A BAUES FIBRATION CATEGORY STRUCTURE ON BANACH AND C*-ALGEBRAS

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Abstract. We show that almost any category suitably cotensored over the full subcategory of topological spaces consisting of disjoint unions of direct products of unit intervals admits a Baues fibration category structure. This has applications to categories occurring in functional analysis, where it shows that e.g. the category of C*-algebras and the category of Banach algebras admit fibration category structures.

We look at the case of C*-algebras in some more detail, examining how the fibration category structure on C*-algebras relates to the cofibration category structure on pointed compact Hausdorff spaces. The analysis for instance reveals that the fibration category structure on C*-algebras cannot be extended to a Quillen model category structure.

1. Introduction

Axiomatic homotopy theory, usually in the form of the homotopical algebra of Quillen [28] (cf. also [16, 19]) have recently had an increasing importance outside homotopy theory, with new categories satifying the axioms appearing yielding important results.

In this paper we demonstrate how many categories occurring in functional analysis (e.g. Banach algebras, C*-algebras, topological *-algebras, pro-C*-algebras and various subcategories thereof) naturally satisfies a weaker set of axioms than those of Quillen, namely those of a fibration category in the sense of Baues [1].

Slightly more precisely, let $\mathcal{I}$ be the (symmetric monoidal [23]) full subcategory of topological spaces, with objects spaces homeomorphic to finite disjoint unions of products, I^n, of unit intervals I ($I^0 = *$). If a category $\mathcal{C}$ is 'cotensored' over $\mathcal{I}$, i.e. if we for $A \in \mathcal{I}$ and $X \in \mathcal{C}$ in a natural way can define an object $\mathrm{map}(A, X) \in \mathcal{C}$, compatible with the structures of the two categories, then the assignment $PX = \mathrm{map}(I, X)$ endows $\mathcal{C}$ with a natural 'path object' (cf. Section 3). We show that this gives $\mathcal{C}$ the structure of a P-category (Theorem 3.7) in the sense of Baues [1]. A P-category structure on $\mathcal{C}$ then automatically gives a fibration category structure on $\mathcal{C}$, with weak equivalences being the ones induced by the path object.

If for instance $\mathcal{C}$ is one of the categories mentioned above, then the topological space of maps, $\mathrm{map}(I^n, X)$ is again an object in $\mathcal{C}$ via 'pointwise operations' and this assignment satisfies the technical requirements.

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Stated briefly, a Baues fibration category structure is an axiomatic framework which allows one to do 'one half' of homotopy theory. In a fibration category one is for instance able to talk about fibrations and construct loop functors, Puppe sequences and various homotopy (invariant) limits (but not the 'dual' constructions). A fibration category structure is roughly one half of the structure of a (proper) model category in the sense of Quillen (cf. [28, 16]) and similar to structures proposed earlier by K. S. Brown [3] and F. Waldhausen [32] among others.

For the category of C^*-algebras the definition of a fibration category structure formalizes an analogy to topology previously treated in a somewhat ad hoc manner (cf. e.g. [30, 29]), and makes an array of results and techniques from homotopy theory readily available (cf. [1]).

The above results raise the question of when the above fibration category structures on $\mathcal{C}$ can be extended to a model category structure in the sense of Quillen. We consider this question in the case of C^*-algebras, where the question of the existence of a model category structure has been raised by Schochet [30]. An adjoint functor argument reduces the question to the case of commutative C^*-algebras, or dually pointed compact Hausdorff spaces, which we then answer it in the negative (Corollary 4.7). Our conclusion just builds on the category of C^*-algebras contains (a suiable subcategory of) the opposite category of the category of pointed compact Hausdorff spaces as a reflective subcategory, and hence remains valid in greater generality (Theorem 4.5).

The analysis indicates that the problem with obtaining a model category structure on the category of C^*-algebras stems not from it, like category of pointed compact Hausdorff spaces, being 'too small'. We conclude by discussing how this problem might possibly be overcome.

2. Notation, prerequisites and relationship with other work

This paper arose out of an attempt to understand the 'topological constructions' used by C^*-algebraists. We hope that it helps tie together the C^*-algebra literature on 'non-commutative topology' (cf. e.g. [30, 29, 25, 26]) and the axiomatic homotopy theory and category theory literature (cf. e.g. [28, 16, 1, 21]). From searching the literature it seems that there is a tendency amongst topologists to overlook examples coming from functional analysis as examples of axiomatic homotopy categories. Conversely it seems like many results from category theory and axiomatic homotopy theory have been partially overlooked by the C^*-algebra community, or at least deserves to be better known.

In our study of non-commutative topology we take the viewpoint of axiomatic homotopy theory, but try to keep the treatment accesible to people outside the field, supplementing with, we hope, pertinent references to the literature. The main prerequisite is basic category theory as describe in the book of Mac Lane [23].

