NOTES ON HOMOTOPY COLIMITS AND HOMOTOPY LIMITS (A WORK IN PROGRESS)

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Contents

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1. Introduction

Given a small category $\mathcal{C}$ and a $\mathcal{C}$-diagram of spaces $\boldsymbol{X}$, the *colimit* of the diagram is a space $\operatorname{colim} \boldsymbol{X}$ such that a map out of $\operatorname{colim} \boldsymbol{X}$ corresponds to a collection of maps, one from each space in the diagram, that "make all the triangles commute". The *homotopy colimit* $\operatorname{hocolim} \boldsymbol{X}$ will be a space such that a map out of $\operatorname{hocolim} \boldsymbol{X}$ corresponds to a collection of maps, one from each space in the diagram, that may not make any triangle commute, but which come along with *homotopies* between the maps which, if this were $\operatorname{colim} \boldsymbol{X}$, would actually commute, together with "homotopies between the homotopies", and "homotopies between the homotopies between the homotopies", etc.

Some simple examples of homotopy colimits:

- If $\mathcal{C}$ is the category $\cdot \to \cdot$ with two objects and a single non-identity map that goes from the first object to the second, then a $\mathcal{C}$-diagram in Top is just a map $f\colon X \to Y$, and the homotopy colimit of that diagram will be the *mapping cylinder* of f, built from $X \times I$ and Y by gluing one end of $X \times I$ to Y along f. If P is a space, then a map from that homotopy colimit to P consists of a map $\alpha_X\colon X \to P$, a map $\alpha_Y\colon Y \to P$, and a homotopy from α_X to the composition $\alpha_Y f$.
- If $\mathcal{C}$ is the category $\cdot \leftarrow \cdot \to \cdot$, then a $\mathcal{C}$-diagram in Top is just a pair of maps with the same domain,
$$Z \xleftarrow{g} X \xrightarrow{f} Y$$
and the homotopy colimit of that diagram will be the union along X of the mapping cylinders of f and g, built from $X \times I \cup_X X \times I$ (which is homeomorphic to $X \times I$), Y, and Z by gluing one end of $X \times I \cup_X X \times I$ to Y along f and the other end to Z along g. If P is a space, then a map from that homotopy colimit to P consists of maps $\alpha_X\colon X \to P$, $\alpha_Y\colon Y \to P$, and $\alpha_Z\colon Z \to P$ together with a homotopy from α_X to the composition $\alpha_Y f$ and a homotopy from α_X to the composition $\alpha_Z g$.
- It gets more interesting when your small category $\mathcal{C}$ has two composable maps neither of which is an identity map. If $\mathcal{C}$ is the category pictured as $\cdot \to \cdot \to \cdot$, then there are *three* non-identity maps: the two pictured maps plus their composition. A $\mathcal{C}$-diagram in Top is a pair of composable maps
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$
and the homotopy colimit of this diagram will have "higher dimension" than the homotopy colimit of the previous examples: to build the homotopy colimit, we start by taking the mapping cylinder of each of the three non-identity maps $f\colon X \to Y$, $g\colon Y \to Z$, and $gf\colon X \to Z$, and then, because

gf is the composition of f and g, we add in $X \times |\Delta[2]|$, with each of the three faces identified with the three cylinders $X \times |\Delta[1]|$, $Y \times |\Delta[1]|$, and $X \times |\Delta[1]|$ (where, for the second of those, we use the map $f \times 1_{|\Delta[1]|}$ to identify that face of $X \times |\Delta[2]|$ with $Y \times |\Delta[1]|$). If P is a space, then a map from that homotopy colimit to P consists of
  - maps $\alpha_X \colon X \to P$, $\alpha_Y \colon Y \to P$, and $\alpha_Z \colon Z \to P$ together with
  - a homotopy $\beta_f \colon X \times |\Delta[1]| \to P$ from α_X to the composition $\alpha_Y f$,
  - a homotopy $\beta_g \colon Y \times |\Delta[1]| \to P$ from α_Y to the composition $\alpha_Z g$,
  - a homotopy $\beta_{gf} \colon X \times |\Delta[1]| \to P$ from α_X to the composition $\alpha_Z(gf)$, and
  - a homotopy $\gamma \colon X \times |\Delta[2]| \to P$ between homotopies whose restrictions to the three faces equal β_f, $\beta_g \circ (f \times 1_{|\Delta[1]|})$, and β_{gf}.

  The restriction of γ to one face is a homotopy from α_X to $\alpha_Y f$, and the restriction to the next face is a homotopy from $\alpha_Y f$ to $(\alpha_Z g)f$. The composition of those two homotopies is a homotopy from α_X to $(\alpha_Z g)f$, and the restriction of γ to the third face is another homotopy from α_X to $\alpha_Z(gf)$; the map γ is a homotopy between those two homotopies.
- As your small category $\mathcal{C}$ has longer strings of composable non-identity maps, the homotopy colimit will be built of "higher dimensional" parts. For example, if your diagram is of the form $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$, then there will be a copy of $W \times |\Delta[3]|$ built into the homotopy colimit, with its four faces identified with the 2-dimensional parts associated with each of $X \xrightarrow{g} Y \xrightarrow{h} Z$, $W \xrightarrow{gf} Y \xrightarrow{h} Z$, $W \xrightarrow{f} X \xrightarrow{hg} Z$, and $W \xrightarrow{f} X \xrightarrow{g} Y$.

If $\mathcal{C}$ is a small category and $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of spaces, then $\operatorname{hocolim} \boldsymbol{X}$ will built from

- the space $\boldsymbol{X}_\alpha$ for every object α of $\mathcal{C}$,
- the space $\boldsymbol{X}_\alpha \times |\Delta[1]|$ for every non-identity map $\alpha \to \beta$ in $\mathcal{C}$ with domain α,
- the space $\boldsymbol{X}_\alpha \times |\Delta[2]|$ for every pair of composable non-identity maps $\alpha \to \beta \to \gamma$ in $\mathcal{C}$ starting at α, and, for all $n > 0$,
- the space $\boldsymbol{X}_\alpha \times |\Delta[n]|$ for every string of n composable non-identity maps in $\mathcal{C}$ starting at α.

We keep track of all of these strings of composable arrows using $(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$, the opposite of the category of *objects of* $\mathcal{C}$ *under* α (see Definition 2.3) and its nerve $\mathrm{N}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}$ (see Example 2.4).

2. Nerves of overcategories and undercategories

Definition 2.1. If $\mathcal{C}$ is a small category, then the *nerve* of $\mathcal{C}$ is the simplicial set $\mathrm{N}\mathcal{C}$ in which an n-simplex σ is a diagram in $\mathcal{C}$ of the form

$$\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n$$

with the face and degeneracy maps defined by

$$(2.2)\quad d_i\sigma = \begin{cases} \alpha_1 \xrightarrow{\sigma_1} \alpha_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n & \text{if } i = 0 \\ \alpha_0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{i-2}} \alpha_{i-1} \xrightarrow{\sigma_i\sigma_{i-1}} \alpha_{i+1} \xrightarrow{\sigma_{i+1}} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n & \text{if } 0 < i < n \\ \alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-2}} \alpha_{n-1} & \text{if } i = n \end{cases}$$

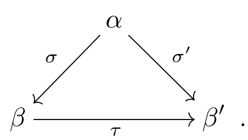
$$s_i\sigma = \alpha_0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{i-1}} \alpha_i \xrightarrow{1_{\alpha_i}} \alpha_i \xrightarrow{\sigma_i} \alpha_{i+1} \xrightarrow{\sigma_{i+1}} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n \ .$$

If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories, then F induces a map of simplicial sets $\mathrm{N}F\colon \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$ defined by

$$\mathrm{N}F(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n) = F\alpha_0 \xrightarrow{F\sigma_0} F\alpha_1 \xrightarrow{F\sigma_1} \cdots \xrightarrow{F\sigma_{n-1}} F\alpha_n \ .$$

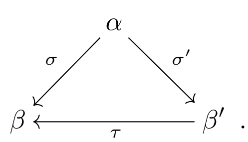
2.1. **Undercategories and their nerves.**

Definition 2.3. If $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then the *category of of objects of* $\mathcal{C}$ *under* α $(\alpha\downarrow\mathcal{C})$ is the category in which an object is a pair (β, σ) where β is an object of $\mathcal{C}$ and σ is a map $\alpha \to \beta$ in $\mathcal{C}$, and a morphism from the object (β, σ) to the object (β', σ') is a map $\tau\colon \beta \to \beta'$ that makes the triangle



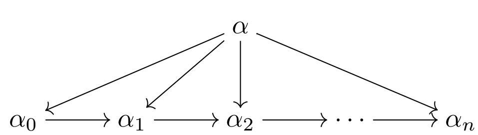
commute.

The category $(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ is the *opposite of the category of objects of* $\mathcal{C}$ *under* α. An object of $(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ is also a pair (β, σ) as above, but a morphism in $(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ from the object (β, σ) to the object (β', σ') is a map $\tau\colon \beta' \to \beta$ that makes the triangle

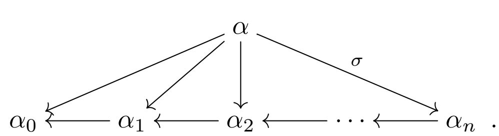


commute.

Example 2.4. Let $\mathcal{C}$ be a small category. If α is an object of $\mathcal{C}$, then an n-simplex of $\mathrm{N}(\alpha\downarrow\mathcal{C})$ is a diagram of the form



and an n-simplex of $\mathrm{N}(a\downarrow\mathcal{C})^{\mathrm{op}}$ is a diagram of the form



We will often denote such a simplex by the ordered pair $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \sigma\colon \alpha \to \alpha_n\big)$, since that pair determines the entire diagram.

Definition 2.5. Let $\mathcal{C}$ be a small category.

(1) $(-\downarrow \mathcal{C})^{\mathrm{op}}$ is the the $\mathcal{C}^{\mathrm{op}}$-diagram of categories that on an object α of $\mathcal{C}$ takes the value $(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$, and that takes the map $\sigma\colon \alpha \to \alpha'$ to the functor $\sigma^*\colon (\alpha'\downarrow \mathcal{C})^{\mathrm{op}} \to (\alpha\downarrow \mathcal{C})^{\mathrm{op}}$ that takes the object $\tau\colon \alpha' \to \beta$ of $(\alpha'\downarrow \mathcal{C})^{\mathrm{op}}$ to the object of $(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$ that is the composition $\alpha \xrightarrow{\sigma} \alpha' \xrightarrow{\tau} \beta$.

(2) $\mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets that on an object α of $\mathcal{C}$ takes the value $\mathrm{N}(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$, and that takes the map $\sigma\colon \alpha \to \alpha'$ to the map of simplicial sets $\sigma^*\colon \mathrm{N}(\alpha'\downarrow \mathcal{C})^{\mathrm{op}} \to \mathrm{N}(\alpha\downarrow \mathcal{C})^{\mathrm{op}}$.

Remark 2.6. Our definition of the homotopy colimit functor will use the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$, rather than the $\mathcal{C}^{\mathrm{op}}$-diagram $\mathrm{N}(-\downarrow \mathcal{C})$. It might have been simpler to define a homotopy colimit functor using the diagram $\mathrm{N}(-\downarrow \mathcal{C})$, but there are technical advantages to using the nerve of the *opposites* of undercategories (see Theorem 11.5).

Note also that both $\mathrm{N}(-\downarrow \mathcal{C})$ and $\mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$ are $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, and not $\mathcal{C}$-diagrams. It's not the fact that we're using $\mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$, the nerve of the *opposites* of the undercategories, that causes the diagram to be contravariant in $\mathcal{C}$.

2.2. **Undercategories of functors.**

Definition 2.7. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories and β is an object of $\mathcal{D}$, then the category $(\beta\downarrow F)$ of *objects of* $\mathcal{C}$ *under* β is the category in which an object is a pair (α, σ) where α is an object of $\mathcal{C}$ and σ is a map $\sigma\colon \beta \to F\alpha$ in $\mathcal{D}$, and a morphism from the object (α, σ) to the object (α', σ') is a map $\tau\colon \alpha \to \alpha'$ in $\mathcal{C}$ that makes the triangle

$$\begin{array}{ccc} & \beta & \\ {}^{\sigma}\swarrow & & \searrow^{\sigma'} \\ F\alpha & \xrightarrow[F\tau]{} & F\alpha' \end{array} .$$

commute.

The category $(\beta\downarrow F)^{\mathrm{op}}$ is the *opposite of the category of objects of* $\mathcal{C}$ *under* β. An object of $(\beta\downarrow F)^{\mathrm{op}}$ is also a pair (α, σ) as above, but a morphism in $(\beta\downarrow F)^{\mathrm{op}}$ from the object (α, σ) to the object (α', σ') is a map $\tau\colon \alpha' \to \alpha$ that makes the triangle

$$\begin{array}{ccc} & \beta & \\ {}^{\sigma}\swarrow & & \searrow^{\sigma'} \\ F\alpha & \xleftarrow[F\tau]{} & F\alpha' \end{array} .$$

commute.

Example 2.8. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If β is an object of $\mathcal{D}$, then an n-simplex of $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$ is determined by an ordered pair $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \sigma\colon \beta \to F\alpha_n\big)$, and we will often use that notation for such a simplex.

Lemma 2.9. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories and α is an object of $\mathcal{C}$, then there is a map of simplicial sets $F_*\colon \mathrm{N}(\alpha\downarrow \mathcal{C})^{\mathrm{op}} \to \mathrm{N}(F\alpha\downarrow F)^{\mathrm{op}}$ that takes the simplex*

$$\big((\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, \tau : \alpha \to \alpha_n\big)$$

of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ *to the simplex*

$$\big((\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, F\tau\colon F\alpha \to F\alpha_n\big)$$

of $\mathrm{N}(F\alpha\downarrow F)^{\mathrm{op}}$.

Proof. This follows directly from the definitions. □

Definition 2.10. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories.

1. $(-\downarrow F)^{\mathrm{op}}$ is the the $\mathcal{D}^{\mathrm{op}}$-diagram of categories that on an object β of $\mathcal{D}$ takes the value $(\beta\downarrow F)^{\mathrm{op}}$, and that takes the map $\sigma\colon \beta \to \beta'$ to the functor $\sigma^*\colon (\beta'\downarrow F)^{\mathrm{op}} \to (\beta\downarrow F)^{\mathrm{op}}$ that takes the object $\tau\colon \beta' \to F\alpha$ of $(\beta'\downarrow F)^{\mathrm{op}}$ to the object of $(\beta\downarrow F)^{\mathrm{op}}$ that is the composition $\beta \xrightarrow{\sigma} \beta' \xrightarrow{\tau} F\alpha$.
2. $\mathrm{N}(-\downarrow F)^{\mathrm{op}}$ is the $\mathcal{D}^{\mathrm{op}}$-diagram of simplicial sets that on an object β of $\mathcal{D}$ takes the value $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$, and that takes the map $\sigma\colon \beta \to \beta'$ to the map of simplicial sets $\sigma^*\colon \mathrm{N}(\beta'\downarrow F)^{\mathrm{op}} \to \mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$.

Example 2.11. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If β is an object of $\mathcal{D}$, then there is a functor $F_*\colon (\beta\downarrow F) \to (\beta\downarrow \mathcal{D})$ that takes the object $(\alpha, \sigma\colon \beta \to F\alpha)$ to the object $(F\alpha, \sigma\colon \beta \to F\alpha)$. This induces a natural map of simplicial sets $F_*\colon \mathrm{N}(\beta\downarrow F)^{\mathrm{op}} \to \mathrm{N}(\beta\downarrow\mathcal{D})^{\mathrm{op}}$ that takes the simplex

$$\big((\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, \sigma\colon \beta \to F\alpha_n\big)$$

of $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$ (see Example 2.8) to the simplex

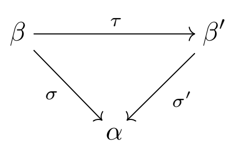
$$\big((F\alpha_0 \xleftarrow{F\sigma_0} F\alpha_1 \xleftarrow{F\sigma_1} \cdots \xleftarrow{F\sigma_{n-1}} F\alpha_n) \in \mathcal{D}, \sigma\colon \beta \to F\alpha_n\big)$$

of $\mathrm{N}(\beta\downarrow\mathcal{D})^{\mathrm{op}}$.

Example 2.12. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories, then the map of simplicial sets $F_*\colon \mathrm{N}(\beta\downarrow F)^{\mathrm{op}} \to \mathrm{N}(\beta\downarrow\mathcal{D})^{\mathrm{op}}$ of Example 2.11 for each object β of $\mathcal{D}$ defines a map of $\mathcal{D}^{\mathrm{op}}$-diagrams of simplicial sets $F_*\colon \mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \mathrm{N}(-\downarrow\mathcal{D})^{\mathrm{op}}$.

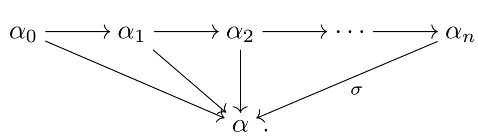
2.3. Overcategories and their nerves.

Definition 2.13. If $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then the *category of objects of* $\mathcal{C}$ *over* α $(\mathcal{C}\downarrow\alpha)$ is the category in which an object is a pair (β, σ) where β is an object of $\mathcal{C}$ and σ is a map $\beta \to \alpha$ in $\mathcal{C}$, and a morphism from the object (β, σ) to the object (β', σ') is a map $\tau : \beta \to \beta'$ that makes the triangle



commute.

Example 2.14. Let $\mathcal{C}$ be a small category. If α is an object of $\mathcal{C}$, then an n-simplex of $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ is a diagram of the form



We will often denote such a simplex by the ordered pair $\big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \sigma\colon \alpha_n \to \alpha\big)$, since that pair determines the entire diagram.

Definition 2.15. Let $\mathcal{C}$ be a small category.

(1) $(\mathcal{C}\downarrow -)$ is the the $\mathcal{C}$-diagram of categories that on an object α of $\mathcal{C}$ takes the value $(\mathcal{C}\downarrow\alpha)$, and that takes the map $\sigma\colon \alpha \to \alpha'$ to the functor $\sigma_*\colon (\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$ that takes the object $\tau\colon \beta \to \alpha$ of $(\mathcal{C}\downarrow\alpha)$ to the object of $(\mathcal{C}\downarrow\alpha')$ that is the composition $\beta \xrightarrow{\tau} \alpha \xrightarrow{\sigma} \alpha'$.

(2) $\mathrm{N}(\mathcal{C}\downarrow -)$ is the $\mathcal{C}$-diagram of simplicial sets that on an object α of $\mathcal{C}$ takes the value $\mathrm{N}(\mathcal{C}\downarrow\alpha)$, and that takes the map $\sigma\colon \alpha \to \alpha'$ to the map of simplicial sets $\sigma_*\colon \mathrm{N}(\mathcal{C}\downarrow\alpha) \to \mathrm{N}(\mathcal{C}\downarrow\alpha')$.

2.4. **Overcategories of functors.**

Definition 2.16. If $\mathcal{C}$ and $\mathcal{D}$ are categories, $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, and β is an object of $\mathcal{D}$, then the *category of objects of* $\mathcal{C}$ *over* β $(F\downarrow\beta)$ is the category in which an object is a pair (α, σ) where α is an object of $\mathcal{C}$ and σ is a map $F\alpha \to \beta$ in $\mathcal{D}$, and a morphism from the object (α, σ) to the object (α', σ') is a map $\tau\colon \alpha \to \alpha'$ in $\mathcal{C}$ such that the triangle

$$\begin{array}{ccc} F\alpha & \xrightarrow{F\tau} & F\alpha' \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma'} & \\ & \beta & \end{array}$$

commutes.

Example 2.17. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If β is an object of $\mathcal{D}$, then an n-simplex of $\mathrm{N}(F\downarrow\beta)$ is determined by an ordered pair $\big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \sigma\colon F\alpha_n \to \beta\big)$, and we will often use that notation for such a simplex.

Example 2.18. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If β is an object of $\mathcal{D}$, then there is a functor $F_*\colon (F\downarrow\beta) \to (\mathcal{D}\downarrow\beta)$ that takes the object $(\alpha, \sigma\colon F\alpha \to \beta)$ to the object $(F\alpha, \sigma\colon F\alpha \to \beta)$. This induces a map of simplicial sets $F_*\colon \mathrm{N}(F\downarrow\beta) \to \mathrm{N}(\mathcal{D}\downarrow\beta)$ that takes the simplex

$$\big((\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, \sigma\colon F\alpha_n \to \beta\big)$$

of $\mathrm{N}(F\downarrow\beta)$ to the simplex

$$\big((F\alpha_0 \xrightarrow{F\sigma_0} F\alpha_1 \xrightarrow{F\sigma_1} \cdots \xrightarrow{F\sigma_{n-1}} F\alpha_n) \in \mathcal{D}, \sigma\colon F\alpha_n \to \beta\big)$$

of $\mathrm{N}(\mathcal{D}\downarrow\beta)$.

Lemma 2.19. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories and α is an object of $\mathcal{C}$, then there is a map of simplicial sets $F_*\colon \mathrm{N}(\mathcal{C}\downarrow\alpha) \to \mathrm{N}(F\downarrow F\alpha)$ that takes the simplex*

$$\big((\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, \tau : \alpha_n \to \alpha\big)$$

of $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ to the simplex

$$\big((\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, F\tau\colon F\alpha_n \to F\alpha\big)$$

of $\mathrm{N}(F\downarrow F\alpha)$.

Proof. This follows directly from the definitions. □

Definition 2.20. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories.

(1) $(F\downarrow -)$ is the the $\mathcal{D}$-diagram of categories that on an object β of $\mathcal{D}$ takes the value $(F\downarrow\beta)$, and that takes the map $\sigma\colon \beta\to\beta'$ to the functor $\sigma_*\colon (F\downarrow\beta)\to(F\downarrow\beta')$ that takes the object $\tau\colon F\alpha\to\beta$ of $(F\downarrow\beta)$ to the object of $(F\downarrow\beta')$ that is the composition $F\alpha \xrightarrow{\tau} \beta \xrightarrow{\sigma} \beta'$.

(2) $\mathrm{N}(F\downarrow -)$ is the $\mathcal{D}$-diagram of simplicial sets that on an object β of $\mathcal{D}$ takes the value $\mathrm{N}(F\downarrow\beta)$, and that takes the map $\sigma\colon\beta\to\beta'$ to the map of simplicial sets $\sigma_*\colon \mathrm{N}(F\downarrow\beta)\to\mathrm{N}(F\downarrow\beta')$.

Example 2.21. If $F\colon \mathcal{C}\to\mathcal{D}$ is a functor between small categories, then the map of simplicial sets $F_*\colon \mathrm{N}(F\downarrow\beta)\to\mathrm{N}(\mathcal{D}\downarrow\beta)$ of Example 2.18 for each object β of $\mathcal{D}$ defines a map of $\mathcal{D}$-diagrams of simplicial sets $F_*\colon \mathrm{N}(F\downarrow -)\to\mathrm{N}(\mathcal{D}\downarrow -)$.

2.5. **Contractible nerves.** The main results of this section are Corollary 2.27 and Corollary 2.29, which assert that if $\mathcal{C}$ is a small category and α is an object of $\mathcal{C}$, then the simplicial sets $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ and $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ are contractible. These are important because

- the definition of the homotopy colimit functor (see Definition 8.1) uses the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ (see Definition 2.5) and
- the definition of the homotopy limit functor (see Definition 11.1) uses the $\mathcal{C}$-diagram of simplicial sets $\mathrm{N}(\mathcal{C}\downarrow -)$ (see Definition 2.15).

The effect of Corollary 2.29 is that the map of $\mathcal{C}^{\mathrm{op}}$-diagrams from $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ to the constant diagram at a single point is an objectwise weak equivalence, and so it is a weak equivalence in the model category of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (see Theorem 5.1). We will show in Proposition 5.13 that the diagram $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ is also a free cell complex (see Definition 5.7), which implies that $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ is a cofibrant object in the model category of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (see Theorem 5.1). Thus, the diagram $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ is a *cofibrant approximation (see* [7, Def. 8.1.2]*) to the constant diagram at a single point.* Since any two cofibrant approximations to the same object are weakly equivalent (see [7, Prop. 8.1.9]), this explains why there are alternative possible definitions of the homotopy colimit functor (using different cofibrant approximations to the constant diagram at a single point) that are naturally weakly equivalent for objectwise cofibrant diagrams in a model category.

Similarly, the effect of Corollary 2.27 is that the map of $\mathcal{C}$-diagrams from $\mathrm{N}(\mathcal{C}\downarrow -)$ to the constant diagram at a single point is an objectwise weak equivalence, and so it is a weak equivalence in the model category of $\mathcal{C}$-diagrams of simplicial sets (see Theorem 5.1). We will show in Proposition 5.13 that the diagram $\mathrm{N}(\mathcal{C}\downarrow -)$ is also a free cell complex (see Definition 5.7), which implies that $\mathrm{N}(\mathcal{C}\downarrow -)$ is a cofibrant object in the model category of $\mathcal{C}$-diagrams of simplicial sets (see Theorem 5.1). Thus, the diagram $\mathrm{N}(\mathcal{C}\downarrow -)$ is a *cofibrant approximation (see* [7, Def. 8.1.2]*) to the constant diagram at a single point.* Since any two cofibrant approximations to the same object are weakly equivalent (see [7, Prop. 8.1.9]), this explains why there are alternative possible definitions of the homotopy limit functor (using different cofibrant approximations to the constant diagram at a single point) that are naturally weakly equivalent for objectwise fibrant diagrams in a model category.

Lemma 2.22. *If $\mathcal{C}$ and $\mathcal{D}$ are small categories, then there is a natural isomorphism of simplicial sets* $\mathrm{N}(\mathcal{C}\times\mathcal{D})\approx\mathrm{N}\mathcal{C}\times\mathrm{N}\mathcal{D}$.

Proof. This follows directly from the definitions. □

Lemma 2.23. *If $[1]$ is the category $0 \to 1$, then for every small category $\mathcal{C}$ there is a natural isomorphism of simplicial sets $\mathrm{N}(\mathcal{C} \times [1]) \approx \mathrm{N}\mathcal{C} \times \Delta[1]$.*

Proof. Since $N[1] \approx \Delta[1]$, this follows from Lemma 2.22. □

Proposition 2.24. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. If $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors, then there exists a natural transformation $\phi\colon F \to G$ if and only if there is a functor $\Phi\colon \mathcal{C} \times [1] \to \mathcal{D}$ such that the restriction of Φ to $\mathcal{C} \times \{0\}$ is F and the restriction of Φ to $\mathcal{C} \times \{1\}$ is G (where $\{0\}$ is the category with one object 0 and its identity map, and $\{1\}$ is the category with one object 1 and its identity map).*

Proof. If $\phi\colon F \to G$ is a natural transformation, then we can define a functor $\Phi\colon \mathcal{C} \times [1] \to \mathcal{D}$ by letting

- $\Phi(\alpha, 0) = F(\alpha)$ and $\Phi(\alpha, 1) = G(\alpha)$ for every object α of $\mathcal{C}$, and
- $\Phi(\sigma, 1_0) = F(\sigma)$, $\Phi(\sigma, 1_1) = G(\sigma)$, and $\Phi(\sigma, 0 \to 1) = \phi(\sigma)$ for every morphism σ of $\mathcal{C}$.

Conversely, if $\Phi\colon \mathcal{C} \times [1] \to \mathcal{D}$ is a functor whose restriction to $\mathcal{C} \times \{0\}$ is F and whose restriction to $\mathcal{C} \times \{1\}$ is G, we can define a natural transformation $\phi\colon F \to G$ by letting $\phi(\alpha) = \Phi(\alpha, 0 \to 1)$ for every object α of $\mathcal{C}$. □

Proposition 2.25. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors from $\mathcal{C}$ to $\mathcal{D}$. If there is a natural transformation from F to G, then the induced maps of nerves $\mathrm{N}F, \mathrm{N}G\colon \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$ are homotopic.*

Proof. Proposition 2.24 implies that there is a functor $\Phi\colon \mathcal{C} \times [1] \to \mathcal{D}$ whose restrictions to $\mathcal{C}\times\{0\}$ and $\mathcal{C}\times\{1\}$ are F and G. Lemma 2.22 implies that $\mathrm{N}(\mathcal{C}\times[1]) \approx \mathrm{N}\mathcal{C} \times \mathrm{N}[1] \approx \mathrm{N}\mathcal{C} \times I$, and so $\mathrm{N}\Phi\colon \mathrm{N}\mathcal{C} \times I \to \mathrm{N}\mathcal{D}$ is a homotopy from $\mathrm{N}F$ to $\mathrm{N}G$. □

Theorem 2.26. *Let $\mathcal{C}$ be a small category. If $\mathcal{C}$ has either an initial object or a terminal object, then the nerve of $\mathcal{C}$ is a contractible simplicial set.*

Proof. If $\mathcal{C}$ has an initial object α, then there is a natural transformation from the constant functor at the object α to the identity functor of $\mathcal{C}$, and so Proposition 2.25 implies that the identity map of $\mathrm{N}\mathcal{C}$ is homotopic to a constant map. Similarly, if $\mathcal{C}$ has a terminal object β, then there is a natural transformation from the identity functor of $\mathcal{C}$ to the constant functor at the object β, and so the identity map of $\mathrm{N}\mathcal{C}$ is homotopic to a constant map. □

Corollary 2.27. *If $\mathcal{C}$ is a small category, then for every object α of $\mathcal{C}$ the simplicial sets $\mathrm{N}(\alpha\downarrow\mathcal{C})$ (see Definition 2.3) and $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ (see Definition 2.16) are contractible.*

Proof. The identity map $1_\alpha\colon \alpha \to \alpha$ is an initial object of $(\alpha\downarrow\mathcal{C})$ and a terminal object of $(\mathcal{C}\downarrow\alpha)$, and so the result follows from Theorem 2.26. □

Lemma 2.28. *If $\mathcal{C}$ is a small category, then there is a natural homeomorphism of topological spaces $|\mathrm{N}\mathcal{C}| \approx |\mathrm{N}\mathcal{C}^{\mathrm{op}}|$.*

Proof. The homeomorphism takes the realization of the simplex $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n$ of $\mathrm{N}\mathcal{C}$ to the realization of the simplex $\alpha_n \leftarrow \alpha_{n-1} \leftarrow \cdots \leftarrow \alpha_0$ of $\mathrm{N}\mathcal{C}^{\mathrm{op}}$. □

Corollary 2.29. *If $\mathcal{C}$ is a small category, then for every object α of $\mathcal{C}$ the simplicial set $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ (see Definition 2.3) is contractible.*

Proof. Corollary 2.27 implies that $\mathrm{N}(\alpha\downarrow\mathcal{C})$ is contractible, and Lemma 2.28 implies that the geometric realization of $\mathrm{N}(\alpha\downarrow\mathcal{C})$ is homeomorphic to that of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$. □

3. Decomposing simplicial sets

Definition 3.1 (The cosimplicial and simplicial indexing categories)**.** If n is a nonnegative integer, we let $[n]$ denote the ordered set $(0, 1, 2, \ldots, n)$.

(1) The *cosimplicial indexing category* $\mathbf{\Delta}$ is the category with objects the $[n]$ for $n \geq 0$ and with morphisms $\mathbf{\Delta}([n], [k])$ the weakly monotone functions $[n] \to [k]$, i.e., the functions $\sigma\colon [n] \to [k]$ such that $\sigma(i) \leq \sigma(j)$ for $0 \leq i \leq j \leq n$.
(2) A cosimplicial object in a category $\mathcal{M}$ is a functor from $\mathbf{\Delta}$ to $\mathcal{M}$.
(3) The *simplicial indexing category* is the category $\mathbf{\Delta}^{\mathrm{op}}$.
(4) A simplicial object in a category $\mathcal{M}$ is a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to $\mathcal{M}$.

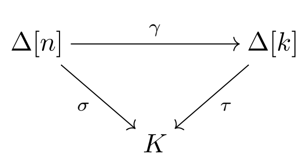
Definition 3.2. If $n \geq 0$, then the *standard n-simplex* $\Delta[n]$ is the simplicial set that has as k-simplices the weakly monotone functions $[k] \to [n]$, i.e., $(\Delta[n])_k = \mathbf{\Delta}([k], [n]) = \mathbf{\Delta}^{\mathrm{op}}([n], [k])$.

Definition 3.3. The *cosimplicial standard simplex* is the cosimplicial simplicial set $\Delta\colon \mathbf{\Delta} \to \mathrm{SS}$ that takes the object $[n]$ of $\mathbf{\Delta}$ to the standard n-simplex $\Delta[n]$. This can be viewed as either a $\mathbf{\Delta}$-diagram of simplicial sets (i.e., a covariant functor of $\mathbf{\Delta}$) or as a $(\mathbf{\Delta}^{\mathrm{op}})^{\mathrm{op}}$-diagram of simplicial sets (i.e., a contravariant functor of $\mathbf{\Delta}^{\mathrm{op}}$).

Proposition 3.4. *If K is a simplicial set and $n \geq 0$, there is a natural isomorphism between the set of n-simplices of K and the set of maps of simplicial sets $\Delta[n] \to K$, under which an n-simplex σ of K corresponds to the map $\Delta[n] \to K$ that takes the nondegenerate n-simplex of $\Delta[n]$ to σ.*

Proof. A simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Sets}$, and a map of simplicial sets is a natural transformation of functors. Since $\big(\Delta[n]\big)_k = \mathbf{\Delta}\big([k], [n]\big) = \mathbf{\Delta}^{\mathrm{op}}\big([n], [k]\big)$, the simplicial set $\Delta[n]$ is the representable functor on $\mathbf{\Delta}^{\mathrm{op}}$ with representing object $[n]$, and the unique nondegenerate n-simplex of $\Delta[n]$ is the identity map of $[n]$. Thus, this is exactly the Yoneda lemma (see, e.g., [9, Lemma 4.2.1], [8, p. 61], or [1, Thm. 1.3.3]). □

Definition 3.5. If K is a simplicial set, then the *category of simplices of K* is the category ΔK that is the overcategory $(\Delta\downarrow K)$, where $\Delta\colon \mathbf{\Delta} \to \mathrm{SS}$ is the cosimplicial standard simplex (see Definition 3.3). That is, an object of ΔK is a map of simplicial sets $\Delta[n] \to K$ for some $n \geq 0$, and a morphism from $\sigma\colon \Delta[n] \to K$ to $\tau\colon \Delta[k] \to K$ is a commutative triangle



for some morphism $\gamma\colon [n] \to [k]$ in $\mathbf{\Delta}$.

Equivalently (see Proposition 3.4), if K is a simplicial set, then the *category of simplices of K* is the category ΔK such that

- the objects of ΔK are the simplices of K and

- if σ and τ are simplices of K, then a morphism from σ to τ in ΔK is a simplicial operator that takes τ to σ. (Note the reversal of direction; if $\partial_k \tau = \sigma$, then ∂_k corresponds to a morphism from σ to τ.)

