How Reye's configuration helps in proving the Bell-Kochen-Specker theorem: a curious geometrical tale

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ABSTRACT

It is shown that the 24 quantum states or "rays" used by Peres (J. Phys. *A24,* 174-8 (1991)) to give a proof of the Bell-Kochen-Specker (BKS) theorem have a close connection with Reye's configuration, a system of twelve points and sixteen lines known to projective geometers for over a century. The interest of this observation stems from the fact that it provides a ready explanation for many of the regularities exhibited by the Peres rays and also permits a systematic construction of all possible non-coloring proofs of the BKS theorem based on these rays. An elementary exposition of the connection between the Peres rays and Reye's configuration is given, following which its applications to the BKS theorem are discussed.

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Some time back Mermin[1] gave a remarkably concise proof of the Bell-Kochen-Specker[2] (BKS) theorem by using nine observables arranged in the form of a 3 x 3 square, as shown in Fig.1. Around the same time Peres[3] gave an alternative proof of this theorem by extracting a set of twenty four rays[4] (see Fig.2) from the square of Fig.1 and showing that it is impossible to color these rays in a certain way. Recently I[5] pointed out how Peres' rays could be used to give an "inequality-free" proof of Bell's nonlocality theorem[6] provided that they were used in conjunction with a specific quantum state (namely, a singlet state of two spin-3/2 particles).

The purpose of this paper is to point out an interesting connection between the 24 rays of Peres and a geometrical entity known as Reye's configuration.[7] Reye's configuration is a system of twelve points and sixteen lines with many remarkable properties that occupies an important place in the field of projective geometry. We will show below that the 24 rays of Peres can be modelled by two Reye's configurations (representing 12 rays each) derived from a pair of dual 24-cells[8] in four-dimensional Euclidean space. The utility of this viewpoint is that it permits a transparent visualization of many of the important relationships among the Peres rays and also facilitates a systematic construction of all possible non-coloring proofs of the BKS theorem that can be based on these rays.[9]

The basic facts about the Peres rays and the few ideas of projective geometry needed in this paper are collected together in Figs.1-6. Very briefly, the story told in these figures and the accompanying captions is the following: (1) it is recalled how the Peres rays arise from the "magic" square of Fig.1; (2) a coordinate notation and numbering scheme are

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[1] N.D.Mermin, "Simple Unified Form for No-Hidden-Variables Theorems", Phys.Rev.Lett. *65*, 3373-6 (1990).

[2] S.Kochen and E.P.Specker, "The problem of hidden variables in quantum mechanics", J. Math. Mech. *17*, 59-88 (1967); J.S.Bell, "On the problem of hidden variables in quantum mechanics", Rev. Mod. Phys. *38*, 447-52 (1966).

[3] A.Peres, "Two simple proofs of the Kochen-Specker theorem", J.Phys. *A24*, 174-8 (1991).

[4] A "ray" is the common representative of all quantum states that differ from each other only in their overall phase. In discussions of the BKS theorem, where interest centers mainly around the orthogonalities between quantum states, it suffices to restrict one's attention to rays alone. Hence the emphasis on rays rather than states throughout this paper.

[5] P.K.Aravind, "Impossible colorings and Bell's theorem", Phys.Lett. *A262*, 282-6 (1999); an elementary version of this argument, specialized to the case of the Peres rays, is given in P.K.Aravind, "The magic tesseracts and Bell's theorem", to appear in Am. J. Phys.

[6] J.S.Bell, "On the Einstein-Podolsky-Rosen paradox", Physics *1*, 195-200 (1964). Reprinted in J.S.Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, New York, 1987).

[7] Theodor Reye (1838-1919) occupied the chair of geometry at Strasbourg from 1872-1909. He wrote a two volume work on synthetic geometry and, in 1878, introduced the configuration of 12 points, 12 planes and 16 lines that now bears his name. For an account of some geometrical problems in which Reye's configuration plays a role, see the book by Hilbert and Cohn-Vossen quoted in ref.8.

[8] A 24-cell is a convex regular polytope in four dimensions with 24 vertices that are distributed symmetrically on the surface of a four-dimensional sphere. It derives its name not from its vertices but from the fact that it possesses 24 regular octahedra for its bounding cells. More about the 24-cell, Reye's configuration and the other geometrical matters discussed here can be found in the book *Geometry and the Imagination* by D.Hilbert and S.Cohn-Vossen (Chelsea, New York, 1983), Ch.III, particularly Secs.22 and 23.

[9] Another set of rays possessing many interesting projective properties is the forty member "Penrose dodecahedron" discussed in J.Zimba and R.Penrose, "On Bell non-locality without probabilities: More curious geometry", Stud. Hist. Phil. Sci. *24*, 697-720 (1993).

introduced for the rays; (3) it is pointed out that the coordinates assigned to the rays can be interpreted as the vertices of two dual 24-cells in four dimensional Euclidean space; and (4) it is shown how the two 24-cells can be projected into three dimensions to yield a pair of Reye's configurations that model the Peres rays. The reader is invited to take a first stroll through Figs.1-6 at this point as a preparation for what lies ahead.

We now embark on the principal demonstrations of this paper. The argument that follows is presented in six sections, with the first four developing all the needed properties of the Peres rays and the last two bringing this knowledge to bear on the BKS theorem. The discussion in each section is organized around a few key statements that are termed either Properties or Theorems,[10] and numbered for easy reference. The proofs of the Properties and Theorems are given in a largely visual fashion by taking the reader back to one or more of the pathways in which he/she strolled earlier and pointing out the relevant features of the landscape.

I. TRANSITIVITY

PROPERTY 1*: Let the absolute value of the inner product of two rays be taken as a quantitative measure of the "relationship" between the rays. Then the transitivity property states that the relationships borne by a given ray to all the others are mirrored by all the rays in the set.*

This property can be verified straightforwardly, but tediously, by using the ray coordinates in Fig.2, but it follows much more directly on noting that any of the 48 vertices of the two 24-cells modelling the rays can be made to pass into any other vertex by a rotation or other[11] symmetry operation that maintains the overall invariance of the vertices. The transistivity property may not be completely obvious in a representation like Fig.4, in which the ray at the center of the cube appears to be different from one at a corner. However a careful examination of the figure always reveals that any essential property of the central ray (such as the number of its nearest neighbors) is also shared by a corner ray.[12] The equivalence of the center and corner rays (or, indeed, any two rays) can also be demonstrated by altering the hyperplane of projection used in the construction of Fig.4 so as to take one of the rays into the position of the other while maintaining the overall invariance of all the rays.

The most obvious use of transitivity is that it allows us to work out all relationships among the rays simply by picking out an arbitrary (and convenient) ray and working out its relationships with the rest (this is done for the Peres rays in the next two sections).