Readers not familiar with the language of axiomatic homotopy theory are however strongly encouraged to take a look in the excellent introduction [16]. Also, for the sake of brevity we shall not try to systematically list the all consequences of a fibration category structure—we refer the reader to the book of Baues [1]. Very recently a new book by Kemps and Porter on axiomatic homotopy has appeared [21], to which we also direct the reader.

There is a tradition in C^*-algebra theory to denote objects constructed analogously to topological constructions by the same name, disregarding the fact that the functor from pointed compact Hausdorff spaces to C*-algebras is contravariant. This causes objects and maps constructed axiomatically to be called by their 'dual' name of what is standard in C*-algebra theory. However in all the cases that we are aware of our analysis shows that, after dualizing, old and new definitions coincide—a reassuring fact in itself.

Unless otherwise specified, all topological spaces will be pointed, although we as customary suppress this from the notation. Also C*-algebras are complex and not assumed to be unital. We refer the reader to Section 4 for a discussion explaining these conventions.

3. A P-category structure on a category $\mathcal{C}$

In this section we show that a category $\mathcal{C}$ which is 'cotensored' over $\mathcal{I}$ as sketched in the introduction can be given the structure of a P-category. (We will only use the word 'cotensored' as an ad hoc term, for the conditions stated in Theorem 3.7, however in many cases of interest, e.g. for many types of topological algebras including C*-algebras, the category will in fact be enriched over compactly generated Hausdorff spaces (cf. [14]) or quasitopological spaces (cf. [15]), and in those cases our 'cotensor' will be a cotensor in the sense of enriched category theory [22, 13] (cf. [15, p. 119] for a discussion of this).)

To motivate the constructions to readers unfamiliar with axiomatic homotopy theory we start out by briefly discussing some basic concepts from homotopy theory. Consider the category of pointed spaces. In the pointed category a homotopy between maps $f_0, f_1 : X \to Y$ is a map $F : X \rtimes I \to Y$ such that $Fi_0 = f_0$ and $Fi_1 = f_1$, where $X \rtimes I = X \times I / * \times I$ is the half-smash product and $i_0, i_1 : X \to X \rtimes I$ are the natural inclusions. By adjunction F corresponds to a pointed map $\tilde{F} : X \to Y^I$, (where Y^I denotes the space of maps $I \to Y$ endowed with the compact-open topology) such that $p_0\tilde{F} = f_0$ and $p_1\tilde{F} = f_1$ where $p_0, p_1 : Y^I \to Y$ are the natural projections. In the language of homotopical algebra (cf. [28, 16]) $X \rtimes I$ is what is known as a cylinder object for X, and Y^I is a path object for Y.

We now abstractly define what we mean by a natural path object. We follow the definition of Baues [1], which is slightly different from the one of Quillen [28] (see however Remark 3.3).

Definition 3.1. *A natural path object on a category $\mathcal{C}$ with finite products is a functor $P : \mathcal{C} \to \mathcal{C}$ together with natural transformations $p_0, p_1 : P \to Id$, $i : Id \to P$ such that for all $X \in \mathcal{C}$ we have that $X \xrightarrow{i} PX \xrightarrow{(p_0,p_1)} X \times X$ is a factorization of the diagonal. Dualizing this definition we get the definition of a natural cylinder object on a category.*

We are now ready to state the definition of a P-category.

Definition 3.2. [1, p. 27] *Let $\mathcal{C}$ be a category with a terminal object $*$ which has all pullbacks. Assume that $\mathcal{C}$ has a natural path object P which preserves pullbacks and takes $*$ to $*$. Assume furthermore that $\mathcal{C}$ has a distinguished class of morphisms called fibrations.*

We say that $\mathcal{C}$ is a P-category if it satisfies the following axioms:

1. *The pullback of a fibration along an arbitrary map is again a fibration.*
2. *The unique map $X \to *$ is fibration for all X. Isomorphisms in $\mathcal{C}$ are fibrations, and fibrations are closed under composition.*

(3) *(Homotopy lifting property) Let $q : X \to Y$ be any fibration, and let A be any object in $\mathcal{C}$. Assume that we have maps $f : A \to X$, $H : A \to PY$ such that the solid arrow part of the following diagram commute.*

$$\begin{tikzcd}
X \arrow[dd, "q"'] & & PX \arrow[ll, "p_0"'] \arrow[dd, "Pq"] \\
& A \arrow[ul, "f"] \arrow[ur, dashed, "\tilde{H}"] \arrow[dr, "H"] & \\
Y & & PY \arrow[ll, "p_0"]
\end{tikzcd}$$

Then there exists a map $\tilde{H} : A \to PX$ making the whole diagram commute.

(4) *For any fibration $q : X \to Y$ the canonical map $PX \to PY \times_{Y \times Y} (X \times X)$ is a fibration.*

(5) *For all objects X there exists an 'exchange map' $T : P^2X \to P^2X$ such that $p_\epsilon \circ T = P(p_\epsilon)$ and $P(p_\epsilon) \circ T = p_\epsilon$, $\epsilon = 0, 1$.*

In the case where PX consists of the maps from the standard unit interval to X, the exchange map of Axiom 5 is just the standard map exchanging the two coordinates.

