WELL-QUASI-ORDERING, THE TREE THEOREM, AND VAZSONYI'S CONJECTURE(1)

BY

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1. **Introduction.** The theory of well-quasi-ordering was first developed by Graham Higman [1] (under the name "finite basis property") and by P. Erdös and R. Rado in an unpublished manuscript. Some hints of the theory had already occurred, however, in B. H. Neumann [4]. The theory was further developed by Rado [5], and by Kruskal ([2; 3] and the present paper).

The theory described in this paper was developed in order to settle the following conjecture due to A. Vazsonyi:

VAZSONYI'S CONJECTURE. *There is no infinite set* $\{\tau_1, \tau_2, \cdots\}$ *of (finite connected) trees such that* τ_i *is not homeomorphically embeddable in* τ_j *for all* $i \neq j$.

The main result of the present paper is the Tree Theorem, which is stated in §2. Roughly speaking, this theorem asserts that if we quasi-order the set $T(X)$ of all functions from all finite trees into a quasi-ordered space X in a natural way, then X well-quasi-ordered implies $T(X)$ well-quasi-ordered. This theorem yields the above conjecture as an easy corollary. It also contains Theorem 1.1 of Higman [1] as a special case.

This paper is self-contained except for a few results quoted from earlier papers. However, the reader unfamiliar with the subject may find the exposition uncomfortably brief.

2. **Basic definitions and the Tree Theorem.** A *quasi-order* (*qo*) is a binary relation which is *transitive* ($x \leqq y \leqq z$ implies $x \leqq z$) and *reflexive* ($x \leqq x$ for all x). A *partial-order* (*po*) is a quasi-order which is *proper* ($x \leqq y \leqq x$ implies $x = y$). We define $x < y$ to mean $x \leqq y$ and $y \nleqq x$.

We suppose from now on that X is qo by $\leqq$. A subset U of X is called an *upper ideal* = *upper set* = *ideal* if x in U and $x \leqq y$ implies y in U. Ideals are obviously closed under intersections and unions. If $A \subset X$, then *upper* A is defined to be the ideal which is the intersection of all the ideals containing A. Clearly upper $A = \{y \mid y \geqq \text{some } x \text{ in } A\}$. If upper $A = U$, we say that A *generates* or *spans* U, and that U is the *ideal generated by* A.

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(1) This paper presents an expanded version of part of my doctoral dissertation at Princeton University. I wish to express my gratitude for the encouragement and direction of R. Lyndon and A. W. Tucker. I also wish to acknowledge my debt to P. Erdös for acquainting me with the problem and to my brother M. Kruskal for his collaboration in the early development of the theory.

A space X is *well-quasi-ordered* (*wqo*) [or *well-partially-ordered* (*wpo*)] if it is qo [or po] and if every ideal has a finite generating set(2).

LEMMA. *Suppose X is qo. Then the following conditions are equivalent*:

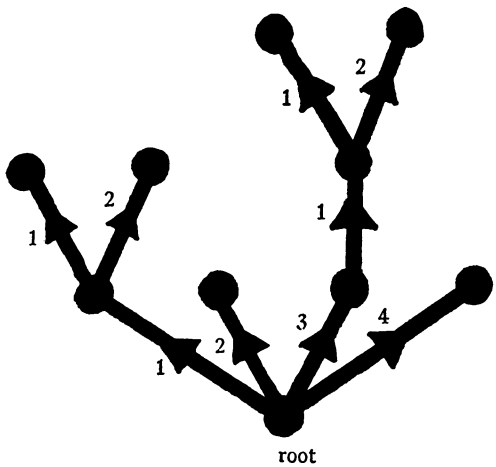
(1) X is *wqo*;

(2) *X satisfies the nowhere ascending chain condition, that is, every sequence $x_1, x_2, \cdots$ of elements from X with the property that $x_i \nleqq x_j$ for $i<j$ is finite*;

(3) *X satisfies both* (a) *the strictly descending chain condition, namely, every sequence $x_1>x_2> \cdots$ is finite, and* (b) *the incomparable chain condition, namely, every sequence $x_1, x_2, \cdots$ with $x_i \nleqq x_j$ for $i \neq j$ is finite*;

(4) *for every x in X, $X-$upper x is wqo.*

The reader may easily supply the proof; see also Theorem 2.5 of [1] or Theorem 1 of [3] for proofs and further conditions.



Let $T^\#$ be the collection of all (finite connected) trees. Write $\tau_1 \leqq \tau_2$ if the tree τ_1 may be homeomorphically embedded in τ_2. Clearly $T^\#$ is qo by $\leqq$. To prove Vazsonyi's Conjecture it suffices to prove that $T^\#$ is wqo. (In fact, Vazsonyi's Conjecture is equivalent to $T^\#$ being wqo, but we do not bother to prove this.)

The proof that $T^\#$ is wqo proceeds by an elaborate induction-like procedure. As frequently happens in such cases, the induction will not work unless we strengthen and generalize the statement to be proved. In the present case we must strengthen and generalize to an extreme degree.

First we replace $T^\#$ by T, the space of (finite connected) *structured trees.* A tree τ is said to be *structured* if:

(1a) a particular vertex, called the *root* of τ, is specified;

(2) Higman [1] calls this "finite basis property." Rado [5] uses "partial well-order." They both deal with po sets rather than qo sets.

(1b) every edge of τ is oriented so that it points away from the root of τ;

(1c) at each vertex v of τ the edges just above v (that is, the edges whose initial vertex is v) are linearly ordered.

We define $\omega: \tau_1 \to \tau_2$ to be a *monomorphism* if:

(2a) ω is a homeomorphic embedding when the structure on τ_1 and τ_2 is disregarded;

(2b) ω takes each vertex of τ_1 into a vertex of τ_2;

(2c) ω maps each edge of τ_1 into an oriented path of τ_2, and does so in an orientation-preserving manner;

(2d) for each vertex v of τ_1, ω maps the edges which initiate at v into paths which initiate at $\omega(v)$ in a manner which is strictly order-preserving (with respect to the linear orders at v and at $\omega(v)$).