Proposition 3.6. *If K is a simplicial set and $G\colon \Delta K \to \mathrm{SS}$ is the ΔK-diagram (see Definition 3.5) of simplicial sets that takes the object $\sigma\colon \Delta[n] \to K$ to the standard simplex $\Delta[n]$, then there is a natural isomorphism $\operatorname{colim}_{\Delta K} G \approx K$.*

Proof. The objects $\sigma\colon \Delta[n] \to K$ of ΔK define natural maps $G(\sigma) \to K$ that commute with the structure maps of G, and these define a natural map $\operatorname{colim}_{\Delta K} G \to K$. Every n-simplex $\sigma\colon \Delta[n] \to K$ (for $n \geq 0$) is an object of ΔK for which the image of the natural map $G(\sigma) = \Delta[n] \to K$ contains the simplex corresponding to σ, and so the natural map $\operatorname{colim}_{\Delta K} G \to K$ is surjective.

To see that the natural map $\operatorname{colim}_{\Delta K} G \to K$ is injective, assume that there are objects $\sigma\colon \Delta[m] \to K$ and $\tau\colon \Delta[n] \to K$ of ΔK together with a k-simplex η of $\Delta[m]$ and a k-simplex μ of $\Delta[n]$ such that the image in K of η under $G(\sigma) \to K$ equals the image in K of μ under $G(\tau) \to K$. Proposition 3.4 implies that there is then a commutative diagram of simplicial sets

$$\begin{array}{ccc} \Delta[k] & \xrightarrow{\mu} & \Delta[n] \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\tau} \\ \Delta[m] & \xrightarrow[\sigma]{} & K \end{array}$$

which we can regard as a diagram in ΔK. The relation that this diagram imposes on $\operatorname{colim}_{\Delta K} G$ implies that the image of η in $\operatorname{colim}_{\Delta K} G$ equals the image of μ in $\operatorname{colim}_{\Delta K} G$, and so the natural map $\operatorname{colim}_{\Delta K} G \to K$ is injective. □

Proposition 3.7. *If K is a simplicial set, then K is naturally isomorphic to the coequalizer of the maps*

$$(3.8) \qquad \Bigl(\coprod_{\substack{n>0\\ 0\leq i\leq n\\ \Delta[n]\to K}} \Delta[n-1]\Bigr) \amalg \Bigl(\coprod_{\substack{n\geq 0\\ 0\leq i\leq n\\ \Delta[n]\to K}} \Delta[n+1]\Bigr) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{n\geq 0\\ \Delta[n]\to K}} \Delta[n]$$

where, on the first summand,

- *ϕ takes the summand $\Delta[n-1]$ indexed by $(n, i, \sigma\colon \Delta[n] \to K)$ by the identity map to the $\Delta[n-1]$ indexed by $(n-1)$ and the composition $\Delta[n-1] \xrightarrow{d^i} \Delta[n] \xrightarrow{\sigma} K$, composed with the injection into the coproduct, and*
- *ψ takes that same summand to the $\Delta[n]$ indexed by (n, σ) by the map $d^i\colon \Delta[n-1] \to \Delta[n]$, composed with the injection into the coproduct*

and, on the second summand,

- *ϕ takes the summand $\Delta[n+1]$ indexed by $(n, i, \sigma\colon \Delta[n] \to K)$ by the identity map to the $\Delta[n+1]$ indexed by $(n+1)$ and the composition $\Delta[n+1] \xrightarrow{s^i} \Delta[n] \xrightarrow{\sigma} K$, composed with the injection into the coproduct, and*
- *ψ takes that same summand to the $\Delta[n]$ indexed by (n, σ) by the map $s^i\colon \Delta[n+1] \to \Delta[n]$, composed with the injection into the coproduct.*

Proof. Proposition 3.6 implies that K is isomorphic to the quotient of the coproduct of standard simplices, one for each simplex of K, in which the various standard simplices are identified according to the action of the simplicial operators on the simplices of K. Since every simplicial operator is a finite composition of face and degeneracy operators, it is sufficient to identify along the relations that come from the face and degeneracy operators. Coequalizing ϕ and ψ on the first summand applies the relations from the face operators, and coequalizing them on the second summand applies the relations from the degeneracy operators. □

Remark 3.9. Proposition 3.7 can also be obtained as a corollary of Proposition 7.24 (see Example 7.25).

3.1. **Degenerate simplices.**

Lemma 3.10. *If $\mathcal{M}$ is a category and $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$, then every iterated degeneracy operator $\boldsymbol{X}_n \to \boldsymbol{X}_{n+k}$ in $\boldsymbol{X}$ has a unique expression in the form $s_{i_1}s_{i_2}\cdots s_{i_k}$ with $i_1 > i_2 > \cdots > i_k$.*

Proof. Such an iterated degeneracy operator corresponds to an epimorphism $\alpha\colon [n+k] \to [n]$ in $\boldsymbol{\Delta}$ (see Definition 3.1), and the set $\{i_1, i_2, \ldots, i_k\}$ is the set of integers i in $[n+k]$ such that $\alpha(i+1) = \alpha(i)$. □

Lemma 3.11. *Let X be a simplicial set, let $n \ge 0$, and let σ and τ be n-simplices of X for which there is an integer k and iterated degeneracy operators $s_{i_1}s_{i_2}\cdots s_{i_k}$ and $s_{j_1}s_{j_2}\cdots s_{j_k}$ such that $s_{i_1}s_{i_2}\cdots s_{i_k}(\sigma) = s_{j_1}s_{j_2}\cdots s_{j_k}(\tau)$. If σ is nondegenerate, then so is τ.*

Proof. If τ is degenerate, then $\tau = s_m\nu$ for some $0 \le m \le n-1$, and so

$$\begin{aligned}\sigma &= d_{i_k}\cdots d_{i_2}d_{i_1}s_{i_1}s_{i_2}\cdots s_{i_k}\sigma\\ &= d_{i_k}\cdots d_{i_2}d_{i_1}s_{j_1}s_{j_2}\cdots s_{j_k}\tau\\ &= d_{i_k}\cdots d_{i_2}d_{i_1}s_{j_1}s_{j_2}\cdots s_{j_k}s_m\nu\ ,\end{aligned}$$

and this last expression for σ has k face operators and $(k+1)$-degeneracy operators. The simplicial identities would then imply that σ was degenerate, which was assumed not to be the case. □

Proposition 3.12. *If X is a simplicial set and μ is a degenerate simplex of X, then there is a unique nondegenerate simplex ν of X and a unique iterated degeneracy operator α such that $\alpha(\nu) = \mu$.*

Proof. For every degenerate simplex μ we can choose a simplex ν of lowest possible degree of which it is a degeneracy; that simplex ν will necessarily be nondegenerate. Lemma 3.11 implies that there is no simplex of degree different from that of ν of which μ is a degeneracy, and so it is sufficient to show that

(1) if $n \ge 0$ and σ and τ are nondegenerate n-simplices such that some degeneracy of σ equals some (possibly different) degeneracy of τ, then $\sigma = \tau$, and
(2) if σ is a nondegenerate simplex and α and β are iterated degeneracy operators such that $\alpha(\sigma) = \beta(\tau)$, then $\alpha = \beta$.

For assertion 1, let k be the smallest positive integer for which there are iterated degeneracy operators $s_{i_1}s_{i_2}\cdots s_{i_k}$ with $i_1 > i_2 > \cdots > i_k$ and $s_{j_1}s_{j_2}\cdots s_{j_k}$ with $j_1 > j_2 > \cdots > j_k$ (see Lemma 3.10) such that $s_{i_1}s_{i_2}\cdots s_{i_k}(\sigma) = s_{j_1}s_{j_2}\cdots s_{j_k}(\tau)$. If we apply the face operator d_{i_1} to both sides of this equation, we obtain

$$s_{i_2}s_{i_3}\cdots s_{i_k}(\sigma) = d_{i_1}s_{j_1}s_{j_2}\cdots s_{j_k}(\tau)\ ,$$

and the simplicial identities imply that the right hand side is either a $(k-1)$-fold iterated degeneracy of τ or a k-fold iterated degeneracy of a face of τ. Lemma 3.11 implies that it cannot be the latter, and so our assumption that k was the smallest positive integer of its type implies that $k = 1$, i.e., $s_{i_1}\sigma = s_{j_1}\tau$. If $i_1 > j_1$, then $\sigma = d_{i_1+1}s_{i_1}\sigma = d_{i_1+1}s_j\tau = s_j d_{i_1}\tau$, which is impossible because σ is nondegenerate. Similarly, we cannot have $i_1 < j_1$. Thus, $i_1 = j_1$, and so $\sigma = \tau$ (because degeneracy operators have left inverses).

For assertion 2, let k be the smallest positive integer for which there are iterated degeneracy operators $s_{i_1}s_{i_2}\cdots s_{i_k}$ with $i_1 > i_2 > \cdots > i_k$ and $s_{j_1}s_{j_2}\cdots s_{j_k}$ with $j_1 > j_2 > \cdots > j_k$ such that $s_{i_1}s_{i_2}\cdots s_{i_k}(\sigma) = s_{j_1}s_{j_2}\cdots s_{j_k}(\sigma)$ (see Lemma 3.10). Because k is the smallest such integer and degeneracy operators have left inverses, we must have $i_1 \neq j_1$. If $i_1 > j_1$, then we can apply d_{i_1+1} to obtain

$$\begin{aligned} s_{i_2}s_{i_3}\cdots s_{i_k}(\sigma) &= d_{i_1+1}s_{j_1}s_{j_2}\cdots s_{j_k}(\sigma) \\ &= s_{j_1}s_{j_2}\cdots s_{j_k}d_{i_1+1-k}(\sigma) \end{aligned}$$

which contradicts Lemma 3.11. Similarly, we cannot have $i_1 < j_1$. Thus, $i_1 = j_1$, and so $s_{i_2}s_{i_3}\cdots s_{i_k}(\sigma) = s_{j_2}s_{j_3}\cdots s_{j_k}(\sigma)$, which implies that $k = 1$ (or else we have contradicted our assumption that k is the smallest positive integer of its type). □

Definition 3.13. If K is a simplicial set and σ is a simplex of K, then the *nondegenerate root* of σ is the unique nondegenerate simplex τ of K for which there is an iterated degeneracy operator D (which will be the identity operator, if σ is nondegenerate) such that $D(\tau) = \sigma$.

3.2. Nondegenerate simplices.

Corollary 3.14. *If K is a simplicial set, then K is naturally isomorphic to the coequalizer of the maps*

$$(3.15)\qquad \coprod_{\substack{n>0 \\ 0\le i\le n \\ \sigma\colon \Delta[n]\to K \\ \sigma \text{ nondegenerate}}} \Delta[n-1] \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{n\ge 0 \\ \sigma\colon \Delta[n]\to K \\ \sigma \text{ nondegenerate}}} \Delta[n]$$

where, on the summand $\Delta[n-1]$ indexed by $(n, i, \sigma\colon \Delta[n] \to K)$, for which the nondegenerate root (see Definition 3.13) of $\partial_i\sigma$ is τ of dimension k and $s_{j_1}s_{j_2}\cdots s_{j_{(n-1-k)}}(\tau) = \partial_i\sigma$ with $j_1 > j_2 > \cdots > j_{(n-1-k)}$,

- *ϕ is the map*

$$s^{j_{(n-1-k)}}\cdots s^{j_2}s^{j_1}\colon \Delta[n-1] \longrightarrow \Delta[k]$$

composed with the injection into the coproduct of the $\Delta[k]$ indexed by (k, τ), where τ is the composition

$$\Delta[k] \xrightarrow{d^{j_1}d^{j_2}\ldots d^{j_{(n-1-k)}}} \Delta[n-1] \xrightarrow{d^i} \Delta[n] \xrightarrow{\sigma} K \ ,$$

and

- *ψ is the map $d^i\colon \Delta[n-1] \to \Delta[n]$ composed with the injection into the coproduct of the summand indexed by (n, σ).*

Proof. **Fixme: Fill this in! This follows from Proposition 3.7.** □

4. Simplicial model categories

If $\mathcal{M}$ is a model category, $i\colon A \to B$ is a cofibration, $p\colon X \to Y$ is a fibration, and at least one of i and p is a weak equivalence, then one of the model category axioms (the *lifting extension axiom*) requires that the dotted arrow exists in every solid arrow diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \quad .$$

Such a solid arrow diagram is an element of the set

$$\mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y)$$

and that model category axiom is exactly the requirement that the natural map of sets

$$\mathcal{M}(B,X) \xrightarrow{i^* \times p_*} \mathcal{M}(A,X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B,Y)$$

be a surjection. If $\mathcal{M}$ is a *simplicial model category*, then for every two objects X and Y we have a simplicial set $\mathrm{Map}(X,Y)$, and one of the axioms (the *homotopy lifting extension axiom*) requires that if $i\colon A \to B$ is a cofibration and $p\colon X \to Y$ is a fibration, then the natural map of simplicial sets

$$\mathrm{Map}(B,X) \xrightarrow{i^* \times p_*} \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)$$

be a fibration that is also a weak equivalence if at least one of i and p is a weak equivalence.

Proposition 4.1. *Let $\mathcal{M}$ be a simplicial model category.*

(1) *If X and Y are objects of $\mathcal{M}$ and K is a simplicial set, then there exist*
- *a simplicial set $\mathrm{Map}(X,Y)$, natural in X and Y, and a natural isomorphism $\big(\mathrm{Map}(X,Y)\big)_0 \approx \mathcal{M}(X,Y)$, and*
- *objects $X \otimes K$ and X^K of $\mathcal{M}$, natural in X and K, together with natural isomorphisms of simplicial sets*

  $$\mathrm{Map}(X \otimes K, Y) \approx \mathrm{Map}\big(K, \mathrm{Map}(X,Y)\big) \approx \mathrm{Map}(X, Y^K)$$

  and, thus, also natural isomorphisms of sets

  $$\mathcal{M}(X \otimes K, Y) \approx \mathcal{M}\big(K, \mathrm{Map}(X,Y)\big) \approx \mathcal{M}(X, Y^K) \quad .$$

(2) *If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $p\colon X \to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets*

$$\mathrm{Map}(B,X) \xrightarrow{i^* \times p_*} \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)$$

is a fibration that is a weak equivalence if either i or p is a weak equivalence.

Proof. This is part of the definition of a simplicial model category (see, e.g., [7, Def. 9.1.6]). □

Lemma 4.2. *Let $\mathcal{M}$ be a simplicial model category.*

(1) *If $\emptyset$ is an initial object of $\mathcal{M}$ and K is a simplicial set, then $\emptyset \otimes K$ is an initial object of $\mathcal{M}$.*
(2) *If $*$ is an terminal object of $\mathcal{M}$ and K is a simplicial sets, then $*^K$ is a terminal object of $\mathcal{M}$.*

Proof. We will prove part 1; the proof of part 2 is similar.

If X is an object of $\mathcal{M}$, then $\mathcal{M}(\emptyset \otimes K, X) \approx \mathcal{M}(\emptyset, X^K)$ (see Proposition 4.1), and $\mathcal{M}(\emptyset, X^K)$ has exactly one element. □

Proposition 4.3. *If $\mathcal{M}$ is a simplicial model category, X is an object of $\mathcal{M}$, and K is a simplicial set, then $X \otimes K$ is naturally isomorphic to the coequalizer of the maps*

$$\Bigl(\coprod_{\substack{n>0\\0\le i\le n\\\Delta[n]\to K}} X\otimes\Delta[n-1]\Bigr) \amalg \Bigl(\coprod_{\substack{n\ge 0\\0\le i\le n\\\Delta[n]\to K}} X\otimes\Delta[n+1]\Bigr) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{n\ge 0\\\Delta[n]\to K}} X\otimes\Delta[n]$$

where, on the first summand,

- *ϕ takes the summand $X \otimes \Delta[n-1]$ indexed by $(n, i, \sigma\colon \Delta[n] \to K)$ by the identity map to the $X \otimes \Delta[n-1]$ indexed by $(n-1)$ and the composition $\Delta[n-1] \xrightarrow{d^i} \Delta[n] \xrightarrow{\sigma} K$, composed with the injection into the coproduct, and*
- *ψ takes that same summand to the $X \otimes \Delta[n]$ indexed by (n, σ) by the map $1_X \otimes d^i\colon \Delta[n-1] \to \Delta[n]$, composed with the injection into the coproduct*

and, on the second summand,

- *ϕ takes the summand $X \otimes \Delta[n+1]$ indexed by $(n, i, \sigma\colon \Delta[n] \to K)$ by the identity map to the $X \otimes \Delta[n+1]$ indexed by $(n+1)$ and the composition $\Delta[n+1] \xrightarrow{s^i} \Delta[n] \xrightarrow{\sigma} K$, composed with the injection into the coproduct, and*
- *ψ takes that same summand to the $X \otimes \Delta[n]$ indexed by (n, σ) by the map $1_X \otimes s^i\colon \Delta[n+1] \to \Delta[n]$, composed with the injection into the coproduct.*

Proof. Since the functor that takes a simplicial set K to $X \otimes K$ is a left adjoint (see Proposition 4.1) and thus preserves colimits, the result follows from Proposition 3.7. □

Proposition 4.4. *If $\mathcal{M}$ is a simplicial model category, X is an object of $\mathcal{M}$, and K is a simplicial set, then X^K is naturally isomorphic to the equalizer of the maps*

$$\prod_{\substack{n\ge 0\\\Delta[n]\to K}} X^{\Delta[n]} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \Bigl(\prod_{\substack{n>0\\0\le i\le n\\\Delta[n]\to K}} X^{\Delta[n-1]}\Bigr) \times \Bigl(\prod_{\substack{n\ge 0\\0\le i\le n\\\Delta[n]\to K}} X^{\Delta[n+1]}\Bigr)$$

where

- *the projection of ϕ onto the factor $X^{\Delta[n-1]}$ indexed by $(n, i, \sigma\colon \Delta[n] \to K)$ is the projection onto the $X^{\Delta[n-1]}$ indexed by the composition $\Delta[n-1] \xrightarrow{d^i} \Delta[n] \xrightarrow{\sigma} K$, and*
- *the projection of ψ onto that same factor is the composition of the projection onto the factor $X^{\Delta[n]}$ indexed by σ with the map $X^{\Delta[n]} \xrightarrow{(1_X)^{d^i}} X^{\Delta[n-1]}$*

and

- *the projection of* ϕ *onto the factor* $X^{\Delta[n+1]}$ *indexed by* $(n, i, \sigma\colon \Delta[n] \to K)$ *is the projection onto the* $X^{\Delta[n+1]}$ *indexed by the composition* $\Delta[n+1] \xrightarrow{s^i} \Delta[n] \to K$*, and*
- *the projection of* ψ *onto that same factor is the composition of the projection onto the* $X^{\Delta[n]}$ *indexed by* σ *with the map* $X^{\Delta[n]} \xrightarrow{(1_X)^{s^i}} X^{\Delta[n+1]}$*.*

Proof. The functor that takes the simplicial set K to the object X^K of $\mathcal{M}^{\mathrm{op}}$ is a left adjoint (see Proposition 4.1), and thus preserves colimits. Thus, as a functor to $\mathcal{M}$ it converts colimits to limits, and so the result follows from Proposition 3.7. □

5. The model category of diagrams of simplicial sets

If $\mathcal{C}$ is a small category, then there is a model category structure on the category of $\mathcal{C}$-diagrams of simplicial sets that is important for studying homotopy colimits and homotopy limits of diagrams in any simplicial model category $\mathcal{M}$, because if $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams in $\mathcal{M}$ and Z is an object of $\mathcal{M}$, then

- the induced map $f^*\colon \mathrm{Map}(\boldsymbol{Y}, Z) \to \mathrm{Map}(\boldsymbol{X}, Z)$ is a map of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets and
- the induced map $f_*\colon \mathrm{Map}(Z, \boldsymbol{X}) \to \mathrm{Map}(Z, \boldsymbol{Y})$ is a map of $\mathcal{C}$-diagrams of simplicial sets.

In addition, if $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then

- the homotopy colimit $\mathrm{hocolim}\, \boldsymbol{X}$ is defined to be the coend $\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$ (see Definition 8.1), where $\mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets (see Definition 2.5), and the fact that the $\mathcal{C}^{\mathrm{op}}$-diagram $\mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is a free cell complex (see Proposition 5.13) (and, thus, a cofibrant $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets; see Theorem 5.1) will be used to prove the homotopy invariance of the homotopy colimit functor (see Theorem 8.4), and
- the homotopy limit $\mathrm{holim}\, \boldsymbol{X}$ is defined to be the end $\hom^{\mathcal{C}}(\mathrm{N}(\mathcal{C}\downarrow -), \boldsymbol{X})$ (see Definition 11.1), where $\mathrm{N}(\mathcal{C}\downarrow -)$ is a $\mathcal{C}$-diagram of simplicial sets (see Definition 2.15), and the fact that the $\mathcal{C}$-diagram $\mathrm{N}(\mathcal{C}\downarrow -)$ is a free cell complex (see Proposition 5.13) (and, thus, a cofibrant $\mathcal{C}$-diagram of simplicial sets; see Theorem 5.1) will be used to prove the homotopy invariance of the homotopy limit functor (see Theorem 11.4).

In fact, the above is true even for model categories $\mathcal{M}$ that may not be simplicial, although to define the homotopy colimit and homotopy limit functors for diagrams in such a model category we must first choose a *framing* (see [7, Def. 16.6.21]) of the model category $\mathcal{M}$, after which the model category of diagrams of simplicial sets plays the same role that it plays here (see [7, Chap. 19]).

There is more than one model category structure on the category of $\mathcal{C}$-diagrams of simplicial sets. The one that is important here is the *Bousfield-Kan* structure [3, p. 314], which is sometimes also called the *projective model category structure*.

Theorem 5.1. *If* $\mathcal{C}$ *is a small category, then there is a model category structure on the category of* $\mathcal{C}$*-diagrams of simplicial sets in which*

- *the weak equivalences are the objectwise weak equivalences,*
- *the fibrations are the objectwise fibrations, and*

- *the cofibrations are the relative free cell complexes (see Definition* 5.7*) and their retracts (and so the cofibrant objects are the free cell complexes and their retracts).*

Proof. See [7, Thm. 11.6.1]. □

5.1. **Relative free cell complexes.** In this section we describe *relative free cell complexes*, which are maps of diagrams of simplicial sets that can be constructed by repeatedly attaching *free cells* to a diagram of simplicial sets (see Definition 5.7). Special cases of attaching a free cell to a diagram of simplicial sets are

- attaching a cell to a simplicial set and
- attaching a free cell to a simplicial set acted upon by a discrete group,

and we begin by describing these familiar special cases in a way that leads to the description of the general case in Definition 5.5.

5.1.1. *Attaching a cell to a simplicial set.* If X is a simplicial set and $n \geq 0$, we *attach an n-cell* to X by choosing an attaching map $f\colon \partial\Delta[n] \to X$ and taking the pushout Y of

$$\Delta[n] \longleftarrow \partial\Delta[n] \xrightarrow{\ f\ } X \ .$$

The simplices of $Y - X$ are the nondegenerate n-simplex of $\Delta[n]$ and its degeneracies.

5.1.2. *Attaching a free cell to a simplicial set with G-action.* Let G be a (discrete) group and let X be a G-simplicial set, i.e., a simplicial set with an action of G. If we view G as a category with a single object, then this is a G-diagram of simplicial sets. If $n \geq 0$, we will *attach a free n-cell* to X to build a G-simplicial set Y, containing X, such that G acts freely on the simplices of $Y - X$.

We begin by choosing an attaching map $f\colon \partial\Delta[n] \to X$, and then attach one n-cell to X for each element of G, as follows: for each element g of G, we attach an n-cell to X using the attaching map $g \cdot f$, which is defined as the composition

$$\partial\Delta[n] \xrightarrow{\ f\ } X \xrightarrow{\ g\ } X$$

(where that second map is the automorphism of X defined by the action of g). Thus, we have a pushout of G-simplicial sets

$$\begin{array}{ccc} \coprod_{g\in G} \partial\Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{g\in G} \Delta[n] & \longrightarrow & Y \end{array} \ .$$

The simplices of $Y - X$ are the nondegenerate n-simplices of the $\Delta[n]$'s in the lower left corner and their degeneracies. The map $X \to Y$ is an example of a *relative free cell complex*, and the set of simplices of $Y - X$ that come from the summand indexed by the identity element of G are the elements of a *basis* of the relative free cell complex (see Definition 5.9). A basis has the following properties:

- A basis is closed under degeneracy operators.
- For every simplex σ of $Y - X$ there are
  - a basis element τ of $Y - X$ and
  - an element g of G such that $g \cdot \tau = \sigma$,

and the pair (τ, g) is unique.

Note that there is only one nondegenerate element of the basis (the nondegenerate n-simplex of the $\Delta[n]$ from the summand indexed by the identity element of G), and by looking at its faces we can recover the attaching map of the free n-cell.

5.1.3. *Attaching a free cell to a diagram of simplicial sets.* If $\mathcal{C}$ is a small category, $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of simplicial sets, $n \geq 0$, and α is an object of $\mathcal{C}$, we will describe how to *attach a free n-cell to $\boldsymbol{X}$ at the object α.* A special case of this is when the category $\mathcal{C}$ is obtained by choosing a (discrete) group G and viewing G as a category with a single object, as in Section 5.1.2. In that special case, there was no need to choose an object α because the category only had one object.

To attach a free n-cell at α, we choose an attaching map $f\colon \partial\Delta[n] \to \boldsymbol{X}_\alpha$, and we will attach one n-cell for each map in $\mathcal{C}$ whose domain is α, that cell being attached to the simplicial set at the target of that map. (Note that in Example 5.1.2 we attached an n-cell for every map, because every map had the unique object as its domain.) That is, for every map $\gamma\colon \alpha \to \beta$ in $\mathcal{C}$ we will attach an n-cell to $\boldsymbol{X}_\beta$ using the attaching map $\gamma_* \circ f$, i.e., the composition

$$\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{\gamma_*} \boldsymbol{X}_\beta \ .$$

If we call the newly created diagram $\boldsymbol{Y}$, then the simplices of $\boldsymbol{Y}_\alpha - \boldsymbol{X}_\alpha$ that were simplices of the cell attached for the identity map of α are the elements of a *basis* of the relative free cell complex $\boldsymbol{X} \to \boldsymbol{Y}$ (see Definition 5.9). A basis has the following properties:

- The basis is closed under degeneracy operators.
- For every object β of $\mathcal{C}$ and every simplex σ of $\boldsymbol{Y}_\beta - \boldsymbol{X}_\beta$ there are
  - an object α of $\mathcal{C}$,
  - a basis element τ of $\boldsymbol{Y}_\alpha - \boldsymbol{X}_\alpha$, and
  - a map $\gamma\colon \alpha \to \beta$ in $\mathcal{C}$ such that $\gamma_*(\tau) = \sigma$,

  and the triple (α, τ, γ) is unique.

Note that there is only one nondegenerate element of the basis (the nondegenerate n-simplex of the $\Delta[n]$ attached for the identity map of α), and by looking at its faces we can recover the attaching map of the free n-cell.

Example 5.2. Let $\mathcal{C}$ be the category $b \xleftarrow{\gamma} a \xrightarrow{\delta} c$ with three objects $\{a, b, c\}$ and two non-identity maps $\gamma\colon a \to b$ and $\delta\colon a \to c$. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of simplicial sets, $n \geq 0$, and we want to attach a free n-cell at a, then we choose an attaching map $f\colon \partial\Delta[n] \to \boldsymbol{X}_a$, and we will be attaching three cells, one for each map in $\mathcal{C}$ whose domain is a:

(1) We attach an n-cell to $\boldsymbol{X}_a$ for the map 1_a, using the attaching map that is the composition $\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_a \xrightarrow{1_{\boldsymbol{X}_a}} \boldsymbol{X}_a$.
(2) We attach an n-cell to $\boldsymbol{X}_b$ for the map γ, using the attaching map that is the composition $\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_a \xrightarrow{\gamma_*} \boldsymbol{X}_b$.
(3) We attach an n-cell to $\boldsymbol{X}_c$ for the map δ, using the attaching map that is the composition $\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_a \xrightarrow{\delta_*} \boldsymbol{X}_c$.

We call the newly created diagram $\boldsymbol{Y}$; the map $\gamma_*\colon \boldsymbol{Y}_a \to \boldsymbol{Y}_b$ takes the simplices newly attached at $\boldsymbol{X}_a$ to the corresponding ones attached at $\boldsymbol{X}_b$, and the map

$\delta_*\colon \boldsymbol{Y}_a \to \boldsymbol{Y}_c$ takes the simplices newly attached at $\boldsymbol{X}_a$ to the corresponding ones attached at $\boldsymbol{X}_c$.

The map $\boldsymbol{X} \to \boldsymbol{Y}$ is a relative free cell complex. A basis consists of the nondegenerate n-simplex that was attached to $\boldsymbol{X}_a$ and its degeneracies (see Definition 5.9). There is only one nondegenerate simplex in the basis, and by looking at its faces we can recover the attaching map of the free cell.

Example 5.3. If we start with $\mathcal{C}$ and $\boldsymbol{Y}$ as in Example 5.2, and we want to attach a k-cell to $\boldsymbol{Y}$ at the object b, then we must choose an attaching map $g\colon \partial\Delta[k] \to \boldsymbol{Y}_b$, and we will be attaching only one k-cell, because there is only one map in $\mathcal{C}$ with domain b: the identity map of b. After we do that, and call the resulting diagram $\boldsymbol{Z}$, the map $\boldsymbol{Y} \to \boldsymbol{Z}$ will be a relative free cell complex, and the map $\boldsymbol{X} \to \boldsymbol{Z}$ will also be a relative free cell complex. A basis for $\boldsymbol{X} \to \boldsymbol{Z}$ will consist of

- the nondegenerate n-simplex of the n-cell attached to $\boldsymbol{X}_a$, and its degeneracies, and
- the nondegenerate k-simplex of the k-cell attached to $\boldsymbol{Y}_b$, and its degeneracies

(see Definition 5.9). Note that there are only two nondegenerate simplices in the basis, and by looking at their faces we can recover the attaching maps of the free cells.

Example 5.4. Let $\mathcal{D}$ be the category

$$\alpha \underset{\delta}{\overset{\gamma}{\rightrightarrows}} \beta$$

with two objects $\{\alpha, \beta\}$ and two non-identity maps $\gamma\colon \alpha \to \beta$ and $\delta\colon \alpha \to \beta$. If $\boldsymbol{X}$ is a $\mathcal{D}$-diagram of simplicial sets, $n \geq 0$, and we want to attach a free n-cell to $\boldsymbol{X}$ at α, then we choose an attaching map $f\colon \partial\Delta[n] \to \boldsymbol{X}_\alpha$ and we will be attaching three n-cells, one for each of the maps whose domain is α:

- We attach an n-cell to $\boldsymbol{X}_\alpha$ for the map 1_α, using the attaching map that is the composition $\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{1_{\boldsymbol{X}_\alpha}} \boldsymbol{X}_\alpha$.
- We attach an n-cell to $\boldsymbol{X}_\beta$ for the map γ, using the attaching map that is the composition $\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{\gamma_*} \boldsymbol{X}_\beta$.
- We attach a second n-cell to $\boldsymbol{X}_\beta$ for the map δ, using the attaching map that is the composition $\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{\delta_*} \boldsymbol{X}_\beta$.

If we call the new diagram $\boldsymbol{Y}$, then we have pushout diagrams

$$\begin{array}{ccc} \coprod_{1_\alpha} \partial\Delta[n] & \longrightarrow & \boldsymbol{X}_\alpha \\ \downarrow & & \downarrow \\ \coprod_{1_\alpha} \Delta[n] & \longrightarrow & \boldsymbol{Y}_\alpha \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \coprod_{\gamma,\delta} \partial\Delta[n] & \longrightarrow & \boldsymbol{X}_\beta \\ \downarrow & & \downarrow \\ \coprod_{\gamma,\delta} \Delta[n] & \longrightarrow & \boldsymbol{Y}_\beta \end{array}$$

where the map

$$\coprod_{\gamma,\delta} \partial\Delta[n] \to \boldsymbol{X}_\beta$$

on the summand indexed by γ is the composition

$$\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{\gamma_*} \boldsymbol{X}_\beta$$

and on the summand indexed by δ is the composition

$$\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{\delta_*} \boldsymbol{X}_\beta \ .$$

A basis for the relative free cell complex consists of the nondegenerate n-simplex attached to $\boldsymbol{X}_\alpha$, and its degeneracies (see Definition 5.9). There is only one nondegenerate simplex in the basis, and by looking at its faces we can recover the attaching map of the free cell.

5.1.4. *Attaching a free cell: general case.*

Definition 5.5. Let $\mathcal{C}$ be a small category and let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram of simplicial sets. If $n \geq 0$, α is an object of $\mathcal{C}$, and $f\colon \partial\Delta[n] \to \boldsymbol{X}_\alpha$ is a map, then the result of *attaching a free n-cell at α along f* is the $\mathcal{C}$-diagram of simplicial sets $\boldsymbol{Y}$ such that

- for an object β of $\mathcal{C}$ the simplicial set $\boldsymbol{Y}_\beta$ is the pushout

$$\begin{array}{ccc} \coprod_{(\sigma\colon \alpha\to\beta)\in\mathcal{C}} \partial\Delta[n] & \longrightarrow & \boldsymbol{X}_\beta \\ \downarrow & & \downarrow \\ \coprod_{(\sigma\colon \alpha\to\beta)\in\mathcal{C}} \Delta[n] & \longrightarrow & \boldsymbol{Y}_\beta \end{array}$$

  where the upper horizontal map on the summand indexed by $(\sigma\colon \alpha \to \beta) \in \mathcal{C}$ is the composition

$$\partial\Delta[n] \xrightarrow{f} \boldsymbol{X}_\alpha \xrightarrow{\sigma_*} \boldsymbol{X}_\beta \ ,$$

  and
- for a map $\tau\colon \beta \to \gamma$ in $\mathcal{C}$ the map $\tau_*\colon \boldsymbol{Y}_\beta \to \boldsymbol{Y}_\gamma$ is defined by sending the summands indexed by $\sigma\colon \alpha \to \beta$ in the pushout that defines $\boldsymbol{Y}_\beta$ to the summands indexed by the composition

$$\sigma \xrightarrow{\sigma} \beta \xrightarrow{\tau} \gamma$$

  in the pushout that defines $\boldsymbol{Y}_\gamma$.

A basis for the relative free cell complex constructed in Definition 5.5 consists of the nondegenerate n-simplex of the $\Delta[n]$ attached for the identity map of α and its degeneracies (see Definition 5.9). There is only one nondegenerate simplex in the basis, and by looking at its faces we can recover the attaching map of the free cell.

