Liouville–Arnold connection for Lefschetz–Kovalev pencils and Eells–Salamon CR twistors

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Abstract

LeBrun's CR twistor space associated to a three-dimensional conformally Riemannian manifold is a real five-dimensional CR manifold, associated in a way similar to the twistor space of an (A)SD four-dimensional Riemannian manifold. LeBrun's original construction is somewhat inexplicit, however. A simpler description of LeBrun's CR structure can be given in spirit of the Koszul–Malgrange complex structure [**KM**], but the proof of its independence on the conformal factor is a nontrivial computation.

We outline a transparent construction of LeBrun's CR twistors for Riemannian three-manifolds along with its dual, which we call the Eells–Salamon's almost CR twistors, and apply them in the theory of Kovalev–Lefschetz pencils on G_2-manifolds.

Contents

1 Preliminaries

1.1 Philosophical preliminaries on linear algebra

Let V be a vector space. The space $W = V \oplus V^*$ carries two canonical forms, the symmetric form g given by $g\,(u \oplus \alpha, v \oplus \beta) = \alpha(v) + \beta(u)$ and the skew-symmetric form ω given by $\omega\,(u \oplus \alpha, v \oplus \beta) = \alpha(v) - \beta(u)$ (the division by 2 may be useful but is not of necessity). The subspace $V \subset W$, as well as V^*, is somehow special w. r. t. these forms: namely, it is maximal isotropic w. r. t. g and Lagrangian w. r. t. ω. If one keeps track of the space W and the form g or ω alone, one would not be able to recover the original subspace V: the special orthogonal group $\mathrm{SO}(W, g)$ acts transitively on the maximal isotropic subspaces, and the symplectic group $\mathrm{Sp}(W, \omega)$ acts transitively on the Lagrangian subspaces.

It seems from this linear algebraic picture that the forms g and ω are interconvertible, and every statement about one of them has a counterpart for the other. However, if one passes to the geometric situation, the skew-symmetric form becomes distinguished by the fact that the total space of the cotangent bundle T^*X for any smooth manifold X carries the canonical symplectic form, but no canonical symmetric form (unless a connection is picked up). This is one of the most intolerable asymmetries in the mathematics, comparable to the fact that there are no de Rham homology calculated via polyvector fields, or the fact that any human being has been born by a woman. The origin of this asymmetry is a mystery to us, but we shall nevertheless concentrate on skew-symmetric forms because of it.

Let us try to generalize the form $\omega \in \Lambda^2(V \oplus V^*)^*$. Let V be as above, $n = \dim V < +\infty$, and p a positive integer no greater than n. The space $V \oplus \Lambda^p V^*$ admits the following $(p+1)$-form ω given by

$$\omega\,(v_0 \oplus \eta_0, \ldots, v_p \oplus \eta_p) = \sum_{i=0}^{p} (-1)^{ip} \eta_i(v_{i+1}, \ldots, v_p, v_0, \ldots, v_{i-1}).$$

For example, when $n = p$, this is simply a nonzero volume form on $(n+1)$-dimensional space, and its automorphism group is $\mathrm{SL}(n+1, \mathbb{R})$. Much like the canonical symplectic form on T^*X, such form exists on the total space of $\Lambda^p T^*X$, given as the exterior differential $d\lambda$ of the p-form λ given at each point $\eta \in \Lambda^p T_x^*$ by the expression

$$\lambda_{(\eta,x)}(u_1, \ldots, u_p) = \eta\,((d\pi)(u_1), \ldots, (d\pi)(u_p))\,,$$

where $u_i \in T_{(\eta,x)}(\Lambda^p T^*X)$ are some vector fields and $\pi \colon \Lambda^p T^*X \to X$ is the projection map. For example, the total space $\Lambda^{\mathrm{top}} T^*X$ of the bundle of volume forms (called in algebraic situation the canonical bundle) on any smooth manifold X carries a canonical volume form.

The meaning of this form ω is far from being clear for $1 < p < \dim V$, but for $k = 2$ this form can be modified in order to get two interesting linear algebraic structures.

Let V be equipped with a volume form ν. For $n = 3$, this gives rise to a map $V \to \Lambda^2(V^*)$, $v \mapsto \iota_v \nu$, which can be extended to a linear complex structure I_ν on $W = V \oplus \Lambda^2 V^*$. Pick up a basis $\langle u, v, w \rangle$ s. t. $\nu(u, v, w) = 1$, and let $\alpha = \iota_u \nu$, $\beta = \iota_v \nu$, $\gamma = \iota_w \nu$. We shall use this as a basis for W, and denote by $u^* \in W^*$ (resp. v^*, w^*, α^* etc.) the forms taking value 1 on u (resp. v, w, α etc.) and vanishing on the other vectors in the basis. What does the form ω looks like in this basis? First note that it vanishes if one plugs into it more than one vector from $\Lambda^2 V^*$ or more than two vectors from V. Hence it is the sum of the monomials of the form $x^* \wedge y^* \wedge \psi^*$, where $x, y \in V$ and $\psi \in \Lambda^2 V^*$. Since one has $\alpha = v^* \wedge w^*$, $\beta = w^* \wedge u^*$ and $\gamma = u^* \wedge v^*$, the following expression holds:

$$\omega = v^* \wedge w^* \wedge \alpha^* + w^* \wedge u^* \wedge \beta^* + u^* \wedge v^* \wedge \gamma^*.$$