To each structured tree τ there corresponds an ordinary tree $\#\tau$ obtained by disregarding the structure. It is clear from (2a) that $\tau_1 \leqq \tau_2$ implies $\#\tau_1 \leqq \#\tau_2$; that is, $\#$ is a homomorphism[3]. It is also clear that $\#$ is onto; that is, $\#$ is an epimorphism[3]. As an epimorphism preserves wqo[3], it suffices to prove that T is wqo in order to prove $T^{\#}$ wqo.

Write $t: \tau \to X$ for a function defined on the vertices of τ into X. We call t a (*structured*) *tree* over X. Intuitively, we may visualize t as τ with each vertex v labelled with an element of X. We shall call τ the *carrier* of t. If τ', τ_i, etc. denote subtrees of τ, we shall denote the function t restricted to these subtrees by t', t_i, etc., and we shall refer to *subtrees of* t in this sense. We shall also use this convention in reverse: if t', t_i, etc. are trees, we shall denote their carriers by τ', τ_i, etc. We may speak of a vertex, root, etc. of t when we mean the corresponding object of τ.

Let $T(X)$ be the collection of all trees over X. If t_1 and t_2 are trees in $T(X)$, define a *monomorphism* $\omega: t_1 \to t_2$ to be a monomorphism $\omega: \tau_1 \to \tau_2$ with

$$\begin{tikzcd} \tau_1 \arrow[rr, "\omega"] \arrow[dr, "t_1"'] & & \tau_2 \arrow[dl, "t_2"] \\ & X & \end{tikzcd}$$

the additional property that

(3) $t_1(v) \leqq t_2\omega(v)$ for every vertex v of τ_1. Intuitively, this requires that ω map each vertex of t_1 into a vertex of t_2 with a greater label. If t_1 and t_2 are in $T(X)$, define $t_1 \leqq t_2$ if and only if there is a monomorphism $\omega: t_1 \to t_2$.

Clearly $T(X)$ is qo. Define $\flat: T(X) \to T$ to be the function which assigns to each tree t over X its carrier τ. It is easy to see that $\flat$ is an epimorphism[3] if X is nonempty. Hence if for some (nonempty) X we show $T(X)$ wqo, it follows that T is wqo. We shall in fact prove the following:

THE TREE THEOREM. *X wqo implies $T(X)$ wqo.*

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[3] See §4 for a fuller statement.

3. **An Equivalent Theorem.** The Tree Theorem is our main result. However it does not seem possible to apply our induction-like proof to it as it stands. To make our proof work, we must apply it to a more complicated proposition which is equivalent to the Tree Theorem.

For any vertex v of a (structured) tree τ we define $d(v)=$ *the degree of* v to be the number of edges whose initial vertex is v. We refer to the edges whose initial vertex is v as the edges which *sprout* from v. We define the *branch of* τ *at* v to consist of v and all vertices which can be reached from v along positively oriented paths starting at v, together with all edges which sprout from any of these vertices. (In other words, the branch at v consists of v and all vertices which lie above v, together with all edges which lie above v.) We define the root of the branch at v to be v itself, and we use in the branch the orientations and linear orders of τ itself. Clearly the branch becomes a structured tree under these definitions. We define the *branch of* t *at* v to be t restricted to the branch of τ at v.

A *forest* means a (finite) ordered sequence of structured trees. A *forest over* X means a (finite) ordered sequence of trees over X. If v is a vertex in τ, let us temporarily denote the edges which sprout from v by $e_1, \cdots, e_p$, where the subscripts are given in accordance with the linear order at v. Let the terminal vertices of these edges be $v_1, \cdots, v_p$. We refer to the branches of t (or of τ) at $v_1, \cdots, v_p$ as the *branches of* t (*or of* τ) *above* v. We refer to the sequence of branches above v as the *forest above* v (where the branches making up the forest are ordered in the obvious way). If the degree of v is 0, the forest above v is the empty sequence.

We define a *graded quasi-ordered space* Q to consist of a qo set Q_+ together with an infinite sequence $Q_0, Q_1, \cdots$ of subsets of Q, possibly overlapping, whose union is Q_+. A graded space Q is said to be wqo if

(4a) Q_+ is wqo, and

(4b) there is some integer N, called the *total degree of* Q, for which $Q_N = Q_{N+1} = \cdots$ but $Q_{N-1} \neq Q_N$.

If a tree $t: \tau \to Q_+$ over Q_+ satisfies the following condition, we call it a *tree over* Q and denote it by $t: \tau \to Q$:

(5) for every v in τ, $t(v)$ is in $Q_{d(v)}$. Intuitively this means that each vertex of t must be labelled with an element from the appropriate subset of Q_+. Define $T(Q)$ to be the collection of all trees over Q. Note that $T(Q)$ is a subset of $T(Q_+)$, and that the obvious qo on $T(Q)$ is precisely that which it receives as a subset of $T(Q_+)$.

THEOREM 1. *Q wqo implies $T(Q)$ wqo.*

This is the proposition that we shall actually prove.

If Q has total degree 0, then $T(Q_0) = T(Q)$ and Q wqo becomes equivalent to Q_0 wqo, so this theorem reduces to the Tree Theorem. On the other hand if we assume the Tree Theorem, we have that Q wqo implies Q_+ wqo, which implies $T(Q_+)$ wqo which implies $T(Q)$ wqo. Thus Theorem 1 is equivalent to the Tree Theorem.

The proof of Theorem 1 consists of three main parts. In the first part, given that Q is wqo and $T(Q)$ is not, we construct another wqo space Q^* which is "smaller" than Q (in a certain technical sense defined later). In the second part we prove that $T(Q^*)$ is not wqo. Thus by repeating this construction, we can obtain an infinite sequence $Q, Q^*, Q^{**}, \cdots$ of wqo spaces, each "smaller" than the preceding one (and such that $T(Q), T(Q^*), \cdots$ are not wqo). In the third part we show that such an infinite sequence of wqo spaces is impossible: the wqo character forbids it.