Remark 5.6. The process of attaching a free cell to a $\mathcal{C}$-diagram of simplicial sets can also be described by defining *free diagrams*, and then taking a pushout in the category of $\mathcal{C}$-diagrams of simplicial sets (see [7, Section 11.5.29]). If α is an object of $\mathcal{C}$ and K is a simplicial set, then the *free $\mathcal{C}$-diagram of simplicial sets on K at α* is the $\mathcal{C}$-diagram F^α_K such that, for every object β of $\mathcal{C}$, $F^\alpha_K(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} K$, with the obvious structure maps. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of simplicial sets, then maps of diagrams $F^\alpha_K \to \boldsymbol{X}$ correspond to maps of simplicial sets $K \to \boldsymbol{X}_\alpha$ (i.e., there is a natural isomorphism $\mathrm{SS}^{\mathcal{C}}(F^{\alpha}_{K}, \boldsymbol{X}) \approx \mathrm{SS}(K, \boldsymbol{X}_{\alpha})$), and we can attach a free n-cell to $\boldsymbol{X}$ at α by forming the pushout in the category of $\mathcal{C}$-diagrams of simplicial sets

$$\begin{array}{ccc} F^{\alpha}_{\partial\Delta[n]} & \longrightarrow & \boldsymbol{X} \\ \downarrow & & \downarrow \\ F^{\alpha}_{\Delta[n]} & \longrightarrow & \boldsymbol{Y} \end{array}$$

where the upper horizontal map corresponds to the chosen attaching map $f\colon \partial\Delta[n] \to \boldsymbol{X}_{\alpha}$.

Definition 5.7. If $\mathcal{C}$ is a small category, then

- a *relative free cell complex* is a map of $\mathcal{C}$-diagrams of simplicial sets that can be constructed by a (possibly transfinite) well ordered process of attaching free cells of various dimensions (see Definition 5.5), and
- a *free cell complex* is a $\mathcal{C}$-diagram of simplicial sets for which the map from the constant $\mathcal{C}$-diagram at the empty simplicial set is a relative free cell complex.

5.2. **Basis of a relative free cell complex.** In this section, we show how to identify a relative free cell complex (see Theorem 5.10).

Definition 5.8. If $\mathcal{C}$ is a category, then $\mathcal{C}^{\mathrm{disc}}$ will denote the *discrete category associated with* $\mathcal{C}$, i.e., the category with the same objects as $\mathcal{C}$ but containing only the identity maps of $\mathcal{C}$. Thus, a $\mathcal{C}^{\mathrm{disc}}$-diagram of sets $\boldsymbol{S}$ consists of a set $\boldsymbol{S}_{\alpha}$ for each object α of $\mathcal{C}$.

Definition 5.9. If $\mathcal{C}$ is a small category and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise inclusion of $\mathcal{C}$-diagrams of simplicial sets, then a *basis* of the map f is a sequence $\boldsymbol{S} = \{\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \ldots\}$ of $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets (see Definition 5.8) such that

(1) for each $n \geq 0$ and object α of $\mathcal{C}$, the set $\boldsymbol{S}^n_{\alpha}$ is a subset of the set of n-simplices of $\boldsymbol{Y}_{\alpha}$ that are not in $\boldsymbol{X}_{\alpha}$,
(2) for $0 \leq i \leq n$ and object α of $\mathcal{C}$, we have $s_i(\boldsymbol{S}^n_{\alpha}) \subset \boldsymbol{S}^{n+1}_{\alpha}$ (i.e., $\boldsymbol{S}$ is closed under degeneracies), and
(3) if $n \geq 0$, β is an object of $\mathcal{C}$, and τ is an n-simplex of $\boldsymbol{Y}_{\beta}$ that is not in $\boldsymbol{X}_{\beta}$, then there exist
   - an object α of $\mathcal{C}$,
   - an element σ of $\boldsymbol{S}^n_{\alpha}$, and
   - a map $\gamma\colon \alpha \to \beta$ in $\mathcal{C}$ such that $\boldsymbol{Y}_{\gamma}(\sigma) = \tau$,

   and such a triple (α, σ, γ) is unique.

Theorem 5.10. *If $\mathcal{C}$ is a small category and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams of simplicial sets that is an objectwise inclusion of simplicial sets, then f is a relative free cell complex if and only if there exists a basis $\boldsymbol{S}$ of f (see Definition 5.9).*

Proof. If f is a relative free cell complex, then $\boldsymbol{Y}$ is constructed, starting from $\boldsymbol{X}$, by a well ordered sequence of attaching free cells. As we saw in the examples in Section 5.1, each time that we attach a new free cell we enlarge the basis to obtain a basis of the enlarged relative free cell complex.

Conversely, suppose that we have an objectwise inclusion $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ of $\mathcal{C}$-diagrams of simplicial sets and a basis $\boldsymbol{S}$ of the inclusion; we will show how to construct $\boldsymbol{Y}$ from $\boldsymbol{X}$ by attaching free cells.

Each nondegenerate basis element corresponds to a free cell to be attached, but we must ensure that the attaching map of each cell factors through the part of $\boldsymbol{Y}$ that has been constructed so far. To ensure that this is the case, we will attach the cells in order of their dimensions. Thus, for each n we choose a well ordering of the nondegenerate elements of $\boldsymbol{S}^n$, and then we order the entire collection of nondegenerate basis elements by ordering them first by dimension and second by the chosen well orderings. If we now proceed to attach all of those cells, in order, then each time that we want to attach an n-cell the entire $(n-1)$-skeleton of $\boldsymbol{Y}$ will already exist, and so the attaching map will factor as needed. □

Corollary 5.11. *Let $\mathcal{C}$ be a small category and let $\boldsymbol{Y}$ be a $\mathcal{C}$-diagram of simplicial sets. If $\boldsymbol{S} = \{\boldsymbol{S}^0, \boldsymbol{S}^1, \boldsymbol{S}^2, \ldots\}$ is a sequence of $\mathcal{C}^{\mathrm{disc}}$-diagrams of sets, then $\boldsymbol{Y}$ is a free cell complex with basis $\boldsymbol{S}$ if and only if:*

1. *for $n \geq 0$ and α an object of $\mathcal{C}$, the set $\boldsymbol{S}^n_\alpha$ is a subset of the set of n-simplices of $\boldsymbol{Y}_\alpha$,*
2. *for $0 \leq i \leq n$ and α an object of $\mathcal{C}$, we have $s_i(\boldsymbol{S}^n_\alpha) \subset \boldsymbol{S}^{n+1}_\alpha$ (i.e., $\boldsymbol{S}$ is closed under degeneracies), and*
3. *if $n \geq 0$, β is an object of $\mathcal{C}$, and τ is an n-simplex of $\boldsymbol{Y}_\beta$, then there exist*
   - *an object α of $\mathcal{C}$,*
   - *an element σ of $\boldsymbol{S}^n_\alpha$, and*
   - *a map $\gamma\colon \alpha \to \beta$ in $\mathcal{C}$ such that $\boldsymbol{X}_\gamma(\sigma) = \tau$,*

   and such a triple (α, σ, γ) is unique.

Proof. This is the case of Theorem 5.10 in which $\boldsymbol{X}$ is the diagram of empty simplicial sets. □

Proposition 5.12. *Let $\mathcal{C}$ be a small category and let $\boldsymbol{K}\colon \mathcal{C} \to \mathrm{SS}$ and $\boldsymbol{L}\colon \mathcal{C} \to \mathrm{SS}$ be $\mathcal{C}$-diagrams of simplicial sets. If $\boldsymbol{K}$ is a free cell complex with basis $\boldsymbol{S}$, $\boldsymbol{L}$ is a free cell complex with basis $\boldsymbol{S}'$, and $f\colon \boldsymbol{K} \to \boldsymbol{L}$ is a map of diagrams that is an objectwise inclusion of simplicial sets and takes every element of $\boldsymbol{S}$ to an element of $\boldsymbol{S}'$, then the map f is a relative free cell complex.*

Proof. Let $\boldsymbol{S}''$ be the elements of $\boldsymbol{S}'$ that are not the image of an element of $\boldsymbol{S}$. For every object β of $\mathcal{C}$ and every simplex t of $\boldsymbol{L}_\beta$ there is a unique triple (α, σ, s) where α is an object of $\mathcal{C}$, $\sigma\colon \alpha \to \beta$ is a map in $\mathcal{C}$, and s is a simplex in $\boldsymbol{S}'_\alpha$ such that $\sigma_*(s) = t$, and such a simplex t is in the image of f if and only if $s \in \boldsymbol{S}_\alpha$. Thus, for every object β of $\mathcal{C}$ and every simplex t of $\boldsymbol{L}_\beta$ that is not in the image of f there is a unique triple (α, σ, s) where α is an object of $\mathcal{C}$, $\sigma\colon \alpha \to \beta$ is a map in $\mathcal{C}$, and s is a simplex of $\boldsymbol{S}''_\alpha$ such that $\sigma_*(s) = t$, and so $f\colon \boldsymbol{K} \to \boldsymbol{L}$ is a relative free cell complex with basis $\boldsymbol{S}''$. □

Proposition 5.13. *Let $\mathcal{C}$ be a small category.*

1. *The $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ is a free cell complex with basis the set of simplices of the form $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$.*
2. *The $\mathcal{C}$-diagram of simplicial sets $\mathrm{N}(\mathcal{C}\downarrow -)$ is a free cell complex with basis the set of simplices of the form $\big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$.*

Proof. This follows from Corollary 5.11. For part 1, if $\tau = \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \gamma\colon \beta \to \alpha_n\big)$ is a simplex of $\mathrm{N}(\beta\downarrow\mathcal{C})^{\mathrm{op}}$, then $\sigma = \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ is a basis element, $\mathrm{N}(\gamma\downarrow\mathcal{C})^{\mathrm{op}}(\sigma) = \tau$, and the triple $(\alpha_n, \sigma, \gamma)$ is unique. For part 2, if $\tau = \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \gamma\colon \alpha_n \to \beta\bigr)$ is a simplex of $\mathrm{N}(\mathcal{C}\downarrow\beta)$, then $\sigma = \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)$ is a basis element, $\mathrm{N}(\mathcal{C}\downarrow\gamma)(\sigma) = \tau$, and the triple $(\alpha_n, \sigma, \gamma)$ is unique. □

The following proposition will be used in Theorem 10.8 to show that a homotopy right cofinal functor (see Definition 10.6) induces a weak equivalence of homotopy colimits for objectwise cofibrant diagrams, and in Theorem 13.7 to show that a homotopy left cofinal functor (see Definition 13.6) induces a weak equivalence of homotopy limits for objectwise fibrant diagrams.

Proposition 5.14. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories.*

(1) *The $\mathcal{D}^{\mathrm{op}}$-diagram of simplicial sets $\mathrm{N}(-\downarrow F)^{\mathrm{op}}$ is a free cell complex with basis the set of simplices of the form $\bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)$.*
(2) *The $\mathcal{D}$-diagram of simplicial sets $\mathrm{N}(F\downarrow -)$ is a free cell complex with basis the set of simplices of the form $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)$.*

Proof. This follows from Corollary 5.11. For part 1, if $\tau = \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \gamma\colon \beta \to F\alpha_n\bigr)$ is a simplex of $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$, then $\sigma = \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)$ is a basis element, $\mathrm{N}(\gamma\downarrow F)^{\mathrm{op}}(\sigma) = \tau$, and the triple $(F\alpha_n, \sigma, \gamma)$ is unique. For part 2, if $\tau = \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \gamma\colon F\alpha_n \to \beta\bigr)$ is a simplex of $\mathrm{N}(F\downarrow\beta)$, then $\sigma = \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)$ is a basis element, $\mathrm{N}(F\downarrow\gamma)(\sigma) = \tau$, and the triple $(F\alpha_n, \sigma, \gamma)$ is unique. □

The following proposition will be used in Proposition 10.9 to show that certain maps of homotopy colimits are cofibrations and in Proposition 13.8 to show that certain maps of homotopy limits are fibrations.

Proposition 5.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If F is the inclusion of a subcategory, then*

(1) *the map of $\mathcal{D}^{\mathrm{op}}$-diagrams of simplicial sets $\mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ is a relative free cell complex, and*
(2) *the map of $\mathcal{D}$-diagrams of simplicial sets $\mathrm{N}(F\downarrow -) \to \mathrm{N}(\mathcal{D}\downarrow -)$ is a relative free cell complex.*

Proof. For part 1, the $\mathcal{D}^{\mathrm{op}}$-diagram $\mathrm{N}(-\downarrow F)^{\mathrm{op}}$ is a free cell complex with basis consisting of the simplices $\bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)$ and the $\mathcal{D}^{\mathrm{op}}$-diagram $\mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ is a free cell complex with basis consisting of the simplices $\bigl((\beta_0 \leftarrow \beta_1 \leftarrow \cdots \leftarrow \beta_n) \in \mathcal{D}, 1\colon \beta_n \to \beta_n\bigr)$. The induced map of diagrams takes that basis element of $\mathrm{N}(-\downarrow F)^{\mathrm{op}}$ to the basis element $\bigl((F\alpha_0 \leftarrow F\alpha_1 \leftarrow \cdots \leftarrow F\alpha_n) \in \mathcal{D}, 1\colon F\alpha_n \to F\alpha_n\bigr)$ of $\mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$, and so the result follows from Proposition 5.12.

For part 2, the $\mathcal{D}$-diagram $\mathrm{N}(F\downarrow -)$ is a free cell complex with basis consisting of the simplices $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon F\alpha_0 \to F\alpha_0\bigr)$ and the $\mathcal{D}$-diagram $\mathrm{N}(\mathcal{D}\downarrow -)$ is a free cell complex with basis consisting of the simplices $\bigl((\beta_0 \to \beta_1 \to \cdots \to \beta_n) \in \mathcal{D}, 1\colon \beta_0 \to \beta_0\bigr)$. The induced map of diagrams takes that basis element of $\mathrm{N}(F\downarrow -)$ to the basis elements $\bigl((F\alpha_0 \to F\alpha_1 \to \cdots \to F\alpha_n) \in \mathcal{D}, 1\colon F\alpha_0 \to F\alpha_0\bigr)$ of $\mathrm{N}(\mathcal{D}\downarrow -)$, and so the result follows from Proposition 5.12. □

6. Free diagrams

6.1. **The free diagram on an object.** Let $\mathcal{C}$ be a small category and let α be an object of $\mathcal{C}$. if $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, we want to define $\mathbf{F}^{\alpha}_{X}$, the *free* $\mathcal{C}$*-diagram in* $\mathcal{M}$ *generated by* X *at* α. This will be a $\mathcal{C}$-diagram that starts with a "generating copy" of X at α, and then adds, as freely as possible, what else is needed in order to have a diagram. Thus, for every map $\sigma\colon \alpha \to \beta$ in $\mathcal{C}$ with domain α, we add in another copy of X at β, and we let σ_* take the generating copy of X, by the identity map, to this new copy. Since we will do this for *every* map with domain α, we will let $\mathbf{F}^{\alpha}_{X}(\beta)$ be the coproduct, indexed by the set of maps from α to β, of copies of X. Each map $\sigma\colon \alpha \to \beta$ in $\mathcal{C}$ from α to β will take the generating copy of X at α to the injection into the coproduct $\mathbf{F}^{\alpha}_{X}(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X$ indexed by the element σ of $\mathcal{C}(\alpha,\beta)$.

Note that $\mathbf{F}^{\alpha}_{X}(\alpha)$ will generally consists of more than just a single copy of X. It will be a coproduct, indexed by $\mathcal{C}(\alpha,\alpha)$, of copies of X; the generating copy of X is the summand of $\mathbf{F}^{\alpha}_{X}(\alpha) = \coprod_{\mathcal{C}(\alpha,\alpha)} X$ indexed by the identity map of α.

Suppose, now, that there is a map $\tau\colon \beta \to \gamma$ in $\mathcal{C}$; that must induce a map $\tau_*\colon \mathbf{F}^{\alpha}_{X}(\beta) \to \mathbf{F}^{\alpha}_{X}(\gamma)$. Since $\mathbf{F}^{\alpha}_{X}(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X$, we will describe what τ_* does to each summand. If $\sigma\colon \alpha \to \beta$ is an element of $\mathcal{C}(\alpha,\beta)$, then τ_* on the summand indexed by σ is the injection into the coproduct $\mathbf{F}^{\alpha}_{X}(\gamma) = \coprod_{\mathcal{C}(\alpha,\gamma)} X$ of the summand indexed by the composition $\tau\circ\sigma\colon \alpha \to \gamma$.

Definition 6.1. Let $\mathcal{C}$ be a small category and let α be an object of $\mathcal{C}$. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, the *free* $\mathcal{C}$*-diagram in* $\mathcal{M}$ *generated by* X *at* α is the diagram $\mathbf{F}^{\alpha}_{X}$ such that for every object β of $\mathcal{C}$

$$\mathbf{F}^{\alpha}_{X}(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X$$

and such that if $\sigma\colon \beta \to \gamma$ is a map in $\mathcal{C}$, then

$$\mathbf{F}^{\alpha}_{X}(\sigma) = \sigma_*\colon \mathbf{F}^{\alpha}_{X}(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X \longrightarrow \mathbf{F}^{\alpha}_{X}(\gamma) = \coprod_{\mathcal{C}(\alpha,\gamma)} X$$

is the map that, on the summand of $\mathbf{F}^{\alpha}_{X}(\beta)$ indexed by $\tau\colon \alpha \to \beta$, is the injection into the coproduct $\mathbf{F}^{\alpha}_{X}(\gamma)$ of the summand indexed by the composition $\sigma\circ\tau\colon \alpha \to \gamma$.

Example 6.2. Let $\mathcal{C}$ be the category $b \xleftarrow{\gamma} a \xrightarrow{\delta} c$ with three objects $\{a,b,c\}$ and two non-identity maps $\gamma\colon a \to b$ and $\delta\colon a \to c$. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, then $\mathbf{F}^{a}_{X}$ is the diagram $X \xleftarrow{\gamma_*} X \xrightarrow{\delta_*} X$, in which both γ_* and δ_* are the identity map. This is because there are three maps in $\mathcal{C}$ with domain a: γ, δ, and 1_a, and so there are a total of three copies of X in the entire diagram.

Example 6.3. Let $\mathcal{C}$ be the category $b \xleftarrow{\gamma} a \xrightarrow{\delta} c$ with three objects $\{a,b,c\}$ and two non-identity maps $\gamma\colon a \to b$ and $\delta\colon a \to c$. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, then $\mathbf{F}^{b}_{X}$ is the diagram $X \xleftarrow{\gamma_*} \emptyset \xrightarrow{\delta_*} \emptyset$, where $\emptyset$ is the initial object of $\mathcal{M}$. This is because there is only one map in $\mathcal{C}$ with domain b, namely 1_b, and so there is only one copy of X in the entire diagram.

Example 6.4. Let G be a discrete group, which we view as a category with a single object, which we will call α. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, then $\mathbf{F}^{\alpha}_{X}$ is the object $\coprod_{G} X$, the coproduct of one copy of X for each element of G, because the maps in our indexing category are the elements of the group G, and they all have the object α as their domain. The elements of G act by permuting the copies of X: the element g of G takes the summand indexed by an element h of G to the injection into the coproduct of the summand indexed by gh.

Example 6.5. Let $\mathcal{D}$ be the category

$$\alpha \underset{\delta}{\overset{\gamma}{\rightrightarrows}} \beta$$

with two objects $\{\alpha, \beta\}$ and two non-identity maps $\gamma\colon \alpha \to \beta$ and $\delta\colon \alpha \to \beta$. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, then $\mathbf{F}^{\alpha}_{X}$ is the diagram

$$X \underset{\delta_*}{\overset{\gamma_*}{\rightrightarrows}} X \amalg X$$

in which γ_* is the injection of the first summand and δ_* is the injection of the second summand.

6.2. **Free diagrams.**

Definition 6.6. Let $\mathcal{C}$ be a small category. If $\mathcal{M}$ is a category, then a *free* $\mathcal{C}$*-diagram in* $\mathcal{M}$ is a diagram that is isomorphic to a coproduct of free diagrams generated by an object (see Definition 6.1).

Remark 6.7. A free $\mathcal{C}$-diagram in $\mathcal{M}$ may be isomorphic to a coproduct of free diagrams generated by an object in more then one way. For example, if A and B are disjoint sets, $\mathcal{C}$ is a small category, and α is an object of $\mathcal{C}$, then $\mathbf{F}^{\alpha}_{A\cup B}$ is isomorphic to $\mathbf{F}^{\alpha}_{A} \amalg \mathbf{F}^{\alpha}_{B}$.

Example 6.8. The diagram of sets $A \to B$ is free if and only if the map $A \to B$ is an inclusion.

Example 6.9. The diagram of sets $A \to C \leftarrow B$ is free if and only if the maps $A \to C$ and $B \to C$ are inclusions with disjoint images in C.

Example 6.10. The diagram of sets $A_1 \to A_2 \to A_3 \to \cdots$ is free if and only if all of the maps in the diagram are inclusions.

Example 6.11. The diagram of sets $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$ is free if and only if all of the maps are inclusions and the inverse limit of the diagram is empty.

Example 6.12. If G is a discrete group viewed as a category with a single object, then a G-diagram of sets is free if and only if it is what is classically called a free G-set.

Example 6.13. If $\mathcal{C}$ is a small category and $\boldsymbol{P}$ is the constant diagram of sets at a single point, then $\boldsymbol{P}$ is free if and only if each connected component of $\mathcal{C}$ has an initial object.

6.3. **Mapping properties of free diagrams.** Let $\mathcal{C}$ be a small category and let α be an object of $\mathcal{C}$. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, then $\mathbf{F}^{\alpha}_{X}$, the free $\mathcal{C}$-diagram generated by X at α, was constructed as "the freest diagram having a copy of X at α" (see Definition 6.1); in this section, we explain exactly what that means.

Suppose that $\boldsymbol{Y}$ is another $\mathcal{C}$-diagram in $\mathcal{M}$, and suppose that we have a map $g\colon X \to \boldsymbol{Y}_{\alpha}$. We will show that this extends uniquely to a map of diagrams $\hat{g}\colon \mathbf{F}^{\alpha}_{X} \to \boldsymbol{Y}$. We will refer to the summand of $\mathbf{F}^{\alpha}_{X} = \coprod_{\mathcal{C}(\alpha,\alpha)} X$ indexed by the identity map 1_α as the *generating copy* of X. Thus, we will be showing that every map $g\colon X \to \boldsymbol{Y}_\alpha$ determines a unique map of diagrams $\mathbf{F}^\alpha_X \to \boldsymbol{Y}$ that on the generating copy of X is the map g.

We begin by showing that there is at most one such extension $\hat{g}$. For each object β of $\mathcal{C}$ we have $\mathbf{F}^\alpha_X(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X$. Thus, each summand of $\mathbf{F}^\alpha_X(\beta)$ is indexed by a map $\sigma\colon \alpha \to \beta$, and $\mathbf{F}^\alpha_X(\sigma) = \sigma_*\colon \mathbf{F}^\alpha_X(\alpha) \to \mathbf{F}^\alpha_X(\beta)$ takes the generating copy of X isomorphically to that summand; thus, since the square

$$\begin{array}{ccc} \mathbf{F}^\alpha_X(\alpha) & \xrightarrow{\hat{g}_\alpha} & \boldsymbol{Y}_\alpha \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow{\scriptstyle \sigma_*} \\ \boldsymbol{F}^\alpha_X(\beta) & \xrightarrow[\hat{g}_\beta]{} & \boldsymbol{Y}_\beta \end{array}$$

must commute, the map $\hat{g}\colon \mathbf{F}^\alpha_X(\beta) \to \boldsymbol{Y}_\beta$ on that summand must equal the composition $X \xrightarrow{g} \boldsymbol{Y}_\alpha \xrightarrow{\sigma_*} \boldsymbol{Y}_\beta$.

To see that this definition does define $\hat{g}$ as a map of diagrams, let $\tau\colon \beta \to \gamma$ be a map in $\mathcal{C}$; we must show that the diagram

$$\begin{array}{ccc} \mathbf{F}^\alpha_X(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X & \xrightarrow{\hat{g}_\beta} & \boldsymbol{Y}_\beta \\ {\scriptstyle \tau_*}\downarrow & & \downarrow{\scriptstyle \tau_*} \\ \mathbf{F}^\alpha_X(\gamma) = \coprod_{\mathcal{C}(\alpha,\gamma)} X & \xrightarrow[\hat{g}_\gamma]{} & \boldsymbol{Y}_\gamma \end{array}$$

commutes. If $\sigma\colon \alpha \to \beta$ is a map in $\mathcal{C}$, then on the summand of $\mathbf{F}^\alpha_X(\beta)$ indexed by σ,

- τ_* is the injection into $\mathbf{F}^\alpha_X(\gamma)$ of the summand indexed by the composition $\tau\sigma$, and $\hat{g}_\gamma$ on that summand is the composition $X \xrightarrow{g} \boldsymbol{Y}_\alpha \xrightarrow{(\tau\sigma)_*} \boldsymbol{Y}_\gamma$, while
- $\hat{g}_\beta$ on that summand is the composition $X \xrightarrow{g} \boldsymbol{Y}_\alpha \xrightarrow{\sigma_*} \boldsymbol{Y}_\beta$, and the composition of that with τ_* is the composition $X \xrightarrow{g} \boldsymbol{Y}_\alpha \xrightarrow{\sigma_*} \boldsymbol{Y}_\beta \xrightarrow{\tau_*} \boldsymbol{Y}_\gamma$.

Since $\boldsymbol{Y}$ is a functor, $(\tau\sigma)_* = \tau_*\sigma_*\colon \boldsymbol{Y}_\alpha \to \boldsymbol{Y}_\gamma$, and so the square commutes.

Thus, we have proved the following theorem:

Theorem 6.14. *Let $\mathcal{C}$ be a small category and let α be an object of $\mathcal{C}$. If $\mathcal{M}$ is a category and X is an object of $\mathcal{M}$, then for every $\mathcal{C}$-diagram $\boldsymbol{Y}$ in $\mathcal{M}$ there is a natural isomorphism*

$$\phi\colon \mathcal{M}^{\mathcal{C}}(\mathbf{F}^\alpha_X, \boldsymbol{Y}) \longrightarrow \mathcal{M}(X, \boldsymbol{Y}_\alpha)$$

that takes a map of diagrams $\hat{g}\colon \mathbf{F}^\alpha_X \to \boldsymbol{Y}$ to the map $g\colon X \to \boldsymbol{Y}_\alpha$ that is the composition

$$X \xrightarrow{i_{(1_\alpha)}} \mathbf{F}^\alpha_X(\alpha) = \coprod_{\mathcal{C}(\alpha,\alpha)} X \xrightarrow{\hat{g}_\alpha} \boldsymbol{Y}_\alpha$$

(where $i_{(1_\alpha)}$ is the injection into the coproduct of the summand indexed by the identity map 1_α).

Proof. The map ϕ is the inverse of the map $\psi\colon \mathcal{M}(X, \boldsymbol{Y}_\alpha) \to \mathcal{M}^{\mathcal{C}}(\mathbf{F}^\alpha_X, \boldsymbol{Y})$ that takes a map $g\colon X \to \boldsymbol{Y}_\alpha$ to the map $\hat{g}\colon \mathbf{F}^\alpha_X \to \boldsymbol{Y}$ such that, for each object β of $\mathcal{C}$,

$\hat{g}_\beta \colon \mathbf{F}^\alpha_X(\beta) = \coprod_{\mathcal{C}(\alpha,\beta)} X \to \boldsymbol{Y}_\beta$ is the map that on the summand of $\mathbf{F}^\alpha_X(\beta)$ indexed by $\sigma\colon \alpha \to \beta$ is the composition $X \xrightarrow{g} \boldsymbol{Y}_\alpha \xrightarrow{\sigma_*} \boldsymbol{Y}_\beta$. □

6.4. **Generating cofibrations and free cell complexes.** Part of the structure of a cofibrantly generated model category $\mathcal{M}$ is a set I of maps called the *generating cofibrations*. These maps are cofibrations in $\mathcal{M}$, and every cofibration in $\mathcal{M}$ is a retract of a map constructed by using the generating cofibrations to "enlarge" a given object by creating a "relative cell complex".

If $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{C}$ is a small category, then there is a cofibrantly generated model category structure on $\mathcal{M}^\mathcal{C}$, the category of $\mathcal{C}$-diagrams in $\mathcal{M}$, and the generating cofibrations for $\mathcal{M}^\mathcal{C}$ are the maps of free diagrams constructed from the generating cofibrations of $\mathcal{M}$. These are cofibrations in $\mathcal{M}^\mathcal{C}$, and every cofibration in $\mathcal{M}^\mathcal{C}$ is a retract of a map constructed by using these to "enlarge" a given diagram by creating a "relative free cell complex". Note that, strictly speaking, since $\mathcal{M}^\mathcal{C}$ is itself a cofibrantly generated model category, the maps of free diagrams constructed using the generating cofibrations of $\mathcal{M}$ are the generating cofibrations of $\mathcal{M}^\mathcal{C}$, and so we could call these maps plain old "relative cell complexes", but we use the phrase "relative free cell complexes" to emphasize that we've just passed from a cofibrantly generated model category $\mathcal{M}$ to some category of diagrams $\mathcal{M}^\mathcal{C}$ over $\mathcal{M}$.

Definition 6.15. If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then *attaching a cell* to an object X is defined as

- choosing an element $i\colon A \to B$ of I,
- choosing a map $f\colon A \to X$, and
- constructing the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array} .$$

The object Y is then said to be obtained from X by attaching a cell. The terminology comes from the category of topological spaces, in which the generating cofibrations are the inclusions $S^{n-1} \to D^n$ for $n \geq 0$; in this case, "attaching a cell" has exactly the classical meaning of that phrase.

Example 6.16. The usual model category structure on Top, the category of topological spaces, has as generating cofibrations the maps $\{S^{n-1} \to D^n\}_{n\geq 0}$. Thus, "attaching a cell" as defined above is exactly what is classically meant by attaching a cell. Note that a relative cell complex in our case is not exactly what is classically called a relative CW-complex, because when we attach multiple cells, there is no restriction that the attaching map of a cell factor through lower dimensional cells.

Example 6.17. The usual model category structure on SS, the category of simplicial sets, has as generating cofibrations the maps $\{\partial\Delta[n] \to \Delta[n]\}_{n\geq 0}$. Thus, "attaching a cell" to a simplicial set X consists of constructing a pushout diagram

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

and the simplicial set Y is then said to be obtained from X by attaching a cell.

Definition 6.18. If $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I, then a *relative cell complex* in $\mathcal{M}$ is a map $X \to Y$ that can be constructed by a well ordered process of attaching cells. A *cell complex* is an object for which the map from the initial object is a relative cell complex.

Example 6.19. In the category of topological spaces, a CW-complex is an example of a cell complex, but not all cell complexes are CW-complexes. This is because in a CW-complex the attaching map of a cell factors through the subcomplex of lower dimensional cells, but this is not required for a cell complex.

Example 6.20. In the category of simplicial sets, every simplicial set is a cell complex, since it can be built from the empty simplicial set by (a possibly infinite process of) attaching nondegenerate simplices one at a time. Every inclusion of simplicial sets is a relative cell complex, for a similar reason.

Fixme: Finish this section.

7. Coends and ends

Lemma 7.1. *A coequalizer is an epimorphism and an equalizer is a monomorphism.*

Proof. We will prove that a coequalizer is an epimorphism; the proof that an equalizer is a monomorphism is dual to that. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{p} C$$

be a coequalizer diagram. If $r\colon C \to W$ and $s\colon C \to W$ are maps such that $rp = sp$, then (since $pf = pg$) we have $rpf = rpg$, and so rp coequalizes f and g. Thus, both r and s are factorizations through C of the map sp, and the uniqueness part of the definition of a coequalizer implies that $r = s$. □

Definition 7.2. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ and $\boldsymbol{K}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, then the *coend* $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is defined to be the object of $\mathcal{M}$ that is the coequalizer of the maps

$$\coprod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_{\alpha'} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha \tag{7.3}$$

where

- the map ϕ on the summand $\boldsymbol{X}_\alpha \otimes \boldsymbol{K}_{\alpha'}$ indexed by $(\sigma\colon \alpha \to \alpha') \in \mathcal{C}$ is the composition of the map

$$\sigma_* \otimes 1_{\boldsymbol{K}_{\alpha'}}\colon \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_{\alpha'} \longrightarrow \boldsymbol{X}_{\alpha'} \otimes \boldsymbol{K}_{\alpha'}$$

  (where $\sigma_*\colon \boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$) with the natural injection into the coproduct, and

- the map ψ on that same summand is the composition of the map
$$1_{\boldsymbol{X}_\alpha} \otimes \sigma^* \colon \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_{\alpha'} \longrightarrow \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha$$
(where $\sigma^* \colon \boldsymbol{K}_{\alpha'} \to \boldsymbol{K}_\alpha$) with the natural injection into the coproduct.

In the notation of [8, pages 222–223], $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} = \int^\alpha \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha$.

Example 7.4. Let K be a simplicial set and let ΔK be the category of simplices of K (see Definition 3.5). If $P \colon (\Delta K)^{\mathrm{op}} \to \mathrm{SS}$ is the $(\Delta K)^{\mathrm{op}}$-diagram of simplicial sets that is a single point for every object of $(\Delta K)^{\mathrm{op}}$ and $G \colon \Delta K \to \mathrm{SS}$ is the functor of Proposition 3.6, then the coend $G \otimes_{\Delta K} P$ is naturally isomorphic to K.

Example 7.5. Let K be a simplicial set and let ΔK be the category of simplices of K (see Definition 3.5). If $F \colon \Delta K \to \mathrm{Top}$ takes an n-simplex of K to $\big|\Delta[n]\big|$ and $P \colon (\Delta K)^{\mathrm{op}} \to \mathrm{SS}$ takes every object of $(\Delta K)^{\mathrm{op}}$ to a point, then the coend $F \otimes_{\Delta K} P$ is the geometric realization of K.

Lemma 7.6. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, $\boldsymbol{K}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, and either*

- *$\boldsymbol{X}$ is the constant diagram at the initial object of $\mathcal{M}$ or*
- *$\boldsymbol{K}$ is the constant diagram at the empty simplicial set,*

then the coend $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is an initial object of $\mathcal{M}$.

Proof. Definition 7.2 and Lemma 4.2 imply that, in either case, $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is the coequalizer of a pair of maps between initial objects of $\mathcal{M}$, and so it is an initial object of $\mathcal{M}$. □

Definition 7.7. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ and $\boldsymbol{K}$ is a $\mathcal{C}$-diagram of simplicial sets, then the *end* $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is defined to be the object of $\mathcal{M}$ that is the equalizer of the maps

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} (\boldsymbol{X}_\alpha)^{\boldsymbol{K}_\alpha} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma \colon \alpha \to \alpha') \in \mathcal{C}} (\boldsymbol{X}_{\alpha'})^{\boldsymbol{K}_\alpha} \tag{7.8}$$

where

- the projection of the map ϕ on the factor $(\boldsymbol{X}_{\alpha'})^{\boldsymbol{K}_\alpha}$ indexed by $(\sigma \colon \alpha \to \alpha') \in \mathcal{C}$ is the composition of a natural projection from the product with the map
$$\sigma_*^{1_{\boldsymbol{K}_\alpha}} \colon (\boldsymbol{X}_\alpha)^{\boldsymbol{K}_\alpha} \longrightarrow (\boldsymbol{X}_{\alpha'})^{\boldsymbol{K}_\alpha}$$
(where $\sigma_* \colon \boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$) and
- the projection of the map ψ on that same factor is the composition of a natural projection from the product with the map
$$(1_{\boldsymbol{X}_{\alpha'}})^{\sigma_*} \colon (\boldsymbol{X}_{\alpha'})^{\boldsymbol{K}_{\alpha'}} \longrightarrow (\boldsymbol{X}_{\alpha'})^{\boldsymbol{K}_\alpha}$$
(where $\sigma_* \colon \boldsymbol{K}_\alpha \to \boldsymbol{K}_{\alpha'}$.