On the other side, note that the space of $(1,0)$-vectors in $W \otimes \mathbb{C}$ w. r. t. the complex structure I_ν is spanned by $u - \sqrt{-1}\alpha$, $v - \sqrt{-1}\beta$ and $w - \sqrt{-1}\gamma$. The holomorphic volume form $\left(u^* - \sqrt{-1}\alpha^*\right) \wedge \left(v^* - \sqrt{-1}\beta^*\right) \wedge \left(w^* - \sqrt{-1}\gamma^*\right)$ can be rewritten as

$$\begin{aligned} & u^* \wedge v^* \wedge w^* - u^* \wedge \beta^* \wedge \gamma^* - \alpha^* \wedge v^* \wedge \gamma^* - \alpha^* \wedge \beta^* \wedge w^* - \\ & \quad - \sqrt{-1} \left(u^* \wedge v^* \wedge \gamma^* + u^* \wedge \beta^* \wedge w^* + \alpha^* \wedge v^* \wedge w^* - \alpha^* \wedge \beta^* \wedge \gamma^*\right). \end{aligned}$$

The form ν identifies the spaces V and $\Lambda^2 V^*$, hence giving rise to a volume form on $\Lambda^2 V^*$, which we shall denote by σ. In our basis it is given by $\sigma = \alpha^* \wedge \beta^* \wedge \gamma^*$. The form σ can be considered as a form on $W = V \oplus \Lambda^2 V^*$ via pullback along the forgetting projection $V \oplus \Lambda^2 V^* \to \Lambda^2 V^*$. It is clear from the above discussion that $\sigma - \omega$ is the imaginary part of a holomorphic volume form on W w. r. t. the complex structure I_ν, and the automorphism group of W preserving $\omega - \sigma$ and I_ν is the group $\mathrm{SL}(3, \mathbb{C})$.

1.2 G_2-linear algebra

There is a similar story in the case $n = 4$ as well. Note that the space $\Lambda^2(V^*)$ has a canonical pseudoconformal structure given by $(\xi, \psi) = \xi \wedge \psi \in \Lambda^4(V^*) \cong \mathbb{R}$. When a volume form on V is chosen, it becomes a genuine pseudo-Euclidean structure. A choise of an Euclidean metric on V corresponds to a choise of a three-dimensional subspace $\Lambda^+ \subset \Lambda^2(V^*)$ on which this scalar product is positive definite (it would be the $+1$-eigensubspace for the Hodge star operator). Let us denote the volume form of the Euclidean structure given by the pairing of 2-forms on Λ^+ by σ as well. Then the form $\rho = \omega|_{V \oplus \Lambda^+} + \sigma \in \Lambda^3(V \oplus \Lambda^+)^*$ **OR MINUS—CHECK CAREFULLY** is called a G_2**-form**. In such setting, the subspace $V \subset V \oplus \Lambda^+$ is called **coassociative**, and $\Lambda^+ \subset V \oplus \Lambda^+$ is called **associative**. For an abstract G_2-form on a space W, a (co)associative subspace is a subspace realizable as the (co)associative subspace for some splitting $W = V \oplus \Lambda^+$ realizing this G_2-form. The property of $V \subset W$ being coassociative is equivalent to $\rho|_V = 0$.

A G_2-space carries a Euclidean structure $(-,-)$ which is given by the Euclidean structure on V corresponding to the choise of the subspace $\Lambda^+ \subset \Lambda^2 V^*$, the Euclidean structure given by the pairing of 2-forms on Λ^+, and the condition $V \perp \Lambda^+$. This Euclidean structure is preserved by the automorphisms preserving the G_2-form $\rho = \omega + \sigma$ **OR MINUS—CHECK CAREFULLY**. The property of $U \subset W$ being associative is equivalent to $\rho_U = \mathrm{Vol}_{(-,-)}|_U$. One can also define the **cross product** $\times$ on a G_2-space by

$$\rho(x, y, z) = (x, y \times z).$$

It can be given by the expression

$$(x \oplus \alpha) \times (y \oplus \beta) = (\alpha(x) - \beta(y))^\sharp \oplus (x^\flat \wedge y^\flat - (\iota_{\alpha\wedge\beta}\sigma)^\star - \iota_{x\wedge y}\nu),$$

where $-^\sharp \colon V^* \to V$ and $-^\flat \colon V \to V^*$ are the raising and lowering of indices given by the Euclidean structure on V, and $-^\star \colon (\Lambda^+)^* \to \Lambda^+$ a lowering of index w. r. t. the Euclidean structure on Λ^+.

For any vector $n \in W$ the operator $I_n \colon x \mapsto x \times n$ has kernel spanned by n, and on its orthogonal complement it is an orthogonal automorphism which squares to $-(n, n)\mathrm{Id}$. Hence if one only keeps track of the cross product $\times$, on can reconstruct the Euclidean structure, knowing that $I_n^2 = -\mathrm{Id}$ iff $(n, n) = 1$. In particular, any cöoriented hyperplane in a G_2-vector space carries a canonical linear complex structure, namely, the cross multiplication by its unit positive normal. A four dimensional subspace is coassociative iff it is complex linear in any ambient hyperplane. Moreover, if V is a coassociative subspace and n, n' and n'' are three unit normals to V orthogonal to each other, then the complex structures $I_n|_V$, $I_{n'}|_V$, $I_{n''}|_V$ commute like unit quaternions. Hence, any coassociative subspace is naturally a one-dimensional quaternionic module.

In more simple terms, consider V as a one-dimensional free module over the quaternions, and $\Lambda^+(V)$ as the space of imaginary quaternions spanned by i, j and k. Then this cross product can be defined as follows:

(i) For $u, v \in \Lambda^+$, one has $u \times v = -\mathrm{Im}(u \cdot v)$, where $\cdot$ stands for the product of quaternions;

(ii) For $u \in \Lambda^+$, $v \in V$, one has $u \times v = u \cdot v$, where $\cdot$ stands for the action of the quaternions on the quaternionic module V;

(iii) For $u, v \in V$, one has $u \times v = w$, where $w \in \Lambda^+$ is such a quaternion that $v = w \cdot u$.