Dualizing the notion of a P-category one obtains an I-category, i.e. a category with a natural cylinder object satisfying the dual requirements (cf. [1, p. 16]). The duality is strict in the sense that if $\mathcal{C}$ is a P-category then $\mathcal{C}^{op}$ is an I-category, and vice versa.

The category of pointed topological spaces possesses both a I- and a P-category structure with the definitions of path and cylinder objects from the beginning of this section, and usual cofibrations and (Hurewicz) fibrations as respectively cofibrations and fibrations. Note that these are exactly the ones given by the lifting property in Axiom 3 of Definition 3.2.

It follows from [1, Thm. I.3a.4] that a P-category structure on $\mathcal{C}$ implies that $\mathcal{C}$ can be given a structure of a fibration category taking as fibrations the fibrations in the P-category structure, and as weak equivalences homotopy equivalence as induced by the path object P. In this structure all objects will be fibrant and cofibrant. Dual to the notion of a fibration category structure is the notion of a cofibration category structure (cf. [1, p. 5]), and hence an I-category structure on $\mathcal{C}$ induces a cofibration category structure on $\mathcal{C}$, in which all objects are fibrant and cofibrant.

Remark 3.3. Just having a path object enables one to define a notion of homotopy. However unless extra assumptions are made on the path object, homotopy will not necessarily be an equivalence relation. If however the path object is part of a P-category structure then homotopy equivalence becomes an equivalence relation preserved under composition, and we say that a map is a weak equivalence if it has a two sided inverse up to homotopy. Furthermore in this case our path object is a path object in the sense of Quillen, i.e. the map $(p_0, p_1) : PX \to X \times X$ is a fibration and $i : X \to PX$ is a weak equivalence. The first claim follows directly from Axiom 4 of the definition of a P-category setting $Y = *$, the second is slightly more complicated (cf. [1, p. 23]).

For use in Section 4 we need the following definition.

Definition 3.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two P-categories. We say that F is a P-functor if F takes $*$ to $*$, pullbacks to pullbacks, fibrations to fibrations, and commute with P.*

When dealing with fibrations it is useful to have made the following observation, which is a direct consequence of the universal property of the pullback.

Observation 3.5. *In the setup of Definition 3.2, a map $q : X \to Y$ has the homotopy lifting property of Axiom 3. with respect to any A iff it has the homotopy lifting property with respect to $X \times_Y PY$, i.e. iff the canonical map $PX \to X \times_Y PY$ has a section.*

Given a category with a natural path object it is possible simply to define fibrations by the homotopy lifting property in Axiom 3 above. In this case several of the axioms above are automatically satisfied:

Lemma 3.6. *Let $\mathcal{C}$ be a category with a terminal object $*$ which has all pullbacks. Assume that $\mathcal{C}$ has a natural path object P which preserves pullbacks and takes $*$ to $*$. Define a distinguished class of morphisms called fibrations by Axiom 3 of Definition 3.2.*

Then $\mathcal{C}$ is a P-category if it satisfies the following two properties:

4. *For any fibration $q : X \to Y$ the canonical map $PX \to PY \times_{Y \times Y} (X \times X)$ is a fibration.*
5. *For all objects X there exists an 'exchange map' $T : P^2X \to P^2X$ such that $p_\epsilon \circ T = P(p_\epsilon)$ and $P(p_\epsilon) \circ T = p_\epsilon$, $\epsilon = 0, 1$.*

Proof. We just have to show that the choice of fibrations given by Axiom 3 will satisfy Axiom 1 and 2, but this is totally straight forward. Axiom 1, that the pullback of a fibration again is a fibration, is a consequence of the fact that P is assumed to preserve pullbacks and the universal property of the pullback construction. We proceed to check Axiom 2. The unique map $X \to *$ is a fibration since P takes $*$ to $*$ and the map $p_0 : PX \to X$ has a right inverse i, by the definition of a path object. That isomorphisms in $\mathcal{C}$ are fibrations follows from the fact that P is a functor, and that fibrations are closed under composition is a direct consequence of the definition. □

If our path object is a reasonably naturally arising one, and our fibrations are chosen to be the ones with the homotopy lifting property, we see that the only axiom which really needs to be checked is Axiom 4. It however turns out that also this axiom is just a formal consequence of the geometry of $I \times I$ in our case of interest.