In the notation of [8, pp. 218–223] or [2, p. 329], $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) = \int_\alpha (\boldsymbol{X}_\alpha)^{\boldsymbol{K}_\alpha}$.

Example 7.9. Let $\mathcal{C}$ be a small category and let $\boldsymbol{K}$ and $\boldsymbol{L}$ be $\mathcal{C}$-diagrams of simplicial sets. If α is an object of $\mathcal{C}$, then $(\boldsymbol{L}_\alpha)^{\boldsymbol{K}_\alpha}$ is the set of maps of simplicial sets from $\boldsymbol{K}_\alpha$ to $\boldsymbol{L}_\alpha$, and so an element of $\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} (\boldsymbol{L}_\alpha)^{\boldsymbol{K}_\alpha}$ is a choice, for each object α of $\mathcal{C}$, of a map of simplicial sets from $\boldsymbol{K}_\alpha$ to $\boldsymbol{L}_\alpha$. Such a choice equalizes the projections of ϕ and ψ onto the factor $(\boldsymbol{L}_{\alpha'})^{\boldsymbol{K}_\alpha}$ indexed by $\sigma\colon \alpha \to \alpha'$ when the square

$$\begin{array}{ccc} \boldsymbol{K}_\alpha & \longrightarrow & \boldsymbol{L}_\alpha \\ {\scriptstyle \boldsymbol{K}(\sigma)}\downarrow & & \downarrow{\scriptstyle \boldsymbol{L}(\sigma)} \\ \boldsymbol{K}_{\alpha'} & \longrightarrow & \boldsymbol{L}_{\alpha'} \end{array}$$

commutes, and so it equalizes ϕ and ψ when that collection of maps commutes with all the morphisms of $\mathcal{C}$. Thus, the end $\hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{L})$ is $\mathrm{SS}^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{L})$, the set of maps of diagrams from $\boldsymbol{K}$ to $\boldsymbol{L}$.

Lemma 7.10. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, $\boldsymbol{K}$ is a $\mathcal{C}$-diagram of simplicial sets, and either*

- *$\boldsymbol{X}$ is the constant diagram at the terminal object of $\mathcal{M}$ or*
- *$\boldsymbol{K}$ is the constant diagram at the empty simplicial set,*

then the end $\hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{X})$ is a terminal object of $\mathcal{M}$.

Proof. Definition 7.2 and Lemma 4.2 imply that, in either case, $\hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{X})$ is the equalizer of a pair of maps between terminal objects of $\mathcal{M}$, and so it is a terminal object of $\mathcal{M}$. □

7.1. **Adjointness.**

Proposition 7.11. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, $\boldsymbol{K}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, and Z is an object of $\mathcal{M}$, then there is a natural isomorphism of sets*

$$\mathcal{M}(\boldsymbol{X}\otimes_{\mathcal{C}}\boldsymbol{K}, Z) \approx \mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{X}, Z)\bigr)$$

(where $\boldsymbol{X}\otimes_{\mathcal{C}}\boldsymbol{K}$ is as in Definition 7.2).

(2) *If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, $\boldsymbol{K}$ is a $\mathcal{C}$-diagram of simplicial sets, and W is an object of $\mathcal{M}$, then there is a natural isomorphism of sets*

$$\mathcal{M}(W, \hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{X})) \approx \mathrm{SS}^{\mathcal{C}}\bigl(\boldsymbol{K}, \mathrm{Map}(W, \boldsymbol{X})\bigr)$$

(where $\hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{X})$ is as in Definition 7.7).

Proof. We will prove part 1; the proof of part 2 is similar.

The object $\boldsymbol{X}\otimes_{\mathcal{C}}\boldsymbol{K}$ is defined as the colimit of (7.3), and so $\mathcal{M}(\boldsymbol{X}\otimes_{\mathcal{C}}\boldsymbol{K}, Z)$ is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \mathcal{M}(\boldsymbol{X}_\alpha\otimes\boldsymbol{K}_\alpha, Z) \underset{\psi^*}{\overset{\phi^*}{\rightrightarrows}} \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathcal{M}(\boldsymbol{X}_\alpha\otimes\boldsymbol{K}_{\alpha'}, Z) \ .$$

Proposition 4.1 implies that this limit is naturally isomorphic to the limit of the diagram

$$\prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \mathrm{SS}\bigl(\boldsymbol{K}_\alpha, \mathrm{Map}(\boldsymbol{X}_\alpha, Z)\bigr) \underset{\psi^*}{\overset{\phi^*}{\rightrightarrows}} \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \mathrm{SS}\bigl(\boldsymbol{K}_{\alpha'}, \mathrm{Map}(\boldsymbol{X}_\alpha, Z)\bigr) \ ,$$

which is the definition of $\mathrm{SS}^{\mathcal{C}^{\mathrm{op}}}\bigl(\boldsymbol{K}, \mathrm{Map}(\boldsymbol{X}, Z)\bigr)$ (see Example 7.9). □

Lemma 7.12. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{A} \to \boldsymbol{B}$ is a map of $\mathcal{C}$-diagrams in $\mathcal{M}$, $\boldsymbol{K} \to \boldsymbol{L}$ is a map of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, and $X \to Y$ is a map of objects in $\mathcal{M}$, then the dotted arrow exists in every solid arrow diagram of the form*

$$\begin{array}{ccc} \boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{L} \amalg_{\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{K}} \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{K} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{L} & \longrightarrow & Y \end{array} \tag{7.13}$$

if and only if the dotted arrow exists in every solid arrow diagram of the form

$$\begin{array}{ccc} \boldsymbol{K} & \longrightarrow & \mathrm{Map}(\boldsymbol{B}, X) \\ \downarrow & \nearrow & \downarrow \\ \boldsymbol{L} & \longrightarrow & \mathrm{Map}(\boldsymbol{A}, X) \times_{\mathrm{Map}(\boldsymbol{A}, Y)} \mathrm{Map}(\boldsymbol{B}, Y) \end{array} . \tag{7.14}$$

(2) *If $\boldsymbol{X} \to \boldsymbol{Y}$ is a map of $\mathcal{C}$-diagrams in $\mathcal{M}$, $\boldsymbol{K} \to \boldsymbol{L}$ is a map of $\mathcal{C}$-diagrams of simplicial sets, and $A \to B$ is a map of objects in $\mathcal{M}$, then the dotted arrow exists in every solid arrow diagram of the form*

$$\begin{array}{ccc} A & \longrightarrow & (\boldsymbol{X})^{\boldsymbol{L}} \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & (\boldsymbol{X})^{\boldsymbol{K}} \times_{(\boldsymbol{Y})^{\boldsymbol{K}}} (\boldsymbol{Y})^{\boldsymbol{L}} \end{array}$$

if and only if the dotted arrow exists in every solid arrow diagram of the form

$$\begin{array}{ccc} \boldsymbol{K} & \longrightarrow & \mathrm{Map}(B, \boldsymbol{X}) \\ \downarrow & \nearrow & \downarrow \\ \boldsymbol{L} & \longrightarrow & \mathrm{Map}(A, \boldsymbol{X}) \times_{\mathrm{Map}(A, \boldsymbol{Y})} \mathrm{Map}(B, \boldsymbol{Y}) \end{array} .$$

Proof. We will prove part 1; the proof of part 2 is similar.

Proposition 7.11 gives a one-to-one correspondence between solid arrow diagrams as in Diagram 7.13 and solid arrow diagrams as in Diagram 7.14, under which a dotted arrow as in Diagram 7.13 corresponds to a dotted arrow as in Diagram 7.14. □

Theorem 7.15. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $j\colon \boldsymbol{A} \to \boldsymbol{B}$ is an objectwise cofibration of $\mathcal{C}$-diagrams in $\mathcal{M}$ and $i\colon \boldsymbol{K} \to \boldsymbol{L}$ is a cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, then the pushout corner map*

$$\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{L} \amalg_{\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{K}} \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{K} \longrightarrow \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{L}$$

is a cofibration in $\mathcal{M}$ that is also a weak equivalence (and, thus, a trivial cofibration) if either j is an objectwise weak equivalence or i is a weak equivalence.

(2) *If* $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ *is an objectwise fibration of* $\mathcal{C}$*-diagrams in* $\mathcal{M}$ *and* $i\colon \boldsymbol{K} \to \boldsymbol{L}$ *is a cofibration of* $\mathcal{C}$*-diagrams of simplicial sets, then the pullback corner map*

$$\hom^{\mathcal{C}}(\boldsymbol{L}, \boldsymbol{X}) \longrightarrow \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) \times_{\hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{Y})} \hom^{\mathcal{C}}(\boldsymbol{L}, \boldsymbol{Y})$$

is a fibration in $\mathcal{M}$ *that is also a weak equivalence (and, thus, a trivial fibration) if either* p *is an objectwise weak equivalence or* i *is a weak equivalence.*

Proof. We will prove part 1; the proof of part 2 is similar.

Note first that we are not assuming that there is any model category structure on the category of $\mathcal{C}$-diagrams in $\mathcal{M}$, and the map $j\colon \boldsymbol{A} \to \boldsymbol{B}$ is just assumed to be an objectwise cofibration. We do, however, use the Bousfield-Kan model category structure on the category of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (in which the weak equivalences are the objectwise weak equivalences and the fibrations are the objectwise fibrations; see Theorem 5.1), and we are assuming that the map $i\colon \boldsymbol{K} \to \boldsymbol{L}$ is a cofibration in this model category; this is a much stronger assumption than just assuming that i is an objectwise cofibration.

To show that the pushout corner map is a cofibration we will show that it has the left lifting property with respect to all trivial fibrations in $\mathcal{M}$. Let $p\colon X \to Y$ be a trivial fibration in $\mathcal{M}$ and assume that we have the solid arrow diagram in Diagram 7.13. Lemma 7.12 implies that the dotted arrow exists in Diagram 7.13 if and only if the dotted arrow exists in Diagram 7.14.

Proposition 4.1 implies that, for every object α of $\mathcal{C}$, the map

$$\mathrm{Map}(\boldsymbol{B}_\alpha, X) \longrightarrow \mathrm{Map}(\boldsymbol{A}_\alpha, X) \times_{\mathrm{Map}(\boldsymbol{A}_\alpha, Y)} \mathrm{Map}(\boldsymbol{B}_\alpha, Y)$$

is a trivial fibration in $\mathcal{M}$, and so the map

$$\mathrm{Map}(\boldsymbol{B}, X) \longrightarrow \mathrm{Map}(\boldsymbol{A}, X) \times_{\mathrm{Map}(\boldsymbol{A}, Y)} \mathrm{Map}(\boldsymbol{B}, Y)$$

is an objectwise trivial fibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets. Thus, this map is a trivial fibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (see Theorem 5.1) and the map $i\colon \boldsymbol{K} \to \boldsymbol{L}$ is a cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, and so the dotted arrow exists in Diagram 7.14, and so the pushout corner map is a cofibration.

Suppose now that $j\colon \boldsymbol{A} \to \boldsymbol{B}$ is also an objectwise weak equivalence, so that it is an objectwise trivial cofibration. To show that the pushout corner map is a trivial cofibration, we will show that it has the left lifting property with respect to all fibrations. Let $p\colon X \to Y$ be a fibration in $\mathcal{M}$ and assume that we have the solid arrow Diagram 7.13. Lemma 7.12 again implies that the dotted arrow exists in Diagram 7.13 if and only if the dotted arrow exists in Diagram 7.14. Proposition 4.1 implies that, for every object α of $\mathcal{C}$, the map

$$\mathrm{Map}(\boldsymbol{B}_\alpha, X) \longrightarrow \mathrm{Map}(\boldsymbol{A}_\alpha, X) \times_{\mathrm{Map}(\boldsymbol{A}_\alpha, Y)} \mathrm{Map}(\boldsymbol{B}_\alpha, Y)$$

is a trivial fibration in $\mathcal{M}$, and so the map

$$\mathrm{Map}(\boldsymbol{B}, X) \longrightarrow \mathrm{Map}(\boldsymbol{A}, X) \times_{\mathrm{Map}(\boldsymbol{A}, Y)} \mathrm{Map}(\boldsymbol{B}, Y)$$

is an objectwise trivial fibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets. Thus, this map is a trivial fibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (see Theorem 5.1) and the map $i\colon \boldsymbol{K} \to \boldsymbol{L}$ is a cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, and so the dotted arrow exists in Diagram 7.14, and so the pushout corner map is a trivial cofibration.

Finally, assume that the map $i\colon \boldsymbol{K} \to \boldsymbol{L}$ is a weak equivalence, so that it is a trivial cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets. To show that the pushout corner map is a trivial cofibration we will show that it has the left lifting property with respect to all fibrations. Let $p\colon X \to Y$ be a fibration in $\mathcal{M}$ and assume that we have the solid arrow Diagram 7.13. Lemma 7.12 implies that the dotted arrow exists in Diagram 7.13 if and only if the dotted arrow exists in Diagram 7.14. Proposition 4.1 implies that, for every object α of $\mathcal{C}$, the map

$$\mathrm{Map}(\boldsymbol{B}_\alpha, X) \longrightarrow \mathrm{Map}(\boldsymbol{A}_\alpha, X) \times_{\mathrm{Map}(\boldsymbol{A}_\alpha, Y)} \mathrm{Map}(\boldsymbol{B}_\alpha, Y)$$

is a fibration in $\mathcal{M}$, and so the map

$$\mathrm{Map}(\boldsymbol{B}, X) \longrightarrow \mathrm{Map}(\boldsymbol{A}, X) \times_{\mathrm{Map}(\boldsymbol{A}, Y)} \mathrm{Map}(\boldsymbol{B}, Y)$$

is an objectwise fibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets. Thus, this map is a fibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets (see Theorem 5.1) and the map $i\colon \boldsymbol{K} \to \boldsymbol{L}$ is a trivial cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, and so the dotted arrow exists in Diagram 7.14, and so the pushout corner map is a trivial cofibration. □

Corollary 7.16. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $F\colon \boldsymbol{K} \to \boldsymbol{L}$ is a cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets and $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ that is objectwise cofibrant, then the induced map of coends $1_{\boldsymbol{X}} \otimes_{\mathcal{C}} F\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{L}$ is a cofibration.*
(2) *If $F\colon \boldsymbol{K} \to \boldsymbol{L}$ is a cofibration of $\mathcal{C}$-diagrams of simplicial sets and $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ that is objectwise fibrant, then the induced map of ends $\hom^{\mathcal{C}}(F,1)\colon \hom^{\mathcal{C}}(\boldsymbol{L},\boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{K},\boldsymbol{X})$ is a fibration.*

Proof. We will prove part 1; the proof of part 2 is similar.

Let $\boldsymbol{A}$ be the constant diagram at the initial object of $\mathcal{M}$ and let $\boldsymbol{B} = \boldsymbol{X}$; the unique map $\boldsymbol{A} \to \boldsymbol{B}$ is then an objectwise cofibration, and so Theorem 7.15 implies that the pushout corner map is a cofibration. Since $\boldsymbol{A}$ is the constant diagram at the initial object, both $\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{K}$ and $\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{L}$ are the initial object of $\mathcal{M}$ (see Lemma 7.6), and so the pushout corner map is isomorphic to the map $1_{\boldsymbol{X}} \otimes_{\mathcal{C}} F\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{L}$. □

Corollary 7.17. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{L}$ is a cofibrant $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets and $j\colon \boldsymbol{A} \to \boldsymbol{B}$ is an objectwise cofibration of $\mathcal{C}$-diagrams in $\mathcal{M}$, then the map $\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{L} \to \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{L}$ is a cofibration in $\mathcal{M}$ that is a weak equivalence if j is an objectwise weak equivalence.*
(2) *If $\boldsymbol{L}$ is a cofibrant $\mathcal{C}$-diagram of simplicial sets and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise fibration of $\mathcal{C}$-diagrams in $\mathcal{M}$, then the map $\hom^{\mathcal{C}}(\boldsymbol{L},\boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{L},\boldsymbol{Y})$ is a fibration in $\mathcal{M}$ that is a weak equivalence if p is an objectwise weak equivalence.*

Proof. We will prove part 1; the proof of part 2 is similar.

Let $\boldsymbol{K}$ be the $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets that is the constant diagram at the empty simplicial set. The unique map $\boldsymbol{K} \to \boldsymbol{L}$ is then a cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, and so Theorem 7.15 implies that the pushout corner map is a cofibration. Since $\boldsymbol{K}$ is the constant diagram at the empty simplicial set, both $\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{K}$ and $\boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{K}$ are the initial object of $\mathcal{M}$ (see Lemma 7.6), and so the pushout corner map is isomorphic to the map $\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{L} \to \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{L}$. □

Corollary 7.18. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{L}$ is a cofibrant $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets and $\boldsymbol{X}$ is an objectwise cofibrant diagram in $\mathcal{M}$, then the coend $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{L}$ is a cofibrant object of $\mathcal{M}$.*
(2) *If $\boldsymbol{L}$ is a cofibrant $\mathcal{C}$-diagram of simplicial sets and $\boldsymbol{X}$ is an objectwise fibrant diagram in $\mathcal{M}$, then the end $\hom^{\mathcal{C}}(\boldsymbol{L}, \boldsymbol{X})$ is a fibrant object of $\mathcal{M}$.*

Proof. We will prove part 1; the proof of part 2 is similar.

Let $\boldsymbol{A}$ be the $\mathcal{C}$-diagram in $\mathcal{M}$ that is the constant diagram at the initial object of $\mathcal{M}$, let $\boldsymbol{B} = \boldsymbol{X}$, and let $j\colon \boldsymbol{A} \to \boldsymbol{B}$ be the unique map. Corollary 7.17 implies that the map $\boldsymbol{A} \otimes_{\mathcal{C}} \boldsymbol{L} \to \boldsymbol{B} \otimes_{\mathcal{C}} \boldsymbol{L}$ is a cofibration, and Lemma 7.6 implies that that is the map from the initial object of $\mathcal{M}$ to $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{L}$. □

7.2. **Homotopy invariance.** Our homotopy invariance results will be consequences of Theorem 7.15. That theorem, though, doesn't directly discuss maps that are merely objectwise weak equivalences; it discusses maps that are objectwise trivial cofibrations or objectwise trivial fibrations. The tool that extends the applicability of this is *Kenny Brown's lemma* (Lemma 7.19), which implies that a functor that takes objectwise trivial cofibrations between objectwise cofibrant objects to weak equivalences also takes all objectwise weak equivalences between objectwise cofibrant objects to weak equivalences, and a functor that takes objectwise trivial fibrations between objectwise fibrant objects to weak equivalences also takes all objectwise weak equivalences between objectwise fibrant objects to weak equivalences (see Corollary 7.20 and Corollary 7.22).

Lemma 7.19 (K. S. Brown, [4])**.** *Let $\mathcal{M}$ be a model category.*

(1) *If $g\colon X \to Y$ is a weak equivalence between cofibrant objects in $\mathcal{M}$ then there is a functorial factorization of g as $g = ji$ where i is a trivial cofibration and j is a trivial fibration that has a right inverse that is a trivial cofibration.*
(2) *If $g\colon X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{M}$ then there is a functorial factorization of g as $g = ji$ where i is a trivial cofibration that has a left inverse that is a trivial fibration and j is a trivial fibration.*

Proof. We will prove part 1; the proof of part 2 is dual.

Because there is a pushout diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \amalg Y \end{array}$$

(where $\emptyset$ is the initial object of $\mathcal{M}$), the fact that X and Y are cofibrant implies that both of the injections $X \to X \amalg Y$ and $Y \to X \amalg Y$ are cofibrations. We can then factor the map $g \amalg 1_Y \colon X \amalg Y \to Y$ as

$$X \amalg Y \xrightarrow{k} Z \xrightarrow{j} Y$$

where k is a cofibration and j is a trivial fibration, and let $i\colon X \to Z$ be the cofibration that is the composition of cofibrations

$$X \longrightarrow X \amalg Y \xrightarrow{k} Z \ .$$

Since g and j are weak equivalences, the "two out of three" property of weak equivalences implies that the cofibration $i\colon X \to Z$ is a weak equivalence, and so i is a trivial cofibration. The composition of cofibrations $Y \to X \amalg Y \to Z$ is a right inverse to the trivial fibration j, and (by the "two out of three" property) is thus also a weak equivalence, and so j has a right inverse that is a trivial cofibration. □

Corollary 7.20. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{K}$ is a cofibrant $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence of objectwise cofibrant $\mathcal{C}$-diagrams in $\mathcal{M}$, then the induced map $f_*\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}$ is a weak equivalence of cofibrant objects.*
(2) *If $\boldsymbol{K}$ is a cofibrant $\mathcal{C}$-diagram of simplicial sets and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence of objectwise fibrant $\mathcal{C}$-diagrams in $\mathcal{M}$, then the induced map $f_*\colon \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{Y})$ is a weak equivalence of fibrant objects.*

Proof. We will prove part 1; the proof of part 2 is similar.

We can use Lemma 7.19 to factor the map of diagrams $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ as $\boldsymbol{X} \xrightarrow{i} \boldsymbol{W} \xrightarrow{j} \boldsymbol{Y}$ where i is an objectwise trivial cofibration and j has a right inverse that is an objectwise trivial cofibration. Corollary 7.17 implies that the map $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{W} \otimes_{\mathcal{C}} \boldsymbol{K}$ is a weak equivalence, Corollary 7.17 together with the "two out of three" property of weak equivalences imply that the map $\boldsymbol{W} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}$ is a weak equivalence, and Corollary 7.18 implies that $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ and $\boldsymbol{Y} \otimes_{\mathcal{C}} \boldsymbol{K}$ are cofibrant. □

Corollary 7.21. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{X}$ is an objectwise cofibrant $\mathcal{C}$-diagram in $\mathcal{M}$ and $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a trivial cofibration of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, then the induced map of coends $f_*\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}'$ is a trivial cofibration.*
(2) *If $\boldsymbol{X}$ is an objectwise fibrant $\mathcal{C}$-diagram in $\mathcal{M}$ and $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a trivial cofibration of $\mathcal{C}$-diagrams of simplicial sets, then the induced map of ends $f^*\colon \hom^{\mathcal{C}}(\boldsymbol{K}', \boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is a trivial fibration.*

Proof. This follows from Theorem 7.15. □

Corollary 7.22. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{X}$ is an objectwise cofibrant $\mathcal{C}$-diagram in $\mathcal{M}$ and $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a weak equivalence of cofibrant $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets, then the induced map $f_*\colon \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}'$ is a weak equivalence of cofibrant objects in $\mathcal{M}$.*
(2) *If $\boldsymbol{X}$ is an objectwise fibrant $\mathcal{C}$-diagram in $\mathcal{M}$ and $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ is a weak equivalence of cofibrant $\mathcal{C}$-diagrams of simplicial sets, then the induced map $f^*\colon \hom^{\mathcal{C}}(\boldsymbol{K}', \boldsymbol{X}) \to \hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is a weak equivalence of fibrant objects in $\mathcal{M}$.*

Proof. We will prove part 1; the proof of part 2 is similar.

We can use Lemma 7.19 to factor the map of diagrams $f\colon \boldsymbol{K} \to \boldsymbol{K}'$ as $\boldsymbol{K} \xrightarrow{i} \boldsymbol{K}'' \xrightarrow{j} \boldsymbol{K}'$ where i is a trivial cofibration and j has a right inverse that is a trivial cofibration. Corollary 7.21 implies that the map $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K} \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}''$ is a weak equivalence, Corollary 7.21 and the "two out of three" property of weak equivalences imply that the map $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}'' \to \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}'$ is a weak equivalence, and Corollary 7.18 implies that $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ and $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}'$ are cofibrant. □

7.3. **Generating sets of maps.** We show here that if S is a set of maps in a small category $\mathcal{C}$ such that every map in $\mathcal{C}$ is a finite composition of elements of S, then you can compute ends and coends over $\mathcal{C}$ using only the maps in S (see Proposition 7.24).

Lemma 7.23. *Let $\mathcal{C}$ be a small category, let $\mathcal{M}$ be a simplicial model category, and let $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ be a $\mathcal{C}$-diagram in $\mathcal{M}$.*

(1) *If $K\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SS}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, $\sigma\colon \alpha \to \beta$ and $\tau\colon \beta \to \gamma$ are maps in $\mathcal{C}$, and $\coprod_{\omega \in \mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\omega \otimes K_\omega \to W$ is a map such that the squares*

$$\begin{array}{ccc} \boldsymbol{X}_\alpha \otimes K_\beta & \xrightarrow{1_{\boldsymbol{X}_\alpha} \otimes \sigma^*} & \boldsymbol{X}_\alpha \otimes K_\alpha \\ {\scriptstyle \sigma_* \otimes 1_{K_\beta}}\Big\downarrow & & \Big\downarrow \\ \boldsymbol{X}_\beta \otimes K_\beta & \longrightarrow & W \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \boldsymbol{X}_\beta \otimes K_\gamma & \xrightarrow{1_{\boldsymbol{X}_\beta} \otimes \tau^*} & \boldsymbol{X}_\beta \otimes K_\beta \\ {\scriptstyle \tau_* \otimes 1_{K_\gamma}}\Big\downarrow & & \Big\downarrow \\ \boldsymbol{X}_\gamma \otimes K_\gamma & \longrightarrow & W \end{array}$$

commute, then the square

$$\begin{array}{ccc} \boldsymbol{X}_\alpha \otimes K_\gamma & \xrightarrow{1_{\boldsymbol{X}_\alpha} \otimes (\tau\sigma)^*} & \boldsymbol{X}_\alpha \otimes K_\alpha \\ {\scriptstyle (\tau\sigma)_* \otimes 1_{K_\gamma}}\Big\downarrow & & \Big\downarrow \\ \boldsymbol{X}_\gamma \otimes K_\gamma & \longrightarrow & W \end{array}$$

also commutes.

(2) *If $K\colon \mathcal{C} \to \mathrm{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, $\sigma\colon \alpha \to \beta$ and $\tau\colon \beta \to \gamma$ are maps in $\mathcal{C}$, and $W \to \prod_{\omega \in \mathrm{Ob}(\mathcal{C})} (X_\omega)^{K_\omega}$ is a map such that the squares*

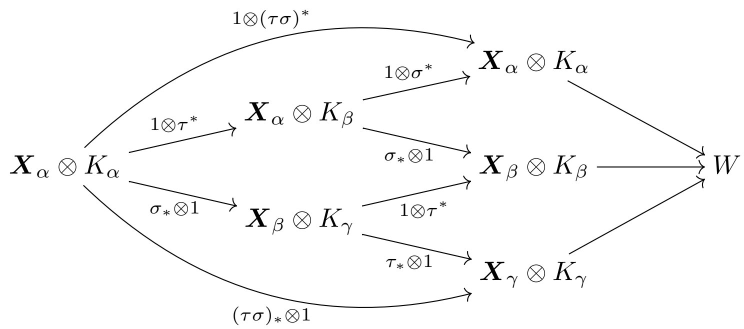
$$\begin{array}{ccc} W & \longrightarrow & (\boldsymbol{X}_\alpha)^{K_\alpha} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle (\sigma_*)^{1_{K_\alpha}}} \\ (\boldsymbol{X}_\beta)^{K_\beta} & \xrightarrow[(1_{\boldsymbol{X}_\beta})^{\sigma^*}]{} & (\boldsymbol{X}_\beta)^{K_\alpha} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W & \longrightarrow & (\boldsymbol{X}_\beta)^{K_\beta} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle (\tau_*)^{1_{K_\beta}}} \\ (\boldsymbol{X}_\gamma)^{K_\gamma} & \xrightarrow[(1_{\boldsymbol{X}_\gamma})^{\tau^*}]{} & (\boldsymbol{X}_\gamma)^{K_\beta} \end{array}$$

commute, then the square

$$\begin{array}{ccc} W & \longrightarrow & (\boldsymbol{X}_\alpha)^{K_\alpha} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle ((\tau\sigma)_*)^{1_{K_\alpha}}} \\ (\boldsymbol{X}_\gamma)^{K_\gamma} & \xrightarrow[(1_{\boldsymbol{X}_\gamma})^{(\tau\sigma)^*}]{} & (\boldsymbol{X}_\gamma)^{K_\alpha} \end{array}$$

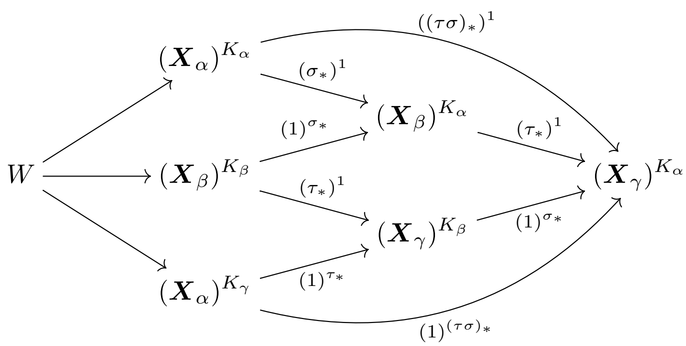
also commutes.

Proof. For part 1, we have the diagram



The two four-sided figures on the right commute by assumption, the four-sided figure to their left commutes because this is a functor of two variables, and the two three-sided figures on the upper and lower left commute because $\tau_*\sigma_* = (\tau\sigma)_*$ and $\sigma^*\tau^* = (\tau\sigma)^*$. Thus, the outer four-sided figure commutes.

For part 2, we have the diagram



The two four-sided figures on the left commute by assumption, the four-sided figure to their right commutes because this is a functor of two variables, and the two three-sided figures on the upper and lower right commute because $(\tau\sigma)_* = \tau_*\sigma_*$. Thus, the outer four-sided figure commutes. $\square$

Proposition 7.24. *Let $\mathcal{C}$ be a small category and let S be a set of maps in $\mathcal{C}$ such that every map in $\mathcal{C}$ is a finite composition of elements of S.*

(1) *If $\mathcal{M}$ is a simplicial model category, $\boldsymbol{X} \colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, and $\boldsymbol{K} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SS}$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets, then the coend $\boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{K}$ is naturally isomorphic to the coequalizer of the maps*

$$\coprod_{(\sigma\colon \alpha \to \alpha') \in S} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_{\alpha'} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\alpha \in \mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \boldsymbol{K}_\alpha$$

where the maps ϕ and ψ are as in Definition 7.2.

(2) *If $\mathcal{M}$ is a simplicial model category, $\boldsymbol{X} \colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, and $\boldsymbol{K} \colon \mathcal{C} \to \mathrm{SS}$ is a $\mathcal{C}$-diagram of simplicial sets, then the end $\hom^{\mathcal{C}}(\boldsymbol{K}, \boldsymbol{X})$ is*

naturally isomorphic to the equalizer of the maps

$$\prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} (\boldsymbol{X}_\alpha)^{\boldsymbol{K}_\alpha} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma\colon \alpha\to\alpha')\in S} (\boldsymbol{X}_\alpha)^{\boldsymbol{K}_{\alpha'}}$$

where the maps ϕ and ψ are as in Definition 7.7.

Proof. This follows from Lemma 7.23. □

Example 7.25. Example 7.4 shows how to reconstruct a simplicial set from the diagram of its simplices, and Proposition 7.24 provides an alternate proof of Proposition 3.7.

8. Homotopy colimits

Definition 8.1. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then the *homotopy colimit* of $\boldsymbol{X}$ (denoted $\operatorname{hocolim}_{\mathcal{C}} \boldsymbol{X}$, or $\operatorname{hocolim} \boldsymbol{X}$) is defined to be the coend $\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ (see Definition 2.5 and Definition 7.2), that is, $\operatorname{hocolim} \boldsymbol{X}$ is the coequalizer of the maps

$$\coprod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \tag{8.2}$$

where

- the map ϕ on the summand $\boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}$ indexed by $(\sigma\colon \alpha\to\alpha')\in\mathcal{C}$ is the composition of the map

$$\sigma_* \otimes 1_{\mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}}\colon \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} \longrightarrow \boldsymbol{X}_{\alpha'} \otimes \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}$$

  (where $\sigma_*\colon \boldsymbol{X}_\alpha \to \boldsymbol{X}_{\alpha'}$) with the injection into the coproduct, and
- the map ψ on that same summand is the composition of the map

$$1_{\boldsymbol{X}_\alpha} \otimes \sigma^*\colon \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} \longrightarrow \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$$

  (where $\sigma^*\colon \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} \to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$) with the injection into the coproduct.

Proposition 8.3. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is an objectwise cofibrant $\mathcal{C}$-diagram in $\mathcal{M}$, then* $\operatorname{hocolim} \boldsymbol{X}$ *is cofibrant.*

Proof. This follows from Corollary 7.18, Proposition 5.13, and Theorem 5.1. □

Theorem 8.4. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are objectwise cofibrant $\mathcal{C}$-diagrams in $\mathcal{M}$ and $f\colon \boldsymbol{X}\to\boldsymbol{Y}$ is an objectwise weak equivalence, then the induced map of homotopy colimits $f_*\colon \boldsymbol{X}\to\boldsymbol{Y}$ is a weak equivalence of cofibrant objects.*

Proof. This follows from Definition 8.1, Proposition 5.13, Theorem 5.1, and Corollary 7.20. □

8.1. **Decomposing homotopy colimits.** In this section, we use Proposition 3.7 to describe the simplicial sets $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ as colimits of diagrams of standard simplices. This allows us to show that the homotopy colimit of a diagram can be constructed as a coequalizer of maps between simpler objects than the ones in the definition of the homotopy colimit (see Definition 8.1 and Proposition 8.5). This will be used in Section 9 to show that the homotopy colimit can be constructed as the realization (see Definition 9.2) of the simplicial replacement (see Definition 9.1) of the diagram.