1.3 G_2-manifolds

A G_2**-structure** on a smooth manifold is a field of structures of G_2-spaces in each tangent space, which is parallel w. r. t. some torsion-free connection. Since the G_2-space structure gives a scalar product, any G_2-manifold carries a Riemannian metric, and the torsion-free connection is its Levi-Civita connection. If X is a G_2-manifold and $Y \subset X$ a cöoriented hypersurface, then Y carries an almost complex structure—namely, cross product with positive unit normal to Y. It is called the **Calabi almost complex structure**. The following Proposition due to Calabi [**C**] tells when this almost complex structure is integrable.

Proposition 1.1. *The Calabi almost complex structure of a hypersurface is integrable iff its second fundamental form is complex linear w. r. t. the Calabi structure.*

1.4 Coassociative submanifolds

Let X be a G_2-manifold and $Y \subset X$ a four-dimensional submanifold. It is called **coassociative** if its tangent spaces at all the points are coassociative subspaces in the tangent spaces to X.

1.5 CR geometry

Let W be a complex vector space, considered as a real vector space W with an endomorphism I s. t. $I^2 = -\mathrm{Id}_W$, and let $V \subset W$ be a real codimension one subspace. Then $U = V \cap IV \subset V$ is a complex subspace in V of codimension one. This motivates the following

Definition 1. A **CR vector space** structure on a real space V is a subspace U and $I \in \mathrm{End}(U)$ s. t. $I^2 = -\mathrm{Id}_U$. It can be equivalently given by the subspace $U^{1,0} \subset V \otimes \mathbb{C}$ s. t. $U^{0,1} = \overline{U^{1,0}}$ does not intersect $U^{1,0}$ by a nonzero vector (but need not to span together with it the whole $V \otimes \mathbb{C}$). In terms of I, the subspace $U^{1,0}$ is just the $\sqrt{-1}$-eigensubspace, and in terms of $U^{1,0}$ the operator I is given by $I|_{U^{1,0}} = \sqrt{-1}\mathrm{Id}$ and $I|_{U^{0,1}} = -\sqrt{-1}\mathrm{Id}$. A linear map of CR spaces is called **CR linear** if it sends $(1,0)$-vectors to $(1,0)$-vectors.

An **almost CR structure** on a smooth manifold X is a field of CR structures on its tangent spaces, and can be given as a codimension one subbundle $F \subset TX$ with an almost complex endomorphism, or a subbundle $F^{1,0} \subset TX \otimes \mathbb{C}$. An almost CR structure is called **integrable**, or simply **CR structure**, if $[F^{1,0}, F^{1,0}] \subseteq F^{1,0}$. For example, any real hypersurface in an almost complex manifold carries an almost CR structure, and if the almost complex manifold was actually a complex manifold, this almost CR structure would be indeed a CR structure. A map between CR manifolds is called **CR holomorphic** if its derivative at each point is a CR linear map of tangent spaces. A question whether a CR manifold can be embedded into some complex manifold CR holomorphically is far from being trivial.

One can consider CR manifolds as contact manifolds by forgetting the complex structure on the distribution F. The Frobenius form $\Phi\colon \Lambda^2 F \to T/F$ of a CR manifold agrees with the complex structure on F: namely, since for any two fields $u, v \in \Gamma F$ one has $[u,v] = [u^{1,0}, v^{1,0}] + [u^{0,1}, v^{1,0}] + [u^{1,0}, v^{0,1}] + [u^{0,1}, v^{0,1}]$. But $[u^{1,0}, v^{1,0}] \in \Gamma F^{1,0} \subset \Gamma F \otimes \mathbb{C}$ and $[u^{0,1}, v^{0,1}] \in \Gamma F^{0,1} \subset \Gamma F \otimes \mathbb{C}$ because of integrability. Hence $\Phi(u,v) = [u,v] \mod F = [u^{1,0}, v^{0,1}] + [u^{0,1}, v^{1,0}] \mod F$, i. e. the Frobenius form Φ is Hermitian. For CR manifolds it is called the **Levi form**.

2 Two kinds of (almost) CR twistors

The main idea of the section is not entirely new, and the main propositions are essentially contained in the paper [**ES**, §7]. However, we state and prove them withount referring to the four-dimensional geometry.

2.1 Characterization of LeBrun's CR twistors via umbilic points

From now on, for an oriented real vector bundle E we shall denote by SE its spherization $E/\mathrm{GL}(1,\mathbb{R})^+$ (i. e. the bundle of rays), and, in the case when E is endowed with a positive definite scalar product, by UE its bundle of unit spheres $\{e \in E\colon \|e\| = 1\}$. For any choise of a metric the corresponding bundle UE can be identified with SE, so we shall use these symbols interchangeably.

It is widely known [**AG**, Ch. 4, § 1.2, example B] that for any smooth oriented manifold X the projectivization $P(T^*X)$ or the spherization $S(T^*X)$ of its cotangent bundle carries a contact distribution. Namely, if $\sigma \in S(T^*X)$ is an oriented hyperplane at point $x \in X$, and $ST^*X \xrightarrow{\pi} X$ is the projection map, the contact hyperplane is given by $N_\sigma \subset T_\sigma(ST^*X)$ as $N_\sigma = (d\pi)^{-1}(\sigma)$.

