us denote the direction cosines of a given vector in terms of the ξ, η, ζ system (ζ being perpendicular to the fundamental plane and positive toward the star) as l_1, l_2, l_3, respectively. In the horizontal system, let us take n_1 as positive to the east, n_2 positive to the north, and n_3 positive upward. We desire the 9 cosines of the angles between the l-axes and the n-axes. These in turn are most easily obtained by way of a third system of direction cosines, m_1, m_2, m_3, giving the components along the u, v, w axes. The 9 components giving the relation of the l's to the first system of m's are shown in Table II. The first two lines of this

TABLE II. SYSTEMS OF DIRECTION COSINES

	m_1	m_2	m_3
l_1	$+\sin\mu_*$	$+\cos\mu_*$	0
l_2	$-\sin\delta_*\cos\mu_*$	$+\sin\delta_*\sin\mu_*$	$+\cos\delta_*$
l_3	$+\cos\delta_*\cos\mu_*$	$-\cos\delta_*\sin\mu_*$	$+\sin\delta_*$
	n_1	n_2	n_3
m_1	$-\sin\lambda$	$-\sin\varphi\cos\lambda$	$+\cos\varphi\cos\lambda$
m_2	$+\cos\lambda$	$-\sin\varphi\sin\lambda$	$+\cos\varphi\sin\lambda$
m_3	0	$+\cos\varphi$	$+\sin\varphi$
	n_1	n_2	n_3
l_1	C_{11}	C_{12}	C_{13}
l_2	C_{21}	C_{22}	C_{23}
l_3	C_{31}	C_{32}	C_{33}

table have already been found in equation (6). The second system gives the relation between the m's and the n's. The third column of this array contains the coefficients used in (4). To form the system relating the l's to the n's it is only necessary to combine the two arrays by the usual rule for matrix multiplication to form the array such as that in the third system of Table II.

If the height of the station above sea-level is h meters, then it is required to make a change h/C_{33} in ζ, without changing ξ or η. The corresponding changes in easting (E) and northing (N) referred to geodetic north are

$$\Delta E = C_{31}h/C_{33}, \quad \Delta N = C_{32}h/C_{33}. \qquad (41)$$

These are usually converted to latitude and longitude before being given to the surveyor. The velocity of the shadow on the fundamental plane, relative to the observer, is

$$x' - \xi' \text{ in the } l_1 \text{ direction,}$$
$$y' - \eta' \text{ in the } l_2 \text{ direction,}$$
$$0 \text{ in the } l_3 \text{ direction,}$$

in the usual *Nautical Almanac* notation, but here, of course, expressed in meters. In order to calculate the components of the velocity E' and N', in the eastward and northward direction, respectively, it is necessary to invert the matrix whose determinant is the minor of C_{33}; i.e., to replace each element by its minor, divided by the determinant and multiplied by -1 raised to a power equal to the number of transpositions required to bring the element to the upper left-hand corner. It happens that the determinant of the minor of C_{33} equals C_{33}; hence the end result can be expressed as follows:

$$E' = \frac{1}{C_{33}} \{C_{22}(x' - \xi') - C_{12}(y' - \eta')\},$$
$$N' = \frac{1}{C_{33}} \{-C_{12}(x' - \xi') + C_{11}(y' - \eta')\}. \quad (42)$$

From (42) the azimuth and speed along the path can be deduced immediately. The equations are in a handy form for computation. If x' and y' are not given in the *Almanac*, they can be deduced by differencing the values of x, y, obtained at the stations, and then dividing by the time differences. They are not required to more than 4 figures.