Proposition 8.5. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then* $\operatorname{hocolim}\boldsymbol{X} = \boldsymbol{X}\otimes_{\mathcal{C}}\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$ *is naturally isomorphic to the coequalizer of the maps*

(8.6)
$$\Bigl(\coprod_{\substack{(\sigma\colon\alpha\to\alpha')\in\mathcal{C}\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha\otimes\Delta[n]\Bigr)\amalg\Bigl(\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n>0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha\otimes\Delta[n-1]\Bigr)\amalg\Bigl(\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha\otimes\Delta[n+1]\Bigr)$$
$$\overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha\otimes\Delta[n]$$

where, on the first summand,

- *the map ϕ on the summand $\boldsymbol{X}_\alpha\otimes\Delta[n]$ indexed by $\bigl((\sigma\colon\alpha\to\alpha')\in\mathcal{C}, n, \bigl((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, \tau\colon\alpha'\to\alpha_n\bigr)\bigr)$ is the composition of the map $\sigma_*\otimes 1_{\Delta[n]}\colon\boldsymbol{X}_\alpha\otimes\Delta[n]\to\boldsymbol{X}_{\alpha'}\otimes\Delta[n]$ with the injection into the coproduct of the summand indexed by $\bigl(\alpha', n, \bigl((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, \tau\colon\alpha'\to\alpha_n\bigr)\bigr)$, and*
- *the map ψ on that same summand is the injection into the coproduct of the summand indexed by $\bigl(\alpha, n, \bigl((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, \tau\sigma\colon\alpha\to\alpha_n\bigr)\bigr)$*

and, on the second summand,

- *the map ϕ on the summand $\boldsymbol{X}_\alpha\otimes\Delta[n-1]$ indexed by $\bigl(\alpha, n, i, \sigma\colon\Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}\bigr)$ is the injection into the coproduct of the summand indexed by α, $(n-1)$, and the composition $\Delta[n-1]\xrightarrow{d^i}\Delta[n]\xrightarrow{\sigma}\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$, and*
- *the map ψ on that same summand is the composition of the map $1_{\boldsymbol{X}_\alpha}\otimes d^i\colon\boldsymbol{X}_\alpha\otimes\Delta[n-1]\to\boldsymbol{X}_\alpha\otimes\Delta[n]$ composed with the injection into the coproduct of the summand indexed by (α, n, σ)*

and, on the third summand,

- *the map ϕ on the summand $\boldsymbol{X}_\alpha\otimes\Delta[n+1]$ indexed by $\bigl(\alpha, n, i, \sigma\colon\Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}\bigr)$ is the injection into the coproduct of the summand indexed by α, $(n-1)$, and the composition $\Delta[n+1]\xrightarrow{s^i}\Delta[n]\xrightarrow{\sigma}\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$, and*
- *the map ψ on that same summand is the composition of the map $1_{\boldsymbol{X}_\alpha}\otimes s^i\colon\boldsymbol{X}_\alpha\otimes\Delta[n+1]\to\boldsymbol{X}_\alpha\otimes\Delta[n]$ composed with the injection into the coproduct of the summand indexed by (α, n, σ).*

Proof. If we let

$$G = \Bigl(\coprod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C} \\ n>0 \\ 0\le i\le n \\ \Delta[n]\to \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n-1] \Bigr) \amalg \Bigl(\coprod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C} \\ n\ge 0 \\ 0\le i\le n \\ \Delta[n]\to \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n+1] \Bigr)$$

and

$$H = \Bigl(\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n>0 \\ 0\le i\le n \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n-1] \Bigr) \amalg \Bigl(\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ 0\le i\le n \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n+1] \Bigr)$$

then in the diagram

$$\begin{array}{ccc}
G & \rightrightarrows & H \\
\downdownarrows & & \downdownarrows \\
\displaystyle\coprod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C} \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] & \rightrightarrows & \displaystyle\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] \\
\downarrow & & \downarrow \\
\displaystyle\coprod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}} & \rightrightarrows & \displaystyle\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}
\end{array}$$

both of the columns are coproducts of coequalizer diagrams (see Proposition 4.3) and are thus coequalizer diagrams, and each of the squares commutes if we use either both upper horizontal arrows or both lower horizontal arrows and either both left vertical arrows or both right vertical arrows.

Since the column on the right is a coequalizer diagram, for every object W of $\mathcal{M}$ maps

$$\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \to W \quad \text{correspond to maps} \quad \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] \to W$$

that coequalize the pair of parallel downward arrows on the right. Since coequalizers are epimorphisms (see Lemma 7.1) and the left column is a coequalizer diagram, a map $\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \to W$ coequalizes the bottom pair of parallel horizontal arrows if and only if the corresponding map $\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] \to W$ coequalizes the middle pair of parallel horizontal arrows. Thus, maps $\coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \to W$ that coequalize the bottom pair of parallel horizontal arrows correspond to maps $\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] \to W$ that coequalize both the middle horizontal pair of parallel arrows and the right vertical pair of parallel arrows, and so an initial object among the former corresponds to an initial object among the latter. □

8.1.1. *Homotopy colimits and basic simplices.* Proposition 8.5 decomposed the objects used in the definition of the homotopy colimit into simpler objects, so that the homotopy colimit was presented as a quotient of a coproduct of objects $\boldsymbol{X}_\alpha \otimes \Delta[n]$, one for each n-simplex of the simplicial set $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$, for each object α of $\mathcal{C}$. The following lemma shows that the map to the coequalizer is actually determined by the map on those summands indexed by the "basic" simplices, where the "basic" simplices are the ones of the form $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ (see Example 2.4), i.e., the simplices of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ for some α in which the map from α to the final vertex of the simplex is the identity map. (The basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$; see Proposition 5.13.) We will use this in Theorem 9.5 to show that the homotopy colimit is isomorphic to the realization (see Definition 9.2) of the simplicial replacement (see Definition 9.1) of the diagram, which is built using only the summands indexed by the basic simplices.

Lemma 8.7. *Let $\mathcal{M}$ be a simplicial model category, let $\mathcal{C}$ be a small category, and let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram in $\mathcal{M}$. If*

$$h\colon \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] \longrightarrow W$$

is a map that coequalizes the maps ϕ and ψ of (8.6)*, then for the summand $\boldsymbol{X}_\alpha \otimes \Delta[n]$ indexed by $\big(\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)\big)$ the diagram*

$$\begin{array}{ccc} \boldsymbol{X}_\alpha \otimes \Delta[n] & \xrightarrow{\tau_*\otimes 1_{\Delta[n]}} & \boldsymbol{X}_{\alpha_n} \otimes \Delta[n] \\ & {\scriptstyle h_\tau}\searrow \quad \swarrow{\scriptstyle h_{(1_{\alpha_n})}} & \\ & W & \end{array}$$

commutes, where h_τ is the composition of h with the injection into the coproduct of the summand indexed by $\big(\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)\big)$ and $h_{(1_{\alpha_n})}$ is the composition of h with the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$.

Proof. The composition of h with the map ϕ on the summand $\boldsymbol{X}_\alpha \otimes \Delta[n]$ indexed by $\big((\tau\colon \alpha \to \alpha_n) \in \mathcal{C}, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ in the first summand on the left of (8.6) is the composition $h_{(1_{\alpha_n})} \circ (\tau_* \otimes 1_{\Delta[n]})$, and the composition of h with the map ψ on that same summand is h_τ. □

9. The realization of the simplicial replacement of a diagram

The main result in this section is Theorem 9.5, which shows that the homotopy colimit of a diagram can be constructed as the realization (see Definition 9.2) of the simplicial replacement (see Definition 9.1) of the diagram. This constructs the homotopy colimit starting with the coproduct of a much smaller collection of objects than the collection used in Proposition 8.5.

A "basic simplex" of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ is one in which the map from the object α to the final vertex of the simplex in $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ is the identity map, i.e., the ones of the form $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ (see Example 2.4). (The basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$; see

Proposition 5.13.) For every object α of $\mathcal{C}$ and every simplex $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)$ of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$, there are

- an object α_n of $\mathcal{C}$,
- a basic simplex $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ in $\mathrm{N}(\alpha_n\downarrow\mathcal{C})^{\mathrm{op}}$, and
- a map $\tau\colon \alpha_n \to \alpha$ in $\mathcal{C}$

such that $\tau^*\colon \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \to \mathrm{N}(\alpha_n\downarrow\mathcal{C})^{\mathrm{op}}$ takes that basic simplex $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ to our simplex $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)$, and such a triple is unique.

The realization of the simplicial replacement is constructed as a quotient of the coproduct of a copy of $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ for every basic simplex $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$. Lemma 8.7 implies that we can construct a map from the homotopy colimit of a diagram $\boldsymbol{X}$ to the realization of the simplicial replacement by mapping the summand $\boldsymbol{X}_\alpha \otimes \Delta[n]$ indexed by $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)$ (see Proposition 8.5) by the composition of $\tau_* \otimes 1_{\Delta[n]}\colon \boldsymbol{X}_\alpha \otimes \Delta[n] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ and the injection into the coproduct of the summand indexed by $\big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ (which, in the proof of Theorem 9.5, is denoted simply $(\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}$).

Definition 9.1. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then the *simplicial replacement* of $\boldsymbol{X}$ is the simplicial object $\coprod_* \boldsymbol{X}$ in $\mathcal{M}$ such that

$$\Big(\coprod\nolimits_* \boldsymbol{X}\Big)_n = \coprod_{(\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}} \boldsymbol{X}_{\alpha_n}$$

and such that

- the face map $d_i\colon \Big(\coprod_* \boldsymbol{X}\Big)_n \to \Big(\coprod_* \boldsymbol{X}\Big)_{n-1}$ on the summand $\boldsymbol{X}_{\alpha_n}$ indexed by $(\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}$ is
  - the injection into the coproduct of the summand indexed by $(\alpha_1 \xleftarrow{\sigma_1} \alpha_2 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}$, if $i = 0$,
  - the injection into the coproduct of the summand indexed by $(\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{i-2}} \alpha_{i-1} \xleftarrow{\sigma_{i-1}\sigma_i} \alpha_{i+1} \xleftarrow{\sigma_{i+1}} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}$, if $0 < i < n$, and
  - the map $(\sigma_{n-1})_*\colon \boldsymbol{X}_{\alpha_n} \to \boldsymbol{X}_{\alpha_{n-1}}$ composed with the injection into the coproduct of the summand $\boldsymbol{X}_{\alpha_{n-1}}$ indexed by $(\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-2}} \alpha_{n-1}) \in \mathcal{C}$, if $i = n$

  and
- the degeneracy map $s^i\colon \Big(\coprod_* \boldsymbol{X}\Big)_n \to \Big(\coprod_* \boldsymbol{X}\Big)_{n+1}$ on the summand $\boldsymbol{X}_{\alpha_n}$ indexed by $(\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}$ is the injection into the coproduct of the summand indexed by $(\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{i-1}} \alpha_i \xleftarrow{1_{\alpha_i}} \alpha_i \xleftarrow{\sigma_i} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}$.

Definition 9.2. Let $\mathcal{M}$ be a simplicial model category. If $\boldsymbol{Y}$ is a simplicial object in $\mathcal{M}$, then its *realization* $\big|\boldsymbol{Y}\big|$ is the coend $\boldsymbol{Y} \otimes_{\boldsymbol{\Delta}^{\mathrm{op}}} \Delta$ (see Definition 3.3 and

Definition 7.2), that is, $\bigl|\boldsymbol{Y}\bigr|$ is the coequalizer of the maps

$$\coprod_{(\sigma\colon [n]\to[k])\in\boldsymbol{\Delta}^{\mathrm{op}}} \boldsymbol{Y}_n \otimes \Delta[k] \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{n\geq 0} \boldsymbol{Y}_n \otimes \Delta[n]$$

where

- the map ϕ on the summand $\boldsymbol{Y}_n \otimes \Delta[k]$ indexed by $(\sigma\colon [n] \to [k]) \in \boldsymbol{\Delta}^{\mathrm{op}}$ is the composition of the map

$$\sigma_* \otimes 1_{\Delta[k]}\colon \boldsymbol{Y}_n \otimes \Delta[k] \longrightarrow \boldsymbol{Y}_k \otimes \Delta[k]$$

  with the injection into the coproduct, and
- the map ψ on that same summand is the composition of the map

$$1_{\boldsymbol{Y}_n} \otimes \sigma^*\colon \boldsymbol{Y}_n \otimes \Delta[k] \longrightarrow \boldsymbol{Y}_n \otimes \Delta[n]$$

  (see Definition 3.3) with the injection into the coproduct.

Proposition 9.3. *Let $\mathcal{M}$ be a simplicial model category. If $\boldsymbol{Y}$ is a simplicial object in $\mathcal{M}$, then $\bigl|\boldsymbol{Y}\bigr|$ is naturally isomorphic to the coequalizer of the maps*

$$(9.4)\qquad \Bigl(\coprod_{\substack{n>0\\0\leq i\leq n}} \boldsymbol{Y}_n \otimes \Delta[n-1]\Bigr) \amalg \Bigl(\coprod_{\substack{n\geq 0\\0\leq i\leq n}} \boldsymbol{Y}_n \otimes \Delta[n+1]\Bigr) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{n\geq 0} \boldsymbol{Y}_n \otimes \Delta[n]$$

where, on the first summand,

- *the map ϕ on the summand $\boldsymbol{Y}_n \otimes \Delta[n-1]$ indexed by (n,i) is the composition of the map $d_i \otimes 1_{\Delta[n-1]}\colon \boldsymbol{Y}_n \otimes \Delta[n-1] \to \boldsymbol{Y}_{n-1} \otimes \Delta[n-1]$ with the injection into the coproduct, and*
- *the map ψ on that same summand is the composition of the map $1_{\boldsymbol{Y}_n} \otimes d^i\colon \boldsymbol{Y}_n \otimes \Delta[n-1] \to \boldsymbol{Y}_n \otimes \Delta[n]$ with the injection into the coproduct*

and, on the second summand,

- *the map ϕ on the summand $\boldsymbol{Y}_n \otimes \Delta[n+1]$ indexed by (n,i) is the composition of the map $s_i \otimes 1_{\Delta[n+1]}\colon \boldsymbol{Y}_n \otimes \Delta[n+1] \to \boldsymbol{Y}_{n+1} \otimes \Delta[n+1]$ with the injection into the coproduct, and*
- *the map ψ on that same summand is the composition of the map $1_{\boldsymbol{Y}_n} \otimes s^i\colon \boldsymbol{Y}_n \otimes \Delta[n+1] \to \boldsymbol{Y}_n \otimes \Delta[n]$ with the injection into the coproduct.*

Proof. Coequalizing ϕ and ψ on the first summand coequalizes the face operators and coequalizing them on the second summand coequalizes the degeneracy operators. Since every morphism in $\boldsymbol{\Delta}^{\mathrm{op}}$ is a finite composition of face and degeneracy operators, the result follows from Proposition 7.24. □

Theorem 9.5. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then* $\operatorname{hocolim} \boldsymbol{X}$*, the homotopy colimit of $\boldsymbol{X}$, is naturally isomorphic to $\Bigl|\coprod_* \boldsymbol{X}\Bigr|$, the realization of the simplicial replacement of $\boldsymbol{X}$.*

Proof. If $\boldsymbol{Y} = \coprod_{*} \boldsymbol{X}$, then (9.4) is naturally isomorphic to

$$(9.6)\quad \Big(\coprod_{\substack{n>0 \\ 0\le i\le n \\ (\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}}} \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1] \Big) \amalg \Big(\coprod_{\substack{n\ge 0 \\ 0\le i\le n \\ (\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}}} \boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1] \Big) \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{n\ge 0 \\ (\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}}} \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$$

We will define a natural isomorphism from the coequalizer of (9.6) to the coequalizer of (8.6). We define

$$P\colon \coprod_{\substack{n\ge 0 \\ (\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}}} \boldsymbol{X}_{\alpha_n} \otimes \Delta[n] \longrightarrow \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{\alpha} \otimes \Delta[n]$$

by defining P on the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ indexed by $\big(n, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$ to be the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$. To show that this induces a map from the coequalizer of (9.6) to the coequalizer of (8.6), let

$$f\colon \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{\alpha} \otimes \Delta[n] \longrightarrow W$$

be a map that coequalizes the maps ϕ and ψ of (8.6); we will show that the composition fP coequalizes the maps ϕ and ψ of (9.6).

There are two summands on the left of (9.6).

- In the first summand, consider the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1]$ indexed by $\big(n, i, (\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}\big)$.
  - If $i < n$, then
    - the composition $P\phi$ on that summand is the injection into the coproduct of the summand indexed by
      - $\big(\alpha_n, (n-1), \big((\alpha_1 \leftarrow \alpha_2 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$, if $i = 0$, and
      - $\big(\alpha_n, (n-1), (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{i-1} \xleftarrow{\sigma_{i-1}\sigma_i} \alpha_{i+1} \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$, if $0 < i < n$, and
    - the composition $P\psi$ on that same summand is the composition of the map $1_{\boldsymbol{X}_{\alpha_n}} \otimes d^i\colon \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$.

    Those two maps from $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1]$ to $\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C}) \\ n\ge 0 \\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{\alpha} \otimes \Delta[n]$ are exactly the maps ϕ and ψ of (8.6) on the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1]$ indexed by $\big(\alpha_n, n, i, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ in the second summand on the left of (8.6), and so they are coequalized by f.
  - If $i = n$, then

- the composition $P\phi$ on that summand is the composition of $(\sigma_{n-1})_* \otimes 1_{\Delta[n-1]} \colon \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1] \to \boldsymbol{X}_{\alpha_{n-1}} \otimes \Delta[n-1]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_{n-1}, (n-1), \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{n-1}) \in \mathcal{C}, 1\colon \alpha_{n-1} \to \alpha_{n-1}\big)\big)$, and
- the composition $P\psi$ on that same summand is the composition of the map $1_{\boldsymbol{X}_{\alpha_n}} \otimes d^n \colon \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$.

On the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1]$ indexed by $\big(\alpha_n, n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ of the second summand of (8.6), the map ϕ is the injection into the coproduct of the summand indexed by $\big(\alpha_{n-1}, (n-1), \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{n-1}) \in \mathcal{C}, \sigma_{n-1}\colon \alpha_n \to \alpha_{n-1}\big)\big)$, and Lemma 8.7 implies that the composition $f\phi$ on that summand is the composition of $(\sigma_{n-1})_* \otimes 1_{\Delta[n-1]} \colon \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1] \to \boldsymbol{X}_{\alpha_{n-1}} \otimes \Delta[n-1]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_{n-1}, (n-1), \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{n-1}) \in \mathcal{C}, 1\colon \alpha_{n-1} \to \alpha_{n-1}\big)\big)$ composed with f. The map ψ maps that same summand to the composition of $1_{\boldsymbol{X}_{\alpha_n}} \otimes d^n \colon \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$, and so $fP\phi = fP\psi$ on the first summand of (9.6).

- In the second summand of (9.6), consider the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1]$ indexed by $\big(n, i, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$.
  - The composition $P\phi$ is the injection into the coproduct of the summand indexed by $\big(\alpha_n, (n+1), \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \alpha_i \xleftarrow{1_{\alpha_i}} \alpha_i \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$, and
  - the composition $P\psi$ is the composition of the map $1_{\boldsymbol{X}_{\alpha_n}} \otimes s^i \colon \boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$.

  Those two maps are exactly the maps ϕ and ψ of (8.6) on the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1]$ indexed by $\big(\alpha_n, n, i, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ in the third summand on the left of (8.6), and so they are coequalized by f.

Thus, the map P induces a map $\widetilde{P}$ from the coequalizer of (9.6) to the coequalizer of (8.6). We will show that $\widetilde{P}$ is an isomorphism by constructing an inverse map $\widetilde{Q}$.

We define

$$Q\colon \coprod_{\substack{\alpha \in \mathrm{Ob}(\mathcal{C}) \\ n \geq 0 \\ \Delta[n] \to \mathrm{N}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_\alpha \otimes \Delta[n] \longrightarrow \coprod_{\substack{n \geq 0 \\ (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}}} \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$$

by defining Q on the summand $\boldsymbol{X}_\alpha \otimes \Delta[n]$ indexed by $\big(\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)\big)$ to be the composition of the map $\tau_* \otimes 1_{\Delta[n]} \colon \boldsymbol{X}_\alpha \otimes \Delta[n] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(n, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$.

To show that this induces a map from the coequalizer of (8.6) to the coequalizer of (9.6), let

$$f\colon \coprod_{\substack{n\geq 0\\ (\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}}} \boldsymbol{X}_{\alpha_n}\otimes\Delta[n] \longrightarrow W$$

be a map that coequalizes the maps ϕ and ψ of (9.6); we will show that the composition fQ coequalizes the maps ϕ and ψ of (8.6).

There are three summands on the left of (8.6).

- Consider the summand $\boldsymbol{X}_\alpha\otimes\Delta[n]$ indexed by $\bigl((\sigma\colon\alpha\to\alpha')\in\mathcal{C}, n, \bigl((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, \tau\colon\alpha'\to\alpha_n\bigr)\bigr)$ in the first summand on the left of (8.6).
  - The composition $Q\phi$ is the composition

    $$\boldsymbol{X}_\alpha\otimes\Delta[n]\xrightarrow{\sigma_*\otimes 1_{\Delta[n]}}\boldsymbol{X}_{\alpha'}\otimes\Delta[n]\xrightarrow{\tau_*\otimes 1_{\Delta[n]}}\boldsymbol{X}_{\alpha_n}\otimes\Delta[n]$$

    composed with the injection into the coproduct of the summand indexed by $\bigl(n,(\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}\bigr)$.
  - The composition $Q\psi$ is the composition of the map $(\tau\sigma)_*\otimes 1_{\Delta[n]}\colon\boldsymbol{X}_\alpha\otimes\Delta[n]\to\boldsymbol{X}_{\alpha_n}\otimes\Delta[n]$ with the injection into the coproduct of the summand indexed by $\bigl(n,(\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}\bigr)$.

  Since $\boldsymbol{X}$ is a functor, $\tau_*\sigma_*=(\tau\sigma)_*$, and so $Q\phi=\ \psi$, and so $fQ\phi=fQ\psi$ on the first summand on the left of (8.6).
- Consider the summand $\boldsymbol{X}_\alpha\otimes\Delta[n-1]$ indexed by $\bigl(\alpha,n,i,\bigl((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C},\tau\colon\alpha\to\alpha_n\bigr)\bigr)$ in the second summand on the left of (8.6).
  - If $i<n$, then
    - the composition $Q\phi$ is the composition of the map $\tau_*\otimes 1_{\Delta[n-1]}\colon\boldsymbol{X}_\alpha\otimes\Delta[n-1]\to\boldsymbol{X}_{\alpha_n}\otimes\Delta[n-1]$ with the injection into the coproduct of the summand indexed by
      - $\bigl((n-1),(\alpha_1\leftarrow\alpha_2\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}\bigr)$, if $i=0$, and
      - $\bigl((n-1),(\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_{i-1}\leftarrow\alpha_{i+1}\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}\bigr)$, if $0<i<n$

      and
    - the composition $Q\psi$ is the composition $(\tau_*\otimes 1_{\Delta[n]})\circ(1_{\boldsymbol{X}_\alpha}\otimes d^i)=(1_{\boldsymbol{X}_{\alpha_n}}\otimes d^i)\circ(\tau_*\otimes 1_{\Delta[n-1]})\colon\boldsymbol{X}_\alpha\otimes\Delta[n-1]\to\boldsymbol{X}_{\alpha_n}\otimes\Delta[n]$ composed with the injection into the coproduct of the summand indexed by $\bigl(n,(\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}\bigr)$.

    Thus, the compositions $Q\phi$ and $Q\psi$ equal the composition of the map $\tau_*\otimes 1_{\Delta[n-1]}\colon\boldsymbol{X}_\alpha\otimes\Delta[n-1]\to\boldsymbol{X}_{\alpha_n}\otimes\Delta[n-1]$ with the maps ϕ and ψ of (9.6) on the summand $\boldsymbol{X}_{\alpha_n}\otimes\Delta[n-1]$ indexed by $\bigl(n,i,(\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}\bigr)$ in the first summand on the left of (9.6), and are thus coequalized by f.
  - If $i=n$, then
    - the composition $Q\phi$ is the composition of the map $(\sigma_{n-1}\tau)_*\otimes 1_{\Delta[n-1]}=\bigl((\sigma_{n-1})_*\otimes 1_{\Delta[n-1]}\bigr)\circ(\tau_*\otimes 1_{\Delta[n-1]})\colon\boldsymbol{X}_\alpha\otimes\Delta[n-1]\to\boldsymbol{X}_{\alpha_{n-1}}\otimes\Delta[n-1]$ with the injection into the coproduct of the summand indexed by $\bigl((n-1),(\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_{n-1})\in\mathcal{C}\bigr)$, and
    - the composition $Q\psi$ is the composition of $(\tau_*\otimes 1_{\Delta[n]})\circ(1_{\boldsymbol{X}_\alpha}\otimes d^i)=(1_{\boldsymbol{X}_{\alpha_n}}\otimes d^i)\circ(\tau_*\otimes 1_{\Delta[n-1]})\colon\boldsymbol{X}_\alpha\otimes\Delta[n-1]\to\boldsymbol{X}_{\alpha_n}\otimes\Delta[n]$

with the injection into the coproduct of the summand indexed by $\big(n, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$.

Thus, the compositions $Q\phi$ and $Q\psi$ equal the composition of the map $\tau_* \otimes 1_{\Delta[n-1]} \colon \boldsymbol{X}_\alpha \otimes \Delta[n-1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1]$ with the maps ϕ and ψ of (9.6) on the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n-1]$ indexed by $\big(n, n, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$ in the first summand of (9.6) and are thus coequalized by f.

- Consider the summand $\boldsymbol{X}_\alpha \otimes \Delta[n+1]$ indexed by $\Big(\alpha, n, i, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau \colon \alpha \to \alpha_n\big)\Big)$ in the third summand on the left of (8.6). On this summand
  - the composition $Q\phi$ is the composition of the map $\tau_* \otimes 1_{\Delta[n-1]} \colon \boldsymbol{X}_\alpha \otimes \Delta[n+1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1]$ with the injection into the coproduct of the summand indexed by $\big((n+1), (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_i \xrightarrow{1_{\alpha_i}} \alpha_i \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$, and
  - the composition $Q\psi$ is the composition of

$$(\tau_* \otimes 1_{\Delta[n]}) \circ (1_{\boldsymbol{X}_\alpha} \otimes s^i) = (1_{\boldsymbol{X}_\alpha} \otimes s^i) \circ (\tau_* \otimes 1_{\Delta[n+1]})$$

composed with the injection into the coproduct of the summand indexed by $\big(n, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$.

Thus, the compositions $Q\phi$ and $Q\psi$ equal the composition of the map $\tau_* \otimes 1_{\Delta[n+1]} \colon \boldsymbol{X}_\alpha \otimes \Delta[n+1] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1]$ with the maps ϕ and ψ of (9.6) on the summand $\boldsymbol{X}_{\alpha_n} \otimes \Delta[n+1]$ indexed by $\big(n, i, (\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}\big)$ in the second summand of (9.6), and are thus coequalized by f.

Thus, the map Q induces a map $\widetilde{Q}$ from the coequalizer of (8.6) to the coequalizer of (9.6).

We will now show that $\widetilde{P}$ and $\widetilde{Q}$ are inverse isomorphisms. The composition $\widetilde{Q}\widetilde{P}$ is the identity because the composition QP is the identity.

The composition PQ takes the summand $\boldsymbol{X}_\alpha \otimes \Delta[n]$ indexed by $\Big(\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau \colon \alpha \to \alpha_n\big)\Big)$ to the composition of the map $\tau_* \otimes 1_{\Delta[n]} \colon \boldsymbol{X}_\alpha \otimes \Delta[n] \to \boldsymbol{X}_{\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\Big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1 \colon \alpha_n \to \alpha_n\big)\Big)$, and Lemma 8.7 implies that the map to the coequalizer of (9.6) coequalizes that and the inclusion into the coproduct of the original summand, and so $\widetilde{P}\widetilde{Q}$ is the identity. □

10. Changing the indexing category of a homotopy colimit

Let $\mathcal{M}$ be a simplicial category and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then there is an induced $\mathcal{C}$-diagram $F^*\boldsymbol{X}$ in $\mathcal{M}$, defined as the composition $F^*\boldsymbol{X} = \boldsymbol{X} \circ F$. In this section, we show that the homotopy colimit $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X}$ of the induced diagram can be constructed as the coend $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}}$ over the category $\mathcal{D}$ (see Definition 2.10 and Theorem 10.4).

The reason this theorem is true is that, although $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X}$ is constructed as a quotient of the (rather large) coproduct of a copy of $\boldsymbol{X}_{F\alpha} \otimes \Delta[n]$ for every object α of $\mathcal{C}$ and every n-simplex of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ (see Proposition 8.5), it can also be constructed from the much smaller coproduct of $\boldsymbol{X}_{F\alpha} \otimes \Delta[n]$ for only certain "basic simplices" of the simplicial sets $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$. (The basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}}$; see Corollary 5.11 and Proposition 5.13.) Similarly, although the coend $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow D)^{\mathrm{op}}$ is defined as a quotient of the (rather large) coproduct of a copy of $\boldsymbol{X}_\beta \otimes \Delta[n]$ for every object β of $\mathcal{D}$ and every n-simplex of $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$ (see Proposition 10.1), it can also be constructed from the much smaller coproduct of $\boldsymbol{X}_\beta \otimes \Delta[n]$ for only certain "basic simplices" of the simplicial sets $\mathrm{N}(-\downarrow F)^{\mathrm{op}}$ (again, the basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(-\downarrow F)^{\mathrm{op}}$; see Corollary 5.11 and Proposition 5.14, and the maps $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}} \to \mathrm{N}(F\alpha\downarrow F)^{\mathrm{op}}$ (see Example 2.11) take basic simplices to basic simplices. Theorem 10.4 shows that the map F_* of Lemma 2.9 defines an isomorphism $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} = F^*\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}} \approx \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}}$. (For a much shorter proof of this that uses the mapping properties of a basis of a free cell complex, see [7, Prop. 19.6.6].)

We begin by showing that the coend $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}}$ has a decomposition similar to that of $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} = F^*\boldsymbol{X} \otimes_{\mathcal{C}} (-\downarrow\mathcal{C})^{\mathrm{op}}$ (see Proposition 8.5).

Proposition 10.1. *Let $\mathcal{M}$ be a simplicial model category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then the coend (see Definition 7.2) $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}}$ is naturally isomorphic to the coequalizer of the maps*

(10.2)
$$\Bigl(\coprod_{\substack{(\sigma\colon \beta\to\beta')\in\mathcal{D}\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\beta'\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_\beta \otimes \Delta[n]\Bigr) \amalg \Bigl(\coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n>0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_\beta \otimes \Delta[n-1]\Bigr) \amalg \Bigl(\coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_\beta \otimes \Delta[n+1]\Bigr)$$
$$\overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_\beta \otimes \Delta[n]$$

where, on the first summand,

- *the map ϕ on the summand $\boldsymbol{X}_\beta \otimes \Delta[n]$ indexed by $\bigl((\sigma\colon \beta \to \beta') \in \mathcal{D}, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta' \to F\alpha_n\bigr)\bigr)$ is the composition of the map $\sigma_* \otimes 1_{\Delta[n]}\colon \boldsymbol{X}_\beta \otimes \Delta[n] \to \boldsymbol{X}_{\beta'} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\bigl(\beta', n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta' \to F\alpha_n\bigr)\bigr)$, and*
- *the map ψ on that same summand is the injection into the coproduct of the summand indexed by $\bigl(\beta, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\sigma\colon \beta \to F\alpha_n\bigr)\bigr)$*

and, on the second summand,

- *the map ϕ on the summand $\boldsymbol{X}_\beta \otimes \Delta[n-1]$ indexed by $\bigl(\beta, n, i, \sigma\colon \Delta[n] \to \mathrm{N}(\beta\downarrow F)^{\mathrm{op}}\bigr)$ is the injection into the coproduct of the summand indexed by β, $(n-1)$, and the composition $\Delta[n-1] \xrightarrow{d^i} \Delta[n] \xrightarrow{\sigma} \mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$, and*
- *the map ψ on that same summand is the composition of the map $1_{\boldsymbol{X}_\beta} \otimes d^i\colon \boldsymbol{X}_\beta \otimes \Delta[n-1] \to \boldsymbol{X}_\beta \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by (β, n, σ)*

and, on the third summand,

- *the map ϕ on the summand $\boldsymbol{X}_\beta \otimes \Delta[n+1]$ indexed by $\bigl(\beta, n, i, \sigma\colon \Delta[n] \to \mathrm{N}(\beta\downarrow F)^{\mathrm{op}}\bigr)$ is the injection into the coproduct of the summand indexed by β, $(n+1)$, and the composition $\Delta[n+1] \xrightarrow{s^i} \Delta[n] \xrightarrow{\sigma} \mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$, and*

- *the map ψ on that same summand is the composition of the map $1_{\boldsymbol{X}_\beta} \otimes s^i \colon \boldsymbol{X}_\beta \otimes \Delta[n+1] \to \boldsymbol{X}_\beta \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by (β, n, σ).*

Proof. This is identical to the proof of Proposition 8.5, changing $\alpha \in \mathrm{Ob}(\mathcal{C})$ to $\beta \in \mathrm{Ob}(\mathcal{D})$, $(\sigma\colon \alpha \to \alpha') \in \mathcal{C}$ to $(\sigma\colon \beta \to \beta') \in \mathcal{D}$, $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ to $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$, and $\mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}$ to $\mathrm{N}(\beta'\downarrow F)^{\mathrm{op}}$. □

Lemma 10.3. *Let $\mathcal{M}$ be a simplicial model category, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories, and let $\boldsymbol{X}$ be a $\mathcal{D}$-diagram in $\mathcal{M}$. If*

$$h\colon \coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_\beta \otimes \Delta[n] \longrightarrow W$$

is a map that coequalizes the maps ϕ and ψ of (10.2), *then for the summand $\boldsymbol{X}_\beta \otimes \Delta[n]$ indexed by $\big(\beta, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta \to F\alpha_n\big)\big)$ the diagram*

$$\begin{array}{ccc} \boldsymbol{X}_\beta \otimes \Delta[n] & \xrightarrow{\tau_*\otimes 1_{\Delta[n]}} & \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n] \\ & {\scriptstyle h_\tau}\searrow \quad \swarrow{\scriptstyle h_{(1_{F\alpha_n})}} & \\ & W & \end{array}$$

commutes, where h_τ is the composition of h with the injection into the coproduct of the summand indexed by $\big(\beta, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta \to F\alpha_n\big)\big)$ and $h_{(1_{F\alpha_n})}$ is the composition of h with the injection into the coproduct of the summand indexed by $\big(F\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\big)\big)$.

Proof. The composition of h with the map ϕ on the summand $\boldsymbol{X}_\beta \otimes \Delta[n]$ indexed by $\big((\tau\colon \beta \to F\alpha_n) \in \mathcal{D}, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\big)\big)$ in the first summand on the left of (10.2) is the composition $h_{(1_{F\alpha_n})} \circ (\tau_* \otimes 1_{\Delta[n]})$, and the composition of h with the map ψ on that same summand is h_τ. □

Theorem 10.4. *If $\mathcal{M}$ is a simplicial model category, $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories, $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, and $F^*\boldsymbol{X} = \boldsymbol{X} \circ F$ is the induced $\mathcal{C}$-diagram in $\mathcal{M}$, then there is a natural isomorphism of coends*

$$\operatorname*{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \quad = \quad \boldsymbol{F}^*\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(\mathcal{C}\downarrow -)^{\mathrm{op}} \quad \approx \quad \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(F\downarrow -)^{\mathrm{op}}$$

and the natural map of homotopy colimits $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \to \operatorname{hocolim}_{\mathcal{D}} \boldsymbol{X}$ is naturally isomorphic to the natural map of coends $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ induced by the natural map of $\mathcal{D}^{\mathrm{op}}$-diagrams $\mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \mathrm{N}(-\downarrow D)^{\mathrm{op}}$ (see Example 2.12).

Proof. Proposition 8.5 implies that $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} = F^*\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}}$ is naturally isomorphic to the coequalizer of the maps

(10.5)
$$\Big(\coprod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\alpha'\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n]\Big) \amalg \Big(\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n>0\\ 0\leq i\leq n\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n-1]\Big) \amalg \Big(\coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ 0\leq i\leq n\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n+1]\Big)$$
$$\overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n]$$

We will define a natural isomorphism from the coequalizer of (10.5) to the coequalizer of (10.2). We define

$$P\colon \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n] \longrightarrow \coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_{\beta}\otimes\Delta[n]$$

by defining P on the summand $\boldsymbol{X}_{F\alpha}\otimes\Delta[n]$ indexed by $\big(\alpha, n, \sigma\colon \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}\big)$ to be the injection into the coproduct of the summand indexed by $(F\alpha, n, F_*\sigma)$ (where F_* is as in Lemma 2.9).