Definition 2. Let X be endowed with a Riemannian metric. Since the Levi-Civita connection is orthogonal, one can consider it as the Ehresmann connection in the unit spheres bundle UTX. It is the splitting $T(UTX) = V \oplus H$, where $V = \ker(d\pi|_{UTX})$ is the vertical subbundle, and H is the horizontal subbundle. The projection $d\pi\colon H_{(v,x)} \to T_x(X)$ is an isomorphism, and $(d\pi)\left(H_{(v,x)} \cap N_{(v,x)}\right) = v^\perp$, where $v \in UT_x(X)$ is a unit tangent vector at x. Let us impose the following complex structure on $N_{(v,x)} = V_{(v,x)} \oplus \left(H_{(v,x)} \cap N_{(v,x)}\right) \cong V_{(v,x)} \oplus v^\perp$. On the subspace $V_{(v,x)}$, we put the complex structure of the fiber UT_x, which is a round 2-sphere, i. e. a complex line. On the subspace $v^\perp$, we put the complex structure I_v given by $I_v(u) = u \times v$, where $\times$ is the cross product on the three-dimensional Euclidean space T_xX. This enhances STX with an integrable almost CR structure I^{LB}, called the **LeBrun's CR structure**.

However, a conformal change of the metric spoils both the Levi-Civita connection and the bundle of unit spheres, and it is not clear at all that this construction depends on the conformal structure alone.

LeBrun's original construction [**LB**] was quite different, and its conformal invariance was clear from his definition, but in the present paper we shall not need that construction, or even the fact that it is equivalent to the Definition 2.

Definition 3. Let (M, g) be a Riemannian manifold, $Z \subset M$ a cöoriented hypersurface, and n a field of positively oriented unit normals of Z. The **second fundamental form** on Z is defined by $\mathrm{II}(u, v) = g(\nabla_u v, n) = \frac{\nabla^M_u v - \nabla^Z_u v}{n}$, where $\nabla = \nabla^M$ is the Levi-Civita connection on M, and ∇^Z is the Levi-Civita connection on Z considered as an abstract Riemannian manifold. The corresponding operator A given by $g\,(A(u), v) = \mathrm{II}(u, v)$ is called the **shape operator**. A point $z \in Z$ is said to be **umbilic** (or, vice versa, Z is said to be umbilic at z) if the shape operator $A\colon T_zZ \to T_zZ$ is a multiplication by a scalar. A hypersurface is called **totally umbilic**, if it is umbilic at all of its points.

Definition 4. We say that an almost CR structure on (UTX, N) is **nice**, if the following conditions are satisfied:

(i) It induces the standard complex structures on the round spheres UT_x.

(ii) For any point $b \in M$ and $Z \ni b$ a real surface passing through b which is umbilic at b, the Gauß map $\gamma\colon Z \to UTM$ is CR holomorphic w. r. t. the standard complex structure on Z at the point b.

Notice that the condition (i) could be viewed as a limiting case of the condition (ii): if a tiny sphere Σ shrinks to a point x, then its image $\gamma(\Sigma)$ under the Gauß map tends to the fiber UT_x; as Σ becomes smaller and smaller, it is closer and closer to total umbilicity, hence its Gauß image needs to be closer and closer to a holomorphic curve, hence the limiting image UT_x should be indeed holomorphic.

Proposition 2.1. *The condition of being an umbilic point on a hypersurface is preserved by the conformal changes of the metric.*

Proof. The Levi-Civita ∇ connection of a metric $g = \langle\cdot,\cdot\rangle$ can be expressed by **Koszul's formula**

$$\langle \nabla_x y, z\rangle = \frac{1}{2}\left(L_x\langle y, z\rangle + \langle [z, x], y\rangle + L_y\langle z, x\rangle + \langle [x, y], z\rangle - L_z\langle x, y\rangle + \langle [y, z], x\rangle\right).$$

If z is perpendicular to the fields x, y and $[x, y]$ at each point, then this formula simplifies to

$$g(\nabla_x y, z) = \frac{1}{2}\left(\langle [z, x], y\rangle + \langle [y, z], x\rangle - L_z\langle x, y\rangle\right).$$

In particular, for a cöoriented hypersurface with unit normal field n one has $\mathrm{II}(u, v) = \frac{1}{2}(\langle [n, u], v\rangle + \langle [v, n], u\rangle - L_n\langle u, v\rangle)$.

Let now $\langle\cdot,\cdot\rangle' = f\langle\cdot,\cdot\rangle$ be a conformal change of the metric, and $n' = n/\sqrt{f}$ be the corresponding unit normal field. By the above formula, one can express the second fundamental form II' w. r. t. the rescaled metric as

$$\begin{aligned}
\mathrm{II}'(u, v) &= \frac{1}{2}(\langle [n', u], v\rangle' + \langle [v, n'], u\rangle' - L_{n'}\langle u, v\rangle') = \\
&= \frac{1}{2}\left(f\left\langle \left[n/\sqrt{f}, u\right], v\right\rangle + f\left\langle \left[v, n/\sqrt{f}\right], u\right\rangle - \frac{1}{\sqrt{f}} L_n(f\langle u, v\rangle)\right) = \\
&= \frac{f}{2}\left(\left\langle \frac{[n, u]}{\sqrt{f}} - nL_u\frac{1}{\sqrt{f}}, v\right\rangle + \left\langle \frac{[v, n]}{\sqrt{f}} + nL_v\frac{1}{\sqrt{f}}, u\right\rangle\right) - \frac{L_n f}{2\sqrt{f}}\langle u, v\rangle - \frac{\sqrt{f}}{2} L_n\langle u, v\rangle = \\
&= \frac{\sqrt{f}}{2}(\langle [n, u], v\rangle + \langle [v, n], u\rangle - L_n\langle u, v\rangle) - \frac{L_n f}{2\sqrt{f}}\langle u, v\rangle = \\
&= \frac{\sqrt{f}}{2}\mathrm{II}(u, v) - \left(L_n\sqrt{f}\right)\langle u, v\rangle.
\end{aligned}$$

In terms of the shape operator, the shape operator of the rescaled metric is given by

$$A'(u) = \frac{\sqrt{f}}{2} A(u) - \left(L_n \sqrt{f}\right) u.$$

Hence the umbilicity condition (i. e. A being scalar multiplication) does not depend on the conformal changes of the metric. $\square$

Perhaps the most classical evidence of this theorem is the *Liouville's theorem on conformal maps*: any conformal isomorphism between two domains in $\mathbb{R}^n$ for $n > 2$ maps pieces of shperes to pieces of spheres. Indeed, in the Euclidean space of dimension at least three the totally umbilic hypersurfaces are precisely the pieces of spheres.