Theorem 3.7. *Let $\mathcal{I}$ be the full subcategory of the category of (unpointed) topological spaces having as objects spaces homeomorphic to finite disjoint unions of products, I^n, of the unit interval I ($I^0 = *$).*

Let $\mathcal{C}$ be a pointed category and suppose that there exists a functor $\mathrm{map} : \mathcal{I}^{op} \times \mathcal{C} \to \mathcal{C}$ *which satisfies a natural equivalence* $\mathrm{map}(-, \mathrm{map}(-, -)) \simeq \mathrm{map}(- \times -, -)$, *as functors $\mathcal{I}^{op} \times \mathcal{I}^{op} \times \mathcal{C} \to \mathcal{C}$.*

Assume that $\mathcal{C}$ has all pullbacks and that the functor $\mathrm{map}(A, -)$ *takes $*$ to $*$ and preserves pullbacks. Also assume that* $\mathrm{map}(-, X)$ *takes the pushouts which exist in $\mathcal{I}$ to pullbacks in $\mathcal{C}$. Finally assume that* $\mathrm{map}(I^0, -)$ *is equivalent to the identity functor.*

Then $\mathrm{map}(I,-) : \mathcal{C} \to \mathcal{C}$ *is a natural path object* P *on* $\mathcal{C}$ *which induces a* P*-category on* $\mathcal{C}$*, taking as fibrations the ones induced by Axiom* 3 *of Definition 3.2.*

Especially $\mathcal{C}$ *is a fibration category in the sense of Baues* [1, p. 10] *with the following structure: Fibrations are the same as in the* P*-category structure. Weak equivalences are the homotopy equivalences induced via the path object* $\mathrm{map}(I,-)$*. In this structure all objects are fibrant and cofibrant.*

Proof. We have that $\mathrm{map}(I,-) : \mathcal{C} \to \mathcal{C}$ is a natural path object P on $\mathcal{C}$, just by the property that $\mathrm{map}(-,-)$ is a functor which takes coproducts to products in the first variable. We check that this path object P satisfies the conditions of Lemma 3.6. By assumption P preserves pullbacks and takes $*$ to $*$.

We now want to show Axiom 4, that for a fibration $q : X \to Y$ the canonical map $PX \to PY \times_{Y \times Y} (X \times X)$ is a fibration. Let s be a section of the canonical map $l : PX \to X \times_Y PY$, which exists by Observation 3.5. We show that the canonical map

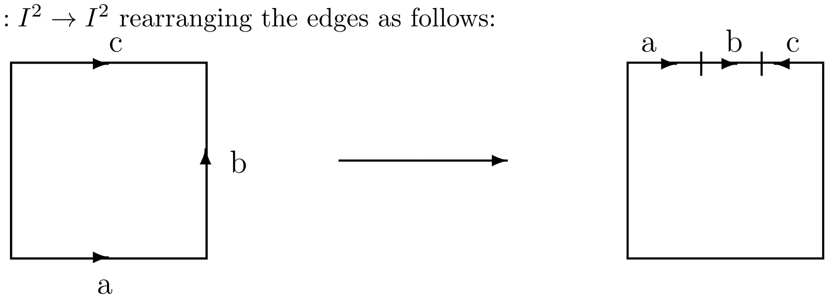
$$P^2X \to PX \times_{PY \times_{Y \times Y} X \times X} P(PY \times_{Y \times Y} X \times X) \tag{3.1}$$

also has a section. This is done by showing that under suitable identifications this map may be identified with $Pl : P^2X \to P(X \times_Y PY)$ which has the section Ps.

The space on the right hand side of (3.1) may be identified as the limit of the diagram

$$\begin{tikzcd}
P^2Y \arrow[d, "Pp_0"'] \arrow[dr, "Pp_1"] \arrow[drr, "p_1P"] & & PX \arrow[dll, "Pq"'] \arrow[dr, "p_1"] & PX \arrow[dll, "Pq"'] \arrow[d, "p_0"] \arrow[dr, "p_1"] & PX \arrow[dll, "Pq"'] \arrow[d, "p_0"] \\
PY & PY & PY & X & X
\end{tikzcd}$$

Since we assume that $\mathrm{map}(-,X)$ takes the pullbacks which exist in $\mathcal{I}$ to pullbacks in $\mathcal{C}$ we can in this diagram identify e.g. $\lim(PX \xrightarrow{p_1} X \xleftarrow{p_0} PX \xrightarrow{p_1} X \xleftarrow{p_0} PX) \simeq PX$. Hence the limit diagram can be simplified to the diagram $P^2Y \to PY \leftarrow PX$, where the maps are somewhat twisted. However consider now a homeomorphism $\alpha : I^2 \to I^2$ rearranging the edges as follows:



Using this as reparametrisation, the pullback becomes $P^2Y \xrightarrow{Pp_1} PY \xleftarrow{Pq} PX$. Moreover the maps $P^2X \to P^2Y$ and $P^2X \to PX$ in this reidentification of (3.1) becomes respectively P^2q and Pp_1. But this means that (3.1) after these identifications is just $Pl : P^2X \to P(X \times_Y PY)$, which has the section Ps. This proves Axiom 4.

That Axiom 5 is satisfied follows from the existence of an 'exchange map' $I^2 \to I^2$ interchanging the two coordinates, using again that $\mathrm{map}(-,-)$ is a functor satisfying the stated adjunction between map and $\times$. This finishes the proof of the existence of a P-category structure.