To show that P induces a map $\widetilde{P}$ from the coequalizer of (10.5) to the coequalizer of (10.2), let

$$f\colon \coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_{\beta}\otimes\Delta[n] \longrightarrow W$$

be a map that coequalizes the maps ϕ and ψ of (10.2); we will show that the composition fP coequalizes the maps ϕ and ψ of (10.5).

There are three summands on the left of (10.5).

- Consider the summand $\boldsymbol{X}_{F\alpha}\otimes\Delta[n]$ indexed by $\big((\sigma\colon \alpha\to\alpha')\in\mathcal{C}, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n)\in\mathcal{C}, \tau\colon \alpha'\to\alpha_n\big)\big)$ in the first summand on the left of (10.5).
  - The composition $P\phi$ on that summand is the composition of the map $(F\sigma)_*\otimes 1_{\Delta[n]}\colon \boldsymbol{X}_{F\alpha}\otimes\Delta[n]\to\boldsymbol{X}_{F\alpha'}\otimes\Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(F\alpha', n, \big((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, F\tau\colon F\alpha'\to F\alpha_n\big)\big)$, and
  - the composition $P\psi$ on that same summand is the injection into the coproduct of the summand indexed by $\big(F\alpha, n, \big((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, F(\tau\sigma)\colon F\alpha\to F\alpha_n\big)\big)$.

  Those compositions $P\phi$ and $P\psi$ exactly equal the maps ϕ and ψ of (10.2) on the summand $\boldsymbol{X}_{F\alpha}\otimes\Delta[n]$ indexed by $\big((F\sigma\colon F\alpha\to F\alpha')\in\mathcal{D}, n, \big((\alpha_0\leftarrow\alpha_1\leftarrow\cdots\leftarrow\alpha_n)\in\mathcal{C}, F\tau\colon F\alpha'\to F\alpha_n\big)\big)$ in the first summand on the left of (10.2), and so they are coequalized by f.
- Consider the summand $\boldsymbol{X}_{F\alpha}\otimes\Delta[n-1]$ indexed by $(\alpha, n, i, \sigma\colon \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}})$ in the second summand on the left of (10.5).

- The composition $P\phi$ on that summand is the injection into the coproduct of the summand indexed by $\bigl(F\alpha, (n-1), F_*(\sigma \circ d^i)\bigr)$, and
- the composition $P\psi$ on that same summand is the composition of the map $1_{\boldsymbol{X}_{F\alpha}} \otimes d^i\colon \boldsymbol{X}_{F\alpha} \otimes \Delta[n-1] \to \boldsymbol{X}_{F\alpha} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $(F\alpha, n, F_*\sigma)$.

Those compositions $P\psi$ and $P\psi$ exactly equal the maps ϕ and ψ of (10.2) on the summand $\boldsymbol{X}_{F\alpha} \otimes \Delta[n-1]$ indexed by $(F\alpha, n, i, F_*\sigma)$, and so they are coequalized by f.

- Consider the summand $\boldsymbol{X}_{F\alpha} \otimes \Delta[n+1]$ indexed by $(\alpha, n, i, \sigma\colon \Delta[n] \to \mathrm{N}(\alpha\downarrow \mathcal{C})^{\mathrm{op}})$ in the third summand on the left of (10.5).
  - The composition $P\phi$ on that summand is the injection into the coproduct of the summand indexed by $\bigl(F\alpha, (n+1), F_*(\sigma \circ s^i)\bigr)$, and
  - the composition $P\psi$ on that same summand is the composition of the map $1_{\boldsymbol{X}_{F\alpha}} \otimes s^i\colon \boldsymbol{X}_{F\alpha} \otimes \Delta[n+1] \to \boldsymbol{X}_{F\alpha} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $(F\alpha, n, F_*\sigma)$.

  Those compositions $P\phi$ and $P\psi$ exactly equal the maps ϕ and ψ of (10.2) on the summand $\boldsymbol{X}_{F\alpha} \otimes \Delta[n+1]$ indexed by $(F\alpha, n, i, F_*\sigma)$, and so they are coequalized by f.

Thus, the map P induces a map $\widetilde{P}$ from the coequalizer of (10.5) to the coequalizer of (10.2). We will show that $\widetilde{P}$ is an isomorphism by constructing an inverse map $\widetilde{Q}$.

We define

$$Q\colon \coprod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}}} \boldsymbol{X}_\beta \otimes \Delta[n] \longrightarrow \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha} \otimes \Delta[n]$$

by defining Q on the summand $\boldsymbol{X}_\beta \otimes \Delta[n]$ indexed by $\bigl(\beta, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta \to F\alpha_n\bigr)\bigr)$ to be the composition of the map $\tau_* \otimes 1_{\Delta[n]}\colon \boldsymbol{X}_\beta \otimes \Delta[n] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$.

To show that Q induces a map $\widetilde{Q}$ from the coequalizer of (10.2) to the coequalizer of (10.5), let

$$g\colon \coprod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha} \otimes \Delta[n] \longrightarrow W$$

be a map that coequalizes the maps ϕ and ψ of (10.5); we will show that the composition gQ coequalizes the maps ϕ and ψ of (10.2).

There are three summands on the left of (10.2).

- Consider the summand $\boldsymbol{X}_\beta \otimes \Delta[n]$ indexed by $\bigl((\sigma\colon \beta \to \beta') \in \mathcal{D}, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta' \to F\alpha_n\bigr)\bigr)$ in the first summand on the left of (10.2).
  - The composition $Q\phi$ on that summand is the composition $\boldsymbol{X}_\beta \otimes \Delta[n] \xrightarrow{\sigma_* \otimes 1_{\Delta[n]}} \boldsymbol{X}_{\beta'} \otimes \Delta[n] \xrightarrow{\tau_* \otimes 1_{\Delta[n]}} \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ composed with the injection into the coproduct of the summand indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$, and

– the composition $Q\psi$ on that summand is the composition of the map $(\tau\sigma)_* \otimes 1_{\Delta[n]}\colon \boldsymbol{X}_\beta \otimes \Delta[n] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$.

Since $\tau_*\sigma_* = (\tau\sigma)_*$, the compositions $Q\phi$ and $Q\psi$ are equal on that summand, and so $gQ\phi = gQ\psi$ on that summand.

- Consider the summand $\boldsymbol{X}_\beta \otimes \Delta[n-1]$ indexed by $\bigl(\beta, n, i, \bigl((\alpha_0 \xleftarrow{\sigma_0} \alpha_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}, \tau\colon \beta \to F\alpha_n\bigr)\bigr)$ in the second summand on the left of (10.2).
  - The composition $Q\phi$ on that summand is
    - the map $\tau_* \otimes 1_{\Delta[n-1]}\colon \boldsymbol{X}_\beta \otimes \Delta[n-1] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n-1]$ composed with the injection into the coproduct of the summand indexed by $\bigl(\alpha_n, (n-1), \bigl((\alpha_1 \leftarrow \alpha_2 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$, if $i = 0$,
    - the map $\tau_* \otimes 1_{\Delta[n-1]}\colon \boldsymbol{X}_\beta \otimes \Delta[n-1] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n-1]$ composed with the injection into the coproduct of the summand indexed by $\bigl(\alpha_n, (n-1), \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{i-1} \xleftarrow{\sigma_{i-1}\sigma_i} \alpha_{i+1} \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$, if $0 < i < n$, and
    - the map $(\sigma_{n-1}\tau)_* \otimes 1_{\Delta[n-1]}\colon \boldsymbol{X}_\beta \otimes \Delta[n-1] \to \boldsymbol{X}_{F\alpha_{n-1}} \otimes \Delta[n-1]$ composed with the injection into the coproduct of the summand indexed by $\bigl(\alpha_{n-1}, (n-1), \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{n-1}) \in \mathcal{C}, 1\colon \alpha_{n-1} \to \alpha_{n-1}\bigr)\bigr)$ if $i = n$,

    and
  - the composition $Q\psi$ on that same summand is the composition $\boldsymbol{X}_\beta \otimes \Delta[n-1] \xrightarrow{1_{\boldsymbol{X}_\beta} \otimes d^i} \boldsymbol{X}_\beta \otimes \Delta[n] \xrightarrow{\tau_* \otimes 1_{\Delta[n]}} \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ composed with the injection into the coproduct of the summand indexed by $\bigl(F\alpha_n, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$.

  Thus, the compositions $Q\phi$ and $Q\psi$ on that summand are the composition of $\tau_* \otimes 1_{\Delta[n-1]}\colon \boldsymbol{X}_\beta \otimes \Delta[n-1] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n-1]$ with the maps ϕ and ψ of (10.5) on the summand $\boldsymbol{X}_{F\alpha_n} \otimes \Delta[n-1]$ indexed by $\bigl(\alpha_n, n, i, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$ in the second summand on the left of (10.5), and are thus coequalized by g.
- Consider the summand $\boldsymbol{X}_\beta \otimes \Delta[n+1]$ indexed by $\bigl(\beta, n, i, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta \to F\alpha_n\bigr)\bigr)$ in the third summand on the left of (10.2).
  - The composition $Q\phi$ on that summand is the map $\tau_* \otimes 1_{\Delta[n+1]}\colon \boldsymbol{X}_\beta \otimes \Delta[n+1] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n+1]$ composed with the injection into the coproduct of the summand indexed by $\bigl(\alpha_n, (n+1), \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{i-1} \leftarrow \alpha_i \xleftarrow{1_{\alpha_i}} \alpha_i \leftarrow \alpha_{i+1} \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$, and
  - the composition $Q\psi$ on that same summand is the composition $\boldsymbol{X}_\beta \otimes \Delta[n+1] \xrightarrow{1_{\boldsymbol{X}_\beta} \otimes s^i} \boldsymbol{X}_\beta \otimes \Delta[n] \xrightarrow{\tau_* \otimes 1_{\Delta[n]}} \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ composed with the injection into the coproduct of the summand indexed by $\bigl(F\alpha_n, n, \bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$.

  Thus, the compositions $Q\phi$ and $Q\psi$ on that summand are the composition of $\tau_* \otimes 1_{\Delta[n+1]}\colon \boldsymbol{X}_\beta \otimes \Delta[n+1] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n+1]$ with the maps ϕ and

ψ of (10.5) on the summand $\boldsymbol{X}_{F\alpha_n} \otimes \Delta[n+1]$ indexed by $\big(\alpha_n, n, i, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ in the third summand on the right of (10.5), and are thus coequalized by g.

Thus, the map Q induces a map $\widetilde{Q}$ from the coequalizer of (10.2) to the coequalizer of (10.5).

We will now show that $\widetilde{P}$ and $\widetilde{Q}$ are inverse isomorphisms. The composition

$$QP\colon \coprod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n] \longrightarrow \coprod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to \mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}}} \boldsymbol{X}_{F\alpha}\otimes\Delta[n]$$

takes the summand $\boldsymbol{X}_{F\alpha} \otimes \Delta[n]$ indexed by $\big(\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)\big)$ to the composition of the map $\tau_* \otimes 1_{\Delta[n]}\colon \boldsymbol{X}_{F\alpha} \otimes \Delta[n] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$. Lemma 8.7 implies that the map to the coequalizer of (10.5) coequalizes that map and the injection into the coproduct, and so $\widetilde{Q}\widetilde{P}$ is the identity map.

The composition PQ takes the summand $\boldsymbol{X}_\beta \otimes \Delta[n]$ indexed by $\big(\beta, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \beta \to F\alpha_n\big)\big)$ to the composition of the map $\tau_* \otimes 1_{\Delta[n]}\colon \boldsymbol{X}_\beta \otimes \Delta[n] \to \boldsymbol{X}_{F\alpha_n} \otimes \Delta[n]$ with the injection into the coproduct of the summand indexed by $\big(F\alpha_n, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\big)\big)$. Lemma 10.3 implies that the map to the coequalizer of (10.2) coequalizes that map and the injection into the coproduct, and so $\widetilde{P}\widetilde{Q}$ is the identity map. Thus, $\widetilde{P}$ and $\widetilde{Q}$ are inverse isomorphisms.

We will now show that the natural map $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \to \operatorname{hocolim}_{\mathcal{D}} \boldsymbol{X}$ equals the composition

$$\operatorname*{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} = F^*\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(-\downarrow\mathcal{C})^{\mathrm{op}} \xrightarrow{\widetilde{P}} \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}} \xrightarrow{1_{\boldsymbol{X}}\otimes_{\mathcal{D}} F_*} \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow\mathcal{D})^{\mathrm{op}} = \operatorname*{hocolim}_{\mathcal{D}} \boldsymbol{X}$$

(where F_* is as in Example 2.12. On the summand $\boldsymbol{X}_{F\alpha} \otimes \Delta[n]$ indexed by $\big(\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, \tau\colon \alpha \to \alpha_n\big)\big)$, the map $\widetilde{P}$ is the injection into the coproduct of the summand indexed y $\big(F\alpha, n, \big((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n) \in \mathcal{C}, F\tau\colon F\alpha \to F\alpha_n\big)\big)$, and $1_{\mathcal{X}} \otimes_{\mathcal{D}} F_*$ takes that to the injection into the coproduct of the summand indexed by $\big(F\alpha\big), n, \big((F\alpha_0 \leftarrow F\alpha_1 \leftarrow \cdots \leftarrow F\alpha_n) \in \mathcal{D}, F\tau\colon F\alpha \to F\alpha_n\big)$, which induces the natural map $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \to \operatorname{hocolim}_{\mathcal{D}} \boldsymbol{X}$. ☐

10.1. **Weak equivalences and cofibrations.**

Definition 10.6. A functor between small categories $F\colon \mathcal{C} \to \mathcal{D}$ is *homotopy right cofinal* (or *homotopy terminal*) if for every object β of $\mathcal{D}$ the simplicial set $\mathrm{N}(\beta\downarrow F)$ (see Definition 2.7) is contractible. If $\mathcal{C}$ is a subcategory of $\mathcal{D}$ and F is the inclusion, then $\mathcal{C}$ is called a *homotopy right cofinal subcategory* (or a *homotopy terminal subcategory*) of $\mathcal{D}$.

Remark 10.7. Lemma 2.28 implies that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is homotopy right cofinal if and only if for every object β of $\mathcal{D}$ the simplicial set $\mathrm{N}(\beta\downarrow F)^{\mathrm{op}}$ is contractible.

Theorem 10.8. *Let $\mathcal{M}$ be a simplicial model category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If F is homotopy right cofinal, $\boldsymbol{X}$ is an objectwise cofibrant $\mathcal{D}$-diagram in $\mathcal{M}$, and $F^*\boldsymbol{X} = \boldsymbol{X} \circ F$ is the induced $\mathcal{C}$-diagram, then the natural map of homotopy colimits $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \to \operatorname{hocolim}_{\mathcal{D}} \boldsymbol{X}$ is a weak equivalence.*

Proof. Theorem 10.4 implies that the natural map of homotopy colimits is isomorphic to the map of coends $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ induced by the map of $\mathcal{D}^{\mathrm{op}}$-diagrams of simplicial sets $F_*\colon \mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ (see Example 2.12). Since both of those $\mathcal{D}^{\mathrm{op}}$-diagrams of simplicial sets are free cell complexes (see Proposition 5.13 and Proposition 5.14) and F is homotopy right cofinal, that map of diagrams is a weak equivalence of cofibrant $\mathcal{D}^{\mathrm{op}}$-diagrams (see Theorem 5.1), and so the result follows from Corollary 7.22. ☐

Proposition 10.9. *Let $\mathcal{M}$ be a simplicial model category, let $\mathcal{D}$ be a small category, and let $F\colon \mathcal{C} \to \mathcal{D}$ be the inclusion of a subcategory. If $\boldsymbol{X}$ is an objectwise cofibrant $\mathcal{D}$-diagram in $\mathcal{M}$ and $F^*\boldsymbol{X} = \boldsymbol{X} \circ F$ is the induced $\mathcal{C}$-diagram, then the natural map of homotopy colimits $\operatorname{hocolim}_{\mathcal{C}} F^*\boldsymbol{X} \to \operatorname{hocolim}_{\mathcal{D}} \boldsymbol{X}$ is a cofibration of cofibrant objects.*

Proof. Theorem 10.4 implies that the natural map of homotopy colimits is isomorphic to the map of coends $\boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \boldsymbol{X} \otimes_{\mathcal{D}} \mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ induced by the map of $\mathcal{D}^{\mathrm{op}}$-diagrams of simplicial sets $F_*\colon \mathrm{N}(-\downarrow F)^{\mathrm{op}} \to \mathrm{N}(-\downarrow \mathcal{D})^{\mathrm{op}}$ (see Example 2.12). Thus, the result follows from Proposition 5.15, Corollary 7.16, and Corollary 7.18. ☐

10.2. **Homotopy colimits as colimits.**

Proposition 10.10. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a objectwise cofibrant $\mathcal{C}$-diagram in $\mathcal{M}$, then there are*

- *a natural objectwise cofibrant $\mathcal{C}$-diagram $\widetilde{\boldsymbol{X}}$ in $\mathcal{M}$,*
- *a natural isomorphism $R\colon \operatorname{colim} \widetilde{\boldsymbol{X}} \approx \operatorname{hocolim} \boldsymbol{X}$, and*
- *a natural objectwise weak equivalence $p\colon \widetilde{\boldsymbol{X}} \to \boldsymbol{X}$*

such that the induced map $\operatorname{colim} p\colon \operatorname{colim} \widetilde{\boldsymbol{X}} \to \operatorname{colim} \boldsymbol{X}$ equals the composition

$$\operatorname{colim} \widetilde{\boldsymbol{X}} \approx \operatorname{hocolim} \boldsymbol{X} = \boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(-\downarrow \mathcal{C})^{\mathrm{op}} \longrightarrow \boldsymbol{X} \otimes_{\mathcal{C}} \boldsymbol{P} \approx \operatorname{colim} \boldsymbol{X}$$

(where $\boldsymbol{P}$ is the constant $\mathcal{C}^{\mathrm{op}}$-diagram at a single point).

Proof. For each object α of $\mathcal{C}$ we have the forgetful functor $F_\alpha\colon (\mathcal{C}\downarrow\alpha) \to \mathcal{C}$ that takes the object $\beta \to \alpha$ of $(\mathcal{C}\downarrow\alpha)$ to β, and this induces a $(\mathcal{C}\downarrow\alpha)$-diagram $F_\alpha^*\boldsymbol{X} = \boldsymbol{X} \circ F_\alpha$ in $\mathcal{M}$; we let $\widetilde{\boldsymbol{X}}_\alpha = \operatorname{hocolim}_{(\mathcal{C}\downarrow\alpha)} F_\alpha^*\boldsymbol{X}$. Proposition 8.3 implies that each $\widetilde{\boldsymbol{X}}_\alpha$ is cofibrant.

If $*$ is the category with one object and no non-identity maps, we let $i_\alpha\colon * \to (\mathcal{C}\downarrow\alpha)$ take that object to the terminal object $1_\alpha\colon \alpha \to \alpha$ of $(\mathcal{C}\downarrow\alpha)$. Since the functor i_α is homotopy right cofinal (see Definition 10.6), the induced map of homotopy colimits $(i_\alpha)_*\colon \boldsymbol{X}_\alpha \to \widetilde{\boldsymbol{X}}_\alpha$ is a weak equivalence (see Theorem 10.8). We can also define a map $p_\alpha\colon \widetilde{\boldsymbol{X}}_\alpha \to \boldsymbol{X}_\alpha$ as the composition

$$\widetilde{\boldsymbol{X}}_\alpha = \operatorname*{hocolim}_{(\mathcal{C}\downarrow\alpha)} i_\alpha^* \boldsymbol{X} = i_\alpha^*\boldsymbol{X} \otimes_{(\mathcal{C}\downarrow\alpha)} \bigl(-\downarrow(\mathcal{C}\downarrow\alpha)\bigr)^{\mathrm{op}} \longrightarrow i_\alpha^*\boldsymbol{X} \otimes_{(\mathcal{C}\downarrow\alpha)} \boldsymbol{P} \approx \operatorname*{colim}_{(\mathcal{C}\downarrow\alpha)} i_\alpha^*\boldsymbol{X} \approx \boldsymbol{X}_\alpha$$

(where $\boldsymbol{P}$ is the constant $(\mathcal{C}\downarrow\alpha)^{\mathrm{op}}$-diagram at a point and that last isomorphism is because $1_\alpha\colon \alpha \to \alpha$ is a terminal object of $(\mathcal{C}\downarrow\alpha)$). Since $p_\alpha \circ (i_\alpha)_*$ is the identity map of $\boldsymbol{X}_\alpha$ and $(i_\alpha)_*$ is a weak equivalence, $p_\alpha\colon \widetilde{\boldsymbol{X}}_\alpha \to \boldsymbol{X}_\alpha$ is a weak equivalence.

If $\sigma\colon \alpha \to \alpha'$ is a map in $\mathcal{C}$, then σ induces a functor $(\mathcal{C}\downarrow\alpha) \to (\mathcal{C}\downarrow\alpha')$ that commutes with F_α and $F_{\alpha'}$ and so induces a map $\sigma_*\colon \widetilde{\boldsymbol{X}}_\alpha \to \widetilde{\boldsymbol{X}}_{\alpha'}$ that makes the square

$$\begin{array}{ccc} \widetilde{\boldsymbol{X}}_\alpha & \longrightarrow & \widetilde{\boldsymbol{X}}_{\alpha'} \\ {\scriptstyle p_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle p_{\alpha'}} \\ \boldsymbol{X}_\alpha & \longrightarrow & \boldsymbol{X}_{\alpha'} \end{array}$$

commute. **Fixme: Add more explanation of why the square commutes!** Thus, we have an objectwise weak equivalence of objectwise cofibrant diagrams $\widetilde{\boldsymbol{X}} \to \boldsymbol{X}$.

We now define $R\colon \operatorname{colim} \widetilde{\boldsymbol{X}} \to \operatorname{hocolim} \boldsymbol{X}$. If α is an object of $\mathcal{C}$, then $\widetilde{\boldsymbol{X}}_\alpha = \operatorname{hocolim}_{(\mathcal{C}\downarrow\alpha)} F_\alpha^*\boldsymbol{X}$. If $\beta \to \alpha$ is an object of $(\mathcal{C}\downarrow\alpha)$, then the forgetful functor

$$\bigl(\beta\downarrow(\mathcal{C}\downarrow\alpha)\bigr)^{\mathrm{op}} \longrightarrow (\beta\downarrow\mathcal{C})^{\mathrm{op}}$$

(which takes the object $\begin{array}{ccc}\beta & \longrightarrow & \gamma \\ & \searrow\ \swarrow & \\ & \alpha & \end{array}$ of $\bigl(\beta\downarrow(\mathcal{C}\downarrow\alpha)\bigr)^{\mathrm{op}}$ to the object $\beta \to \gamma$ of $(\beta\downarrow\mathcal{C})^{\mathrm{op}}$) induces a map

$$\boldsymbol{X}_\beta \otimes \mathrm{N}\bigl(\beta\downarrow(\mathcal{C}\downarrow\alpha)\bigr)^{\mathrm{op}} \longrightarrow \boldsymbol{X}_\beta \otimes \mathrm{N}(\beta\downarrow\mathcal{C})^{\mathrm{op}}$$

such that the composition

$$\boldsymbol{X}_\beta \otimes \mathrm{N}\bigl(\beta\downarrow(\mathcal{C}\downarrow\alpha)\bigr)^{\mathrm{op}} \longrightarrow \boldsymbol{X}_\beta \otimes \mathrm{N}(\beta\downarrow\mathcal{C})^{\mathrm{op}} \longrightarrow \operatorname{hocolim} \boldsymbol{X}$$

coequalizes the maps ϕ and ψ of (8.2). Thus, we have maps $\widetilde{\boldsymbol{X}}_\alpha \to \operatorname{hocolim} \boldsymbol{X}$ such that if $\sigma\colon \alpha \to \alpha'$ is a map in $\mathcal{C}$, then

$$\begin{array}{ccc} \widetilde{\boldsymbol{X}}_\alpha & \xrightarrow{\ \sigma_*\ } & \widetilde{\boldsymbol{X}}_{\alpha'} \\ & \searrow \quad \swarrow & \\ & \operatorname{hocolim} \boldsymbol{X} & \end{array}$$

commutes, and so we have an induced map $R\colon \operatorname{colim} \widetilde{\boldsymbol{X}} \to \operatorname{hocolim} \boldsymbol{X}$.

To show that R is an isomorphism, we'll construct an inverse map $S\colon \operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \widetilde{\boldsymbol{X}}$. For this, we'll use the decomposition of $\operatorname{hocolim} \boldsymbol{X}$ in Proposition 8.5. For each object α of $\mathcal{C}$ and simplex $\bigl((\alpha_0 \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_n), \sigma\colon \alpha \to \alpha_n\bigr)$ of $\mathrm{N}(\alpha\downarrow\mathcal{C})^{\mathrm{op}}$ (see Example 2.4), we map the summand $\boldsymbol{X}_\alpha \otimes \Delta[n]$ to $\widetilde{\boldsymbol{X}}_{\alpha_0} = \operatorname{hocolim}_{(\mathcal{C}\downarrow\alpha_0)} F_{\alpha_0}^* \boldsymbol{X}$ by viewing that simplex as a simplex of $\mathrm{N}\bigl((\alpha \to \alpha_0)\downarrow(\mathcal{C}\downarrow\alpha_0)\bigr)^{\mathrm{op}}$, and then compose with the injection $\widetilde{\boldsymbol{X}}_{\alpha_0} \to \operatorname{colim} \widetilde{\boldsymbol{X}}$. That defines a map from the right hand side of (8.6) to $\operatorname{colim} \widetilde{\boldsymbol{X}}$ that coequalizes the maps ϕ and ψ of (8.6), and thus defines our map $S\colon \operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \widetilde{\boldsymbol{X}}$. Using the decomposition of Proposition 8.5

on both $\operatorname{hocolim} \boldsymbol{X}$ and each $\widetilde{\boldsymbol{X}}_\alpha = \operatorname{hocolim} F_\alpha^* \boldsymbol{X}$ one can show that R and S are inverse isomorphisms.

Finally, since each category $(\mathcal{C}\downarrow\alpha)$ has the terminal object $1_\alpha \colon \alpha \to \alpha$, and each square

$$\begin{array}{ccc} \widetilde{\boldsymbol{X}}_\alpha = \operatorname*{hocolim}_{(\mathcal{C}\downarrow\alpha)} F_\alpha^* \boldsymbol{X} & \longrightarrow & \operatorname{colim} \operatorname*{hocolim}_{(\mathcal{C}\downarrow\alpha)} F_\alpha^* \boldsymbol{X} = \operatorname{hocolim} \boldsymbol{X} \\ \downarrow & & \downarrow \\ \boldsymbol{X}_\alpha = \operatorname*{colim}_{(\mathcal{C}\downarrow\alpha)} F_\alpha^* \boldsymbol{X} & \longrightarrow & \operatorname{colim} \operatorname*{colim}_{(\mathcal{C}\downarrow\alpha)} F_\alpha^* \boldsymbol{X} = \operatorname{colim} \boldsymbol{X} \end{array}$$

commutes, the map $\operatorname{colim} p \colon \operatorname{colim} \widetilde{\boldsymbol{X}} \to \operatorname{colim} \boldsymbol{X}$ has the required factorization. **Fixme: This paragraph needs more explanation!** □

11. Homotopy limits

Definition 11.1. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X} \colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then the *homotopy limit* $\operatorname{holim} \boldsymbol{X}$ of $\boldsymbol{X}$ is defined to be the end $\hom^{\mathcal{C}}(\mathrm{N}(\mathcal{C}\downarrow -), \boldsymbol{X})$ (see Definition 2.15 and Definition 7.7), that is, $\operatorname{holim} \boldsymbol{X}$ is the equalizer of the maps

$$\prod_{\alpha \in \mathrm{Ob}(\mathcal{C})} (\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(\sigma \colon \alpha \to \alpha') \in \mathcal{C}} (\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \tag{11.2}$$

where

- the projection of the map ϕ on the factor $(\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}$ indexed by $\sigma \colon \alpha \to \alpha'$ is the composition of a natural projection from the product with the map
$$(\sigma_*)^{1_{\mathrm{N}(\mathcal{C}\downarrow\alpha)}} \colon (\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \longrightarrow (\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}$$
and
- the projection of the map ψ on that same factor is the composition of a natural projection from the product with the map
$$(1_{\boldsymbol{X}_{\alpha'}})^{\mathrm{N}(\sigma_*)} \colon (\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha')} \longrightarrow (\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \quad .$$

Proposition 11.3. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is an objectwise fibrant $\mathcal{C}$-diagram in $\mathcal{M}$, then* $\operatorname{holim} \boldsymbol{X}$ *is fibrant.*

Proof. This follows from Corollary 7.18, Proposition 5.13, and Theorem 5.1. □

Theorem 11.4. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are objectwise fibrant $\mathcal{C}$-diagrams in $\mathcal{M}$ and $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence, then the induced map of homotopy limits $f_* \colon \operatorname{holim} \boldsymbol{X} \to \operatorname{holim} \boldsymbol{Y}$ is a weak equivalence of fibrant objects.*

Proof. This follows from Definition 11.1, Proposition 5.13, Theorem 5.1, and Corollary 7.20. □

11.1. **The space of maps from a homotopy colimit.**

Theorem 11.5. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ and Y is an object of $\mathcal{M}$, then $\mathrm{Map}(\boldsymbol{X}, Y)$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets and there is a natural isomorphism of simplicial sets $\mathrm{Map}(\mathrm{hocolim}_{\mathcal{C}} \boldsymbol{X}, Y) \approx \mathrm{holim}_{\mathcal{C}^{\mathrm{op}}} \mathrm{Map}(\boldsymbol{X}, Y)$.*

Proof. **Fixme: Fill this in.** □

11.2. **Decomposing homotopy limits.** In this section, we use Proposition 3.7 to describe the simplicial sets $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ as colimits of diagrams of standard simplices. This allows us to show that the homotopy limit of a diagram can be constructed as an equalizer of maps between simpler objects than the ones in the definition of the homotopy limit (see Definition 11.1 and Proposition 11.6). This will be used in Section 12 to show that the homotopy limit can be constructed as the total object (see Definition 12.2) of the cosimplicial replacement (see Definition 12.1) of the diagram.

Proposition 11.6. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then $\mathrm{holim}\, \boldsymbol{X}$ is naturally isomorphic to the equalizer of the maps*

$$\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \Big(\prod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{\alpha'})^{\Delta[n]}\Big) \times \Big(\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n>0\\ 0\leq i\leq n\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n-1]}\Big) \times \Big(\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ 0\leq i\leq n\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n+1]}\Big) \tag{11.7}$$

where the projections of the maps ϕ and ψ onto the first factor are such that

- *the projection of ϕ onto the factor $(\boldsymbol{X}_{\alpha'})^{\Delta[n]}$ indexed by $\big((\sigma\colon \alpha\to\alpha')\in \mathcal{C}, n, \big((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon \alpha_n\to\alpha\big)\big)$ is the composition of the projection from the product onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon\alpha_n\to\alpha\big)\big)$ with the map $(\sigma_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_\alpha)^{\Delta[n]}\to(\boldsymbol{X}_{\alpha'})^{\Delta[n]}$, and*
- *the projection of ψ onto that same factor is the projection from the product onto the factor $(\boldsymbol{X}_{\alpha'})^{\Delta[n]}$ indexed by $\big(\alpha', n, \big((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \sigma\tau\colon\alpha_n\to\alpha'\big)\big)$,*

the projections onto the second factor are such that

- *the projection of ϕ onto the $(\boldsymbol{X}_\alpha)^{\Delta[n-1]}$ indexed by $\big(\alpha, n, i, \sigma\colon\Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)\big)$ is the projection onto the $(\boldsymbol{X}_\alpha)^{\Delta[n-1]}$ indexed by α, $(n-1)$, and the composition $\Delta[n-1]\xrightarrow{d^i}\Delta[n]\xrightarrow{\sigma}\mathrm{N}(\mathcal{C}\downarrow)$, and*
- *the projection of ψ onto that same factor is the composition of the projection onto the $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by (α, n, σ) with the map $(1_{\boldsymbol{X}_\alpha})^{d^i}\colon(\boldsymbol{X}_\alpha)^{\Delta[n]}\to(\boldsymbol{X}_\alpha)^{\Delta[n-1]}$,*

and the projections onto the third factor are such that

- *the projection of ϕ onto the $(\boldsymbol{X}_\alpha)^{\Delta[n+1]}$ indexed by $\big(\alpha, n, i, \sigma\colon \Delta[n] \to \mathrm{N}(\mathcal{C}\downarrow\alpha)\big)$ is the projection onto the $(\boldsymbol{X}_\alpha)^{\Delta[n+1]}$ indexed by α, $(n+1)$, and the composition $\Delta[n+1] \xrightarrow{s^i} \Delta[n] \xrightarrow{\sigma} \mathrm{N}(\mathcal{C}\downarrow\alpha)$, and*
- *the projection of ψ onto that same factor is the composition of the projection onto the $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by σ with the map $(1_{\boldsymbol{X}_\alpha})^{s^i}\colon (\boldsymbol{X}_\alpha)^{\Delta[n]} \to (\boldsymbol{X}_\alpha)^{\Delta[n+1]}$.*

Proof. If we let

$$G = \Big(\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n>0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n-1]}\Big) \times \Big(\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n+1]}\Big)$$

and

$$H = \Big(\prod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}\\ n>0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{\alpha'})^{\Delta[n-1]}\Big) \times \Big(\prod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}\\ n\ge 0\\ 0\le i\le n\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{\alpha'})^{\Delta[n+1]}\Big)$$

then in the diagram

$$\begin{array}{ccc}
\displaystyle\prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} (\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} & \rightrightarrows & \displaystyle\prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}} (\boldsymbol{X}'_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \\
\downarrow & & \downarrow \\
\displaystyle\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]} & \rightrightarrows & \displaystyle\prod_{\substack{(\sigma\colon \alpha\to\alpha')\in\mathcal{C}\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{\alpha'})^{\Delta[n]} \\
\downarrow\downarrow & & \downarrow\downarrow \\
G & \rightrightarrows & H
\end{array}$$

both of the columns are products of equalizer diagrams (see Proposition 4.4) and are thus equalizer diagrams, and each of the squares commutes if you use either both upper horizontal arrows or both lower horizontal arrows and either both left vertical arrows or both right vertical arrows.