[10] The term Theorem is used to distinguish those Properties that are a little less obvious and also play a key role in the proofs of the BKS theorem presented in Sec.VI.

[11] Rotations alone suffice to take any vertex of a 24-cell into any other vertex of that 24-cell, but a new type of operation is needed to exchange vertices between dual 24-cells: it is a combination of a Clifford displacement and a magnification ; see H.S.M.Coxeter, "Regular Polytopes" (Dover, New York 1973), p.239.

[12] Let us verify that both the center and corner rays have eight nearest neighbors each. This is easily verified for the center ray, whose nearest neighbors are just the points at the cube vertices. For a corner ray the nearest neighbors are the three ideal points X, Y and Z, the cube center, the three cube vertices nearest to it, and the cube vertex opposite to it. The last mentioned point might seem like an absurdity, since it is separted from the corner point by the point at the cube center. However this contradiction disappears if it is realized that one is in a projective space where it is quite possible for three points on a line to all be adjacent to one another (one can get from one "extreme" point to the other without bumping into the point "in between" by proceeding via the ideal point on that line).

However transitivity also has wider implications, because it applies not only to individual rays but also to collections of rays (such as the “lines”, “triangles” and “tetrads” to be discussed later) that can be made to pass into each other by suitable symmetry operations. This wider use of transitivity, which is absolutely crucial, is invoked frequently in what follows.

The reason we use the Reye’s configurations to model the Peres rays is that they allow many of the relationships between the rays to be easily visualized, certainly more so than if the rays are viewed in their original four-dimensional setting. But an even more compelling reason for this choice is that the lines and incidences of Reye’s configuration turn out to have a surprising relevance for the BKS theorem, as will be demonstrated in Secs.V and VI below.

II. PROXIMITY

PROPERTY 2: *Each ray is equally inclined to its eight nearest neighbors, which lie on the four lines passing through it.*

If one focusses on ray 4 (at the center of the cube in Fig.4) and uses the correspondences in Fig.3, one readily sees that it has the same inner product with the eight rays at the corners of the cube. Transitivity then implies that the same is true for all the other rays.

Two neighboring rays in the same configuration will be referred to as a “segment” (see Fig.7 for a glossary of this and other terms used frequently in this paper). Three rays from the same configuration that are nearest neighbors of each other (and hence equally inclined to each other) either lie on a “line” or a “triangle”, where the former has been defined in the captions to Figs.4 and 5 and the latter consists of three non-collinear segments.

Any ray lies inside an octahedral cage formed by its six nearest neighbors in the other configuration, with the edges of the cage being segments of the other configuration. The relationship of a ray to its nearest neighbors in the other configuration is summarized in

PROPERTY 3: *Each ray is equally inclined to its six nearest neighbors in the other configuration.*

This property is most easily verified by focussing on ray 4, for which the nearest neighbors are the six octahedron vertices in Fig.5. We will refer to a pair of nearest neighbor rays in opposite configurations as a “couple”.

III. ORTHOGONALITY

PROPERTY 4: *Each ray is orthogonal to exactly nine other rays: the three next nearest neighbors in its own configuration and the six next nearest neighbors in the other configuration.*

Again considering ray 4, one sees that the rays deemed to be orthogonal to it by the above rule are the ones corresponding to the nine ideal points in Fig.6. Use of the correspondences in Fig.3 readily allows this claim to be verified. Transitivity then guarantees that the same is true for all the other rays. This rule makes it possible to pick out all rays orthogonal to a given one simply by inspection of Figs.3, 4 or 5 (this is an engaging game, but played on a somewhat slippery slope).

An orthogonal pair of rays from the same configuration will be referred to as a “pure pair”. An orthogonal pair of rays from different configurations will be referred to as a “mixed pair”. Any pair of rays from the same configuration must form either a pure pair or a segment. Any pair of rays from different configurations must form either a mixed pair or a couple.

IV. COMMUNITY

(The reader impatient to get to the BKS theorem as quickly as possible can read Property 5, look at Fig.8, and then proceed to the next section).

PROPERTY 5: *The Peres rays form exactly 24 tetrads (a tetrad being defined as a mutually orthogonal set of four rays). Six of these tetrads are “pure” tetrads involving only rays from the same configuration, while the remaining eighteen are “mixed tetrads” involving two rays from one configuration and two from the other. There are no orthogonalites between the Peres rays that are not represented in these tetrads.*

Focussing on ray 4, we see that the three rays orthogonal to it in its own configuration, which are 1, 2 and 3, join with it to form a “pure” tetrad. Transitivity then implies that any ray must join with three others in its own configuration to form a pure tetrad and, since a ray can clearly belong to no more than one such tetrad, it follows that the pure tetrads must be mutually disjoint. The 24 rays must therefore divide up into six pure tetrads, with three pure tetrads spanning each configuration. Each pure tetrad contains six distinct pure pairs within it, so the six pure tetrads contain a total of thirty six pure pairs within them. But we know from Sec.III that each ray is orthogonal to three others in its own configuration, and so we can calculate that the twenty four rays will give rise to (24 x 3)/2 = 36 distinct pure pairs. The agreement between these two tallies demonstrates that all the pure pairs are completely accounted for (and, further, that each occurs only once) in the six pure tetrads.

We turn next to mixed tetrads. A mixed tetrad could consist of three rays from one configuration and one from the other, or of two rays from each configuration. However the former possibility can be ruled out because the presence of three rays from the same configuration would automatically lead to a pure tetrad. So a mixed tetrad must consist of two pure pairs, with one coming from each configuration. Each configuration has eighteen pure pairs within it (six for each of the three pure tetrads), and the mixed tetrads must result from combining these pairs together in all possible ways. However it turns out that the pure pairs from different configurations can only be matched up in one way,[13] and so the total number of mixed tetrads is eighteen.

Let us verify that all possible mixed pairs occur in the mixed tetrads. Since each ray is orthogonal to six others in the other configuration, the total number of distinct mixed pairs is (24 x 6)/2 = 72. But each mixed tetrad contains four mixed pairs (after excluding the two pure pairs), and the eighteen mixed tetrads therefore contain a total of 18 x 4 = 72 mixed pairs. This proves the claim, provided that one can demonstrate that no duplication of

[13] This is easily seen by considering a specific case, whence one finds that the common intersection of the two sets of six orthogonal rays corresponding to the two members of a pure pair is a unique pair from the other configuration.

mixed pairs occurs in the mixed tetrads, but this is easily done.[14] We have now verified the statement, made at the end of Property 5, that all orthogonalities between the Peres rays (represented by all possible pure and mixed pairs) occur among the 24 tetrads.