The last part of the theorem is by [1, Thm. I.3a.4] a formal consequence of the first. □

Remark 3.8. The dual of Theorem 3.7 is also true, i.e. if a category $\mathcal{C}$ is 'tensored' over $\mathcal{I}$, then the assignment $IX = I \times X$ endows $\mathcal{C}$ with cylinder objects which induces an I-category structure on $\mathcal{C}$. We leave the exact statement and proof of this theorem to the reader (use e.g. duality).

As mentioned in the introduction it is obvious that the assumptions in Theorem 3.7 are satisfied for e.g. Banach algebras and C*-algebras. For convenience we state the last part of Theorem 3.7 in the category of C*-algebras:

Corollary 3.9. *The category of* C**-algebras admits a fibration category structure with weak equivalences being the usual homotopy equivalences and fibrations being what is usually known as* C**-algebra cofibrations. In this structure all objects are fibrant and cofibrant.* □

Example 3.10. Note that $C_0(X \wedge Y) = C_0(X) \otimes C_0(Y)$ and that

$$\begin{aligned} C_0(X \times Y) &= \{f \in C(X \times Y) | f(*) = 0\} = \{f \in C(X) \otimes C(Y) | f(*) = 0\} \\ &= \ker(C_0(X)^+ \otimes C_0(Y)^+ \overset{aug}{\to} \mathbf{C}). \end{aligned}$$

This indicates that it might be worth studying $A \boxtimes B := \ker(A^+ \otimes B^+ \overset{aug}{\to} \mathbf{C})$ for general C*-algebras A, B and some tensor product $\otimes$ (or free product!). Indeed this can also be put into an axiomatic framework since $\boxtimes$ defines a fat product on the category of C*-algebras in the sense of Baues (cf. [1, p. 155]). A fat product allows one to define a cosmash product as the homotopy fiber of the canonical map $A \boxtimes B \to A \prod B$. The product, fat product and cosmash product relates via the formulas generalizing well known splitting formulas from topology (cf. [1, p. 156]). In the case where $A \boxtimes B \to A \prod B$ is already a fibration, the cosmash product coincides with the tensor product, and we for instance obtain the splitting formula

$$\Sigma(A \boxtimes B) \simeq \Sigma(A \otimes B) \oplus \Sigma A \oplus \Sigma B,$$

where Σ denotes the C*-algebraic suspension functor, i.e. the loop functor in the fibration category structure.

Remark 3.11. It is our hope that also other fibration category structures can be defined on the category of C*-algebras, $\mathbf{C^*}$, e.g. with asymptotic equivalences as weak equivalences. This would provide a good setup for studying noncommutative shape theory, a theory of current interest (cf. e.g. [8, 7, 9, 10, 5]) sparked by the definition of E-theory (a generalized version of KK-theory) via asymptotic morphisms by Connes-Higson [6]. (For information about shape theory in the framework of cofibration categories see [27] and the references given therein.)

For separable C*-algebras A, B, E-theory is defined as

$$E(A, B) := [[\Sigma A \otimes \mathbb{K}, \Sigma B \otimes \mathbb{K}]],$$

where $[[-,-]]$ denotes homotopy classes of asymptotic morphisms, Σ denotes the C*-algebraic suspension functor (which would be a loop functor in the notation of this paper), and $\mathbb{K}$ denotes the compact operators.

Apart from understanding asymptotic morphisms, one should in order to understand E-theory from an abstract homotopical point of view also need to understand the role of the suspension (suspension stabilization) and the tensoring with the compact operators (matrix stabilization).

By generalized Bott periodicity we have $E(A,B) = \mathrm{colim}_n[[\Sigma^n A \otimes \mathbb{K}, \Sigma^n B \otimes \mathbb{K}]]$. In order to understand the suspension stabilization it seems natural to study 'suspension spectra' of C*-algebras. In the 'stable homotopy category' of C*-algebras the functor, Σ, which we will show does not have an adjoint in $\mathrm{Ho}(\mathbf{C^*})$, would have an adjoint, Σ^{-1}, by the construction of the stable category. This stable category would satisfy some of the axioms of a axiomatic stable homotopy category of [20], and hence might admit treatment from a homotopical viewpoint.

Tensoring with the compact operators is a localization functor (i.e. a coaugmented idempotent functor on $\mathrm{Ho}(\mathbf{C^*})$), and these have been much studied in category theory and topology (cf. e.g. [2, 18]). It moreover has the intriguing property that in the $(\mathbb{K} \otimes -)$-local homotopy category of C*-algebras finite products and coproducts coincide [29, Prop. 3.4].

We end this section with an observation we will need in the next section, when examining extensions of fibration category structures to model category structures.