Since the column on the left is an equalizer diagram, for every object W of $\mathcal{M}$ maps

$$W \to \prod_{\alpha\in\mathrm{Ob}(\mathcal{C})} (\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \qquad \text{correspond to maps} \qquad W \to \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$$

that equalize the pair of parallel downward arrows on the left. Since equalizers are monomorphisms (see Lemma 7.1), such a map $W \to \prod_{\alpha\in\mathrm{Ob}(\mathcal{C})}(\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}$ equalizes the upper pair of parallel horizontal arrows if and only if the corresponding map equalizes the middle pair of parallel horizontal arrows. Thus, maps $W \to \prod_{\alpha\in\mathrm{Ob}(\mathcal{C})}(\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}$ that equalize the top pair of parallel horizontal arrows correspond to maps $W \to \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$ that equalize both the middle horizontal pair of parallel arrows the the left pair of vertical parallel arrows, and so a terminal object among the former corresponds to a terminal object among the latter. □

11.2.1. *Homotopy limits and basic simplices.* Proposition 13.1 decomposed the objects used in the definition of the homotopy limit into simpler objects, so that the homotopy limit was presented as a subobject of a product of objects $\boldsymbol{X}_\alpha^{\Delta[n]}$, one for each n-simplex of the simplicial set $\mathrm{N}(\mathcal{C}\downarrow\alpha)$, for each object α of $\mathcal{C}$. The following lemma shows that the map from the equalizer is actually determined by the map to those factors indexed by the "basic" simplices, where the "basic" simplices are the ones of the form $\big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)$ (see Example 2.14), i.e., the simplices of $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ for some α in which the map from the final vertex of the simplex to α is the identity map. (The basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(\mathcal{C}\downarrow -)$; see Proposition 5.13.) We will use this in Theorem 12.5 to show that the homotopy limit is isomorphic to the total object (see Definition 12.2) of the cosimplicial replacement (see Definition 12.1) of the diagram, which is built using only the summands indexed by the basic simplices.

Lemma 11.8. *Let $\mathcal{M}$ be a simplicial model category, let $\mathcal{C}$ be a small category, and let $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ be a $\mathcal{C}$-diagram in $\mathcal{M}$. If*

$$h\colon E \longrightarrow \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$$

is a map that equalizes the maps ϕ and ψ of (11.7), then for every factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\big)\big)$ the diagram

$$\begin{array}{ccc} & E & \\ {\scriptstyle h_{(1_{(\alpha_n)})}}\swarrow & & \searrow{\scriptstyle h_\tau} \\ (\boldsymbol{X}_{(\alpha_n)})^{\Delta[n]} & \xrightarrow[(\tau_*)^{(1_{\Delta[n]})}]{} & (\boldsymbol{X}_\alpha)^{\Delta[n]} \end{array}$$

commutes, where h_τ is the composition of h with the projection onto that factor and $h_{(1_{(\alpha_n)})}$ is the composition of h with the projection onto the factor indexed by $\big(\alpha_n, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$.

Proof. The projection of ϕ onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\big((\tau\colon \alpha_n \to \alpha) \in \mathcal{C}, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ in the first factor on the right of (11.7) is the composition $(\tau_*)^{(1_{\Delta[n]})} \circ h_{(1_{(\alpha_n)})}$, and the projection of ψ onto that same factor is h_τ. □

12. The total object of the cosimplicial replacement of a diagram

The main result in this section is Theorem 12.5, which shows that the homotopy limit of a diagram can be constructed as the total object (see Definition 12.2) of the cosimplicial replacement (see Definition 12.1) of the diagram. This constructs the homotopy limit starting with the product of a much smaller collection of objects than the collection used in Proposition 11.6.

A "basic simplex" of $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ is one in which the map from the final vertex of the simplex in $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ to α is the identity map, i.e., the ones of the form $\big((\alpha_0 \to$

$\alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)$ (see Example 2.14). (The basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(\mathcal{C}\downarrow -)$; see Proposition 5.13.) For every object α of $\mathcal{C}$ and every simplex $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\bigr)$ of $\mathrm{N}(\mathcal{C}\downarrow\alpha)$, there are

- an object α_n of $\mathcal{C}$,
- a basic simplex $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)$ in $\mathrm{N}(\mathcal{C}\downarrow\alpha_n)$, and
- a map $\tau\colon \alpha_n \to \alpha$ in $\mathcal{C}$ such that $\tau_*\colon \mathrm{N}(\mathcal{C}\downarrow\alpha_n) \to \mathrm{N}(\mathcal{C}\downarrow\alpha)$ takes that basic simplex $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)$ to our simplex $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\bigr)$,

and such a triple is unique.

The total object of the cosimplicial replacement is constructed as a subobject of the product of a copy of $\boldsymbol{X}_{\alpha_n}^{\Delta[n]}$ for every basic simplex $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)$. Lemma 11.8 implies that we can construct a map from the total object of the cosimplicial replacement of a diagram to the homotopy limit by mapping to the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\bigr)$ (see Proposition 11.6) by the composition of the map to the factor indexed by $\bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)$ (which, in the proof of Theorem 12.5, is denoted simply $(\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}$) and $(\tau_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_\alpha)^{\Delta[n]}$.

Definition 12.1. Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then the *cosimplicial replacement* of $\boldsymbol{X}$ is the cosimplicial object $\prod^* \boldsymbol{X}$ in $\mathcal{M}$ such that

$$\Bigl(\prod\nolimits^* \boldsymbol{X}\Bigr)^n = \prod_{(\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}} \boldsymbol{X}_{\alpha_n}$$

and such that

- the projection of the coface map $d^i\colon \bigl(\prod^* \boldsymbol{X}\bigr)^n \to \bigl(\prod^* \boldsymbol{X}\bigr)^{n+1}$ onto the factor $\boldsymbol{X}_{\alpha_{n+1}}$ indexed by $(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_n} \alpha_{n+1}) \in \mathcal{C}$ is the composition of a projection from the product with
  - the identity map from the factor $\boldsymbol{X}_{\alpha_{n+1}}$ indexed by $(\alpha_1 \xrightarrow{\sigma_1} \alpha_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} \alpha_{n+1}) \in \mathcal{C}$ if $i = 0$,
  - the identity map from the factor $\boldsymbol{X}_{\alpha_{n+1}}$ indexed by $(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{i-2}} \alpha_{i-1} \xrightarrow{\sigma_i\sigma_{i-1}} \alpha_{i+1} \xrightarrow{\sigma_{i+1}} \cdots \xrightarrow{\sigma_n} \alpha_{n+1}) \in \mathcal{C}$ if $0 < i < n+1$, and
  - the map $(\sigma_n)_*\colon \boldsymbol{X}_{\alpha_n} \to \boldsymbol{X}_{\alpha_{n+1}}$ from the factor $\boldsymbol{X}_{\alpha_n}$ indexed by $(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n) \in \mathcal{C}$ if $i = n+1$

  and
- the projection of the codegeneracy map $s^i\colon \bigl(\prod^* \boldsymbol{X}\bigr)^n \to \bigl(\prod^* \boldsymbol{X}\bigr)^{n-1}$ onto the factor $\boldsymbol{X}_{\alpha_{n-1}}$ indexed by $(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-2}} \alpha_{n-1}) \in \mathcal{C}$ is the composition of a projection from the product with the identity map from the factor $\boldsymbol{X}_{\alpha_{n-1}}$ indexed by $(\alpha_0 \xrightarrow{\sigma_0} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{i-1}} \alpha_i \xrightarrow{1_{\alpha_i}} \alpha_i \xrightarrow{\sigma_i} \cdots \xrightarrow{\sigma_{n-2}} \alpha_{n-1}) \in \mathcal{C}$ for $0 \le i \le n-1$.

Definition 12.2. Let $\mathcal{M}$ be a simplicial model category. If $\boldsymbol{Y}$ is a cosimplicial object in $\mathcal{M}$, then its *total object* (or *corealization*) $\operatorname{Tot}\boldsymbol{Y}$ is defined to be the end $\hom^{\boldsymbol{\Delta}}(\Delta, \boldsymbol{Y})$ (see Definition 3.3 and Definition 7.7), that is, $\operatorname{Tot}\boldsymbol{Y}$ is the equalizer of the maps

$$\prod_{n\geq 0} (\boldsymbol{Y}^n)^{\Delta[n]} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{(\sigma\colon [n]\to[k])\in\boldsymbol{\Delta}} (\boldsymbol{Y}^k)^{\Delta[n]}$$

where

- the projection of the map ϕ onto the factor $(\boldsymbol{X}^k)^{\Delta[n]}$ indexed by $\sigma\colon [n] \to [k]$ is the composition of a projection from the product with the map
$$(\sigma_*)^{1_{\Delta[n]}}\colon (\boldsymbol{Y}^n)^{\Delta[n]} \longrightarrow (\boldsymbol{Y}^k)^{\Delta[n]}$$
and
- the projection of the map ψ onto that same factor is the composition of a projection from the product with the map
$$(1_{\boldsymbol{Y}_k})^{\sigma_*}\colon (\boldsymbol{Y}^k)^{\Delta[k]} \longrightarrow (\boldsymbol{Y}^k)^{\Delta[n]} \quad .$$

Proposition 12.3. *Let $\mathcal{M}$ be a simplicial model category. If $\boldsymbol{Y}$ is a cosimplicial object in $\mathcal{M}$, then $\operatorname{Tot}\boldsymbol{Y}$ is naturally isomorphic to the equalizer of the maps*

$$\prod_{n\geq 0} (\boldsymbol{Y}_n)^{\Delta[n]} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \Big(\prod_{\substack{n>0\\0\leq i\leq n}} (\boldsymbol{Y}_n)^{\Delta[n-1]}\Big) \times \Big(\prod_{\substack{n\geq 0\\0\leq i\leq n}} (\boldsymbol{Y}_n)^{\Delta[n+1]}\Big) \tag{12.4}$$

where the projections of ϕ and ψ onto the first factor are such that

- *the projection of ϕ onto the $(\boldsymbol{Y}_n)^{\Delta[n-1]}$ indexed by (n,i) is the composition of a projection from the product with the map $(d^i)^{1_{\Delta[n-1]}}\colon (\boldsymbol{Y}_{n-1})^{\Delta[n-1]} \to (\boldsymbol{Y}_n)^{\Delta[n-1]}$, and*
- *the projection of ψ onto that same factor is the composition of a projection from the product with the map $(1_{\boldsymbol{Y}_n})^{d^i}\colon (\boldsymbol{Y}_n)^{\Delta[n]} \to (\boldsymbol{Y}_{n-1})^{\Delta[n-1]}$*

and the projections onto the second factor are such that

- *the projection of ϕ onto the $(\boldsymbol{Y}_n)^{\Delta[n+1]}$ indexed by (n,i) is the composition of a projection from the product with the map $(s^i)^{1_{\Delta[n+1]}}\colon (\boldsymbol{Y}_{n+1})^{\Delta[n+1]} \to (\boldsymbol{Y}_n)^{\Delta[n+1]}$, and*
- *the projection of ψ onto that same factor is the composition of a projection from the product with the map $(1_{\boldsymbol{Y}_n})^{s^i}\colon (\boldsymbol{Y}_n)^{\Delta[n]} \to (\boldsymbol{Y}_n)^{\Delta[n+1]}$.*

Proof. Equalizing the projections of ϕ and ψ onto the first factor equalizes the coface operators and equalizing the projections onto the second factor equalizes the codegeneracy operators. Since every morphism in the cosimplicial indexing category is a finite composition of coface and codegeneracy operators, the result follows from Proposition 7.24. □

Theorem 12.5. *Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{C}$ be a small category. If $\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$, then $\operatorname{holim}\boldsymbol{X}$, the homotopy limit of $\boldsymbol{X}$, is naturally isomorphic to $\operatorname{Tot}\prod^{*}\boldsymbol{X}$, the total object of the cosimplicial replacement of $\boldsymbol{X}$.*

Proof. We will define a natural isomorphism from the equalizer of (12.4) for $\boldsymbol{Y} = \prod^{*} \boldsymbol{X}$ to the equalizer of (11.7). We define

$$P\colon \prod_{n\geq 0} (\boldsymbol{Y}_n)^{\Delta[n]} \longrightarrow \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$$

by letting the projection of P onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\bigl(\alpha, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \sigma\colon \alpha_n \to \alpha\bigr)\bigr)$ be the projection onto the $(\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$ indexed by $(\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}$ composed with the map $(\sigma_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_\alpha)^{\Delta[n]}$. To show that this defines a map from the equalizer of (12.4) to the equalizer of (11.7) we must show that if $f\colon E \to \prod_{n\geq 0}(\boldsymbol{Y}_n)^{\Delta[n]}$ is a map that equalizes the maps ϕ and ψ of (12.4), then the composition Pf equalizes the maps ϕ and ψ of (11.7).

Let $f\colon E \to \prod_{n\geq 0}(\boldsymbol{Y}_n)^{\Delta[n]}$ be a map that equalizes the maps ϕ and ψ of (12.4). The projection of ϕ onto the first factor on the right of (11.7), projected onto the factor $(\boldsymbol{X}_{\alpha'})^{\Delta[n]}$ indexed by $\bigl((\sigma\colon \alpha \to \alpha') \in \mathcal{C}, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n), \tau\colon \alpha_n \to \alpha\bigr)\bigr)$, when composed with P, is the projection onto the $(\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$ indexed by $(\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}$ composed with the map $(\tau_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_\alpha)^{\Delta[n]}$ composed with the map $(\sigma_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_\alpha)^{\Delta[n]} \to (\boldsymbol{X}_{\alpha'})^{\Delta[n]}$. The projection of ψ onto that same factor, when composed with P, is the projection onto the $(\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$ indexed by $(\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}$ composed with the map $\bigl((\sigma\tau)_*\bigr)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_{\alpha'})^{\Delta[n]}$, composed with the identity map. Since $\sigma_*\tau_* = (\sigma\tau)_*$, the map P equalizes the projection onto the first factor of (11.7) of ϕ and ψ, an so the composition Pf equalizes them as well.

Since f equalizes the maps ϕ and ψ of (12.4), Pf equalizes the projections of ϕ and ψ of (11.7) onto the second and third factors of the right side of (11.7), and so P defines a map $\widetilde{P}$ from the equalizer of (12.4) to the equalizer of (11.7).

To show that $\widetilde{P}$ is an isomorphism, we'll define an inverse map $\widetilde{Q}$. We begin by defining

$$Q\colon \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]} \longrightarrow \prod_{n\geq 0} (\boldsymbol{Y}_n)^{\Delta[n]} \approx \prod_{\substack{n\geq 0\\ (\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}}} (\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$$

by letting the projection of Q onto the factor $(\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$ indexed by $\bigl(n, (\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}\bigr)$ be the projection onto the $(\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$ indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\bigr)\bigr)$. To show that this defines a map from the equalizer of (11.7) to the equalizer of (12.4), we let $g\colon F \to \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\geq 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$ be a map that equalizes the maps ϕ and ψ of (11.7). The fact that g equalizes the projections of ϕ and ψ onto the second and third factors of (11.7) implies that the composition Qg equalizes (12.4). Thus, Q defines a map $\widetilde{Q}$ from the equalizer of (11.7) to the equalizer of (12.4).

We will now show that $\widetilde{Q}$ is a inverse for $\widetilde{P}$. The composition $\widetilde{Q}\widetilde{P}$ is the identity of the equalizer of (12.4) because QP is the identity of $\prod_{n\geq 0}(\boldsymbol{Y}_n)^{\Delta[n]}$. To see that

$\widetilde{P}\widetilde{Q}$ is the identity, let $g\colon F \to \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$ equalize the maps ϕ and ψ of (11.7). The composition $PQ\colon \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]} \to \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$ is such that its projection onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\big)\big)$ is the composition of the projection onto the $(\boldsymbol{X}_{\alpha_n})^{\Delta[n]}$ indexed by $\big(\alpha_n, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ composed with the map $(\tau_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_\alpha)^{\Delta[n]}$, and since g equalizes the maps ϕ and ψ of (11.7), this is also true of the map PQg (see Lemma 11.8). Thus, $PQg = g$, and so $\widetilde{P}\widetilde{Q}$ is the identity of the equalizer of (11.7). □

13. Changing the indexing category of a homotopy limit

Let $\mathcal{M}$ be a simplicial category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then there is an induced $\mathcal{C}$-diagram $F^*\boldsymbol{X}$ in $\mathcal{M}$, defined as the composition $F^*\boldsymbol{X} = \boldsymbol{X} \circ F$. In this section, we show that the homotopy limit $\mathrm{holim}_{\mathcal{C}} F^*\boldsymbol{X}$ of the induced diagram can be constructed as the end $\hom^{\mathcal{D}}(\mathrm{N}(F\downarrow -), \boldsymbol{X})$ over the category $\mathcal{D}$ (see Definition 2.20 and Theorem 13.4).

The reason this theorem is true is that, although $\mathrm{holim}_{\mathcal{C}} F^*\boldsymbol{X}$ is constructed as a subobject of the (rather large) product of a copy of $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ for every object α of $\mathcal{C}$ and every n-simplex of $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ (see Proposition 11.6), it can also be constructed from the much smaller product of $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ for only certain "basic simplices" of the simplicial sets $\mathrm{N}(\mathcal{C}\downarrow -)$. (The basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(\mathcal{C}\downarrow -)$; see Corollary 5.11 and Proposition 5.13.) Similarly, although the end $\hom^{\mathcal{D}}(\mathrm{N}(D\downarrow -), \boldsymbol{X})$ is defined as a subobject of the (rather large) product of a copy of $(\boldsymbol{X}_\beta)^{\Delta[n]}$ for every object β of $\mathcal{D}$ and every n-simplex of $\mathrm{N}(F\downarrow\beta)$ (see Proposition 13.1), it can also be constructed from the much smaller product of $(\boldsymbol{X}_\beta)^{\Delta[n]}$ for only certain "basic simplices" of the simplicial sets $\mathrm{N}(F\downarrow -)$ (again, the basic simplices are in fact the elements of a basis for the free cell complex $\mathrm{N}(F\downarrow -)$; see Corollary 5.11 and Proposition 5.14, and the maps $\mathrm{N}(\mathcal{C}\downarrow\alpha) \to \mathrm{N}(F\downarrow F\alpha)$ (see Example 2.19) take basic simplices to basic simplices. Theorem 13.4 shows that the map F_* of Lemma 2.19 defines an isomorphism $\mathrm{holim}_{\mathcal{C}} F^*\boldsymbol{X} = \hom^{\mathcal{C}}(\mathrm{N}(\mathcal{C}\downarrow -), F^*\boldsymbol{X}) \approx \hom^{\mathcal{D}}(\mathrm{N}(F\downarrow -), \boldsymbol{X})$. (For a much shorter proof of this that uses the mapping properties of a basis of a free cell complex, see [7, Prop. 19.6.6].) We begin by showing that the end $\hom^{\mathcal{D}}(\mathrm{N}(F\downarrow -), \boldsymbol{X})$ has a decomposition similar to that of $\mathrm{holim}_{\mathcal{C}} F^*\boldsymbol{X} = \hom^{\mathcal{C}}((\mathcal{C}\downarrow -), F^*\boldsymbol{X})$ (see Proposition 11.6).

Proposition 13.1. *Let $\mathcal{M}$ be a simplicial model category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, then the end*

$\hom^{\mathcal{C}}(\mathrm{N}(F\downarrow -), \boldsymbol{X})$ *is naturally isomorphic to the equalizer of the maps*

$$\prod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(F\downarrow\beta)}} (\boldsymbol{X}_\beta)^{\Delta[n]} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \tag{13.2}$$

$$\Bigl(\prod_{\substack{(\sigma\colon \beta\to\beta')\in\mathcal{D}\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(F\downarrow\beta)}} (\boldsymbol{X}_{\beta'})^{\Delta[n]}\Bigr) \times \Bigl(\prod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ 0\leq i\leq n\\ \Delta[n]\to\mathrm{N}(F\downarrow\beta)}} (\boldsymbol{X}_\beta)^{\Delta[n-1]}\Bigr) \times \Bigl(\prod_{\substack{\beta\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ 0\leq i\leq n\\ \Delta[n]\to\mathrm{N}(F\downarrow\beta)}} (\boldsymbol{X}_\beta)^{\Delta[n+1]}\Bigr)$$

where the projections of the maps ϕ and ψ onto the first factor are such that

- *the projection of ϕ onto the factor $(\boldsymbol{X}_{\beta'})^{\Delta[n]}$ indexed by $\bigl((\sigma\colon \beta\to\beta')\in \mathcal{D}, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon F\alpha_n\to\beta\bigr)\bigr)$ is the composition of the projection from the product onto the factor $(\boldsymbol{X}_\beta)^{\Delta[n]}$ indexed by $\bigl(\beta, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon F\alpha_n\to\beta\bigr)\bigr)$ with the map $(\sigma_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_\beta)^{\Delta[n]}\to(\boldsymbol{X}_{\beta'})^{\Delta[n]}$, and*
- *the projection of ψ onto that same factor is the projection from the product onto the factor $(\boldsymbol{X}_{\beta'})^{\Delta[n]}$ indexed by $\bigl(\beta', n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in \mathcal{C}, \sigma\tau\colon F\alpha_n\to\beta'\bigr)\bigr)$,*

the projections onto the second factor are such that

- *the projection of ϕ onto the $(\boldsymbol{X}_\beta)^{\Delta[n-1]}$ indexed by $\bigl(\beta, n, i, \sigma\colon \Delta[n]\to \mathrm{N}(F\downarrow\beta)\bigr)$ is the projection onto the $(\boldsymbol{X}_\beta)^{\Delta[n-1]}$ indexed by β, $(n-1)$, and the composition $\Delta[n-1]\xrightarrow{d^i}\Delta[n]\xrightarrow{\sigma}\mathrm{N}(F\downarrow\beta)$, and*
- *the projection of ψ onto that same factor is the composition of the projection onto the $(\boldsymbol{X}_\beta)^{\Delta[n]}$ indexed by (β, n, σ) with the map $(1_{\boldsymbol{X}_\beta})^{d^i}\colon (\boldsymbol{X}_\beta)^{\Delta[n]}\to(\boldsymbol{X}_\beta)^{\Delta[n-1]}$,*

and the projections onto the third factor are such that

- *the projection of ϕ onto the $(\boldsymbol{X}_\beta)^{\Delta[n+1]}$ indexed by $\bigl(\beta, n, i, \sigma\colon \Delta[n]\to \mathrm{N}(F\downarrow\beta)\bigr)$ is the projection onto the $(\boldsymbol{X}_\beta)^{\Delta[n+1]}$ indexed by β, $(n+1)$, and the composition $\Delta[n+1]\xrightarrow{s^i}\Delta[n]\xrightarrow{\sigma}\mathrm{N}(F\downarrow\beta)$, and*
- *the projection of ψ onto that same factor is the composition of the projection onto the $(\boldsymbol{X}_\beta)^{\Delta[n]}$ indexed by (β, n, σ) with the map $(1_{\boldsymbol{X}_\beta})^{s^i}\colon (\boldsymbol{X}_\beta)^{\Delta[n]}\to(\boldsymbol{X}_\beta)^{\Delta[n+1]}$.*

Proof. This is identical to the proof of Proposition 11.6, changing $\alpha\in\mathrm{Ob}(\mathcal{C})$ to $\beta\in\mathrm{Ob}(\mathcal{D})$, $(\sigma\colon\alpha\to\alpha')\in\mathcal{C}$ to $(\sigma\colon\beta\to\beta')\in\mathcal{D}$, $\mathrm{N}(\mathcal{C}\downarrow\alpha)$ to $\mathrm{N}(F\downarrow\beta)$, and $\mathrm{N}(\mathcal{C}\downarrow\alpha')$ to $\mathrm{N}(F\downarrow\beta')$. □

Lemma 13.3. *Let $\mathcal{M}$ be a simplicial model category, let $F\colon \mathcal{C}\to\mathcal{D}$ be a functor between small categories, and let $\boldsymbol{X}\colon\mathcal{D}\to\mathcal{M}$ be a $\mathcal{D}$-diagram in $\mathcal{M}$. If*

$$h\colon E\longrightarrow \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{D})\\ n\geq 0\\ \Delta[n]\to\mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$$

is a map that equalizes the maps ϕ and ψ of (13.2)*, then for every factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\bigl(\alpha, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon F\alpha_n \to \alpha\bigr)\bigr)$ the diagram*

$$\begin{array}{ccc} & E & \\ {}^{h_{(1_{(F\alpha_n)})}}\swarrow & & \searrow^{h_\tau} \\ (\boldsymbol{X}_{(F\alpha_n)})^{\Delta[n]} & \xrightarrow[(\tau_*)^{(1_{\Delta[n]})}]{} & (\boldsymbol{X}_\alpha)^{\Delta[n]} \end{array}$$

commutes, where h_τ is the projection onto that factor and $h_{(1_{(F\alpha_n)})}$ is the projection onto the factor indexed by $\bigl(F\alpha_n, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)\bigr)$.

Proof. The projection of ϕ onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\bigl((\tau\colon F\alpha_n \to \alpha) \in \mathcal{D}, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\bigr)\bigr)$ in the first factor on the right of (13.2) is the composition $(\tau_*)^{(1_{\Delta[n]})} \circ h_{(1_{(F\alpha_n)})}$, and the projection of ψ onto that same factor is h_τ. □

Theorem 13.4. *If $\mathcal{M}$ is a simplicial model category, $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between small categories, $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in $\mathcal{M}$, and $F^*\boldsymbol{X} = \boldsymbol{X} \circ F\colon \mathcal{C} \to \mathcal{M}$ is the induced $\mathcal{C}$-diagram in $\mathcal{M}$, then there is a natural isomorphism of ends*

$$\hom^{\mathcal{D}}(\mathrm{N}(F\downarrow -), \boldsymbol{X}) \quad \approx \quad \hom^{\mathcal{C}}(\mathrm{N}(\mathcal{C}\downarrow -), F^*\boldsymbol{X}) = \operatorname*{holim}_{\mathcal{C}} F^*\boldsymbol{X}$$

and the natural map of homotopy limits $\operatorname{holim}_{\mathcal{D}} \boldsymbol{X} \to \operatorname{holim}_{\mathcal{C}} F^*\boldsymbol{X}$ is isomorphic to the map of ends induced by the map of $\mathcal{D}$-diagrams $\mathrm{N}(F\downarrow -) \to \mathrm{N}(D\downarrow -)$ (see Example 2.21*).*

Proof. Proposition 11.6 applied to the $\mathcal{C}$-diagram $F^*\boldsymbol{X}$ implies that $\operatorname{holim}_{\mathcal{C}} F^*\boldsymbol{X} = \hom^{\mathcal{C}}(\mathrm{N}(\mathcal{C}\downarrow -), F^*\boldsymbol{X})$ is naturally isomorphic to the equalizer of the maps

$$(13.5) \qquad \prod_{\substack{\alpha \in \mathrm{Ob}(\mathcal{C}) \\ n \geq 0 \\ \Delta[n] \to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]} \quad \overset{\phi}{\underset{\psi}{\rightrightarrows}}$$

$$\Bigl(\prod_{\substack{(\sigma\colon \alpha \to \alpha') \in \mathcal{C} \\ n \geq 0 \\ \Delta[n] \to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha'})^{\Delta[n]}\Bigr) \times \Bigl(\prod_{\substack{\alpha \in \mathrm{Ob}(\mathcal{C}) \\ n > 0 \\ 0 \leq i \leq n \\ \Delta[n] \to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n-1]}\Bigr) \times \Bigl(\prod_{\substack{\alpha \in \mathrm{Ob}(\mathcal{C}) \\ n \geq 0 \\ 0 \leq i \leq n \\ \Delta[n] \to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n+1]}\Bigr)$$

where the maps ϕ and ψ are as described in Proposition 11.6. We will define a natural isomorphism from the equalizer of (13.2) to the equalizer of (13.5).

We begin by defining

$$R\colon \prod_{\substack{\alpha \in \mathrm{Ob}(\mathcal{D}) \\ n \geq 0 \\ \Delta[n] \to \mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]} \longrightarrow \prod_{\substack{\alpha \in \mathrm{Ob}(\mathcal{C}) \\ n \geq 0 \\ \Delta[n] \to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]}$$

by letting the projection of R onto the factor $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ indexed by $\bigl(\alpha, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\bigr)\bigr)$ be the projection from the product onto the factor $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ indexed by $\bigl(F\alpha, n, \bigl((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, F\tau\colon F\alpha_n \to F\alpha\bigr)\bigr)$.

To show that R induces a map from the equalizer of (13.2) to the equalizer of (13.5), let $f\colon E \to \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$ be a map that equalizes the maps ϕ and ψ of (13.2). The projection of ϕ onto the first factor on the right of (13.5), projected onto the factor $(\boldsymbol{X}_{F\alpha'})^{\Delta[n]}$ indexed by $\bigl((\sigma\colon \alpha\to\alpha')\in\mathcal{C}, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon \alpha_n\to\alpha\bigr)\bigr)$, when composed with R, is the projection onto the factor $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ indexed by $\bigl(F\alpha, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, F\tau\colon F\alpha_n\to F\alpha\bigr)\bigr)$ composed with the map $((F\sigma)_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{F\alpha})^{\Delta[n]}\to(\boldsymbol{X}_{F\alpha'})^{\Delta[n]}$. The projection of ψ onto that same factor, when composed with R, is the projection onto the factor $(\boldsymbol{X}_{F\alpha'})^{\Delta[n]}$ indexed by $\bigl(F\alpha', n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, F(\sigma\tau)\colon F\alpha_n\to F\alpha'\bigr)\bigr)$. Since the map f equalizes the maps ϕ and ψ of (13.2), Rf equalizes the projections of ϕ and ψ of (13.5) onto the first factor on the right (see Lemma 13.3).

Similarly, since f equalizes the projections of ϕ and ψ of (13.2) onto the second and third factors on the right of (13.2), Rf equalizes the projections of ϕ and ψ of (13.5) onto the second and third factors of (13.5). Thus, R defines a map $\widetilde{R}$ from the equalizer of (13.2) to the equalizer of (13.5).

To show that $\widetilde{R}$ is an isomorphism, we'll define an inverse map $\widetilde{S}$. We begin by defining

$$S\colon \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]} \longrightarrow \prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$$

by letting the projection of S onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\bigl(\alpha, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon F\alpha_n\to\alpha\bigr)\bigr)$ be the projection onto the factor $(\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}$ indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, 1\colon \alpha_n\to\alpha_n\bigr)\bigr)$ composed with the map $(\tau_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}\to(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$. To show that this defines a map from the equalizer of (13.5) to the equalizer of (13.2), we let $g\colon E\to\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ be a map that equalizes the maps ϕ and ψ of (13.5). The projection of ϕ onto the first factor on the right of (13.2), projected onto the factor $(\boldsymbol{X}_{\alpha'})^{\Delta[n]}$ indexed by $\bigl((\sigma\colon\alpha\to\alpha')\in\mathcal{D}, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, \tau\colon F\alpha_n\to\alpha\bigr)\bigr)$, when composed with S, is the projection onto the factor $(\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}$ indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, 1\colon\alpha_n\to\alpha_n\bigr)\bigr)$ composed with the map $(\tau_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}\to(\boldsymbol{X}_\alpha)^{\Delta[n]}$, composed with the map $(\sigma_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_\alpha)^{\Delta[n]}\to(\boldsymbol{X}_{\alpha'})^{\Delta[n]}$. The projection of ψ onto that same factor is the projection onto the factor $(\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}$ indexed by $\bigl(\alpha_n, n, \bigl((\alpha_0\to\alpha_1\to\cdots\to\alpha_n)\in\mathcal{C}, 1\colon\alpha_n\to\alpha_n\bigr)\bigr)$ composed with the map $\bigl((\sigma\tau)_*\bigr)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}\to(\boldsymbol{X}_{F\alpha'})^{\Delta[n]}$. Since $(\sigma\tau)_* = \sigma_*\tau_*$, the map S equalizes the projections of ϕ and ψ onto the first factor on the right of (13.2), and so the composition Sg also equalizes them.

Since g equalizes the projections onto the second and third factors on the right of (13.5), Sg equalizes the projections onto the second and third factors on the right of (13.2), and so S defines a map $\widetilde{S}$ from the equalizer of (13.5) to the equalizer of (13.2).

We will now show that $\widetilde{S}$ is an inverse for $\widetilde{R}$. To see that $\widetilde{S}\widetilde{R}$ is the identity, let $f\colon E\to\prod_{\substack{\alpha\in\mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ \Delta[n]\to\mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$ be a map that equalizes the maps ϕ and ψ of

(13.2). The composition

$$SR\colon \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]} \longrightarrow \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{D})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(F\downarrow\alpha)}} (\boldsymbol{X}_\alpha)^{\Delta[n]}$$

is such that its projection onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon F\alpha_n \to \alpha\big)\big)$ is the composition of the projection onto the $(\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}$ indexed by $\big(F\alpha_n, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon F\alpha_n \to F\alpha_n\big)\big)$ with the map $(\tau_*)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{F\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_\alpha)^{\Delta[n]}$. Since f equalizes the maps ϕ and ψ of (13.2), that composition equals the projection onto the factor $(\boldsymbol{X}_\alpha)^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon F\alpha_n \to \alpha\big)\big)$ (see Lemma 13.3), and so $\widetilde{S}\widetilde{R}$ is the identity.

To see that $\widetilde{R}\widetilde{S}$ is the identity, let $g\colon E \to \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ equalize the maps ϕ and ψ of (13.5). The composition

$$RS\colon \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]} \longrightarrow \prod_{\substack{\alpha\in \mathrm{Ob}(\mathcal{C})\\ n\ge 0\\ \Delta[n]\to \mathrm{N}(\mathcal{C}\downarrow\alpha)}} (\boldsymbol{X}_{F\alpha})^{\Delta[n]}$$

is such that its projection onto the factor $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\big)\big)$ is the composition of the projection onto the $(\boldsymbol{X}_{F\alpha_n})^{\Delta[n]}$ indexed by $\big(\alpha_n, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, 1\colon \alpha_n \to \alpha_n\big)\big)$ with the map $\big((F\tau)_*\big)^{1_{\Delta[n]}}\colon (\boldsymbol{X}_{F\alpha_n})^{\Delta[n]} \to (\boldsymbol{X}_{F\alpha})^{\Delta[n]}$. Since g equalizes the maps ϕ and ψ of (13.5), that composition equals the projection onto the factor $(\boldsymbol{X}_{F\alpha})^{\Delta[n]}$ indexed by $\big(\alpha, n, \big((\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n) \in \mathcal{C}, \tau\colon \alpha_n \to \alpha\big)\big)$ (see Lemma 11.8), and so $\widetilde{R}\widetilde{S}$ is the identity. □

13.1. **Weak equivalences and fibrations.**

Definition 13.6. A functor between small categories $F\colon \mathcal{C} \to \mathcal{D}$ is *homotopy left cofinal* (or *homotopy initial*) if for every object β of $\mathcal{D}$ the simplicial set $\mathrm{N}(F\downarrow\beta)$ (see Definition 2.16) is contractible. If $\mathcal{C}$ is a subcategory of $\mathcal{D}$ and F is the inclusion, then $\mathcal{C}$ is called a *homotopy left cofinal subcategory* (or a *homotopy initial subcategory*) of $\mathcal{D}$.