The 24 tetrads are listed (and labelled) in lexicographic order in Fig.8. The pure tetrads are easily picked out visually in Figs.4 and 5. Two of them appear as dual tetrahedra inscribed in the cube of Fig.4 and the third as a tetrahedron with one vertex at the center of the cube and the other three vertices at the ideal points. The three remaining pure tetrads appear as identical tetrahedra in Fig.5, with the vertices of each consisting of a pair of opposite octahedron vertices and a pair of ideal points. The mixed tetrads are not as visually obvious, but each can be picked out in Fig.6 as a tetrahedron that shares an edge with a pure tetrad in one configuration and a second, skew edge with a pure tetrad in the other configuration.

A clearer picture of the interconnectedness of the tetrads can be obtained by tracing out all the closed cycles of tetrahedra formed by them, as done in Fig.9. A *cycle* is traced out by beginning from an edge (or orthogonal pair) of an arbitrary tetrahedron and proceeding via a chain of skew edges through neighboring tetrahedra until one arrives back at the edge one started from. There are exactly nine cycles, each consisting of four tetrads, that can be traced out in this way. Each cycle links a pure tetrad in one configuration to a pure tertrad in the other (hence the total of nine cycles), and it does so with the minimum number of tetrahedra (four) permitted by the constraint that pure and mixed tetrahedra alternate within a cycle.

V. NONCOLORABILITY AND CRITICALITY : PRELIMINARIES

A non-coloring proof of the BKS theorem using a set of rays requires us to show that it is impossible to color each of the rays either red or green in such a way that (i) no two orthogonal rays are both colored green, and (ii) no complete set of orthogonal rays (four in the present case) has all its members colored red. An uncolorable set of rays is further said to be critical if the deletion of even a single ray from it converts it into a colorable set. The central problem we wish to address is that of identifying all subsets (proper and improper) of the 24 Peres rays that constitute uncolorable, and also possibly critical, sets. This problem has already been solved earlier by means of a computer search,[15,16,17] and it is now known that there are many 18- and 20-ray critical sets. Our purpose here is to show how all the earlier conclusions, and more, follow as a logical consequence of the properties of the Peres rays discussed above.

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[14] Suppose that two mixed tetrads share the mixed pair A,x in common and that their other mixed pairs are B, y (for one) and B',y' (for the other) , where we use capital and small letters to distinguish rays from the two configurations. (It is tacitly assumed that B differs from B' and y from y', since, if either assumption were untrue, the two tetrads would coincide). The fact that B,y,B' and y' are all orthogonal to A and x implies that all the former rays lie in the two-dimensional plane orthogonal to both A and x. Now B and B' are orthogonal (since they belong to the same pure tetrad as A) and B and y are orthogonal (since they belong to the same mixed tetrad), so it follows that B' and y must be parallel. But this is absurd, since these are rays from different configurations.

[15] M.Kernaghan, "Bell-Kochen-Specker theorem for 20 vectors", J. Phys. A: Math. Gen. 27, L829-30 (1994).

[16] M.Kernaghan and A.Peres, "Kochen-Specker theorem for eight-dimensional space", Phys.Lett. *A198*, 1-5 (1995).

[17] A.Cabello, J.M.Estebaranz and G.Garcia-Alcaine, "Bell-Kochen-Specker theorem: A proof with 18 vectors", Phys. Lett. *A212*, 183-7 (1996).

As a first step towards identifying uncolorable sets, we note the existence of the following two types of colorable sets:

THEOREM 1: *Any subset of the Peres rays from which a pure pair is missing is colorable.*

THEOREM 2: *Any subset of the Peres rays from which a couple is missing is colorable.*

We prove these theorems in Figs.10(a) and 10(b) by deleting a pure pair or a couple from the 24 rays and showing that the 22 rays left behind in each case are colorable. Since all pure pairs are transitively equivalent to each other, as are all couples, this single demonstration suffices. Of course, an arbitrary set lacking a pure pair or a couple can always be derived from one of the two considered here by further deletions, and so is colorable. The proofs of Figs.10(a) and 10(b) are carried out using the 24 tetrads of Fig.8, because these tetrads incorporate all the orthogonalities between the rays and so include all the vital information needed for a successful coloring (or non-coloring) demonstration. It is also worth noting that the rules for a successful coloring of a subset of the Peres rays, as attempted in Figs.10(a) and 10(b), can be stated as follows: (i') any tetrad from which no rays are deleted must contain exactly one green ray and three red rays, and (ii') any tetrad from which at least one ray has been deleted can contain at most one green ray.

Theorems 1 and 2 state sufficient conditions for colorability. By turning them around, we can obtain necessary conditions for uncolorability. We proceed to examine this point.

We first take up Theorem 1, but before considering what it implies about uncolorability, we note two of its simple consequences for colorability and criticality contained in the following theorems.

THEOREM 3: *Any subset of the Peres rays containing 17 or less rays is colorable.*

THEOREM 4: *Any uncolorable 18-ray set is critical.*

The proof of Theorem 3 follows on noting that, since the six pure tetrads are mutually disjoint and span all the rays, the deletion of seven or more rays must necessarily lead to the deletion of a pure pair from one of these tetrads and hence to a colorable set by Theorem 1. Theorem 4 follows as a direct corollary of Theorem 3.

Now for uncolorability. The necessary conditions implied by Theorem 1 for an uncolorable set of rays can be stated as

THEOREM 5: *Any uncolorable set consists of two deleted sets of rays, one from each configuration, each of which can only be one of the following: (a) the null set, (b) a point, (c) a segment, (d) a line or (e) a triangle.*

The truth of this theorem is evident on noting that the above collections of rays are the only ones that can be deleted from a configuration without deleting a pure pair from it.

We turn next to Theorem 2. This theorem implies that the two deleted sets of rays (one from each configuration) in an uncolorable set must be such that every ray in one is orthogonal to every ray in the other (recalling that every pair of rays from opposite configurations must either be orthogonal or form a couple). This requirement can be further refined into the following two theorems that characterize all the ways in which the deleted sets from the two configurations can be related to each other.

THEOREM 6: *If an uncolorable set has a missing line (or segment) from one configuration, the only points of the other configuration that can be missing from it are the ones lying on a unique line, each of whose points is orthogonal to every point on the original line (or segment). This unique line in the second configuration will be termed the "mate" of the line (or segment) in the first configuration.*

Proof: Consider the line passing through the points (4,5,9) of the "cube" configuration. Each of these points has six orthogonal companions in the "octahedron" configuration, and the common intersection of these three sets of six companions is the line (16,19,22) of the octahedron configuration, which is therefore the (unique) mate of the original line. Transitivity then guarantees that any line in either configuration has a unique mate in the other. An examination shows that the mate of any cube edge is the further of the two octahedron edges that shares a twofold axis with it, while the mate of a cube diagonal is one of the ideal lines of the octahedron configuration. If one replaces the deleted line at the beginning of this proof by a deleted segment, the parallel result follows.