The star's altitude ψ_* and azimuth A_* are given, respectively, by

$$\sin \psi_* = C_{33}, \quad (43)$$

and

$$\tan A_* = \frac{C_{31}}{C_{32}}. \quad (44)$$

The correction made in (41) is applied by the office computer and based on a map elevation obtained at the estimated position of the site. A further small correction, to take up the second approximation, is best obtained by specifying the displacement ΔT perpendicular to the path. To obtain ΔT we resolve ΔE and ΔN from (41) perpendicular to the path, as follows:

$$\Delta T = K\Delta h,$$

where

$$K = \frac{C_{31}}{C_{33}} \cos A_p - \frac{C_{32}}{C_{33}} \sin A_p,$$

Δh being the height correction and A_p being the azimuth of the path. The quantity K should be supplied to the surveyor. Much trouble will be avoided if the path of the chosen lunar feature is drawn on a large-scale map, beginning with a line of azimuth A_p through the calculated initial position and then displacing by the amount ΔT on the side toward the star's calculated position at occultation if the assumed height is too small; on the other side if it is too large. The result will be an irregular line passing over the terrain, from which the site can be chosen with some assurance that it will not be seriously changed by the survey.

Reductions. The 9 tapes which form the basis of this paper are reproduced in Figures 2 and 3. In examining Figures 2 and 3 it will be found useful to hold up the page so as to look along the graph, in order to see the drops more distinctly. It should be noted that the star 996 is 228 B Aurigae, not previously known to be double; but evidently, on the basis of the last 3 tapes, a close double with a separation of about 0″.05 at a position angle of 255°. The other component of the separation cannot be determined from these measurements. With these figures plus the arm corrections and the positions of the stations and their heights, it is possible to recalculate everything in this paper and verify the results. The procedure in measuring the tapes was to measure the time of the drop at a point ⅓ of the distance between the mean level before the occultation and the mean level after the occultation. Where the point appeared to be in doubt due to the irregularity of the curve, a smooth curve was drawn free-hand and measurements made to it. The difference between the two arms was measured at the end of the tape. The cut-off of the current provokes a simultaneous movement of both arms; this is measured at a point ⅓ of the way from the level before the cut-off to the level after the cut-off. An exception was made in the case of Antonito, where the cut-off was measured at the level of the time ticks themselves, because the tape was not accurately centered in the guides; and hence the curved ordinates drawn on the tape did not correspond to the actual path of the pen.

Table III summarizes the computation. In the first section, the fourth column gives the time as read from the tape. The next three columns are corrections: (a) taken from the published values of the time correction;[11] (b) the arm correction; (c) the transmission time, calculated from the map distance using a value of 300,000 km/sec for the velocity of transmission. The sum of these corrections applied to the tape time is given in the eighth column. The last two columns give, respectively, the latitude and longitude on the International Ellipsoid. In the central section of the table, Column 2 gives the height as obtained by the survey party. The levelling was done by second-order methods from a first-order level point of the U. S. Coast and Geodetic Survey level net. The error of this net is probably less than 30 cm at every point in the U. S.; and the error of the connection is also less than 30 cm. The height so obtained represents the height of the terrain above the geoid. The four columns of corrections give (d) the height of the geoid above the Clarke spheroid of 1866, obtained from