Lemma 3.12. *If $\mathcal{C}$ possesses a fibration category structure which is extendable to a model category structure in the sense of Quillen. Then the, by the fibration category induced, loop functor will naturally descend to a loop functor on* $\mathrm{Ho}(\mathcal{C})$ *in the sense of Quillen, and will hence especially have a left adjoint. Likewise the dual statement for cofibration categories hold.*

Proof. The homotopy categories of Baues and Quillen agree by [1, I.§2], [1, II.3.6], and [28, Thm. I.1.1], and loop functors are defined in the same way in [1, II.§6] and [28, I.§2], and therefore agree on the homotopy category. The fact that the loop functor has an adjoint in a model category is stated as [28, Thm. I.2.2]. The statement about cofibration categories is a categorical dual and hence also hold. □

4. Axiomatic homotopy theory in the category of C*-algebras

In this section we discuss homotopy theory in the category of compact Hausdorff spaces and how it relates to fibration category structures on C*-algebras. One result of our analysis will be that the fibration category structure on C*-algebras and various naturally occurring subcategories thereof cannot be extended to model category structures.

It is well known that ${\mathbf{Comp_*}}^{op}$ is equivalent to the category of commutative C*-algebras via Gelfand duality. More precisely the contravariant functor $F : \mathbf{Comp_*} \to \mathbf{C^*}$, on objects given by $F(X) = C_0(X)$, embeds ${\mathbf{Comp_*}}^{op}$ in $\mathbf{C^*}$ as the full subcategory of commutative C*-algebras. Here $C_0(X)$ denotes the algebra of complex valued continuous functions $f : X \to \mathbf{C}$ which vanish at the base point. The inverse is given by to A associating the maximal ideal spectrum of A^+, where A^+ denotes the forced unitization of A. (Note that S^0 corresponds to $\mathbf{C}$.)

By Theorem 3.7 the category of commutative C*-algebras, $\mathbf{CommC^*}$, possesses a P-category structure via the path object $(-)^I$. Likewise $\mathbf{Comp_*}$ possesses a structure as an I-category using the cylinder object $-\rtimes I$, which induces a P-category structure on ${\mathbf{Comp_*}}^{op}$. Under the equivalence above $F(X \rtimes I) = C_0(X \rtimes I) = C_0(((X \setminus *) \times I)^+) = C_0(X)^I = F(X)^I$, where $^+$ denotes the one point compactification. Hence the equivalence of categories ${\mathbf{Comp_*}}^{op} \simeq \mathbf{CommC^*}$ is in fact an equivalence of P-categories, which allows us to simply identify the two categories without possibility for confusion. However note also that the P-category structure on pointed topological spaces, $\mathbf{Top_*}$, does not immediately restrict to $\mathbf{Comp_*}$, since in general X^I is quite far from being compact.

In relating structures transferring results between the categories $\mathbf{Comp_*}$ and $\mathbf{C^*}$, it is most useful to make the following observation:

Lemma 4.1. *We have that* $\mathbf{Comp_*}^{op}$ *is a full and reflective subcategory of* $\mathbf{C^*}$*, meaning that it is a full subcategory where the inclusion functor has a left adjoint (cf.* [23, p. 88ff]*). The left adjoint (reflector) of the inclusion is given by the commutator functor which on objects is given by* $A \mapsto A/[A, A]$ *and applying Gelfand duality. This adjunction descends to the homotopy categories and gives that* $\mathrm{Ho}(\mathbf{Comp_*}^{op})$ *is a reflective subcategory of* $\mathrm{Ho}(\mathbf{C^*})$*.*

Proof. The first statement is obvious from the definition. The second follows directly from the definition of homotopy via the path object. □

The above lemma guarantees that if a functor $F : \mathbf{C^*} \to \mathbf{C^*}$ restricts to a functor $F : \mathbf{Comp_*}^{op} \to \mathbf{Comp_*}^{op}$, and if F has a left adjoint on $\mathbf{C^*}$, then the restriction of F to $\mathbf{Comp_*}^{op}$ will likewise have a left adjoint, obtained by composing and precomposing with the reflector and the inclusion. Combining this observation with Lemma 3.12, we have reduced the question of proving that the fibration category structure on $\mathbf{C^*}$ does not extend to a model category structure, to showing that Σ does not have a right adjoint on $\mathrm{Ho}(\mathbf{Comp_*})$.

Before doing this we study homotopy theory in $\mathbf{Comp_*}$. The results are all well known to topologists. However, since we have been unable to find suitable references, and also feel that the discussion is illuminating, we proceed in some detail—the impatient reader may skip right to Proposition 4.3

The category $\mathbf{Comp_*}$ is a reflective subcategory of the category of all pointed topological spaces, $\mathbf{Top_*}$. The left adjoint is given by the Stone-Čech compactification β (cf. e.g. [23, 2]). Especially it follows (cf. [23]) that $\mathbf{Comp_*}$ not only has arbitrary limits (coinciding with the limits in $\mathbf{Top_*}$) but also also arbitrary colimits given by first taking the colimit in $\mathbf{Top_*}$ and then applying β.