Theorem 13.7. *Let $\mathcal{M}$ be a simplicial model category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If F is homotopy left cofinal, $\boldsymbol{X}$ is an objectwise fibrant $\mathcal{D}$-diagram in $\mathcal{M}$, and $F^*\boldsymbol{X} = \boldsymbol{X}\circ F$ is the induced $\mathcal{C}$-diagram, then the natural map of homotopy limits* $\mathrm{holim}_{\mathcal{D}} \boldsymbol{X} \to \mathrm{holim}_{\mathcal{C}} F^*\boldsymbol{X}$ *is a weak equivalence.*

Proof. Theorem 13.4 implies that the natural map of homotopy limits is isomorphic to the map of ends $\hom^{\mathcal{D}}\big(\mathrm{N}(\mathcal{D}\downarrow -), \boldsymbol{X}\big) \to \hom^{\mathcal{D}}\big(\mathrm{N}(F\downarrow -), \boldsymbol{X}\big)$ induced by the map of $\mathcal{D}$-diagrams $F_*\colon \mathrm{N}(F\downarrow -) \to \mathrm{N}(\mathcal{D}\downarrow -)$ (see Example 2.21). Since F is homotopy left cofinal, that map of diagrams is a weak equivalence of cofibrant $\mathcal{D}$-diagrams, and so the result follows from Corollary 7.22. □

Proposition 13.8. *Let $\mathcal{M}$ be a simplicial model category, let $\mathcal{D}$ be a small category, and let $F\colon \mathcal{C} \to \mathcal{D}$ be the inclusion of a subcategory. If $\boldsymbol{X}$ is an objectwise fibrant*

$\mathcal{D}$-diagram in $\mathcal{M}$ and $F^*\boldsymbol{X} = \boldsymbol{X} \circ F$ is the induced $\mathcal{C}$-diagram, then the natural map of homotopy limits $\operatorname{holim}_{\mathcal{D}} \boldsymbol{X} \to \operatorname{holim}_{\mathcal{C}} F^*\boldsymbol{X}$ *is a fibration of fibrant objects.*

Proof. Theorem 13.4 implies that the natural map of homotopy limits is naturally isomorphic to the map of ends $\hom^{\mathcal{D}}\big(\mathrm{N}(\mathcal{D}\downarrow -), \boldsymbol{X}\big) \to \hom^{\mathcal{D}}\big(\mathrm{N}(F\downarrow -), \boldsymbol{X}\big)$ induced by the map of $\mathcal{D}$-diagrams $F_*\colon \mathrm{N}(F\downarrow -) \to \mathrm{N}(\mathcal{D}\downarrow -)$ (see Example 2.21). Thus, the result follows from Proposition 5.15, Corollary 7.16, and Corollary 7.18. □

14. Diagrams of spaces

14.1. **The geometric realization and total singular complex functors.** In this section we show that the homotopy colimit and homotopy limit functors behave well with respect to the total singular complex and geometric realization functors (see Theorem 14.2 and Theorem 14.3).

Lemma 14.1. *Let K be a simplicial set.*

1. *If X is a simplicial set, then there is a natural isomorphism $\big|X \otimes K\big| \approx \big|X\big| \otimes K$.*
2. *If X is a topological space, then there is a natural isomorphism $\mathrm{Sing}(X^K) \approx (\mathrm{Sing}\, X)^K$.*

Proof. For part 1, we have natural isomorphisms

$$\big|X \otimes K\big| = \big|X \times K\big| \approx \big|X\big| \times \big|K\big| = \big|X\big| \otimes K \ .$$

For part 2, for every $n \geq 0$ there are natural isomorphisms

$$\begin{aligned}
\big(\mathrm{Sing}(X^K)\big)_n &= \mathrm{Top}\big(\big|\Delta[n]\big|, X^K\big) \\
&= \mathrm{Top}\big(\big|\Delta[n]\big|, X^{|K|}\big) \\
&\approx \mathrm{Top}\big(\big|\Delta[n]\big| \times \big|K\big|, X\big) \\
&\approx \mathrm{Top}\big(\big|\Delta[n] \times K\big|, X\big) \\
&\approx \mathrm{SS}(\Delta[n] \times K, \mathrm{Sing}\, X) \\
&\approx \mathrm{SS}\big(\Delta[n], (\mathrm{Sing}\, X)^K\big) \\
&\approx \big((\mathrm{Sing}\, X)^K\big)_n
\end{aligned}$$

where $X^{|K|}$ is the topological space of continuous functions from $\big|K\big|$ to X. Thus, we have a natural isomorphism $\mathrm{Sing}(X^K) \approx (\mathrm{Sing}\, X)^K$. □

Theorem 14.2. *Let $\mathcal{C}$ be a small category.*

1. *If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of simplicial sets, then there is a natural isomorphism*
$$\big|\operatorname{hocolim} \boldsymbol{X}\big| \approx \operatorname{hocolim}\big|\boldsymbol{X}\big|$$
of topological spaces.
2. *If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of topological spaces, then there is a natural isomorphism*
$$\mathrm{Sing}(\operatorname{holim} \boldsymbol{X}) \approx \operatorname{holim}(\mathrm{Sing}(\boldsymbol{X}))$$
of simplicial sets.

Proof. For part 1, the geometric realization functor is a left adjoint, and so it commutes with colimits. Lemma 14.1 thus implies that we have natural isomorphisms

$$\begin{aligned}
\left|\operatorname{hocolim} \boldsymbol{X}\right| &= \left|\boldsymbol{X} \otimes_{\mathcal{C}} \mathrm{N}(- \downarrow \mathcal{C})^{\mathrm{op}}\right| \\
&= \left|\operatorname{colim}\Big(\coprod \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \rightrightarrows \coprod \boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}\Big)\right| \\
&\approx \operatorname{colim}\Big(\coprod \left|\boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}}\right| \rightrightarrows \coprod \left|\boldsymbol{X}_\alpha \otimes \mathrm{N}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}\right|\Big) \\
&\approx \operatorname{colim}\Big(\coprod \left|\boldsymbol{X}_\alpha\right| \otimes \mathrm{N}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \rightrightarrows \coprod \left|\boldsymbol{X}_\alpha\right| \otimes \mathrm{N}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}\Big) \\
&= \left|X\right| \otimes_{\mathcal{C}} \mathrm{N}(- \downarrow \mathcal{C})^{\mathrm{op}} \\
&= \operatorname{hocolim}\left|\boldsymbol{X}\right| \ .
\end{aligned}$$

Part 2 is similar: the total singular complex functor is a right adjoint, and so it commutes with limits. Lemma 14.1 thus implies that we have natural isomorphisms

$$\begin{aligned}
\operatorname{Sing}(\operatorname{holim} \boldsymbol{X}) &= \operatorname{Sing}\big(\hom^{\mathcal{C}}(\mathrm{N}(\mathcal{C} \downarrow -), \boldsymbol{X})\big) \\
&= \operatorname{Sing}\Big(\lim\Big(\prod (\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \rightrightarrows \prod (\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}\Big)\Big) \\
&\approx \lim\Big(\prod \operatorname{Sing}\big((\boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}\big) \rightrightarrows \prod \operatorname{Sing}\big((\boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}\big)\Big) \\
&\approx \lim\Big(\prod (\operatorname{Sing} \boldsymbol{X}_\alpha)^{\mathrm{N}(\mathcal{C}\downarrow\alpha)} \rightrightarrows \prod (\operatorname{Sing} \boldsymbol{X}_{\alpha'})^{\mathrm{N}(\mathcal{C}\downarrow\alpha)}\Big) \\
&= \hom^{\mathcal{C}}\big(\mathrm{N}(\mathcal{C} \downarrow -), \operatorname{Sing} \boldsymbol{X}\big) \\
&= \operatorname{holim}(\operatorname{Sing} \boldsymbol{X}) \ .
\end{aligned}$$

□

Theorem 14.3. *Let $\mathcal{C}$ be a small category.*

(1) *If $\boldsymbol{X}$ is an objectwise cofibrant $\mathcal{C}$-diagram of topological spaces, then there is a natural weak equivalence of simplicial sets*

$$\operatorname{hocolim}(\operatorname{Sing} \boldsymbol{X}) \longrightarrow \operatorname{Sing}(\operatorname{hocolim} \boldsymbol{X}) \ .$$

(2) *If $\boldsymbol{X}$ is an objectwise fibrant $\mathcal{C}$-diagram of simplicial sets, then there is a natural weak equivalence of topological spaces*

$$\left|\operatorname{holim} \boldsymbol{X}\right| \longrightarrow \operatorname{holim}\left|\boldsymbol{X}\right| \ .$$

Proof. For part 1, the natural objectwise weak equivalence of objectwise cofibrant $\mathcal{C}$-diagrams of topological spaces $\left|\operatorname{Sing} \boldsymbol{X}\right| \to \boldsymbol{X}$ induces a natural weak equivalence $\operatorname{hocolim}\left|\operatorname{Sing} \boldsymbol{X}\right| \to \operatorname{hocolim} \boldsymbol{X}$ (see Theorem 8.4). Theorem 14.2 implies that this is naturally isomorphic to a natural weak equivalence $\left|\operatorname{hocolim}(\operatorname{Sing} \boldsymbol{X})\right| \to \operatorname{hocolim} \boldsymbol{X}$, and this corresponds under the standard adjunction to a natural weak equivalence $\operatorname{hocolim}(\operatorname{Sing} \boldsymbol{X}) \to \operatorname{Sing}(\operatorname{hocolim} \boldsymbol{X})$.

Part 2 is similar: the natural objectwise weak equivalence of objectwise fibrant $\mathcal{C}$-diagrams of simplicial sets $\boldsymbol{X} \to \operatorname{Sing}\left|\boldsymbol{X}\right|$ induces a natural weak equivalence $\operatorname{holim} \boldsymbol{X} \to \operatorname{holim}(\operatorname{Sing}\left|\boldsymbol{X}\right|)$ (see Theorem 11.4). Theorem 14.2 implies that this is naturally isomorphic to a natural weak equivalence $\operatorname{holim} \boldsymbol{X} \to \operatorname{Sing}(\operatorname{holim}\left|\boldsymbol{X}\right|)$ and this corresponds under the standard adjunction to a natural weak equivalence $\left|\operatorname{holim} \boldsymbol{X}\right| \to \operatorname{holim}\left|\boldsymbol{X}\right|$. □

14.2. **Filtered diagrams of simplicial sets.** The main result of this section is Theorem 14.17.

Lemma 14.4. *Let $\mathcal{C}$ be a small filtered category and let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram of simplicial sets. If K is a simplicial set with finitely many nondegenerate simplices, then the natural map*

$$\operatorname*{colim}_{\alpha} \mathrm{SS}(K, \boldsymbol{X}_\alpha) \longrightarrow \mathrm{SS}(K, \operatorname*{colim}_{\alpha} \boldsymbol{X}_\alpha)$$

is an isomorphism of sets.

Proof. We first show that the natural map is surjective. Let $f\colon K \to \operatorname{colim} \boldsymbol{X}$ be a map. For each nondegenerate simplex $\sigma\colon \Delta[n] \to K$ of K there is an object α_σ of $\mathcal{C}$ such that the composition $\Delta[n] \xrightarrow{\sigma} K \xrightarrow{f} \operatorname{colim} \boldsymbol{X}$ factors through $\boldsymbol{X}_{\alpha_\sigma}$; since there are only finitely many of those and $\mathcal{C}$ is filtered, there exists an object β of $\mathcal{C}$ such that all of those compositions factor through $\boldsymbol{X}_\beta$. Corollary 3.14 implies that there are only finitely many relations to be imposed upon these factorizations through $\boldsymbol{X}_\beta$ in order for the map σ to factor, and so there is a map $\beta \to \gamma$ in $\mathcal{C}$ such that the map $\boldsymbol{X}_\beta \to \boldsymbol{X}_\gamma$ imposes those relations, and so the map f factors through $\boldsymbol{X}_\gamma$. Thus, the natural map is surjective.

To see that it is injective, let α and β be objects of $\mathcal{C}$ and let $f\colon K \to \boldsymbol{X}_\alpha$ and $g\colon K \to \boldsymbol{X}_\beta$ be maps such that the compositions $K \xrightarrow{f} \boldsymbol{X}_\alpha \to \operatorname{colim} \boldsymbol{X}$ and $K \xrightarrow{g} \boldsymbol{X}_\beta \to \operatorname{colim} \boldsymbol{X}$ are equal. Corollary 3.14 implies that it is sufficient to find an object γ of $\mathcal{C}$ and maps $s\colon \alpha \to \gamma$ and $t\colon \beta \to \gamma$ in $\mathcal{C}$ such that, for each nondegenerate simplex $\sigma\colon \Delta[n] \to K$ of K, the compositions

$$\Delta[n] \xrightarrow{\ \sigma\ } K \xrightarrow{\ f\ } \boldsymbol{X}_\alpha \xrightarrow{\ s_*\ } \boldsymbol{X}_\gamma$$

and

$$\Delta[n] \xrightarrow{\ \sigma\ } K \xrightarrow{\ g\ } \boldsymbol{X}_\beta \xrightarrow{\ t_*\ } \boldsymbol{X}_\gamma$$

are equal; these exist because there are only finitely many such nondegenerate simplices and $\mathcal{C}$ is filtered. □

Lemma 14.5. *Let $\mathcal{C}$ be a small filtered category and let $i\colon K \to L$ be a map between simplicial sets with finitely many nondegenerate simplices. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathcal{C}$-diagrams of simplicial sets and for every object α of $\mathcal{C}$ the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ has the right lifting property with respect to i, then the induced map of colimits* $\operatorname{colim} f\colon \operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{Y}$ *has the right lifting property with respect to i.*

Proof. Given a solid arrow diagram

$$\begin{array}{ccc} K & \xrightarrow{\ s\ } & \operatorname{colim} \boldsymbol{X} \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow \\ L & \xrightarrow[\ t\ ]{} & \operatorname{colim} \boldsymbol{Y} \end{array} \qquad (14.6)$$

Lemma 14.4 implies that there are objects α and β of $\mathcal{C}$ and factorizations

$$K \xrightarrow{\ s_\alpha\ } \boldsymbol{X}_\alpha \longrightarrow \operatorname{colim} \boldsymbol{X} \quad \text{of } s \text{ and}$$

$$L \xrightarrow{\ t_\beta\ } \boldsymbol{Y}_\beta \longrightarrow \operatorname{colim} \boldsymbol{Y} \quad \text{of } t.$$

We can then choose an object γ of $\mathcal{C}$ for which there are maps $\alpha \to \gamma$ and $\beta \to \gamma$ in $\mathcal{C}$, and we will have the not necessarily commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{s_\alpha} & \boldsymbol{X}_\alpha & \longrightarrow & \boldsymbol{X}_\gamma \\ {\scriptstyle i}\downarrow & & & & \downarrow{\scriptstyle f_\gamma} \\ L & \xrightarrow[t_\beta]{} & \boldsymbol{Y}_\beta & \longrightarrow & \boldsymbol{Y}_\gamma \end{array} ,$$

and since K has only finitely many nondegenerate simplices, there is a map $\gamma \to \delta$ in $\mathcal{C}$ such that the solid arrow diagram

$$\begin{array}{ccccccc} K & \xrightarrow{s_\alpha} & \boldsymbol{X}_\alpha & \longrightarrow & \boldsymbol{X}_\gamma & \longrightarrow & \boldsymbol{X}_\delta \\ {\scriptstyle i}\downarrow & & & & & \nearrow & \downarrow \\ L & \xrightarrow[t_\beta]{} & \boldsymbol{Y}_\beta & \longrightarrow & \boldsymbol{Y}_\gamma & \longrightarrow & \boldsymbol{Y}_\delta \end{array}$$

does commute. Since $f_\delta \colon \boldsymbol{X}_\delta \to \boldsymbol{Y}_\delta$ has the right lifting property with respect to i, there exists a dotted arrow making the diagram commute, and that dotted arrow defines the required dotted arrow in Diagram 14.6. □

Proposition 14.7. *Let $\mathcal{C}$ be a small filtered category, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\mathcal{C}$-diagrams of simplicial sets, and let $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of diagrams.*

(1) *If the map f is an objectwise fibration, then the induced map of colimits $\operatorname{colim} f \colon \operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{Y}$ is a fibration.*
(2) *If the map f is an objectwise trivial fibration, then the induced map of colimits $\operatorname{colim} f \colon \operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{Y}$ is a trivial fibration.*

Proof. A map of simplicial sets is

- a fibration if it has the right lifting property with respect to the maps $\Lambda[n,k] \to \Delta[n]$ for all $n > 0$ and $0 \le k \le n$ and
- a trivial fibration if it has the right lifting property with respect to the maps $\partial\Delta[n] \to \Delta[n]$ for all $n \ge 0$,

and so the result follows from Lemma 14.5. □

Corollary 14.8. *If $\mathcal{C}$ is a small filtered category and $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of fibrant simplicial sets, then $\operatorname{colim} \boldsymbol{X}$ is fibrant.*

Proof. This follows from Proposition 14.7, letting $\boldsymbol{Y}$ be the constant diagram at a point. □

Lemma 14.9. *Let $\mathcal{C}$ be a small filtered category and let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram of fibrant simplicial sets. If K is a simplicial set with finitely many nondegenerate simplices, then the $\mathcal{C}$-diagram of simplicial sets $\operatorname{Map}(K, \boldsymbol{X})$ (which on an object α of $\mathcal{C}$ is $\operatorname{Map}(K, \boldsymbol{X}_\alpha)$) is a $\mathcal{C}$-diagram of fibrant simplicial sets and the natural map*

$$\operatorname*{colim}_\alpha \operatorname{Map}(K, \boldsymbol{X}_\alpha) \longrightarrow \operatorname{Map}(K, \operatorname*{colim}_\alpha \boldsymbol{X}_\alpha)$$

is an isomorphism of fibrant simplicial sets.

Proof. Since every simplicial set is cofibrant and each $\boldsymbol{X}_\alpha$ is fibrant, each simplicial set $\operatorname{Map}(K, \boldsymbol{X}_\alpha)$ is fibrant.

The colimit of a diagram of simplicial sets is constructed dimensionwise, and so it is sufficient to show that for every $n \geq 0$ the map of sets

$$\operatorname*{colim}_{\alpha} \operatorname{Map}_n(K, \boldsymbol{X}_\alpha) \longrightarrow \operatorname{Map}_n(K, \operatorname*{colim}_{\alpha} \boldsymbol{X}_\alpha)$$

is an isomorphism. Since

$$\operatorname{Map}_n(K, \boldsymbol{X}_\alpha) = \operatorname{SS}(K \times \Delta[n], \boldsymbol{X}_\alpha) \ ,$$
$$\operatorname{Map}_n(K, \operatorname*{colim}_{\alpha} \boldsymbol{X}_\alpha) = \operatorname{SS}(K \times \Delta[n], \operatorname*{colim}_{\alpha} \boldsymbol{X}_\alpha) \ ,$$

and $K \times \Delta[n]$ has only finitely many nondegenerate simplices, the isomorphism follows from Lemma 14.4, and Corollary 14.8 implies that these are fibrant simplicial sets. □

Theorem 14.10. *Let $\mathcal{C}$ be a small filtered category. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathcal{C}$-diagrams of simplicial sets and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence, then the induced map of colimits* $\operatorname{colim} f\colon \operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{Y}$ *is a weak equivalence.*

Proof. Factor the map of diagrams $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ as $\boldsymbol{X} \xrightarrow{i} \boldsymbol{W} \xrightarrow{p} \boldsymbol{Y}$ where i is a trivial cofibration and p is a fibration (in the Bousfield-Kan model structure on $\mathcal{C}$-diagrams of simplicial sets). The map i is then also an objectwise trivial cofibration. Since f is an objectwise weak equivalence, the two-out-of-three property implies that the objectwise fibration p is actually an objectwise trivial fibration, and so Proposition 14.7 implies that $\operatorname{colim} p\colon \operatorname{colim} \boldsymbol{W} \to \operatorname{colim} \boldsymbol{Y}$ is a trivial fibration.

Since colimit is a left Quillen functor (because the constant diagram functor (its right adjoint) preserves fibrations and trivial fibrations), $\operatorname{colim} i\colon \operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{W}$ is a trivial cofibration, and so the composition $\operatorname{colim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{W} \to \operatorname{colim} \boldsymbol{Y}$ is a weak equivalence. □

Proposition 14.11. *Let $\mathcal{C}$ be a small filtered category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of simplicial sets, then the natural map* $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$ *is a weak equivalence.*

Proof. Proposition 10.10 implies that there is a $\mathcal{C}$-diagram $\widetilde{\boldsymbol{X}}$ and an objectwise weak equivalence of diagrams $\widetilde{\boldsymbol{X}} \to \boldsymbol{X}$ for which the induced map of colimits $\operatorname{colim} \widetilde{\boldsymbol{X}} \to \operatorname{colim} \boldsymbol{X}$ is isomorphic to the natural map $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$, and Theorem 14.10 implies that the induced map of colimits is a weak equivalence. □

Corollary 14.12. *Let $\mathcal{C}$ be a small filtered category. If $\boldsymbol{X}$ is a $\mathcal{C}$-diagram of fibrant simplicial sets then the natural map* $\operatorname{hocolim} \boldsymbol{X} \to \operatorname{colim} \boldsymbol{X}$ *is a fibrant approximation to* $\operatorname{hocolim} \boldsymbol{X}$.

Proof. Proposition 14.11 implies that the map is a weak equivalence and Corollary 14.8 implies that $\operatorname{colim} \boldsymbol{X}$ is fibrant. □

14.3. **Commuting hocolim and holim: simplicial sets.**

Lemma 14.13. *If $\mathcal{C}$ is a small filtered category, $\mathcal{D}$ is a finite category (i.e., $\mathcal{D}$ has finitely many objects and finitely many morphisms), and $\boldsymbol{X}$ is a $\mathcal{C} \times \mathcal{D}$-diagram of sets, then the natural map*

$$\operatorname*{colim}_{\mathcal{C}} \lim_{\mathcal{D}} \boldsymbol{X} \longrightarrow \lim_{\mathcal{D}} \operatorname*{colim}_{\mathcal{C}} \boldsymbol{X}$$

is an isomorphism of sets.

Proof. See, e.g., [8, p. 211] or [1, Thm. 2.13.4] or [9, Thm. 9.5.2]. □

Corollary 14.14. *If $\mathcal{C}$ is a small filtered category, $\mathcal{D}$ is a finite category (i.e., $\mathcal{D}$ has finitely many objects and finitely many morphisms), and $\boldsymbol{X}$ is a $\mathcal{C}\times\mathcal{D}$-diagram of simplicial sets, then the natural map*

$$\operatorname*{colim}_{\mathcal{C}} \lim_{\mathcal{D}} \boldsymbol{X} \longrightarrow \lim_{\mathcal{D}} \operatorname*{colim}_{\mathcal{C}} \boldsymbol{X}$$

is an isomorphism of simplicial sets.

Proof. Since limits and colimits of diagrams of simplicial sets are constructed dimensionwise, this follow from Lemma 14.13. □

Definition 14.15. A category $\mathcal{D}$ is *finite and acyclic* if

- $\mathcal{D}$ has finitely many objects,
- $\mathcal{D}$ has finitely many morphisms, and
- there is no composable sequence of non-identity maps that starts and ends at the same object.

This is equivalent to saying that the nerve $\mathrm{N}\mathcal{D}$ of $\mathcal{D}$ has only finitely many nondegenerate simplices. This is also equivalent to what Dwyer and Spaliński [5, 10.13] have called a *very small* category.

Proposition 14.16. *If $\mathcal{D}$ is a finite and acyclic category (see Definition 14.15) and α is an object of $\mathcal{D}$, then $\mathrm{N}(\mathcal{D}\downarrow\alpha)$, the nerve of the overcategory of α, has only finitely many nondegenerate simplices.*

Proof. The conditions on $\mathcal{D}$ imply that there are only finitely many composable strings of non-identity maps. □

Theorem 14.17. *Let $\mathcal{C}$ be a small filtered category and let $\mathcal{D}$ be a finite and acyclic category (see Definition 14.15). If $\boldsymbol{X}$ is an objectwise fibrant $\mathcal{C}\times\mathcal{D}$-diagram of simplicial sets, then there is a natural isomorphism*

$$\operatorname*{holim}_{\mathcal{D}} \mathrm{Fib}_{\mathcal{D}}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X}) \approx \mathrm{Fib}(\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{X})$$

where $\mathrm{Fib}_{\mathcal{D}}(\mathrm{hocolim}_{\mathcal{C}} \boldsymbol{X})$ is an objectwise fibrant approximation to the $\mathcal{D}$-diagram $\mathrm{hocolim}_{\mathcal{C}} \boldsymbol{X}$ and $\mathrm{Fib}(\mathrm{hocolim}_{\mathcal{C}} \mathrm{holim}_{\mathcal{D}} \boldsymbol{X})$ is a fibrant approximation to $\mathrm{hocolim}_{\mathcal{C}} \mathrm{holim}_{\mathcal{D}} \boldsymbol{X}$.

Proof. Let $\mathcal{L}$ be the category $a \overset{\phi}{\underset{\psi}{\rightrightarrows}} b$ with two objects $\{a, b\}$ and two non-identity maps $\phi, \psi\colon a \to b$. Let $\boldsymbol{Y}$ be the $\mathcal{C}\times\mathcal{L}$-diagram of simplicial sets such that

- $\boldsymbol{Y}_{\beta,a} = \prod_{\alpha\in\mathrm{Ob}(\mathcal{D})} (\boldsymbol{X}_{\beta,\alpha})^{\mathrm{N}(\mathcal{D}\downarrow\alpha)}$,
- $\boldsymbol{Y}_{\beta,b} = \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{D}} (\boldsymbol{X}_{\beta,\alpha'})^{\mathrm{N}(\mathcal{D}\downarrow\alpha)}$,
- the maps ϕ and ψ are as in Definition 11.1, and
- if $\tau\colon \beta \to \beta'$ is a map in $\mathcal{C}$, then the maps $\boldsymbol{Y}_{\beta,a} \to \boldsymbol{Y}_{\beta',a}$ and $\boldsymbol{Y}_{\beta,b} \to \boldsymbol{Y}_{\beta',b}$ are induced by the maps $\boldsymbol{X}_{\beta,\alpha} \to \boldsymbol{X}_{\beta',\alpha}$ and $\boldsymbol{X}_{\beta,\alpha'} \to \boldsymbol{X}_{\beta',\alpha'}$.

Corollary 14.14 implies that the natural map $\mathrm{colim}_{\mathcal{C}} \lim_{\mathcal{L}} \boldsymbol{Y} \to \lim_{\mathcal{L}} \mathrm{colim}_{\mathcal{C}} \boldsymbol{Y}$ is an isomorphism, and so it remains only to identify these colimits and limits.

Since $\boldsymbol{Y}_{\beta,a}$ and $\boldsymbol{Y}_{\beta,b}$ are defined as finite products, $X^K = \mathrm{Map}(K, X)$ for all simplicial sets X and K, and each $\mathrm{N}(\mathcal{D}\downarrow\alpha)$ has finitely many nondegenerate simplices (see Proposition 14.16), Lemma 14.9 implies that

$$\operatorname*{colim}_{\mathcal{C}} \boldsymbol{Y}_{\beta,a} \approx \prod_{\alpha\in\mathrm{Ob}(\mathcal{D})} \operatorname*{colim}_{\mathcal{C}}\bigl((\boldsymbol{X}_{\beta,\alpha})^{\mathrm{N}(\mathcal{D}\downarrow\alpha)}\bigr) \approx \prod_{\alpha\in\mathrm{Ob}(\mathcal{D})} (\operatorname*{colim}_{\mathcal{C}} \boldsymbol{X}_{\beta,\alpha})^{\mathrm{N}(\mathcal{D}\downarrow\alpha)}$$

and

$$\operatorname*{colim}_{\mathcal{C}} \boldsymbol{Y}_{\beta,b} \approx \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{D}} \operatorname*{colim}_{\mathcal{C}}\bigl((\boldsymbol{X}_{\beta,\alpha'})^{\mathrm{N}(\mathcal{D}\downarrow\alpha)}\bigr) \approx \prod_{(\sigma\colon \alpha\to\alpha')\in\mathcal{D}} (\operatorname*{colim}_{\mathcal{C}} \boldsymbol{X}_{\beta,\alpha'})^{\mathrm{N}(\mathcal{D}\downarrow\alpha)} \ .$$

Corollary 14.12 thus implies that for every object β of $\mathcal{C}$ the limit of the diagram

$$\operatorname*{colim}_{\mathcal{C}} \boldsymbol{Y}_{\beta,a} \quad \overset{\phi}{\underset{\psi}{\rightrightarrows}} \quad \operatorname*{colim}_{\mathcal{C}} \boldsymbol{Y}_{\beta,b}$$

is the homotopy limit of an objectwise fibrant approximation to the $\mathcal{D}$-diagram $\operatorname{hocolim}_{\mathcal{C}} \boldsymbol{X}$. That is, $\lim_{\mathcal{L}} \operatorname{colim}_{\mathcal{C}} \boldsymbol{Y}$ is $\operatorname{holim}_{\mathcal{D}} \operatorname{Fib}_{\mathcal{D}}(\operatorname{hocolim}_{\mathcal{C}} \boldsymbol{X})$.

To identify $\operatorname{colim}_{\mathcal{C}} \lim_{\mathcal{L}} \boldsymbol{Y}$, note that for each object β of $\boldsymbol{C}$ the limit $\lim_{\mathcal{L}} \boldsymbol{Y}_{\beta,*}$ is $\operatorname{holim}_{\mathcal{D}} \boldsymbol{X}_{\beta,*}$. Proposition 11.3 implies that these are all fibrant simplicial sets, and so Corollary 14.12 implies that $\operatorname{colim}_{\mathcal{C}} \lim_{\mathcal{L}} \boldsymbol{Y} = \operatorname{colim}_{\mathcal{C}} \operatorname{holim}_{\mathcal{D}} \boldsymbol{X}$ is a fibrant approximation to $\operatorname{hocolim}_{\mathcal{C}} \operatorname{holim}_{\mathcal{D}} \boldsymbol{X}$. □

14.4. **Commuting hocolim and holim: topological spaces.** The main result of this section is Theorem 14.19.

The following theorem shows that, for diagrams in the category of topological spaces, the homotopy colimit functor preserves objectwise weak equivalences even if the objects are not assumed to be cofibrant.

Theorem 14.18. *Let $\mathcal{C}$ be a small category. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\mathcal{C}$-diagrams of topological spaces and $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is an objectwise weak equivalence, then the induced map of homotopy colimits $f_*\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a weak equivalence.*

Proof. See [6, Appendix A]. □

Theorem 14.19. *Let $\mathcal{C}$ be a small filtered category and let $\mathcal{D}$ be a finite and acyclic category (see Definition 14.15). If $\boldsymbol{Y}$ is a $\mathcal{C}\times\mathcal{D}$-diagram of topological spaces, then there is a natural zig-zag of weak equivalences between* $\operatorname{holim}_{\mathcal{D}} \operatorname{hocolim}_{\mathcal{C}} \boldsymbol{Y}$ *and* $\operatorname{hocolim}_{\mathcal{C}} \operatorname{holim}_{\mathcal{D}} \boldsymbol{Y}$.

Proof. Let $\boldsymbol{X}$ be the (objectwise) total singular complex of $\boldsymbol{Y}$, so that $\boldsymbol{X}$ is a $\mathcal{C}\times\mathcal{D}$-diagram of fibrant simplicial sets. Theorem 14.17 implies that there is a natural isomorphism

$$\operatorname*{holim}_{\mathcal{D}} \operatorname{Fib}_{\mathcal{D}}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X}) \approx \operatorname{Fib}(\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{X}) \ ,$$

and so we have a natural isomorphism of topological spaces

$$\bigl|\operatorname*{holim}_{\mathcal{D}} \operatorname{Fib}_{\mathcal{D}}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X})\bigr| \approx \bigl|\operatorname{Fib}(\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{X})\bigr| \ . \tag{14.20}$$

We will show that there is a natural zig-zag of weak equivalences between the left hand side of (14.20) and $\operatorname{holim}_{\mathcal{D}} \operatorname{hocolim}_{\mathcal{C}} \boldsymbol{Y}$ and a natural zig-zag of weak equivalences between the right hand side of (14.20) and $\operatorname{hocolim}_{\mathcal{C}} \operatorname{holim}_{\mathcal{D}} \boldsymbol{Y}$.

We first examine the left hand side of (14.20). Theorem 14.3 implies that there is a natural weak equivalence

$$\bigl|\operatorname*{holim}_{\mathcal{D}} \operatorname{Fib}_{\mathcal{D}}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X})\bigr| \longrightarrow \operatorname*{holim}_{\mathcal{D}}\bigl|\operatorname{Fib}_{\mathcal{D}}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X})\bigr| \ ,$$

Theorem 11.4 implies that we have a natural weak equivalence

$$\operatorname*{holim}_{\mathcal{D}}\bigl|\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X}\bigr| \longrightarrow \operatorname*{holim}_{\mathcal{D}}\bigl|\operatorname{Fib}_{\mathcal{D}}(\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X})\bigr| \ ,$$

and Theorem 14.2 gives us a natural isomorphism

$$\operatorname*{holim}_{\mathcal{D}}\bigl|\operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{X}\bigr| \approx \operatorname*{holim}_{\mathcal{D}} \operatorname*{hocolim}_{\mathcal{C}}\bigl|\boldsymbol{X}\bigr| \ .$$

Since $\boldsymbol{X} = \operatorname{Sing} \boldsymbol{Y}$, we have a natural objectwise weak equivalence $\bigl|\boldsymbol{X}\bigr| = \bigl|\operatorname{Sing} \boldsymbol{Y}\bigr| \longrightarrow \boldsymbol{Y}$, and so Theorem 14.18 and Theorem 11.4 give us a natural weak equivalence

$$\operatorname*{holim}_{\mathcal{D}} \operatorname*{hocolim}_{\mathcal{C}}\bigl|\boldsymbol{X}\bigr| \longrightarrow \operatorname*{holim}_{\mathcal{D}} \operatorname*{hocolim}_{\mathcal{C}} \boldsymbol{Y} \ .$$

We next examine the right hand side of (14.20). We have a natural weak equivalence

$$\bigl|\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{X}\bigr| \longrightarrow \bigl|\operatorname{Fib}(\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{X})\bigr| \ ,$$

Theorem 14.2 gives us a natural isomorphism

$$\bigl|\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{X}\bigr| \approx \operatorname*{hocolim}_{\mathcal{C}}\bigl|\operatorname*{holim}_{\mathcal{D}} \boldsymbol{X}\bigr| \ ,$$

Theorem 14.3 and Theorem 14.18 give us a natural weak equivalence

$$\operatorname*{hocolim}_{\mathcal{C}}\bigl|\operatorname*{holim}_{\mathcal{D}} \boldsymbol{X}\bigr| \longrightarrow \operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}}\bigl|\boldsymbol{X}\bigr| \ ,$$

and since we have a natural objectwise weak equivalence $\bigl|\boldsymbol{X}\bigr| = \bigl|\operatorname{Sing} \boldsymbol{Y}\bigr| \to \boldsymbol{Y}$, Theorem 14.18 and Theorem 11.4 give us a natural weak equivalence

$$\operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}}\bigl|\boldsymbol{X}\bigr| \longrightarrow \operatorname*{hocolim}_{\mathcal{C}} \operatorname*{holim}_{\mathcal{D}} \boldsymbol{Y} \ . \qquad \square$$

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