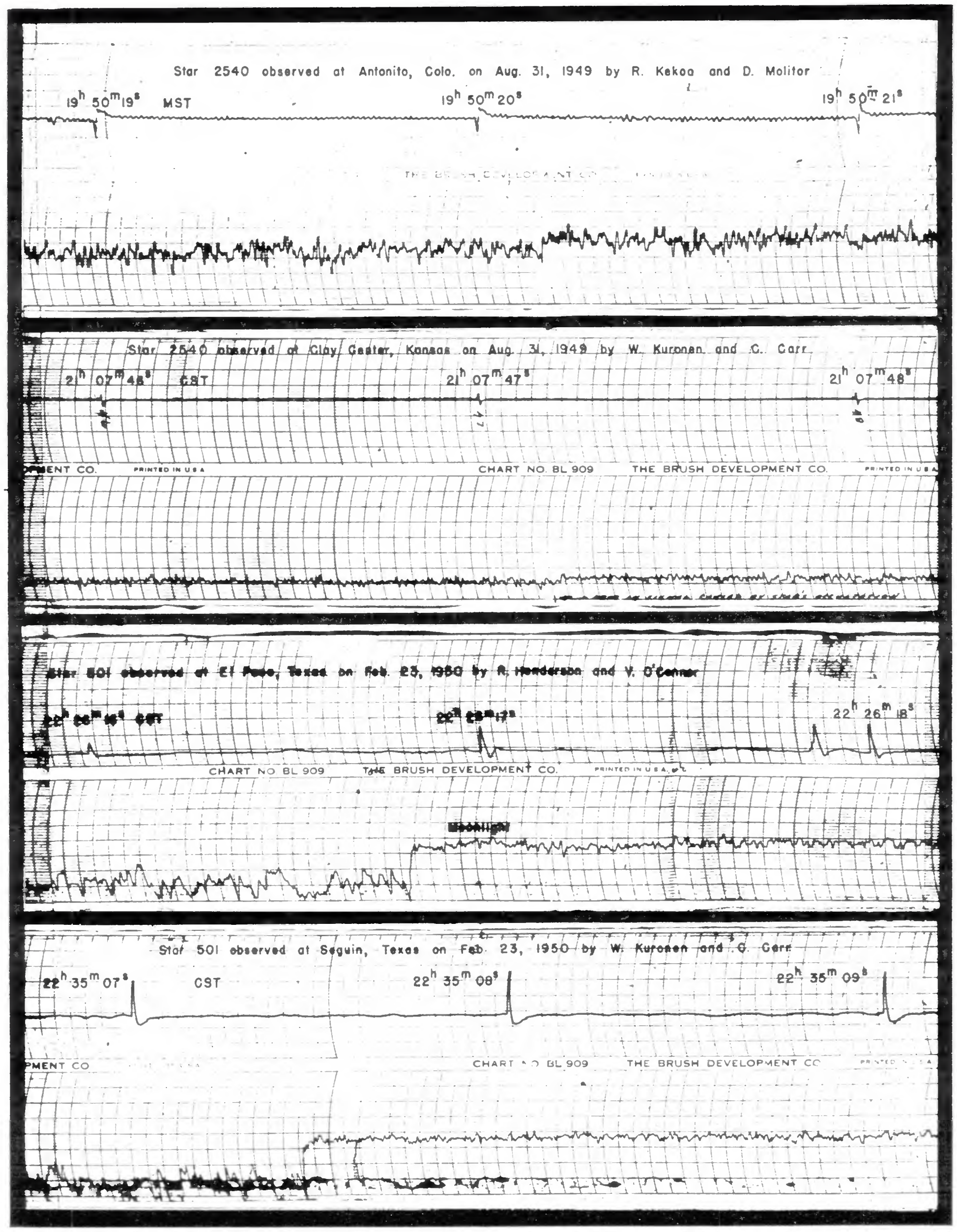

Figure 2. Occultation tapes. Upper channel, time by radio (WWV); lower channel, photocell output, increasing downward.

Duerksen's deflections[12] starting from Hayford's[4] geoidal contours; (e) the height of the Clarke spheroid of 1866 above the International, the two being assumed tangent at Meade's Ranch; (f) the height of the center of motion of the instrument above the ground. At the moment of the occultation, the instrument's declination axis is parallel to the x-axis of the fundamental plane, since it is perpendicular to the polar axis and perpendicular to the direction to the star. Hence, the position of the instrument was fully taken into account by adding the height of the center of motion to the ground height and correcting the value of ξ by the distance from the center of motion to the axis of the instrument, $+$ if the telescope is east, $-$ if the telescope is west. The fourth correction (g) is the refraction height.[10] The other columns are self-explanatory