4. **Some lemmas and definitions.** Suppose that X and Y are qo spaces. Suppose $h: X \rightarrow Y$ is a function. We call h a *homomorphism* if h preserves $\leqq$, that is, if $x_1 \leqq x_2$ implies $h(x_1) \leqq h(x_2)$. We call h a *monomorphism* if it is 1-1 and if $x_1 \leqq x_2$ if and only if $h(x_1) \leqq h(x_2)$. We call h an *epimorphism* if it is onto and a homomorphism. We call h an *isomorphism* it is onto and a monomorphism. The following simple lemmas will be used many times:

EPIMORPHISM LEMMA. *If $h: X \rightarrow Y$ is a homomorphism, then X wqo implies $h(X)$ wqo. In particular, if h is an epimorphism, then X wqo implies Y wqo.*

MONOMORPHISM LEMMA. *If $h: X \rightarrow Y$ is a monomorphism, then Y wqo implies X wqo. Every subspace of a wqo space is wqo.*

Suppose Q and Q' are graded qo spaces. By a *function* $h: Q \rightarrow Q'$ we mean a function $h: Q_+ \rightarrow Q'_+$ which satisfies the following:

$$\text{(6)} \qquad \text{for every } i \geqq 0, \qquad h(Q_i) \subset Q'_i.$$

If $h: Q_+ \rightarrow Q'_+$ is also a homomorphism, we call $h: Q \rightarrow Q'$ a *homomorphism.* If in addition $h(Q_i) = Q'_i$, then we call h an *epimorphism.* It is easy to verify the following:

SECOND EPIMORPHISM LEMMA. *If $h: Q \rightarrow Q'$ is a homomorphism, then Q wqo implies $h(Q)$ wqo. In particular, if h is an epimorphism, then Q wqo implies Q' wqo.*

Note that the total degree of $h(Q)$ necessarily exists and is less than or equal to that of Q.

If $X_1, \cdots, X_k$ are disjoint qo spaces, we define their *direct union*, written

$$\dot{\bigcup} X_i \text{ or } X_1 \dot{\cup} \cdots \dot{\cup} X_k,$$

to be the qo space whose set is the set-theoretic union of the sets X_i and whose qo is the union of the quasi-orders of the X_i; that is, an ordering relation exists between two elements in the direct union if and only if the two elements belong to the same X_i and an ordering relation exists between them there. If $X_1, \cdots, X_k$ are qo spaces we define their *Cartesian product* (or *direct* *product*) to be the space whose set is the Cartesian product of the X_i's with $\leqq$ defined component-wise: $(x_1, \cdots, x_k) \leqq (y_1, \cdots, y_k)$ if and only if $x_i \leqq y_i$ for every i. Both the direct union and Cartesian product are clearly qo spaces.

FINITE UNION LEMMA. *If a qo space is the union of a finite number of wqo subspaces, then the space is wqo. The direct union of a finite number of disjoint wqo spaces is wqo.*

FINITE CARTESIAN PRODUCT LEMMA. *The Cartesian product of a finite number of wqo spaces is wqo.*

The first lemma is trivial. The second lemma, which is Theorem 2.3 of [1] and the Finite Product Lemma of [3] may easily be proved using condition (2) of the lemma is §2.

Let $F = \{\lambda\}$ be the collection of all finite linearly ordered sets λ, including the empty set. If X is qo, define $F(X)$ to be the set of all functions $f: \lambda \rightarrow X$ using any λ in F. Each element of $F(X)$ may be thought of as a finite sequence of elements from X, and this picture is only slightly inaccurate. Define a monomorphism $\omega: f_1 \rightarrow f_2$ to mean a monomorphism $\omega: \lambda_1 \rightarrow \lambda_2$ (where λ_1 is the domain of f_1 and λ_2 is the domain of f_2) with the additional property that

$$\begin{tikzcd} \lambda_1 \arrow[rr, "\omega"] \arrow[dr, "f_1"'] & & \lambda_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

$f_1(i) \leqq f_2\omega(i)$ for every i in λ_1. Define $f_1 \leqq f_2$ if and only if there is a monomorphism $\omega: f_1 \rightarrow f_2$. Intuitively, one sequence is less than another if some subsequence of the greater sequence majorizes the smaller sequence term by term. We agree that the empty sequence (that is, the unique function from the empty set into X) is $\leqq$ every sequence. Intuitively, $F(X)$ is the set of all finite sequences of elements of X, qo in a natural though unfamiliar way.

Clearly $F(X)$ is qo. We shall generally denote an element f of $F(X)$ as a sequence. Thus for example, (x_1, x_2, x_4) denotes the function f: $\{1, 2, 4\} \rightarrow X$ whose values are x_1, x_2, and x_4. We shall seldom mention the set λ involved as it is not important.

Notice that the natural embedding of $X \times \cdots \times X$ in $F(X)$ is a monomorphism. Thus the following theorem is closely related to the Finite Cartesian Product Lemma:

FINITE SEQUENCE THEOREM. *X wqo implies $F(X)$ wqo.*

For a proof of this theorem, see Theorem 4.3 of [1] or §4 of [3]. Rado [5] and Kruskal [2] contain related but much more general theorems. If we consider only unbranched (structured) trees, $T(X)$ reduces to $F(X)$ so the Tree Theorem may be considered an extension of the Finite Sequence Theorem.

If Q is any graded space, define $+Q$ to be the graded space of total degree 0 given as follows:

$$+Q_i \equiv (+Q)_i = Q_+,$$
$$+Q_+ \equiv (+Q)_+ = Q_+.$$

Obviously $+Q$ is wqo if and only if Q_+ is wqo. Also, $T(Q)$ is a subset of $T(+Q)$, which is the same as $T(Q_+)$.