Proposition 2.2. *The set of nice almost CR structures on (STX, N) for a conformally Riemannian threefold X does not depend on the representative of the conformal class.*

Proof. The Gauß map does not depend on the conformal factor, unless we identify the bundle of unit vectors UTX for both metrics with the spherization STX. The induced complex structure on any surface $Z \subset X$ does not depend on the conformal factor. The condition for a point to be umbilic does not depend on the conformal factor by the Proposition 2.1. $\square$

Proposition 2.3. *A nice almost CR structure on (STX, N) is the LeBrun's CR structure.*

Proof. The coinicidence on the vertical subbundle is automatic. Hence one should only show that a nice almost CR structure on the Levi-Civita horizontal vectors is the LeBrun's one. Consider a unit vector $u \in T_x$, and let Z be the image of the oriented plane $u^\perp \subset T_x X$ under the exponential map. Its second fundamental form vanishes at x, hence Gauß map $\gamma\colon Z \to UTX$ is holomorphic at x. However, since Z is flat up to second order at x, its differential at x is exactly the isomorphism $u^\perp \cong T_x Z \to \pi^{-1}\left(u^\perp\right) \cap H_{(u,x)}$ which inverts the differential of the projection. By construction, this is a complex linear map. $\square$

Proposition 2.4. *The LeBrun's CR structure is nice.*

This is essentially the first-order version of the Proposition 7.1 (i) in [**ES**].

Proof. If $Z \subset X$ is a surface passing through x with positive unit normal $u \in T_x X$, then the tangent space $T_{(v,x)}\gamma(Z) \subset T_{(v,x)}(STX)$ to its image under the Gauß map $\gamma(Z) \subset STX$ can be considered as a real linear map $H_{(v,x)} \to V_{(v,x)}$. It goes as follows. The Levi-Civita connection determines a trivialization of X in a first-order formal neighborhood of the point x. The first order formal neighborhood of Z at x is the space $H_{(v,x)}$, and its Gauß map maps to the first order formal neighborhood of u inside TU_x, i. e. $V_{(v,x)}$. The Gauß map is holomorphic at a point iff the point is umbilic. Hence for Z umbilic at x the map $H_{(v,x)} \to V_{(v,x)}$ is holomorphic, and its graph is a complex line. Since Z was arbitrary, this implies what we desired. $\square$

Proposition 2.5. *The LeBrun's CR stucture does not depend on a conformal factor.*

Proof. Indeed, it is the only nice almost CR structure, and the set of nice CR structures does not depend on a conformal factor due to the Proposition 2.2. □

Proposition 2.6. *LeBrun's almost CR structure is really integrable.*

Proof. somehow □

Proposition 2.7. *The Levi form of LeBrun's CR structure is nondegenerate.*

Proof. somehow □

2.2 Eells–Salamon's almost CR structure

The LeBrun's CR structure, as we have shown, is an invariant of the conformal structure. In the present section we introduce a new CR structure on the bundle of unit tangent vectors to a Riemannian threefold, which is no longer conformally invariant, but is more convenient for the rest of the discussion. It is motivated by the following

Proposition 2.8. *Let M be a Riemannian threefold, $Z \subset M$ a cöoriented surface and $z \in Z$ such point that the* **mean curvature** $\mu(z) = \mathrm{Tr}\,(\mathrm{II}_z)/2$ *vanishes at z. Denote the positive unit normal to Z at z by u. Then the subspace $T_{(z,u)}\gamma(Z)$ is a graph of a complex-antilinear map $H_{(z,u)} \cap N_{(z,u)} \to V_{(z,u)}$.*

Proof. This is a statement involving first derivatives only, so it can be proved in a first-order formal neighborhood of the point z. However, it is identified with a first-order formal neighborhood in a Euclidean space via the Levi-Civita connection, and in the Euclidean space this is a classical theorem going back to Chern [**Ch**] or probably (in three dimensions) even Gauß himself. □

Definition 5. Let (M, g) be a Riemannian threefold, and UTM the bundle of unit tangent vectors. The Levi-Civita connection regarded as an Ehresmann connection gives the splitting $N = H \cap N \oplus V$ of the standard contact distribution $N \subset T(UTM)$, which is complex linear w. r. t. the LeBrun's CR structure I. Let us endow the contact subbundle with a new complex structure I^{ES}, given by $I^{ES}|_V = I^{LB}|_V$, $I^{ES}|_{H\cap N} = -I^{LB}|_{H\cap N}$. We shall call this almost complex structure the **Eells–Salamon's almost CR structure**. We shall refer to the maps from the LeBrun's twistor space which are CR holomorphic w. r. t. the Eells–Salamon's almost CR structure as to **twsited CR holomorphic**.