We will speak of two sets of points (such as a line and a line, or a point and a line) as being orthogonal if each member of one is orthogonal to each member of the other.

THEOREM 7: *If an uncolorable set has a lone point missing from one configuration, the only points from the other configuration that can be missing from it are one of the following:*

(a) any or all of the points on one of the four lines that are the mates of the four lines through the point.

(b) any or all of the points on one of the four triangles defined by the mates of three out of four of the lines through the point.

Proof: Taking the lone missing point to be ray 19 of the octahedron configuration, the six points of the cube configuration that are orthogonal to it are 6,9,8,12,1 and 2. These six points yield exactly four lines (each consisting of a cube edge and the ideal point lying on it) and four triangles (each consisting of a cube edge and the ideal point not lying on it), any one of which can be partly or entirely deleted without sacrificing uncolorability. The deletion of the lines can also be understood by noting that any point is orthogonal to the mates of the four lines through it, thus making it safe to remove part, or all, of any of these mates. Similarly, if one views a triangle as arising from the mates of three out of the four lines through a point, it is evident that the triangle is orthogonal to the point and can be partly, or wholly, deleted.

VI. NONCOLORABILITY AND CRITICALITY : FINALITIES

We are now in possession of all the tools we need to identify uncolorable and critical sets. We will begin by looking at 18-ray sets and work our way upwards to the full 24-ray set, identifying all uncolorable and critical sets along the way. For any n-ray set ($n \geq 18$), we will first identify candidate uncolorable sets by using Theorems 5,6 and 7. We use the descriptor "candidate" because Theorems 5-7 have been demonstrated to provide only necessary, but not sufficient, conditions for uncolorability. However we will demonstrate that the conditions are in fact sufficient, so that all candidates are indeed uncolorable. A further examination will then reveal which of the uncolorable sets is critical.

Let us first consider 18-ray sets. Using Theorem 5 alone, any potentially uncolorable set can be characterized in terms of its missing rays as either a “line-line” set, a “line-triangle” set or a “triangle-triangle” set (a line-triangle set, for example, is one that is missing a line from one configuration and a triangle from the other). However an application of Theorem 6 rules out the line-triangle and triangle-triangle sets, and also a large number of the line-line ones. The line-triangle and triangle-triangle sets get ruled out because a line (or triangle) can never be completely orthogonal to a triangle in the other configuration (recall that a line can only be orthogonal to the points on its mate, and that the opposing triangle can have at most two of its vertices on this mate). Of the many line-line sets that are possible, Theorem 6 allows only those for which the two lines are the mates of each other. It remains to be seen whether this last type of line-line set (the only “candidate” allowed by both Theorems 5 and 6) is indeed uncolorable.

There are sixteen line-line sets to be considered, since there are sixteen lines in each configuration and each line can be paired up only with its mate in the other configuration. But each line-line set is transitively equivalent to all the others, so it suffices to show that any one is uncolorable (or colorable) to demonstrate that the rest are too. We consider the particular set defined by the excluded lines (1,5,10) and (16,20,24) of the cube and octahedron configurations, respectively. If we consider only those tetrads not involving any of these six excluded rays, we are left with the nine boldface tetrads of Figure 11. These boldface tetrads involve only the 18 surviving rays of the set, with each occuring exactly twice. The coloring rules (stated below Theorem 2) require these 18 rays to be colored either red or green in such a way that each boldface tetrad contains exactly one green ray and three red rays. But this is impossible since, on the one hand, the total number of green rays is required to be odd (as there can only be one green ray per boldface tetrad) while, on the other hand, it is required to be even (because each green ray is repeated twice). This proves that this particular line-line set is indeed uncolorable, and the same then follows for all the others.

Figure 12 lists all the sixteen uncolorable 18-ray sets, with the six deleted points shown first, followed by the nine tetrads involving only the 18 surviving rays. It follows from Theorem 4 that all these sets are critical. These results agree with those found earlier in ref.17 by means of a computer search.

We next consider 19-ray sets. Theorem 5 allows only segment-line and segment-triangle sets, but Theorem 6 rules out the latter and allows only segment-line sets for which the line is the mate of the segment; however such sets contain uncolorable 18-ray sets as subsets and so are uncolorable but not critical.

Passing on to 20-ray sets, the only candidates allowed by Theorems 5,6 and 7 are segment-segment sets (with the two segments orthogonal to each other), point-line sets (with the point and line orthogonal) and point-triangle sets (with the point and triangle orthogonal). The former two sets contain critical line-line sets as subsets and so are uncolorable but not critical. The point-triangle sets require further examination to determine their status. Let us consider the particular point-triangle set with point 19 deleted from the octahedron configuration and the triangle (2,6,12) deleted from the cube configuration. If we consider only the tetrads with none of these four deleted points occuring in them, we obtain the eleven boldface tetrads shown in Figure 13. These boldface tetrads involve only the 20 survivng rays, with eighteen of them occuring twice and the remaining two occuring four times. It follows from this that these 20 rays cannot be properly colored because, on the one hand, the eleven boldface tetrads are required to contain an odd number of green rays (one per tetrad) while, on the other, they are required to contain an even number of green rays (since each green ray is repeated two or four times). This establishes that this particular 20-ray set is uncolorable. However this set can be made to yield three others (with the same point but different triangles) by rotations about the fourfold axis through point 19, and these four sets can be replicated twenty four times by varying the choice of the lone deleted point, so the total number of uncolorable sets of the point-triangle type is 4 x 24 = 96, matching the result found earlier in refs.16 and 17.

Are these 96 uncolorable sets critical? To determine this, we need to look at all possible deletions of a single ray from one of these sets and see if the resulting sets are colorable. Deletion of an additional point from the configuration containing the lone deleted point would lead to a colorable set, because the lone point is the only one in its configuration that is orthogonal to all three triangle vertices.[18] Also, from Theorem 5, the deletion of a fourth point from the configuration containing the triangle would lead to a colorable set. This establishes that these 96 sets are critical.

We next pass to 21-ray sets, among which the only uncolorable ones are null-line sets, null-triangle sets and point-segment sets (with the point and segment orthogonal). The first and third of these are not critical because they contain uncolorable 18-ray sets as subsets, while the second is not critical because it contains an uncolorable 20-ray set as a subset. The only uncolorable (but not critical) 22-ray sets are the ones with a segment or a mixed pair missing, while all 23-ray sets are uncolorable but not critical. Finally, the full 24-ray set is uncolorable but not critical.