A space Q of total degree N is called *neutral* if $Q_0, \cdots, Q_N$ are pairwise disjoint and if there are no ordering relations between elements belonging to different sets of this collection. We wish to associate a neutral space $\natural Q$ with any space Q which has some total degree (which we shall call N). Let $\{\psi_0, \psi_1, \cdots\}$ be an infinite set with the "discrete" ordering: $\psi_i \leqq \psi_j$ if and only if $i=j$. Define

$$\natural Q_i \equiv (\natural Q)_i = \begin{cases} Q_i \times \{\psi_i\}, & \text{for } 0 \leqq i \leqq N-1, \\ Q_i \times \{\psi_N\}, & \text{for } i \geqq N, \end{cases}$$

$$\natural Q_+ \equiv (\natural Q)_+ = \dot{\bigcup_0^N} \natural Q_i.$$

$\natural Q_+$ acquires a natural order as a subset of $Q_+ \times \{\psi_0, \cdots, \psi_N\}$. It is easy to see that $\natural Q$ is a neutral graded space of the same total degree as Q. Using the Finite Union Lemma, we see that $\natural Q$ is wqo if and only if Q is wqo.

There is a natural epimorphism $\natural^{-1}: \natural Q \to Q$ which is defined by

$$\natural^{-1}(x, \psi_i) = x.$$

It is easy to verify that $\natural^{-1}$ is an epimorphism.

Suppose that Y is a qo space, and that $(y_1, \cdots, y_r)$ is a fixed element of $F(Y)$. Let $(z_1, \cdots, z_j)$ be any other element of $F(Y)$ such that

$$(y_1, \cdots, y_r) \nleqq (z_1, \cdots, z_j).$$

We shall need the following construction in §5.

Let the *degree of inequality* of $(z_1, \cdots, z_j)$ (with respect to $(y_1, \cdots, y_r)$) be the largest integer ρ such that

$$(y_1, \cdots, y_\rho) \leqq (z_1, \cdots, z_j).$$

Clearly $0 \leqq \rho \leqq r-1$. Let the *first residual term* of $(z_1, \cdots, z_j)$ be the first term z_i (if any) such that $y_1 \leqq z_i$. Recursively, let the *kth residual term* of $(z_1, \cdots, z_j)$ be the first term z_i (if any) such that z_i follows the $(k-1)$st residual term and $y_k \leqq z_i$. Clearly $(z_1, \cdots, z_j)$ has precisely ρ residual terms if and only if its degree of inequality equals ρ. Let the *residual sequence* be the sequence $(z_{i_1}, \cdots, z_{i_\rho})$ of residual terms.

If z_{i_k} and $z_{i_{k+1}}$ are the kth and $(k+1)$st residual terms, then clearly $y_{k+1} \nleqq z_i$ for $i_k < i < i_{k+1}$. Let

$$((z_1, \cdots, z_{i_1-1}), (z_{i_1+1}, \cdots, z_{i_2-1}), \cdots, (z_{i_\rho+1}, \cdots, z_j))$$

be the *nonresidual double sequence* associated with $(z_1, \cdots, z_j)$. This double sequence clearly belongs to

$$F(Y - \text{upper } y_1) \times \cdots \times F(Y - \text{upper } y_{\rho+1}).$$

If X and X^* are qo spaces, we call X^* *smaller than* X if there is a monomorphism from X^* into X. We call X^* *essentially smaller* than X if there is an x in X such that X^* is smaller than $X-$upper x. By condition (4) of the lemma in §2, if X is not wqo, then there is an essentially smaller space X^* such that X^* is not wqo.

If Q and Q^* are two graded spaces of total degrees N and N^*, we define Q^* to be *smaller than Q by descent at R* if

(7a) Q_R^* is essentially smaller than Q_R;

(7b) Q_n^* is smaller than Q_n for all $n > R$, and

(7c) $N^* = \max (N, R)$.

The integer R is called the *index of descent.* If Q^* is smaller than Q by descent at some unspecified integer, we say merely that *Q^* is smaller than Q.*

The reason for this peculiar definition is that it describes the result of the construction in a later section. Another way of expressing (7c) is this: $N^* \geqq N$; and furthermore $R < N^*$ in case $N^* = N$, but $R = N^*$ in case $N^* > N$. It should be pointed out that the relation "smaller" for graded spaces is *not* transitive, though "smaller by descent at R" is transitive.

5. **First part of proof. Construction of Q^*.** It will suffice to prove Theorem 1 for neutral graded spaces. To see this first notice that any function h: $Q \rightarrow Q'$ induces a function Th: $T(Q) \rightarrow T(Q')$ by the following natural definition:

$$Th(t\colon \tau \rightarrow Q) = (ht\colon \tau \rightarrow Q').$$

$$\begin{tikzcd} & \tau \arrow[dl, "t"'] \arrow[dr, "ht"] & \\ Q \arrow[rr, "h"] & & Q' \end{tikzcd}$$

It is easy to see that if h is a homomorphism or an epimorphism, then Th is also. We have already seen that for any graded space Q there is a neutral space $\natural Q$ and an epimorphism $\natural^{-1}$: $\natural Q \rightarrow Q$. Then $T\natural^{-1}$: $T(\natural Q) \rightarrow T(Q)$ is also an epimorphism. Thus if Theorem 1 holds for neutral graded spaces, we have that Q wqo implies $\natural Q$ wqo, which implies $T(\natural Q)$ wqo, which implies $T(Q)$ wqo.

Now suppose that Q is some wqo neutral space and that $T(Q)$ is not wqo. Q will remain fixed throughout the remainder of this and the following section. We wish to construct a wqo neutral space Q^* which is smaller than Q such that $T(Q^*)$ is also not wqo. Throughout this and the following section, N and N^* will always refer to the total degrees of Q and Q^*.