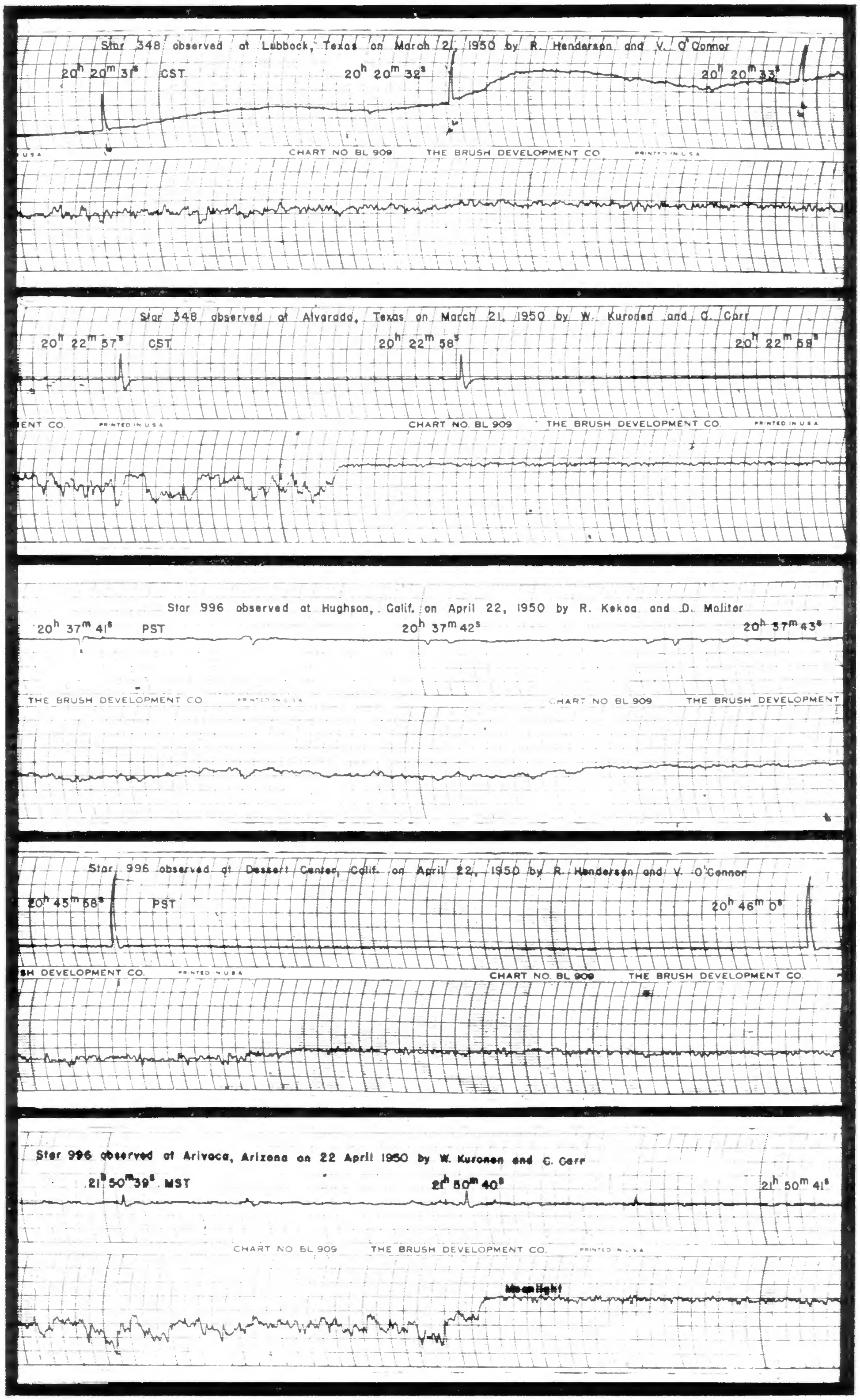

Figure 3. Occultation tapes. Upper channel, time by radio (WWV); lower channel, photocell output, increasing downward.