A direct proof of the uncolorability of the full 24-ray set can be given by *reductio ad absurdum* as follows. Assuming that this set is colorable, there must be exactly one green ray in each of the six pure tetrads since these tetrads are mutually disjoint and span all the rays. The three green rays in each configuration must form either a line or a triangle, so the properly colored rays must consist of a green line in each configuration, or a green triangle in each configuration, or a green line in one configuration and a green triangle in the other. However it is not hard to see that none of these possibilities can be realized. If there is a green line in a configuration, then ten of the lines of this configuration (the one containing all the green rays and the three remaining lines through each of the green rays) are partially green and the other six are wholly red; the mates of the ten partially green lines are forced to be wholly red while the mates of the other six lines can be partially green. In a similar fashion, if there is a green triangle in a configuration, nine of the lines of this configuration (the three lines along the sides of the triangle and the two other lines through each of its vertices) are partially green and the remaining seven lines are wholly red; this forces the nine mates of the partially green lines to be wholly red but allows the mates of the seven other lines to be partially green. Thus, whether there is a green line or a green triangle in one configuration, one sees that there are at most seven partially green lines in the other configuration, which is too few to accommodate either a green line or a green triangle.

To conclude, all the different uncolorable subsets of the Peres rays are listed and categorized in Fig.14.

[18] This follows from the fact that the lone point is the unique intersection of the mates to the three triangle sides.

$1 \otimes \sigma_z$	$\sigma_z \otimes 1$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes 1$	$1 \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$\sigma_x \otimes \sigma_z$	$\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Fig.1. Mermin's "magic" square, an array of nine observables used by Mermin (ref.1) to prove the Bell-Kochen-Specker theorem. Each entry in this square is the tensor product of observables for a pair of spin-1/2 particles, with 1, σ_x, σ_y and σ_z representing the identity operator and the Pauli operators. The observables in each row and column of this array constitute a complete commuting set, a fact used in the construction of the Peres rays in Fig.2.

ROW 1:	**(2,0,0,0)**	**(0,2,0,0)**	**(0,0,2,0)**	**(0,0,0,2)**	**Rays 1-4**
ROW 2:	**(1,1,1,1)**	**(-1,1,-1,1)**	**(-1,-1,1,1)**	**(1,-1,-1,1)**	**Rays 5-8**
ROW 3:	**(-1,-1,-1,1)**	**(-1,1,1,1)**	**(1,-1,1,1)**	**(1,1,-1,1)**	**Rays 9-12**
COL 1:	**(1,0,1,0)**	**(0,1,0,1)**	**(1,0,-1,0)**	**(0,1,0,-1)**	**Rays 13-16**
COL 2:	**(1,1,0,0)**	**(1,-1,0,0)**	**(0,0,1,1)**	**(0,0,1,-1)**	**Rays 17-20**
COL 3:	**(1,0,0,1)**	**(0,1,1,0)**	**(1,0,0,-1)**	**(0,1,-1,0)**	**Rays 21-24**

Fig.2. The 24 rays of Peres. The four entries in each row are the simultaneous eigenstates of the complete set of commuting observables in one of the rows or columns of Fig.1; the relevant row or column is indicated to the left and a numbering scheme for the states (or rays) is introduced on the right, it being understood that the ray numbers in each row increase from left to right. Each ray is denoted by a set of numbers (x_1, x_2, x_3, x_4) that serve as a shorthand for the quantum mechanical state $x_1 \uparrow\uparrow + x_2 \uparrow\downarrow + x_3 \downarrow\uparrow + x_4 \downarrow\downarrow$, where $\uparrow$ and $\downarrow$ represent spin-up and spin-down states of the two particles. Note that the rays are not normalized. Each ray can also be interpreted as a real, four-dimensional vector with coordinates (x_1, x_2, x_3, x_4). If the vectors corresponding to rays 1-12 are taken together with their inverses, they yield the vertices of a regular four-dimensional solid known as a 24-cell. Similarly, the vectors corresponding to rays 13-24, together with their inverses, yield another 24-cell that is the “dual” of the first. (See the book quoted in ref.8 for more details about the 24-cell and for the meaning of duality).

VERTEX	IMAGE	VERTEX	IMAGE
1 = (2,0,0,0)	**X**	**13 = (1,0,1,0)**	**zx**
2 = (0,2,0,0)	**Y**	**14 = (0,1,0,1)**	**(0,2,0)**
3 = (0,0,2,0)	**Z**	**15 = (1,0,-1,0)**	**zx’**
4 = (0,0,0,2)	**(0,0,0)**	**16 = (0,1,0,-1)**	**(0,-2,0)**
5 = (1,1,1,1)	**(2,2,2)**	**17 = (1,1,0,0)**	**xy**
6 = (-1,1,-1,1)	**(-2,2,-2)**	**18 = (1,-1,0,0)**	**xy’**
7 = (-1,-1,1,1)	**(-2,-2,2)**	**19 = (0,0,1,1)**	**(0,0,2)**
8 = (1,-1,-1,1)	**(2,-2,-2)**	**20 = (0,0,1,-1)**	**(0,0,-2)**
9 = (-1,-1,-1,1)	**(-2,-2,-2)**	**21 = (1,0,0,1)**	**(2,0,0)**
10 = (-1,1,1,1)	**(-2,2,2)**	**22 = (0,1,1,0)**	**yz**
11 = (1,-1,1,1)	**(2,-2,2)**	**23 = (1,0,0,-1)**	**(-2,0,0)**
12 = (1,1,-1,1)	**(2,2,-2)**	**24 = (0,1,-1,0)**	**yz’**

Fig.3. Showing how the two 24-cells of Fig.2 can be projected into three dimensions to yield a pair of Reye’s configurations that model the Peres rays. The projection is done through the common center (0,0,0,0) of the two 24-cells and onto the hyperplane $x_4 = 2$, which touches a vertex of one of the 24-cells and contains a bounding cell of the other. This is a projective mapping, in which any pair of antipodal vertices in the four-dimensional object space gets mapped into a single point in the three-dimensional image space. The first column shows (half) the vertices of one 24-cell and the second column their images, while the third and fourth columns do the same for the second 24-cell. Note that some 24-cell vertices (namely, those lying on hyperplanes parallel to the image hyperplane) get mapped onto “ideal” points in the image space. An ideal point, by definition, is the (unique) point at which a set of parallel lines meets. The ideal points X, Y and Z lie on the coordinate axes while the other ideal points lie in the three coordinate planes and along the bisectors of the coordinate axes. The distinction between the ideal points xy and xy’ is that the former lies on the bisector of the positive x and positive y axes, while latter lies on the bisector of the positive x and negative y axes.