If τ is a structured tree, then the *height* of τ is the number of edges in the longest oriented path in τ. As $T(Q)$ is not wqo, it must (according to the lemma of §2) contain some tree t such that $T(Q)-\text{upper } t$ is not wqo. Among all such trees t choose one, which we shall call $s:\sigma\rightarrow Q$, whose carrier σ has minimum height. The tree s will play a leading role throughout this and the following section. Let the degree of the root of s be called r, that is, s has r branches just above its root. Let these branches be called $s_1, \cdots, s_r$ where the subscripts have been assigned in accordance with the linear ordering at the root of s. (According to our previous convention, the carrier of each s_i is called σ_i. In the future we shall use this convention without further notice.) Each tree s_i has height strictly smaller than the height of s. It follows from the manner in which s was chosen that $T(Q)-\text{upper } s_i$ is wqo for $i=1$ to r. This fact is of central importance. Finally we set $u=s(\text{root of } s)$, that is, u is the label at the root of s. Thus u is in Q_r.

In case s has height 0, much of the preceding paragraph is vacuous, and also unnecessary. For if s has height 0, define Q^* thus:

$$Q_i^* = Q_i - \text{upper } u,$$
$$Q_+^* = \bigcup Q_i^*,$$
$$N^* = N.$$

It is easy to verify that Q^* is a wqo neutral space, that $T(Q^*)=T(Q)-\text{upper } s$ and is therefore not wqo, and that Q^* is smaller than Q with the index of descent being zero. Thus if s has height 0 the construction of Q^* is elementary. We shall hereafter assume that the height of s is greater than zero, which in turn implies that $r>0$.

Our construction of Q^* is conceptually a little different in the cases $r<N$ and $r\geqq N$, though no formal distinction need be made. In case $r<N$, we shall have Q_r^* essentially smaller than Q_r, but $Q_i^*=Q_i$ for $i\geqq r$. In case $r\geqq N$, we shall have Q_i^* essentially smaller than Q_i both for $i=r$ and for $i>r$. The reader is advised to keep these two different cases in mind, though no further reference to this distinction will be made. In both cases Q_i^* is relatively complicated for $i<r$.

Recalling from before that F is the operator that forms the space of finite sequences, first define

$$F_i = F[T(Q) - \text{upper } s_i], \qquad \text{for } i = 1 \text{ to } r,$$
$$W_i = F_1 \times \cdots \times F_{i+1}, \qquad \text{for } i = 0 \text{ to } r-1.$$

Next define Q' by

$$Q_i' = \begin{cases} Q_+ \mathbin{\dot\cup} [Q_+ \times W_i], & \text{for } i = 0 \text{ to } r-1, \\ Q_i - \text{upper } u, & \text{for } i \geqq r, \end{cases}$$

$$Q' = \bigcup Q_i' = Q_+ \mathbin{\dot\cup} \dot{\bigcup_0^{r-1}} [Q_+ \times W_i],$$

$$N' = \max(N, r).$$

We define Q^* to be $\natural Q'$. (If $r>N$, Q' is not neutral.)

It is easy to see that Q^* is a neutral graded space, that it has total degree $N^*=\max(N, r)$, and that Q^* is smaller than Q by descent at r. To verify that Q^* is wqo, we note that $T(Q)-$upper s_i is wqo by a previous remark, F_i is wqo by the Finite Sequence Theorem, W_i and also $Q_+\times W_i$ are wqo by the Finite Cartesian Product Lemma, Q_i' is wqo either by the Finite Union Lemma or trivially depending on i, each $Q_i^*=Q_i'\times\{\psi_i\}$ with $0\leqq i\leqq N^*$ is obviously wqo, and finally Q_+^* is wqo by the Finite Union Lemma. Thus we have proved that Q^* is a wqo neutral space.

It is convenient to define another graded-space Q'', which is larger than Q', for use in the next section. Let

$$Q_i'' = \begin{cases} Q_+ \dot{\cup} [Q_+\times W_i], & \text{for } i = 0 \text{ to } r-1, \\ Q_+, & \text{for } i \geqq r, \end{cases}$$

$$Q_+'' = \bigcup Q_i'' = Q_+ \dot{\cup} \dot{\bigcup_{0}^{r-1}} [Q_+ \times W_i],$$

$$N'' = \max(N, r).$$

Q'' is clearly a graded space of total degree N'', though not necessarily neutral. As a matter of fact it is also wqo, though we have no need of this fact. Furthermore, $T(Q'')$ contains $T(Q)$, $T(Q')$, and $T(+Q)$.

6. **Second part of proof. $T(Q^*)$ is not wqo.** To prove that $T(Q^*)$ is not wqo we shall define a function H: $T(Q')\to T(+Q)$. We shall prove that H is a homomorphism and covers $T(Q)-$upper s. The Second Epimorphism Lemma then shows that $T(Q^*)$ is not wqo, as we see from this diagram:

$$T(Q^*) \xrightarrow{T\natural^{-1}} T(Q') \xrightarrow{H} T(+Q) \supset T(Q) - \text{upper } s.$$

The definition of H is a little complicated and requires certain preliminaries. Suppose t: $\tau\to Q$ is in $T(Q'')$. We classify the vertices of t according to which part of Q'' their labels lie in. If $t(v)$ is in Q_+, we call v an *ordinary vertex* of t. If $t(v)$ is in $Q_+\times W_\rho$ we call v *special of degree* ρ. (Clearly if v is special of degree ρ, it has degree ρ in the ordinary sense.)

