A REMARK ON THE STRØM MODEL CATEGORY

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Let **Top** denote the model category of topological spaces with the Strøm model structure. The cofibrations are the closed Hurewicz cofibrations, the fibrations are the Hurewicz fibrations, and the weak equivalences are the homotopy equivalences (see [2]). Every object is both fibrant and cofibrant in **Top**.

Proposition. **Top** *is not cofibrantly generated.*

Proof. The proof here is an improved and simplified version of the argument sketched in [1, Remark 4.7]. We will show that the class of cofibrations in **Top** does not have a generating set.

Suppose that there is a generating set $\mathcal{I}$ of cofibrations in **Top**. Let κ be an infinite cardinal which is greater than the cardinalities of the topological spaces that appear in the set of maps $\mathcal{I}$. We denote by κ^{op} the underlying poset with the reverse order. We consider the topological space X_κ whose underlying set is that of κ^{op} (or κ), and the topology is the order topology on κ^{op}. More specifically, the topology of X_κ is generated by the "upward-closed" subsets:

$$U_c\colon = \{x \in X_\kappa \colon x \geq c \text{ in } \kappa^{\mathrm{op}}\},$$

for each $c \in X_\kappa$. In this order topology, the open subsets are exactly the "upward-closed" subsets and U_c is smallest open set containing $c \in X_\kappa$. (Note: The opposite conventions also appear in the literature.)

Since X_κ is cofibrant in **Top**, it must be a retract of an $\mathcal{I}$-cellular object. That is, there is a topological space $\widetilde{X}_\kappa$ which contains X_κ as a retract,

$$X_\kappa \overset{i}{\hookrightarrow} \widetilde{X}_\kappa \overset{r}{\to} X_\kappa,$$

and $\widetilde{X}_\kappa$ is the transfinite composition of an μ-sequence

$$\varnothing = Z_0 \rightarrowtail Z_1 \rightarrowtail \cdots \rightarrowtail Z_{\lambda<\mu} \rightarrowtail \cdots \rightarrowtail \widetilde{X}_\kappa$$

where each map $Z_i \to Z_{i+1}$, $i+1<\mu$, is a pushout along a map in $\mathcal{I}$. In particular, the inclusion $Z_i \rightarrowtail \widetilde{X}_\kappa$ is identified with a closed subset of $\widetilde{X}_\kappa$ for every $i<\mu$. Then the collection of subspaces

$$Y_i\colon = i^{-1}(Z_i), \quad i<\mu,$$

defines a filtration of X_κ by closed subsets,

$$\varnothing = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{\lambda<\mu} \subseteq \cdots \subseteq X_\kappa,$$

such that the cardinality of $Y_{i+1} \setminus Y_i$ is less than κ for each $i+1<\mu$. On the other hand, by construction, every non-empty closed subset of X_κ has cardinality κ, hence we reach a contradiction. □

Remark. We emphasize that the proof applies in the case where **Top** consists of *all* topological spaces.

Remark. Every cofibrantly generated model category $\mathbf{M}$ with a set of generating cofibrations between cofibrant objects has the following property: the homotopy category $\mathrm{Ho}(\mathbf{M})$ admits a set of objects $\mathbf{S}$ which jointly detect the terminal object, that is, $X \in \mathbf{M}$ is weakly equivalent to the terminal object in $\mathbf{M}$ if and only if $\mathrm{Ho}(\mathbf{M})(G, X) = *$ for every $G \in \mathbf{S}$. It would be interesting to know if the Strøm model category has this property or not.

References

[1] G. Raptis, *Homotopy theory of posets.* Homology Homotopy Appl. 12 (2010), no. 2, 211–230.
[2] A. Strøm, *The homotopy category is a homotopy category.* Arch. Math. (Basel) 23 (1972), 435–441.