Proposition 2.9 (first-order version of the Proposition 7.1 (ii) in [**ES**])**.** *The non-vertical complex tangent lines w. r. t. the Eells–Salamon's almost CR structure are the subspaces tangent to the images of surfaces under the Gauß map at the points where their mean curvature vanishes.*

Proof. Immediate from the Proposition 2.8. □

Proposition 2.10. *The Eells–Salamon's almost CR structure is never integrable.*

Proof. The Levi-Civita-horizontal fields have type $(1,0)$ w. r. t. the Eells–Salamon's almost CR structure iff they have type $(0,1)$ w. r. t. the LeBrun's almost CR structure, and vice versa. By contrast, the notion of type coinicides for vertical fields w. r. t. these scructures. On a vertical field of type $(1,0)$, say $v^{1,0} = v \in \Gamma(V \otimes \mathbb{C})$, the Levi form of the LeBrun's CR structure vanishes, since such fields are tangent to the complex curves. On the other hand, the Levi form of the LeBrun's CR structure is nondegenerate by the Proposition 2.7, hence there exists a horizontal field $h \in \Gamma(H \otimes \mathbb{C})$ s. t. the commutator $[h, v]$ is not tangent to the contact distribution. By the Proposition 2.6, the LeBrun's almost CR structure is integrable, hence $[h^{1,0}, v] = [h^{1,0}, v^{1,0}] \in \Gamma^{1,0}(N \otimes \mathbb{C})$ and $[h^{0,1}, v] = [h, v] \notin \Gamma(N \otimes \mathbb{C})$. However, both fields $h^{0,1}$ and v has type $(1,0)$ w. r. t. the Eells–Salamon's almost CR structure. Therefore the commutator of $(1,0)$-type fields can be not of the type $(1,0)$ w. r. t. it, which means that the Eells–Salamon's almost CR structure is not integrable. $\square$

Definition 6. Let X be a Riemannian manifold. A hypersurface $Z \subset X$ is called **minimal** if its mean curvature vanishes.

These are precisely the surfaces for which the Gauß map is holomorphic w. r. t. the Eells–Salamon's almost CR structure. In the following discussion, we shall need the following lemma about minimal hypersurfaces.

Proposition 2.11. *Let $p\colon X \to M$ be a Riemannian submersion with minimal fibers, and $Z \subset M$ a hypersurface. Then the following are equivalent:*

(i) $Z \subset M$ is minimal;

(ii) $p^{-1}(Z) \subset X$ is minimal.

Proof. From [**G**, §$\frac{1}{2}$] we know that for a hypersurface W the condition of minimality is equivalent to $\mathrm{Area}(U_\varepsilon) = \mathrm{const}$ for any $U \subseteq W$, where Area stands for the Riemannian volume of codimension one, and the subscript ε stands for the ε-equidistant deformation. Since the fibers are minimal, they have the same volume, and moreover the volume of any domain in the fiber does not change while being transported along any vector field lifted from the base orthogonally to the fiber. For a Riemannian submersion, the equidistant deformation commutes with projection, which yields the Proposition. $\square$

3 Kovalev–Lefschetz pencils

3.1 Preliminaries

Let us remind that if W is a G_2-space, and $V \subset W$ a coassociative subspace, then $V^\perp$ is acting on W by the cross product as imaginary quaternions, hence V carries a quaternionic structure. Therefore, any coassociative submanifold carries an almost quaternionic structure.

Proposition 3.1. *Let $X \to B$ be a fibration (possibly with some degenerate fibers) whose fibers are coassociative. Then the almost quaternionic structure on these fibers is really hyperkähler.*

Proof. The bundle spanned by three almost complex strucutre is isomorphic to the normal bundle by the very construction, and the normal bundle of a fiber of a locally trivial fibration is trivial. The trivialization is given by lifting the basic fields to orthogonal normal fields. Let $v \in T_b B$ be a vector, and $\widetilde{v}$ be its horizontal lift. Then the form ω_{I_v} on the fiber $\pi^{-1}(b)$ is given by $(\iota_{\widetilde{v}}\rho)\,|_{\pi^{-1}(b)}$, where ρ is the G_2 3-form on X. One has $d\omega_{I_v} = d\left(\iota_{\widetilde{v}}\rho|_{\pi^{-1}(b)}\right) = (d\iota_{\widetilde{v}}\rho)|_{\pi^{-1}(b)}$. Since $d\rho = 0$, one has $d\iota_{\widetilde{v}}\rho = d\iota_{\widetilde{v}}\rho + \iota_{\widetilde{v}}d\rho = \mathrm{Lie}_{\widetilde{v}}\rho$, hence $d\omega_{I_v}$ is equal to $(\mathrm{Lie}_{\widetilde{v}}\rho)\,|_{\pi^{-1}(b)}$. If one extends the vector v to a local vector field v on the base, and identify the fibers in different points along the trajectory of v through b by the flow of the lift $\widetilde{v}$, this could be further rewritten as $\mathrm{Lie}_{\widetilde{v}}\left(\rho|_{\pi^{-1}(b)}\right)$. However, all the fibers are coassociative, i. e. $\rho|_{\pi^{-1}(b)} = 0$. Therefore one has $d\omega_{I_v} = \mathrm{Lie}_{\widetilde{v}}0 = 0$. Hence the three fundamental forms on the fiber are closed, and the corresponding complex structures are integrable. Therefore the fiber is a hyperkähler manifold. □

Definition 7. A fibration (maybe with some degenerate fibers) on a G_2-manifold is called a **Kovalev–Lefschetz pencil** if its fibers are coassociative.

3.2 Liouville–Arnold connection

Donaldson [**D**] taught us that one should think of coassociative fibrations (with some degenerate fibers maybe) on G_2-manifolds as of elliptic fibrations on K3 surfaces (hence the name 'Kovalev–Lefschetz pencils'). One of the principal features of the elliptic fibration on a K3 surface is the integral affine structure on the base of such fibration away from points underlying singular fibers. It is constructed through the identification between the tangent bundle of the base and the bundle of first cohomology of the fibers as the Gauß–Manin connection on the latter. In the present paragraph, we seek for an analogous connection in the tangent bundle of the base of a Kovalev–Lefschetz pencil.