We have already defined a forest to be an ordered sequence of trees. Call the *length* of a forest the number of trees in it. Then the forest above a vertex which is of degree ρ has length ρ. Define a *double forest* to be an ordered sequence of forests. Thus the elements of W_i are double forests. The *length* of a double forest is the number of forests in it. If

$$((t_{11}, \cdots, t_{1\phi_1}), \cdots, (t_{\rho+1,1}, \cdots, t_{\rho+1,\phi_{\rho+1}}))$$

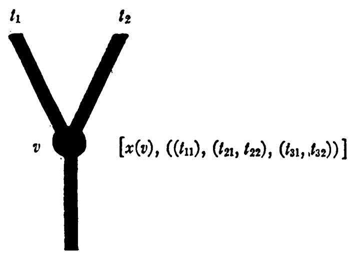
is a double forest of length $\rho+1$ and

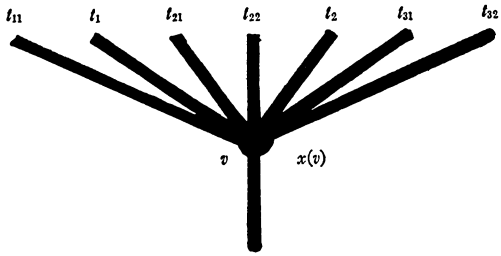
$$(t_1, \cdots, t_\rho)$$

is a forest of length ρ, we call the following forest their *intermingling*:

$$(t_{11}, \cdots, t_{1\phi_1}, t_1, t_{21}, \cdots, t_{2\phi_2}, t_2, \cdots, t_\rho, t_{\rho+1,1}, \cdots, t_{\rho+1,\phi_\rho+1}).$$

If t is any element of $T(Q'')$ and v is any vertex of t, we shall define another tree E_vt, which we call the *expansion of t at v*, which also belongs to $T(Q'')$. If v is an ordinary vertex of t, define E_vt to be t itself. If v is special of degree ρ in t, write the forest above v as $(t_1, \cdots, t_\rho)$ and write $t(v) = [x(v), w(v)]$ where of course $x(v)$ is in Q_+ and $w(v)$ is a double forest of length $\rho+1$ and belongs to W_ρ. Define E_vt to be the same as t with two changes: the label at v is changed from $[x(v), w(v)]$ to $x(v)$; and the forest above v is changed from $(t_1, \cdots, t_\rho)$ to the intermingling of $(t_1, \cdots, t_\rho)$ with $w(v)$. This expansion process is illustrated by the accompanying diagram.





It is clear that E_vt belongs to $T(Q'')$, and that if v is special then E_vt has one less special vertex than t, for the vertex v is not special in E_vt, and the many new vertices introduced into E_vt are all necessarily ordinary.

Now we define H: $T(Q') \to T(+Q)$. If t is in $T(Q')$, let the vertices of t be $v^1, \cdots, v^\nu$. Define

$$Ht = E_{v^\nu} \cdots E_{v^2}E_{v^1}t.$$

It is clear that every vertex of Ht is ordinary, so Ht belongs to $T(+Q)$. It is also clear that Ht is independent of the order in which the expansion operators are applied; thus Ht is well-defined. We note that H could be extended to apply to every tree in $T(Q'')$, but we do not wish to make this extension.

We wish to prove that H is a homomorphism. To do so we need two homomorphism-like properties of the expansion operator. First, if t and t' are in $T(Q'')$, if $\omega: t \to t'$ is a monomorphism, and if v is a vertex of t, then there exists a monomorphism $E_v\omega$: $E_vt \to E_{\omega(v)}t'$. Second, if t and t' are in $T(Q'')$, if ω: $t \to t'$ is a monomorphism, and if v' is a vertex of t' which is not in the image of ω, then $\omega: t \to E_{v'}t'$ is a monomorphism.

The second of these statements is obvious. To prove the first, we note that v is ordinary if and only if $\omega(v)$ is ordinary, and that v is special of degree ρ if and only if $\omega(v)$ is special of degree ρ, because $t(v) \leqq t'\omega(v)$.

In case v and $\omega(v)$ are ordinary, $E_vt = t$ and $E_{\omega(v)}t' = t'$, so we may define $E_v\omega$ to be ω itself. In case v and $\omega(v)$ are special of degree ρ, write

$$t(v) = [x(v), (f_1, \cdots, f_{\rho+1})],$$

$$t'\omega(v) = [x'(v), (f_1', \cdots, f_{\rho+1}')],$$

where f_i and f_i' are each forests which belong to F_i. Then $t(v) \leqq t'\omega(v)$ implies that $x(v) \leqq x'(v)$ and that $f_i \leqq f_i'$ for $i=1$ to $\rho+1$. Thus for each i there is a monomorphism $\theta_i: f_i \rightarrow f_i'$. Now we put together ω and the θ_i to form $E_v\omega$. For those vertices of E_vt which come from trees of f_i we define $E_v\omega$ with the aid of θ_i; for every other vertex we define $E_v\omega$ to agree with ω. It is not hard to see that $E_v\omega$ is a monomorphism.

Now we prove that H is a homomorphism. Suppose that t and t^* are in $T(Q'')$, and that $t \leqq t^*$. Then there is a monomorphism $\omega: t \rightarrow t^*$. Let $v^1, \cdots, v^\nu$ be the vertices of t. Let $v^{*1}, \cdots, v^{*\mu}$ be those vertices of t^* which are not in the image of ω. Then

$$E_{v^\nu} \cdots E_{v^1}\omega: E_{v^\nu} \cdots E_{v^1}t \rightarrow E_{v^{*\mu}} \cdots E_{v^{*1}}E_{\omega(v^\nu)} \cdots E_{\omega(v^1)}t^*$$

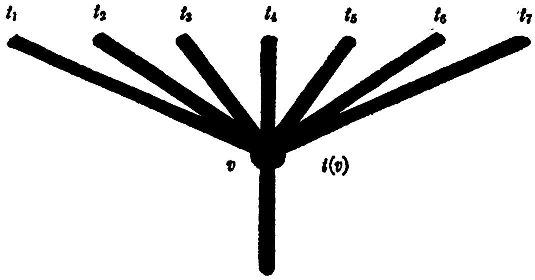
is a monomorphism from Ht into Ht^*, so $Ht \leqq Ht^*$. This proves that H is a homomorphism.