Consider the from ι induced map on homotopy categories $\mathrm{Ho}(\mathbf{Comp_*}) \to \mathrm{Ho}(\mathbf{Top_*})$, where homotopy equivalence is induced via the cylinder object $-\rtimes I$.

In order to relate homotopy theory in $\mathrm{Ho}(\mathbf{Comp_*})$ to homotopy theory in $\mathrm{Ho}(\mathbf{Top_*})$ we have to look at properties of this inclusion. The category $\mathrm{Ho}(\mathbf{Top_*})$ of course has arbitrary coproducts and products, since the cylinder object $-\rtimes I$ commutes with arbitrary products and coproducts in $\mathbf{Top_*}$. However $\mathrm{Ho}(\beta)$ is no longer left adjoint to $\mathrm{Ho}(\iota)$, so the conclusion does not carry over to $\mathrm{Ho}(\mathbf{Comp_*})$. Even stronger, we have the following:

Proposition 4.2. *Let* $\{X_i\}_i$ *be an infinite collection of nontrivial spaces in* $\mathrm{Ho}(\mathbf{Comp_*})$*. Then* $\beta(\vee_i X_i)$ *is never a coproduct in* $\mathrm{Ho}(\mathbf{Comp_*})$

Proof. For $\beta(\vee_i X_i)$ to be the coproduct of $\{X_i\}_i$ means per definition that $[\beta \vee_i X_i, Y] = \prod_i [X_i, Y] (= [\vee_i X_i, Y])$, via the natural maps, for all $Y \in \mathrm{Ho}(\mathbf{Comp_*})$. But setting $Y = S^1$ we get by a straightforward calculation due to Dowker [11] that

$$\ker(\beta^* : [\beta \vee_i X_i, S^1] \to [\vee_i X_i, S^1]) = C(\vee_i X_i, \mathbf{R})/C_b(\vee_i X_i, \mathbf{R})$$

where $C(\vee_i X_i, \mathbf{R})$ denotes continuous real valued functions on $\vee_i X_i$ and $C_b(\vee_i X_i, \mathbf{R})$ is the subgroup of bounded such functions. This kernel is nonzero since by the assumptions $\vee_i X_i$ is non-compact. $\square$

The bad behavior of the Stone-Čech compactification with respect to homotopy (essentially originating from the fact that $\beta X \times I \neq \beta(X \times I)$ for X non-compact), and the resulting lack of coproducts is one of the problems with doing homotopy theory in $\mathbf{Comp}_*$. (See e.g. [4], as well as other papers by those authors, for more information about how homotopy relates to the Stone-Čech compactification.) Another, more serious problem is the lack of function objects. Minimally one would want the suspension functor to have a right adjoint [29, p. 157], at least in the homotopy category, but this is not the case:

Proposition 4.3. *The suspension functor Σ, induced by the cylinder object $-\rtimes I$ does not have a right adjoint neither as a functor on $\mathbf{Comp}_*$ nor as a functor on $\mathrm{Ho}(\mathbf{Comp}_*)$.*

Proof. To see that Σ does not have an adjoint in $\mathbf{Comp}_*$ one can note that Σ does not commute with infinite coproducts in $\mathbf{Comp}_*$, in fact not even up to homotopy. For example we have that $[\Sigma\beta(\vee_{\mathbf{N}} I), S^1] = [\beta(\vee_{\mathbf{N}} I), \Omega S^1] = 0$ whereas, by the formula in the proof of Proposition 4.2, $[\beta(\vee_{\mathbf{N}} \Sigma I), S^1] = C(\vee_{\mathbf{N}} \Sigma I, \mathbf{R})/C_b(\vee_{\mathbf{N}} \Sigma I, \mathbf{R})$ which is uncountable.

To see that Σ does not have an adjoint on $\mathrm{Ho}(\mathbf{C}^*)$, we argue by contradiction. By [23, p. 83] Σ has an adjoint iff for all $Y \in \mathrm{Ho}(\mathbf{Comp}_*)$, $[\Sigma -, Y]$ is a representable functor on $\mathrm{Ho}(\mathbf{Comp}_*)$. That is if we for all $Y \in \mathrm{Ho}(\mathbf{Comp}_*)$ can find $X \in \mathrm{Ho}(\mathbf{Comp}_*)$ such that $[\Sigma -, Y] = [-, \Omega Y]$ is naturally equivalent to $[-, X]$ as functors on $\mathrm{Ho}(\mathbf{Comp}_*)$. Especially we get a canonical isomorphism $[X, \Omega Y] \simeq [X, X]$. Let $f \in [X, \Omega Y]$ be the element corresponding to $\mathrm{id}_X \in [X, X]$. By writing up the standard diagram from the Yoneda lemma, we see that the natural equivalence is in fact given by the natural transformation $[-, f] : [-, X] \to [-, \Omega Y]$. In particular we see that $f : X \to \Omega Y$ induces an isomorphism on homotopy groups. Now take $Y = S^1$. We have that ΩS^1 is homotopy equivalent to $\mathbf{Z}$. However since X is assumed to be compact, $f(X)$ has to be a compact subset of ΩS^1. Especially f cannot possibly be surjective on π_0, and we get the desired contradiction. $\square$