Proposition 3.2. *Let $X \xrightarrow{p} B$ be a Kovalev–Lefschetz pencil with smooth fibers. Then there exists a canonical identification $TB \to R^2p_*(\mathbb{R})^+$, where R^2p_* is the bundle of second de Rham cohomology of the fibers, and the superscript $^+$ stands for the positive subspace spanned by the classes of three symplectic forms of the fiber, i. e. the bundle of self-dual classes in the second cohomology.*

Proof. should be simple from the definitions □

Definition 8. Consider the Poincaré pairing $H^2(S,\mathbb{R}) \times H^2(S,\mathbb{R}) \to \mathbb{R}$ as a scalar product on the bundle $R^2p_*(\mathbb{R})$, and restrict it to the subbundle $R^2p_*(\mathbb{R})^+$. It becomes a positive definite scalar product there. Considering it as a metric in the tangent bundle TB by the means of the above Proposition, we call it the **Liouville—Arnold metric**.

Definition 9. Consider the Gauß–Manin connection ∇^{GM} in $R^2p_*(\mathbb{R})$, and let $\varpi\colon R^2p_*(\mathbb{R}) \to R^2p_*(\mathbb{R})^+$ be the orthogonal projector w. r. t. the Poincaré pairing on the second cohomology. Define the connection ∇^{LA} on $R^2p_*(\mathbb{R})^+$ by $\nabla^{LA}_u v = \varpi\left(\nabla^{GM}_u v\right)$. Considering it as a connection ∇^{LA} in the tangent bundle $TB \xrightarrow{\sim} R^2p_*(\mathbb{R})^+$ by the means of the above Proposition, we call it the **Liouville–Arnold connection**.

The reason for this name is that the integral flat connection on a base of the Lagrangian fibration is described by the Liouville–Arnold theorem. However, the analogous connection on the base of a Kovalev–Lefschetz pencil can be not flat, hence the usual name is inappropriate.

Proposition 3.3. *For a Kovalev–Lefschetz pencil the projection map is a Riemannian submersion w. r. t. the Liouville—Arnold metric.*

Proof. should be simple from the definitions ◻

Proposition 3.4. *The Liouville–Arnold connection is the Levi-Civita connection of the Liouville—Arnold metric.*

Proof. Let us denote both the Liouville—Arnold metric and Poincaré pairing by $(-,-)$. One has $\left(\nabla^{LA}_u(-,-)\right)(v,w) = \left(\nabla^{LA}_u v, w\right) + \left(v, \nabla^{LA}_u w\right) - L_u(v,w) = \left(\nabla^{GM}_u v, w\right) + \left(v, \nabla^{GM}_u w\right) - L_u(v,w) = \left(\nabla^{GM}_u(-,-)\right)(v,w) = 0$, since the Poincaré pairing is defined on the lattice of integral classes in cohomology on the differentiable level and is independent on the hypercomplex structure.

The Gauß—Manin connection can be given in terms of the differential forms by $\nabla^{GM}_u[\alpha] = [\mathrm{Lie}_{\widetilde{u}}\alpha]$, where α is a form defined along fibers and extended in the perpendicular direction as zero on the horizontal vectors, and $\widetilde{u}$ is the horizontal lift of the field u. Let us remind that if ρ is the G_2 3-form on the total space, then the form ω_{I_v} on a coassociative fiber F is given by $(\iota_{\widetilde{v}}\rho)|_F$. Then one has $\mathrm{Tors}^{\nabla^{LA}}(u,v) = \varpi\left([\mathrm{Lie}_{\widetilde{u}}\iota_{\widetilde{v}}\rho - \mathrm{Lie}_{\widetilde{v}}\iota_{\widetilde{u}}\rho]\right) - [u,v] = \varpi\left([\mathrm{Lie}_{\widetilde{u}}\iota_{\widetilde{v}}\rho - d\iota_{\widetilde{v}}\iota_{\widetilde{u}}\rho - \iota_{\widetilde{v}}d\iota_{\widetilde{u}}\rho]\right) - [u,v] = \varpi\left(\left[\mathrm{Lie}_{\widetilde{u}}\iota_{\widetilde{v}}\rho - \iota_{\widetilde{v}}\mathrm{Lie}_{\widetilde{u}}\rho - \iota_{\widetilde{[u,v]}}\rho\right]\right) = \varpi\left(\left[\iota_{[\widetilde{u},\widetilde{v}]-\widetilde{[u,v]}}\rho\right]\right) = 0$, since the field $[\widetilde{u},\widetilde{v}]-\widetilde{[u,v]}$ is vertical, and the form ρ vanishes on fibers (we have also used that $d\rho = 0$ in the identity $d\iota_{\widetilde{u}}\rho = \mathrm{Lie}_{\widetilde{u}}\rho$). Hence the Liouville—Arnold connection is torsion-free and hence the Levi-Civita connection for the Liouville—Arnold metric. ◻

3.3 The period map

Let $p\colon X \to B$ be a Kovalev–Lefschetz pencil with smooth fibers over a ball. Let $UTB \xrightarrow{\pi} B$ be its LeBrun's CR twistor space.

Definition 10. Consider the pullback $\pi^*(X) \to UTB$. Its fibers can be endowed with a complex structure as follows. Namely, on the fiber $X_{(u,b)}$ one puts the complex structure defined by $u \in T_bB$ on the fiber $p^{-1}(b)$. Since the space UTB is simply connected, this gives rise to a map $per\colon UTB \to \mathfrak{Per}$, where $\mathfrak{Per}$ is the period space of the fiber. We shall call it the **period map**.