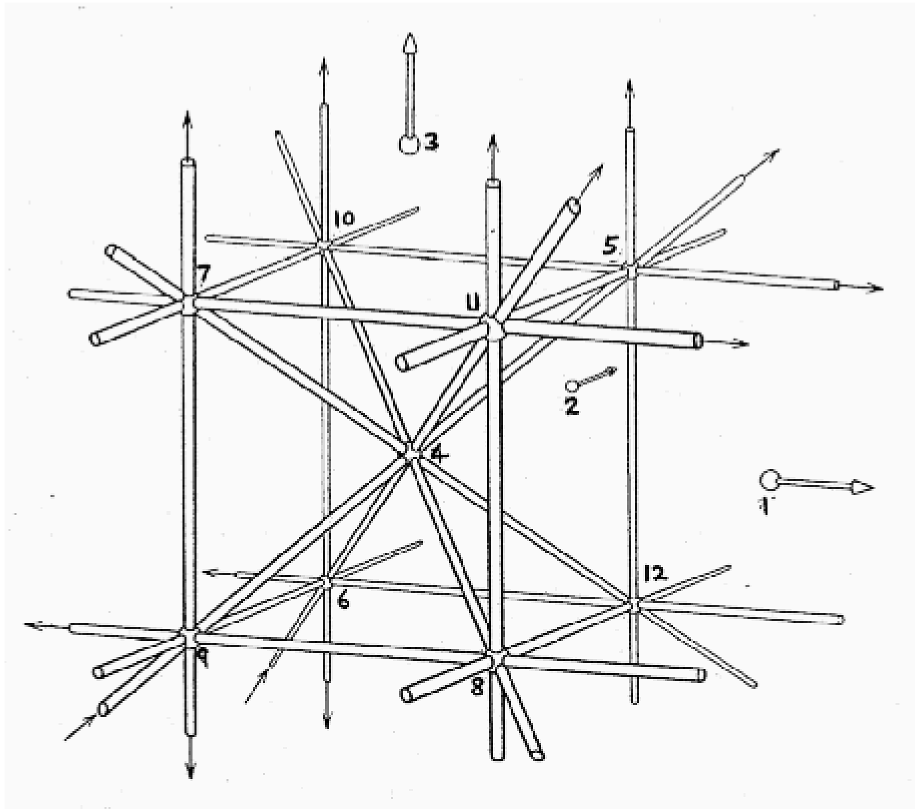

Fig.4. Reye's configuration corresponding to rays 1-12, constructed from their projections in the second column of Fig.3. The rays appear in this representation as the eight vertices of a cube, the center of the cube and the three ideal points in which parallel sets of cube edges meet. The sixteen lines of this configuration are defined to be (the extensions of) the twelve cube edges and the four cube body diagonals. Note that each line has three points (rays) on it, while each point (ray) has four lines passing through it. The eight nearest neighbors of any ray, which lie on the four lines through it, are all inclined at equal angles to it. The three next nearest neighbors of any ray are orthogonal to it. (A rod model of Reye's configuration is on display at the Boston Museum of Science, with the cube edges suitably distorted so that their extensions converge on ideal points located at finite distances from the cube center).

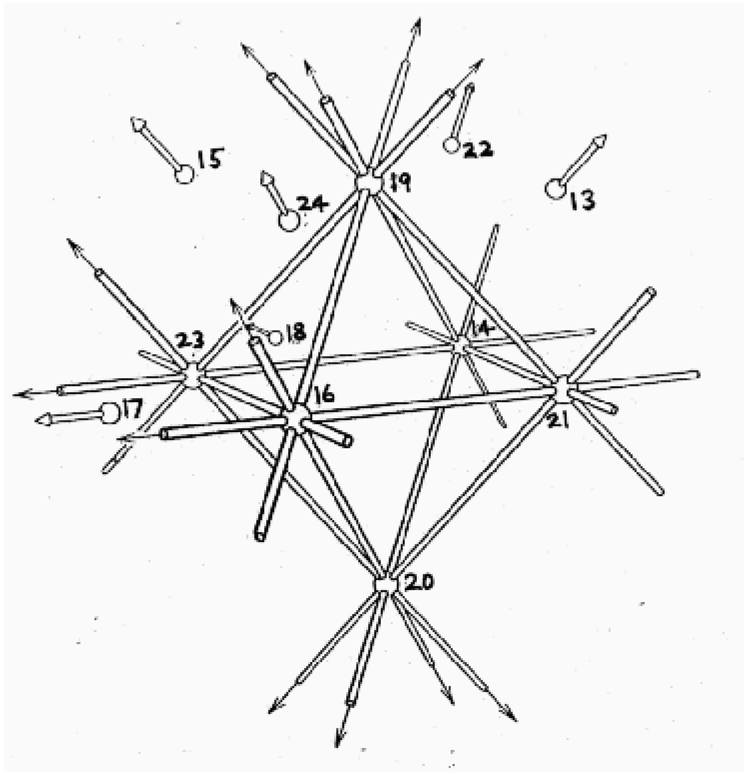

Fig.5. Reye's configuration corresponding to rays 13-24, constructed from their projections in the fourth column of Fig.3. The rays appear in this representation as the six vertices of an octahedron and the six ideal points in which pairs of parallel octahedron edges meet. The sixteen lines of this configuration are defined to be the twelve edges of the octahedron and the four ideal lines each passing through the three ideal points (13,17,24), (13,18,22), (15,18,24) or (15,17,22). Note that each line has three points (rays) on it, while each point (ray) has four lines passing through it. The eight nearest neighbors of any ray, which lie on the four lines through it, are all inclined at equal angles to it. The three next nearest neighbors of any ray are orthogonal to it.