TABLE III. OBSERVATIONAL AND REDUCTION DATA

Station	Star	Date	Tape Time	Time Corrections (a)	(b)	(c)	Corrected Time	Latitude	Longitude W
1	2540	1949 Sept. 1	$2^h 50^m 20^s.177$	$+0^s.032$	$-0^s.012$	$+0^s.006$	$2^h 50^m 20^s.203$	36° 57′ 24″.137	106° 3′ 5″.023
2	2540	1949 Sept. 1	3 7 47.213	+ 32	+ 2	+ 8	3 7 47.255	39 22 39.817	96 48 24.974
3	501	1950 Feb. 24	4 26 16.823	− 35	− 2	+ 9	4 26 16.795	31 40 33.382	106 16 59.842
4	501	1950 Feb. 24	4 35 7.449	− 35	− 6	+ 8	4 35 7.416	29 36 59.767	98 0 50.176
5	348	1950 Mar. 22	2 20 31.991	− 12	− 2	+ 8	2 20 31.985	33 9 8.536	101 37 54.583
6	348	1950 Mar. 22	2 22 57.628	− 12	− 7	+ 7	2 22 57.616	32 19 33.927	97 12 10.453
7	996	1950 Apr. 23	4 37 42.441	+ 18	+ 7	+ 13	4 37 42.479	37 34 24.186	120 49 19.622
8	996	1950 Apr. 23	4 45 58.509	+ 18	0	+ 11	4 45 58.538	34 3 39.901	115 14 19.455
9	996	1950 Apr. 23	4 50 40.044	+ 18	− 8	+ 11	4 50 40.065	31 38 39.369	111 32 2.359

Station	Surveyed Height (meters)	Height Corrections (d) (meters)	(e) (meters)	(f) (meters)	(g) (meters)	Total Height (meters)	ξ (meters)	η (meters)	x (meters)	y (meters)
1	2530.32	12.80	−1.00	1.28	12.89	2556.29	1 076 598	5 751 784	−536 571	5 099 297
2	413.44	4.17	− .21	1.52	24.99	443.91	2 148 390	5 672 321	+534 979	5 020 389
3	1115.12	26.00	−3.90	1.28	6.70	1145.20	4 815 415	2 104 719	3 305 044	1 238 866
4	178.40	22.00	−3.70	1.52	12.30	210.52	5 303 806	2 263 376	3 793 508	1 397 410
5	904.83	18.01	−2.01	1.28	21.00	943.11	5 221 126	2 993 965	3 490 415	2 813 744
6	216.45	20.00	−3.00	1.52	34.00	268.97	5 350 525	3 050 539	3 619 792	2 870 267
7	37.83	23.00	−5.60	1.28	5.80	62.31	4 488 093	2 299 570	2 800 378	1 876 777
8	370.48	29.00	−4.70	1.28	8.90	404.96	4 974 602	2 277 530	3 286 960	1 854 475
9	1099.08	30.00	−4.00	1.52	13.50	1140.10	5 250 714	2 265 061	3 563 105	1 841 819

Station	σ (meters)	$\Delta\sigma$ (meters)	$\Delta\sigma'$ (meters)	Θ	Coefficient of Δa	Constant (meters)	Δa (meters)	Residuals (meters)
1	1 740 130	2 142	2 134	−.226740	−.026854	− 6	+ 215	+ 4
2	1 740 147	2 159	2 146	−.173032	+.026854	+ 6	+ 215	− 4
3	1 740 954	2 966	2 951	−.218319	−.019377	+ 5	− 244	− 6
4	1 740 947	2 960	2 942	−.179565	+.019377	− 5	− 244	+ 6
5	1 740 068	2 081	2 072	−.364183	−.0062735	−14	+2161	+13
6	1 740 096	2 109	2 099	−.351636	+.0062735	+14	+2161	−13
7	1 739 867	1 879	1 875	−.409422	−.037457	+ 3	− 74	− 5
8	1 739 860	1 872	1 865	−.365826	+.006139	− 7	−1143	+ 7
9	1 739 873	1 885	1 877	−.340648	+.031317	+ 4	+ 136	− 2

(a) Adopted correction WWV, (b) Arm correction, (c) Transmission time, (d) Geoid above ellipsoid, (e) Clarke 1866 above International Ellipsoid, (f) Height of instrument, (g) Refraction height.

1. Antonito, Colo.
2. Clay Center, Kans.
3. El Paso, Tex.
4. Sequin, Tex.
5. Lubbock, Tex.
6. Alvarado, Tex.
7. Hughson, Calif.
8. Desert Center, Calif.
9. Arivoca, Ariz.

except those referring to Δa and the constant term in the third section of the table.