If $Z \subset B$ is a surface, one can construct a manifold foliated into K3 surfaces (or tori) over Z it two ways. First, one can consider the preimage $Y = p^{-1}(Z)$ with its Calabi almost complex structure. Second, one can lift Z to a surface $\widetilde{Z} \subset UTB$ by the Gauß map, send it into a (not necessarily holomorphic) submanifold $\mathfrak{Z} = per\left(\widetilde{Z}\right) \subset \mathfrak{Per}$ in the period space, and then consider the tautological family of K3 surfaces (or tori) over $\mathfrak{Z}$. Note that the first one has an almost complex structure inducing the complex structure on each fiber, while the second has no reasonable almost complex structure. Our goal is to relate these two somehow.

Proposition 3.5. *Let $p\colon X \to B$ be a Kovalev–Lefschetz pencil with smooth fibers over a ball, $Z \subset B$ a real surface. Then the following are equivalent:*

(i) The Calabi almost complex structure on the preimage $p^{-1}(Z)$ is integrable;

(ii) Mean curvature of the preimage $p^{-1}(Z)$ vanishes;

(iii) Z is a minimal surface w. r. t. the Poincaré metric.

Proof. The assertion (i) implies obviously implies (ii) because of the Calabi condition, i. e. the Proposition 1.1. To see the contrary, notice that the fibers are automatically complex, hence the only pair of directions to check the complex linearity of the second fundamental form is the direction perpendicular to the fiber, which is real two-dimensional, hence the complex linearity is indeed equivalent to the vanishing of the mean curvature.

Since the fibers are calibrated, they are minimal submanifolds, and by Proposition 3.3 the projection p is a Riemannian submersion, hence by the Proposition 2.11 the assertions (ii) and (iii) are equivalent, as well. □

Proposition 3.6. *Let $p\colon X \to B$ be a Kovalev–Lefschetz pencil with smooth fibers over a ball, $Z \subset B$ a real surface and $z \in Z$ a point. Consider the hypersurface $p^{-1}(Z)$ with its Calabi complex structure. Then the following are equivalent:*

(i) Nijenhuis tensor of $p^{-1}(Z)$ vanishes identically along the fiber $p^{-1}(z)$;

(ii) The mean curvature of Z vanishes at z.

Proof. This is just the infinitesimal version of the above Proposition. □

Proposition 3.7. *The period map for a Kovalev–Lefschetz pencil is twisted CR holomorphic.*

Proof. Clearly, the fibers of the fibration $UTB \to B$ are mapped by the period map to the twistor lines, which are holomorphic. This shows that the period map is twisted CR holomorphic on the vertical subbundle $V \subset T(UTB)$.

To show this outside of the vertical subbundle, consider the tangent 2-subspace $W \subset N_{(u,b)}$ which is mapped to a complex line by the differential of the period map, and let $Z \subset B$ be a germ of a surface containing the point b

with unit normal vector u s. t. its image under the Gauß map is tangent to W. Since its image is a 1-jet of a complex curve at the point $per(u, b)$, the inverse image of the tautological bundle over this image is a first-order jet of a holomorphic variation over this point. However, its total space is exactly the same as the preimage $p^{-1}(Z) \subset X$ of $Z \subset B$ with its Calabi structure. Therefore the Nijenhuis tensor of $p^{-1}(Z)$ vanishes along the fiber $p^{-1}(b)$, which is by the Proposition 3.6 equivalent to the vanishing of the mean curvature of the surface Z at the point b. By the Proposition 2.9 this is equivalent to the subspace W being a complex line w. r. t. the Eells–Salamon's almost CR structure.

To sum the things up, the condition of being sent to a complex line by the differential of the period map is equivalent to the condition of being a complex line w. r. t. the Eells–Salamon's almost CR structure. This is precisely the twisted holomorphicity of the period map, which we have desired. □

Proposition 3.8. *The image of the period map is either a twistorial line or a complex surface.*

Proof. The image is no less than a twistorial line, hence it sends the vertical spheres to the twistorial lines biholomorphically. The Eells–Salamon's almost CR structure is not integrable by the Proposition 2.10, hence it cannot be an immersion. If its image is neither a curve nor a surface, it is a three-dimensional CR manifold. For each point, the image contains some twistorial lines through this point, hence its CR bundle is integrable with twistorial lines being the integral submanifolds. The fibers of the mapping onto such a manifold would be complex curves since the map is CR holomorphic, and the preimages of the twistorial lines are the integral hypersurfaces for the CR bundle on the Eells–Salamon's twistor space. In particular, the standard contact distribution on STB would be involutive, which is widely known to be impossible [**AG**]. □

This Proposition suggests the two cases for a Kovalev–Lefschetz pencils, which we shall now examine. These two cases resemble the two possibilities for the elliptic fibrations on a complex surface with zero first Chern class: it is either a product of a trivial family with fiber isomorphic to a fixed elliptic curve over another elliptic curve, or a Lefschetz pencil on a K3 surface.

Proposition 3.9. *If the image of the period map is a twistorial line, the Kovalev–Lefschetz pencil is locally an orthogonal Cartesian product, and the Poincaré metric on the base is flat.*

Proof. The CR holomorphic projection from the Eells–Salamon's twistor space STB of a ball B to a rational curve, which is a biholomorphism when restricted to each fiber, is nothing but the foliation on STB, which identifies all the fibers, i. e. a flat connection. Hence the Poincaré metric in the base is flat in this case, and no degenerations of the fibers can occur, since there are no degenerations over the twistorial line. □

In what follows, we shall assume that the image of the period map is a surface.

Proposition 3.10. *The fibers of the period map are the trajectories of the Liouville–Arnold geodesic flow.*

Proof. The geodesic equation $\nabla_v^{LA} v = 0$ means exactly that the covariant derivative of v (which we consider as the complex structure on the fiber given by the cross product with its horizontal lift) vanishes along v (which we consider as a tangent field of the curve in the base). $\square$

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