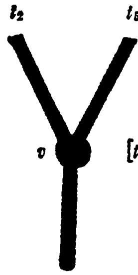
To prove that H covers $T(Q)$ — upper s is a little complicated and requires certain preliminaries. If t is a tree in $T(Q'')$ and v is a vertex of t, we call v *prunable* if

(8a) the branch of t at v belongs to $T(+Q)$, and

(8b) the forest $(s_1, \cdots, s_r) \nleqq$ the forest of t above v. We call v *minimal prunable* if it is prunable and if it is not above any other prunable vertices, that is, if v cannot be reached from any other prunable vertex along a positively oriented path of t. If v is prunable we shall define another tree P_vt, which we call *t pruned at v*, which also belongs to $T(Q'')$. (The process of pruning is inverse to the process of expanding in a sense that we shall soon make precise.) Recall from an earlier section the concepts of degree of inequality, residual sequence, and nonresidual double sequence. By condition (8b) we see that the forest $(t_1, \cdots, t_n)$ of t above v must have some degree of inequality ρ with respect to the forest $(s_1, \cdots, s_r)$. Of course $0 \leqq \rho \leqq r-1$. Define the *residual forest of t above v* to be the residual sequence of $(t_1, \cdots, t_n)$ with respect to $(s_1, \cdots, s_r)$. Define the *nonresidual double forest of t at v* to be the nonresidual double sequence of $(t_1, \cdots, t_n)$ with respect to $(s_1, \cdots, s_r)$. Denote the nonresidual double forest at v by $w(v)$. Define P_vt to be the same as t but with two changes: the label at v is changed from $t(v)$ to $[t(v), w(v)]$; and the forest above v is changed from $(t_1, \cdots, t_n)$ to the residual forest of t above v. This pruning process is illustrated by the accompanying diagram.

From the definition of degree of inequality we see that v has degree ρ in P_vt. From condition (8a) we see that the nonresidual double forest $w(v)$ is made of trees lying in $T(+Q)$. From the definition of nonresidual double sequence we see that $w(v)$ lies in W_ρ. Thus we have shown that P_vt lies in





$[t(v), ((t_1), (t_3, t_4), (t_6, t_7))]$

$T(Q'')$. We also note that v is not prunable in P_vt because the label at v in P_vt is not in Q_+; so P_vt has strictly fewer prunable vertices than t.

We prove that if v is any prunable vertex of any tree t in $T(Q'')$, then $E_vP_vt=t$. As P_v and E_v only change the trees on which they operate at and above v, E_vP_vt clearly is the same as t everywhere else than at and above v. P_v changes the label at v from $t(v)$ to $[t(v), w(v)]$ where $w(v)$ is the nonresidual double forest associated with the forest above v in t. E_v then changes the label $[t(v), w(v)]$ back to $t(v)$. As for the forest above v, P_v changes the original forest above v to its residual forest by removing the trees which belong to the nonresidual double forest $w(v)$. E_v then intermingles this new forest above v with the double forest $w(v)$, thus obviously restoring the original forest above v.

The preceding paragraph demonstrates that the pruning operators are inverse to the expansion operators. As H is a product of expansion operators, we can show that H covers $T(Q)-$upper s by showing that each tree in $T(Q)-$upper s may be transformed into a tree of $T(Q')$ by a suitable product of pruning operators. The following three paragraphs are technical steps necessary to prove that the pruning operators may be applied often enough.

If t is in $T(Q'')-$upper s, and if v in t is prunable, then P_vt is also in $T(Q'')-$upper s. For suppose ω: $s\to P_vt$ is a monomorphism. As $s(v')$ is in Q_+ for every vertex v' of s, and as $s(v')\leqq P_vt(\omega(v'))$ according to the definition of monomorphism, we must have $P_vt(\omega(v'))$ in Q_+, which implies that v cannot be in the image of ω. Then by an earlier remark, ω: $s\to E_vP_vt$ is a monomorphism, so $s\leqq t$, which contradicts the assumption made about t.

Recall that u is the label at the root of s. If t is in $T(Q'')-$upper s, if v is a vertex of t such that $u\leqq t(v)$, and if the branch of t at v is in $T(+Q)$, then v is prunable. We only need to prove that condition (8b) holds. Suppose that it fails. Let

$$\omega\colon\ (s_1, \cdots, s_r) \to \text{the forest above } v$$

be a monomorphism. Then it is possible to define a monomorphism ω': $s\to t$ by defining ω' (root of s) $=v$ and defining ω' for the other vertices of s with the aid of ω. But such a monomorphism ω' cannot exist by the assumption concerning t, so ω also cannot exist, so condition (8b) holds.

Suppose t is in $T(Q'')-$upper s. Suppose v is minimal prunable in t. Suppose that t has the property that for every vertex v' of t, $u \leqq t(v')$ implies v' prunable. Then $P_v t$ has the same property. For let v' be in $P_v t$ and suppose $u \leqq P_v t(v')$. This implies $v \neq v'$, so $P_v t(v') = t(v')$. Thus $u \leqq t(v')$. Then as t is assumed to have the property we are considering, v' is prunable in t. As v was assumed minimal prunable in t, we have that v' is not below v. Therefore the branch of t at v' is the same as the branch of $P_v t$ at v'. As v' is prunable in t, and as prunability at a vertex depends only on the nature of the branch at that vertex, v' is prunable in $P_v t$. Thus $P_v t$ does indeed have the property we claimed for it.