Corollary 4.4. *The natural cofibration category structure on* $\mathrm{Ho}(\mathbf{Comp}_*)$ *cannot be extended to a model category structure in the sense of Quillen* [28]. $\square$

Using Lemma 3.12, the remarks following Lemma 4.1, together with an extraction of what was used about $\mathbf{Comp}_*$ in the proof of Proposition 4.3 allows us to obtain the following general theorem:

Theorem 4.5. *Let $\mathcal{D}$ be any full subcategory of $\mathbf{Comp}_*$ which contains $*$ and S^0 and which is closed under applying $-\rtimes I$ an forming pushouts (and hence possesses the structure of an I-category).*

Let $\mathcal{C}$ be some P-category, which contains $\mathcal{D}^{op}$ as a reflective *subcategory and assume that the inclusion functor is a P-functor in the sense of Definition 3.4.*

Then the, from the fibration category induced, loop functor on $\mathrm{Ho}(\mathcal{C})$ *does not have a left adjoint, and hence the fibration category structure cannot be extended to a model category structure.* $\square$

Remark 4.6. Note that the condition that the inclusion be a P-functor, in the case of a reflective subcategory, just amounts to requiring P to commute with the inclusion, the rest of the conditions being guaranteed by the definition of reflectiveness and Observation 3.5.

Corollary 4.7. *The fibration category structure on* C**-algebras of Corollary 3.9 does not extend to a model category structure. The same holds for various naturally occurring subcategories, ('admissible subcategories'* [30]*), such as separable* C**-algebras or separable nuclear* C**-algebras.* □

Remark 4.8. We have so far considered non-unital C*-algebras. We now explain how this relates to the category of unital C*-algebras and unital maps, $\mathbf{uC^*}$. (Note that $\mathbf{uC^*}$ has initial object $\mathbf{C}$ and terminal object 0.) Consider the category of augmented C*-algebras, i.e. the over-category (cf. [23, p. 46]) $\mathbf{uC^*} \downarrow \mathbf{C}$. The functor $\mathbf{C^*} \to (\mathbf{uC^*} \downarrow \mathbf{C})$, given by forced unitization is easily seen to be an equivalence of categories. Hence keeping this identification in mind we see that $\mathbf{C^*}$ *simeq*$(\mathbf{uC^*} \downarrow \mathbf{C})$ relates to $\mathbf{uC^*}$ just as $\mathbf{Comp_*}$ relates to $\mathbf{Comp}$. Passing to commutative C*-algebras this of course fits with ordinary Gelfand duality expressing a categorical duality between unpointed compact Hausdorff spaces and commutative unital C*-algebras via $X \mapsto C(X)$. (Note also that forced unitization, as a functor $\mathbf{uC^*} \to (\mathbf{uC^*} \downarrow \mathbf{C})$ is right adjoint to the forgetful functor $(\mathbf{uC^*} \downarrow \mathbf{C}) \to \mathbf{uC^*}$, and as a functor $\mathbf{C^*} \to \mathbf{uC^*}$ left adjoint to the inclusion $\mathbf{uC^*} \to \mathbf{C^*}$.)

In particular we see that a (not nessesarily unital) C*-algebra should be defined as well-pointed if the augmentation map $A^+ \to \mathbf{C}$ is a fibration as defined via the path object $P = \mathrm{map}(I, -)$ on $\mathbf{uC^*}$, and likewise we se that every object in C* is freely homotopic (i.e. homotopic in $\mathbf{uC^*}$) to a well-pointed C*-algebra.

Finally we remark that it should be kept in mind that one in topology usually would take a weak equivalences the maps which induce isomorphisms on homotopy groups. This only coincides with homotopy equivalence induced via the cylinder object on well-pointed CW complexes. (However, for an analysis of model category structures on (well pointed) topological spaces taking weak equivalences to be strong equivalences see Strøm [31].) This does not have a good noncommutative analog, since neither completely satisfactory notions of homotopy groups nor noncommutative CW complex have yet been found, although effort has been put into doing this and partial steps have been made (cf. [29, 30, 24, 17]). One way of circumventing this problem could be to instead find a fibration category structure with weak equivalences being asymptotic equivalences generalizing topological shape equivalence as remarked on in Remark 3.11, hence avoiding pathologies in a different way.

5. Possibilities for homotopy theory in other categories of topological algebras

The problem with C^*-algebras is that the category is somehow too small. There are several ways around this:

- [14, 12, 15, 25, 26]
- Relation to the topological case (quasi-topological spaces, compactly generated...); Say why σ-C^*-algebras do not give these problems, since topology

is metrizeable, and hence compactly generated. Wiedner's example, and the explanation via Dubuc-Porta, that the Kellyfication is $C([0,1])$.

- P has an adjoint functor (categorial considerations, or refer to Phillips?).

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