The solution for Δa was made by least squares, utilizing a device of Gauss's. We wish to obtain the value of Δa; the values of U are uninteresting. We therefore sum the 2 or 3 equations for each occultation; divide the resulting equation through by 2 or 3, and subtract from each of the original equations. In effect, we form the mean for each term of the equations of condition, and write down the deviations from the mean. The coefficients of Δa after this reduction are shown in column six and the constant term in column seven of the third section of the table.

The resulting values of Δa are written in column eight of the third section of the table. Each value has a weight which is proportional to the square of the coefficient of Δa, on the assumption that each equation has the same weight.

Results. The weighted mean value of Δa is +60 meters, its mean error is ±169 meters; thus the resulting value of a is

$$6\,378\,448 \pm 169 \text{ meters (m.e.)}.$$

The mean error of a single measurement of $\Delta\sigma$ is

$$\pm 11.21 \text{ meters}.$$

The value of $\pi_{☾}$ is, from Eq. (40)

$$3422''.70 + 0.179 \times 0.060 - 0.025 = 3422''.686.$$

It may be considered, however, that in bringing in these considerations of the dynamical parallax, the radius of the parallel, and the gravity corrections at Meade's Ranch, we have usurped the functions of the theoreticians. We therefore present the results which are obtained from a simpler machinery. Calling the results above Solution I, we find Solution II. Ignoring the corrections at Meade's Ranch but retaining the dynamical considerations and those of the equatorial radius of the earth,

$$a = 6\,378\,428 \pm 166 \text{ meters},$$

$$\pi = 3422''.682 \pm 0''.030.$$

Solution III. Ignoring all corrections, and setting

$$\Delta\sigma = b_6 \Delta p,$$
$$p = 384\,407.6 \pm 4.7 \text{ km},$$
$$\pi = 3422{\cdot}662 \pm 0''.042.$$

The difference between Solution I and Solution II is negligible; the choice is between scrupulousness and perspicuity. Solution III is markedly different from either I or II. This is because we have assumed the earth's form as the International Ellipsoid. The choice between Solution III and either I or II will depend on the degree of reliance which the reader feels can be placed in the International Ellipsoid as describing the earth's figure.

Discussion. The residuals are unexpectedly insensitive to changes in the assumed value of the earth's semi-major axis. Fundamentally, they are related to the linear speed of the moon's motion over the earth's surface. Now if we imagine the earth's semi-major axis slightly increased, the effect on the linear speed of the moon's shadow on the fundamental plane will not be proportional to the increase, but only to the ⅔ power of the increase. This is because the change of the moon's parallax will partly offset the increased semi-major axis in the calculation of the distance. When we come to the linear speed of the moon's shadow on the earth's surface, there is another reduction of about 50 per cent, since the assumed linear speed of the earth's surface, due to its rotation, is also increased. The net result is that the sensitivity of the residuals to the value of a is approximately ⅓ of what was expected when this work was first reported.[14] The sensitivity to the value of the parallax is also low, for similar reasons. It follows that even with mediocre values of the basic constants it is possible to employ the occultation technique for geodetic measurements.

The determination of the earth's equatorial radius here presented, though not of the highest precision, has a certain value because it is independent of the deflections of the vertical. The calculation of the figure of the earth by extrapolating measurements of its radius of curvature made on land has led to the following dilemma:

a. On the isostatic assumption, the geoid should be systematically lower over the oceans; and hence, relatively convex over the continents. This assumption leads to the International value of the earth's equatorial radius of 6 378 388 meters, in round figures. It predicts negative anomalies at sea.

b. The measurements of gravity at sea in submarines, on the other hand, indicate a slight excess of positive anomalies at sea. This implies that the sea-bottom is supported in some manner other than hydrostatic equilibrium; i.e., there is a direct contradiction of the isostatic assumption. It leads to Jeffreys' value of 6 378 097.[15] The measurements here described support the isostatic assumption; they have a certain usefulness in this connection because the value of a which is found is not dependent on measurements of the earth's radius of curvature; and it is only weakly related to any considerations bearing on the earth's gravitational field.

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