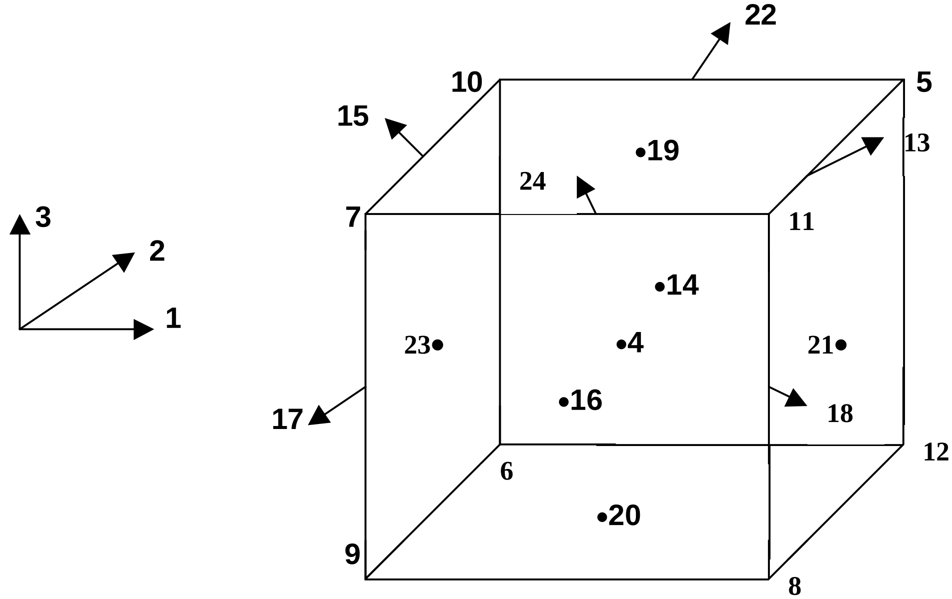

Fig.6. A representation of the 24 Peres rays obtained by superposing the Reye's configurations in Figs.4 and 5. The rays appear in this figure as the center of the cube, its eight vertices, its six face centers, and nine ideal points lying along the twofold and fourfold axes of the cube. The ideal points 1,2 and 3, which lie along the x,y and z axes, are shown by arrows to the left of the cube, while the other ideal points are indicated by arrows next to the midpoints of the appropriate cube edges. The numbering of the rays is the same as in Figs.4 and 5. Note that each ray has six nearest neighbors in the other configuration (compared to eight in its own) and six next nearest neighbors in the other configuration (compared to three in its own).

TERM	MEANING
Point	**An arbitrary ray**
Segment	**A pair of nearest neighbor rays in the same configuration**
Couple	**A ray and one of its nearest neighbors in the other configuration**
Pure pair	**An orthogonal pair of rays from the same configuration**
Mixed pair	**An orthogonal pair of rays from different configurations**
Line	**Three rays lying on a line of a configuration**
Triangle	**Three nearest neighbor rays of the same configuration not lying on a line**
Pure tetrad	**A set of four mutually orthogonal rays from the same configuration**
Mixed tetrad	**A set of four mutually orthogonal rays, with two rays from one configuration and two from the other**

Fig.7. Some sets of two, three and four rays that occur frequently in this paper.

T01	**1**	**2**	**3**	**4**	**T07**	**3**	**4**	**17**	**18**	**T13**	**6**	**8**	**17**	**19**	**T19**	**10**	**11**	**17**	**20**
T02	**1**	**2**	**19**	**20**	**T08**	**5**	**6**	**7**	**8**	**T14**	**7**	**8**	**13**	**14**	**T20**	**10**	**12**	**13**	**16**
T03	**1**	**3**	**14**	**16**	**T09**	**5**	**6**	**15**	**16**	**T15**	**9**	**10**	**11**	**12**	**T21**	**11**	**12**	**22**	**23**
T04	**1**	**4**	**22**	**24**	**T10**	**5**	**7**	**18**	**20**	**T16**	**9**	**10**	**21**	**24**	**T22**	**13**	**14**	**15**	**16**
T05	**2**	**3**	**21**	**23**	**T11**	**5**	**8**	**23**	**24**	**T17**	**9**	**11**	**14**	**15**	**T23**	**17**	**18**	**19**	**20**
T06	**2**	**4**	**13**	**15**	**T12**	**6**	**7**	**21**	**22**	**T18**	**9**	**12**	**18**	**19**	**T24**	**21**	**22**	**23**	**24**

Fig.8. The 24 sets of four mutually orthogonal rays ("tetrads") formed by the Peres rays. The "pure" tetrads, labelled T01,T08,T15, T22,T23 and T24, consist only of rays from a single configuration, while the "mixed" tetrads (all the rest) each consist of two rays from the "cube" configuration and two rays from the "octahedron" configuration. Each ray occurs in exactly one pure tetrad and three mixed tetrads. Each ray is orthogonal to nine other rays, which are the remaining members of the four tetrads that that ray occurs in.

C1	**T01, T07,T23,T02**	**C4**	**T08, T14,T22,T09**	**C7**	**T15, T21,T24,T16**
C2	**T01, T06,T22,T03**	**C5**	**T08, T13,T23,T10**	**C8**	**T15, T20,T22,T17**
C3	**T01, T05,T24,T04**	**C6**	**T08, T12,T24,T11**	**C9**	**T15, T19,T23,T18**

Fig.9. The nine cycles of tetrahedra, C1-C9, formed by the tetrads of Fig.8. Each cycle consists of four tetrads, with pure and mixed tetrads alternating.

T01	1	2	**3**	**4**	**T07**	**3**	**4**	$\underline{\mathbf{17}}$	**18**	**T13**	**6**	**8**	$\underline{\mathbf{17}}$	**19**	**T19**	**10**	**11**	$\underline{\mathbf{17}}$	**20**
T02	1	2	**19**	**20**	**T08**	$\underline{\mathbf{5}}$	**6**	**7**	**8**	**T14**	**7**	**8**	**13**	$\underline{\mathbf{14}}$	**T20**	**10**	$\underline{\mathbf{12}}$	**13**	**16**
T03	1	**3**	$\underline{\mathbf{14}}$	**16**	**T09**	$\underline{\mathbf{5}}$	**6**	**15**	**16**	**T15**	**9**	**10**	**11**	$\underline{\mathbf{12}}$	**T21**	**11**	$\underline{\mathbf{12}}$	**22**	**23**
T04	1	**4**	**22**	**24**	**T10**	$\underline{\mathbf{5}}$	**7**	**18**	**20**	**T16**	**9**	**10**	$\underline{\mathbf{21}}$	**24**	**T22**	**13**	$\underline{\mathbf{14}}$	**15**	**16**
T05	2	**3**	$\underline{\mathbf{21}}$	**23**	**T11**	$\underline{\mathbf{5}}$	**8**	**23**	**24**	**T17**	**9**	**11**	$\underline{\mathbf{14}}$	**15**	**T23**	$\underline{\mathbf{17}}$	**18**	**19**	**20**
T06	2	**4**	**13**	**15**	**T12**	**6**	**7**	$\underline{\mathbf{21}}$	**22**	**T18**	**9**	$\underline{\mathbf{12}}$	**18**	**19**	**T24**	$\underline{\mathbf{21}}$	**22**	**23**	**24**

Fig.10(a). Proof that the 22-ray set remaining after the deletion of the pure pair (1,2) is colorable. The red rays are shown in boldface, the green rays in boldface with an underscore and the two deleted rays in ordinary type.