Now we can prove that H covers $T(Q)-$upper s. Let t be any tree in $T(Q)-$upper s. Every branch of t belongs to $T(+Q)$ of course. Thus by the second preceding paragraph, $u \leqq t(v')$ implies that v' is prunable. Let v^1 be any minimal prunable vertex of t (if t has any prunable vertices). Form $P_{v^1}t$. By the third preceding paragraph, $P_{v^1}t$ belongs to $T(Q'')-$upper s. By the immediately preceding paragraph, $u \leqq P_{v^1}t(v')$ implies v' prunable. Let v^2 be any minimal prunable vertex of $P_{v^1}t$ (if this tree has any prunable vertices). Form $P_{v^2}P_{v^1}t$. As before we see that this tree belongs to $T(Q'')-$upper s, and that $u \leqq P_{v^2}P_{v^1}t(v')$ implies v' is prunable. We continue the process of picking minimal prunable vertices $v^3, v^4, \cdots$ and forming new trees as long as we can. By an earlier remark, each successive tree formed has strictly fewer prunable vertices than the tree from which it was formed. Therefore this process must stop after a finite number of steps. Let v^ν be the last minimal prunable vertex which occurs. Let $t' = P_{v^\nu} \cdots P_{v^2}P_{v^1}t$. Then t' belongs to $T(Q'')-$upper s and has the property that $u \leqq t'(v')$ implies v' prunable. But t' has no prunable vertices. Therefore $u \nleqq t'(v')$ for every vertex v' of t'. Therefore t' belongs to $T(Q')-$upper s. Furthermore it is clear that $Ht' = t$. For let $v^{\nu+1}, \cdots, v^\mu$ be the vertices of t' other than $v^1, \cdots, v^\nu$. Then

$$Ht' = E_{v^1} \cdots E_{v^\mu}t' = E_{v^1} \cdots E_{v^\nu}P_{v^\nu} \cdots P_{v^1}t = t.$$

Thus H does indeed cover $T(Q)-$upper s. This completes the proof that $T(Q^*)$ is not wqo.

7. Third part of the proof. In this section we prove the following:

LEMMA. *If $Q(1), Q(2), \cdots$ is a sequence of wqo neutral (graded) spaces, and $Q(k+1)$ is smaller than $Q(k)$ for every k, then the sequence is finite.*

Let the total degree of $Q(k)$ be $N(k)$, and let the index of descent from $N(k)$ to $N(k+1)$ be $R(k)$. For convenience, let $Q_{N(k)}(k)$ be denoted by $Q_\infty(k)$; the mnemonic significance of this is obvious.

Our basic tool is the following:

LEMMA. *If $X(1)$, $X(2)$, $\cdots$ is a sequence of qo spaces, each smaller than the preceding one, and if $X(1)$ is wqo, then there are only a finite number of values of k for which $X(k+1)$ is essentially smaller than $X(k)$.*

This is essentially Theorem 2.4 of [1] and condition (3b) of Theorem 1 of [3]. The reader may prove this very simple but useful lemma for himself.

We prove the sequence $Q(1)$, $Q(2)$, $\cdots$ must be finite by repeated use of the lemma just stated. First note that $Q_\infty(1)$, $Q_\infty(2)$, $\cdots$ is a sequence of qo spaces satisfying the hypotheses of the lemma. Hence $Q_\infty(k+1)$ is essentially smaller than $Q_\infty(k)$ for only a finite number of values of k. But whenever $R(k) \geqq N(k)$, we have $Q_\infty(k+1)$ essentially smaller than $Q_\infty(k)$. Thus $R(k) \geqq N(k)$ only a finite number of times. Let k_0 be so large that $R(k) < N(k)$ for all $k \geqq k_0$. Let $M = N(k_0)$. By condition (7c) in the definition of descent, we see that $M = N(k_0) = N(k_0+1) = N(k_0+2) = \cdots$. Thus for $k \geqq k_0$ we have $R(k) < M$.

Now $Q_{M-1}(k_0)$, $Q_{M-1}(k_0+1)$, $\cdots$ satisfy the hypotheses of the lemma. But whenever $R(k) = M-1$, $Q_{M-1}(k+1)$ is essentially smaller than $Q_{M-1}(k)$. Hence by the lemma we may pick a k_1 so large that for $k \geqq k_1$, $R(k) < M-1$.

Using the argument in the preceding paragraph M times, we eventually find a k_M so large that for $k \geqq k_M$ we have $R(k) < M - M = 0$, which is impossible. That is to say, k cannot be greater than k_M. Therefore the sequence $Q(1)$, $Q(2)$, $\cdots$ is finite. This completes the proof of the Tree Theorem.

8. **Two conjectures.** If g_1 and g_2 are graphs, define $g_1 \leqq g_2$ if g_1 can be homeomorphically embedded in g_2. This relation is a qo on the set of all graphs. The set of all graphs is not wqo, for infinite, pairwise incomparable sets of elements are known.

CONJECTURE 1. *The set of all (connected) trees, finite or infinite, is wqo.*

CONJECTURE 2. *The set of all finite graphs of maximum degree $\leqq 3$ is wqo.*

9. **Connections with Higman's theory.** The main result of Higman [1], namely Theorem 1.1, is equivalent to our Theorem 1 with an extra hypothesis: namely that Q_N (and hence Q_{N+1}, Q_{N+2}, $\cdots$) is empty, where N is the total degree of Q. Geometrically this means that we consider only structured trees with vertices of degree $\leqq N-1$. However the language and viewpoint of [1] are so different from the present paper that a brief glossary might be useful.

Higman [1]	*Present Paper*
Quasi-order	Quasi-order
Quasi-order with finite basis property	Well-quasi-order
Algebra A	Space containing $T(Q)$
Set of operations M	Set Q
Set of n-ary operations M_n	Set Q_n
Minimal algebra	Space $T(Q)$
Closure of a set B	Upper B
Operator V	Operator F.

Actually not any minimal algebra corresponds strictly to some $T(Q)$ but only a minimal algebra which is "free" in the sense that different expressions do not represent the same elements of the algebra. However any algebra is the epimorphic image of a free algebra, and epimorphism preserves wqo, so it is sufficient to deal with free algebras. The tree spaces may be considered as explicit constructions of the free (minimal) algebras in the same way that word groups are used to give explicit constructions of free groups and free products.

Bibliography

1. Graham Higman, *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc. vol. 2 (1952) pp. 326–336.

2. Joseph Kruskal, *Well-partial-order and Rado's conjecture*, submitted to the Proc. London Math. Soc.

3. ———, *The theory of well-partially-ordered sets*, June, 1954, Princeton University, doctoral thesis.

4. B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 202–346.

5. Richard Rado, *Partial well-ordering of sets of vectors*, Mathematika vo'. 1 (1954) pp. 89–95.

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