T01	1	$\underline{\mathbf{2}}$	**3**	**4**	**T07**	**3**	**4**	$\underline{\mathbf{17}}$	**18**	**T13**	**6**	**8**	$\underline{\mathbf{17}}$	**19**	**T19**	**10**	**11**	$\underline{\mathbf{17}}$	**20**
T02	1	$\underline{\mathbf{2}}$	**19**	**20**	**T08**	$\underline{\mathbf{5}}$	**6**	**7**	**8**	**T14**	**7**	**8**	13	**14**	**T20**	**10**	**12**	13	**16**
T03	1	**3**	**14**	**16**	**T09**	$\underline{\mathbf{5}}$	**6**	**15**	**16**	**T15**	$\underline{\mathbf{9}}$	**10**	**11**	**12**	**T21**	**11**	**12**	$\underline{\mathbf{22}}$	**23**
T04	1	**4**	$\underline{\mathbf{22}}$	**24**	**T10**	$\underline{\mathbf{5}}$	**7**	**18**	**20**	**T16**	$\underline{\mathbf{9}}$	**10**	**21**	**24**	**T22**	13	**14**	**15**	**16**
T05	$\underline{\mathbf{2}}$	**3**	**21**	**23**	**T11**	$\underline{\mathbf{5}}$	**8**	**23**	**24**	**T17**	$\underline{\mathbf{9}}$	**11**	**14**	**15**	**T23**	$\underline{\mathbf{17}}$	**18**	**19**	**20**
T06	$\underline{\mathbf{2}}$	**4**	13	**15**	**T12**	**6**	**7**	**21**	$\underline{\mathbf{22}}$	**T18**	$\underline{\mathbf{9}}$	**12**	**18**	**19**	**T24**	**21**	$\underline{\mathbf{22}}$	**23**	**24**

Fig.10(b). Proof that the 22-ray set remaining after the deletion of the couple (1,13) is colorable. The red rays are shown in boldface, the green rays in boldface with an underscore and the two deleted rays in ordinary type.

T01	1	2	3	4	**T07**	**3**	**4**	**17**	**18**	**T13**	**6**	**8**	**17**	**19**	T19	10	11	17	20
T02	1	2	19	20	T08	5	6	7	8	**T14**	**7**	**8**	**13**	**14**	T20	10	12	13	16
T03	1	3	14	16	T09	5	6	15	16	T15	9	10	11	12	**T21**	**11**	**12**	**22**	**23**
T04	1	4	22	24	T10	5	7	18	20	T16	9	10	21	24	T22	13	14	15	16
T05	**2**	**3**	**21**	**23**	T11	5	8	23	24	**T17**	**9**	**11**	**14**	**15**	T23	17	18	19	20
T06	**2**	**4**	**13**	**15**	**T12**	**6**	**7**	**21**	**22**	**T18**	**9**	**12**	**18**	**19**	T24	21	22	23	24

Fig.11. When the 6 rays on the lines (1,5,10) and (16,20,24) are deleted, the remaining 18 rays are the exclusive members of the nine boldface tetrads. Each of the 18 rays occurs exactly twice in the boldface tetrads, making them impossible to color properly. This demonstrates that these 18 rays constitute an uncolorable set.

6 MISSING RAYS		NINE TETRADS CONTAINING ONLY THE 18 SURVIVING RAYS
1,5,10	16,20,24	T05,T06,T07,T12,T13,T14,T17,T18,T21
1,6,12	16,19,22	T05,T06,T07,T10,T11,T14,T16,T17,T19
1,7,11	14,20,22	T05,T06,T07,T09,T11,T13,T16,T18,T20
1,8,9	14,19,24	T05,T06,T07,T09,T10,T12,T19,T20,T21
2,7,10	13,20,21	T03,T04,T07,T09,T11,T13,T17,T18,T21
2,5,11	15,20,23	T03,T04,T07,T12,T13,T14,T16,T18,T20
2,6,9	15,19,21	T03,T04,T07,T10,T11,T14,T19,T20,T21
2,8,12	13,19,23	T03,T04,T07,T09,T10,T12,T16,T17,T19
3,7,9	14,18,21	T02,T04,T06,T09,T11,T13,T19,T20,T21
3,6,10	16,17,21	T02,T04,T06,T10,T11,T14,T17,T18,T21
3,5,12	16,18,23	T02,T04,T06,T12,T13,T14,T16,T17,T19
3,8,11	14,17,23	T02,T04,T06,T09,T10,T12,T16,T18,T20
4,5,9	15,18,24	T02,T03,T05,T12,T13,T14,T19,T20,T21
4,8,10	13,17,24	T02,T03,T05,T09,T10,T12,T17,T18,T21
4,7,12	13,18,22	T02,T03,T05,T09,T11,T13,T16,T17,T19
4,6,11	15,17,22	T02,T03,T05,T10,T11,T14,T16,T18,T20

Fig.12. The sixteen 18-ray subsets of the Peres rays that constitute the smallest uncolorable, critical sets. The six rays missing from each set are shown first, with the first three belonging to a line of the "cube" configuration of Fig.4 and the second three to a line of the "octahedron" configuration of Fig.5. The last column lists the nine tetrads containing only the18 surviving rays (and not involving any deleted rays).

T01	1	2	3	4	**T07**	**3**	**4**	**17**	**18**	T13	6	8	17	19	**T19**	**10**	**11**	**17**	**20**
T02	1	2	19	20	T08	5	6	7	8	**T14**	**7**	**8**	**13**	**14**	T20	10	12	13	16
T03	**1**	**3**	**14**	**16**	T09	5	6	15	16	T15	9	10	11	12	T21	11	12	22	23
T04	**1**	**4**	**22**	**24**	**T10**	**5**	**7**	**18**	**20**	**T16**	**9**	**10**	**21**	**24**	**T22**	**13**	**14**	**15**	**16**
T05	2	3	21	23	**T11**	**5**	**8**	**23**	**24**	**T17**	**9**	**11**	**14**	**15**	T23	17	18	19	20
T06	2	4	13	15	T12	6	7	21	22	T18	9	12	18	19	**T24**	**21**	**22**	**23**	**24**

Fig.13. When the rays 2,6,12 corresponding to a triangle in the “cube” configuration and the lone ray 19 in the “octahedron” configuration are deleted, the remaining 20 rays are the exclusive members of the eleven boldface tetrads. Because each of these 20 rays occurs an even number of times in these tetrads, it is impossible to color these tetrads properly. This proves that this 20-ray set is uncolorable.

DELETED RAYS	TYPES OF DELETIONS	NUMBER OF UNCOLORABLE SETS	CRITICAL?
0	Null-null	1	NO
1	Null-point	24	NO
2	Null-segment	96	NO
2	Point-point	72	NO
3	Null-line	32	NO
3	Null-triangle	96	NO
3	Point-segment	288	NO
4	Point-triangle	96	YES
4	Segment-segment	144	NO
4	Point-line	96	NO
5	Segment-line	96	NO
6	Line-line	16	YES

Fig.14. All possible uncolorable subsets of the 24 Peres rays. The first column lists the number of deleted rays, the second column the types of deletions from the two configurations, the third column the number of uncolorable sets of this type, and the fourth column whether this type of uncolorable set is critical. It is understood, in the second column, that the deleted sets in the two configurations are orthogonal. The total number of uncolorable sets of all types is 1057.