3 Matchings

Hall's Theorem

Matching: A *matching* in G is a subset $M \subseteq E(G)$ so that no edge in M is a loop, and no two edges in M are incident with a common vertex. A matching M is *maximal* if there is no matching M' with $M \subset M'$ and *maximum* if there is no matching M'' with $|M| < |M''|$.

Alternating & Augmenting Paths: If M is a matching in G, a path $P \subseteq G$ is M-*alternating* if the edges of P belong alternately to M and to $E(G) \setminus M$ (in other words, for every $v \in V(P)$ with degree 2 in P, some edge of P incident with v is in M). The path P is M-*augmenting* if it is M-alternating, has distinct ends, say u, v, and no edge of M is incident with u or v in G (not just in P).

Theorem 3.1 (Berge) *A matching M in G is maximum if and only if there is no M-augmenting path.*

Proof: For the "only if" direction we prove the contrapositive. Assuming G contains an M-augmenting path P, the set $(M \setminus E(P)) \cup (E(P) \setminus M)$ is a matching with larger cardinality than M, so M is not maximum.

For the "if" direction, we also prove the contrapositive, so we shall assume that M is not maximum, and show there is an augmenting path. Since M is not maximum, there exists a matching M' with $|M'| > |M|$. Consider the subgraph $H \subseteq G$ with $V(H) = V(G)$ and $E(H) = M \cup M'$. Every component of this graph is either a cycle of even length with edges alternately in M and M', a path with edges alternately in M and M', or a path consisting of one edge e with $e \in M \cap M'$. Since $|M'| > |M|$, there is a component of H which is a path with more edges in M' than M. Then P is an M-augmenting path. $\square$

Neighbors: If $X \subseteq V(G)$, the *neighbors* of X, is the set

$$N(X) = \{v \in V(G) \setminus X : v \text{ is adjacent to some point in } X\}.$$

For $x \in X$, we define $N(x) = N(\{x\})$.

Cover: We say that a set of edges $S \subseteq E(G)$ *covers* a set of vertices X if every $x \in X$ is incident with some edge in S. Similarly, a set of vertices $X \subseteq V(G)$ *covers* a set of edges $S \subseteq E(G)$ if every edge in S is incident with some point in X.

Theorem 3.2 (Hall's Marriage Theorem) *Let G be a bipartite graph with bipartition (A, B). Then, there is a matching $M \subseteq G$ which covers A if and only if $|N(X)| \geq |X|$ for every $X \subseteq A$.*

Proof: The "only if" condition is obvious: if there exists $X \subseteq A$ with $|N(X)| < |X|$, then no matching can cover A.

For the "if" direction, let M be a maximum matching, and suppose that M does not cover A. Choose a vertex $u \in A$ not covered by M, and define the sets X, Y as follows:

$$\begin{aligned} X &= \{x \in A : \text{there is an } M\text{-alternating path from } u \text{ to } x\} \\ Y &= \{y \in B : \text{there is an } M\text{-alternating path from } u \text{ to } y\} \end{aligned}$$

Let $M' \subseteq M$ be the set of edges in M which are incident with a point in $X \cup Y$. By parity, every M-alternating path which begins at u and ends at a point in $X \setminus \{u\}$ must have its last edge in M, so every point in $X \setminus \{u\}$ is incident with an edge in M'. If there is a point $y \in Y$ not incident with an edge in M', then there is an M-alternating path from u to y which is M-augmenting, contradicting the previous theorem. Thus, every point in Y is incident with some edge in M'. It follows from this that $|X \setminus \{u\}| = |M'| = |Y|$.

Let $x \in X$ and let $y \in N(x)$. Since $x \in A$, we may choose an M-alternating path from u to x. Note (as before) that the last edge of this path is in M'. If y appears in this path, then $y \in Y$. Otherwise, we may extend this path by the edge xy to a new M-alternating path. Thus, in either case, we find that $y \in Y$. It follows from this that $N(X) \subseteq Y$. But then, $|N(X)| \leq |Y| = |M'| < |X|$. This completes the proof. $\square$

Regular A graph G is *k-regular* if every vertex of G has degree k. We say that G is *regular* if it is k-regular for some k.

Perfect Matchings: A matching M is *perfect* if it covers every vertex.

Corollary 3.3 *Every regular bipartite graph has a perfect matching.*

Proof: Let G be a k-regular bipartite graph with bipartition (A, B). Let $X \subseteq A$ and let t be the number of edges with one end in X. Since every vertex in X has degree k, it follows that $k|X| = t$. Similarly, every vertex in $N(X)$ has degree k, so t is less than or equal to $k|N(X)|$. It follows that $|X|$ is at most $|N(X)|$. Thus, by Hall's Theorem, there is a matching covering A, or equivalently, every maximum matching covers A. By a similar argument, we find that every maximum matching covers B, and this completes the proof. $\square$

Stable Marriages

System of Preferences: If G is a graph, a *system of preferences* for G is a family $\{>_v\}_{v \in V(G)}$ so that each $>_v$ is a linear ordering of $N(v)$. If $u, u' \in N(v)$ and $u >_v u'$, we say that v *prefers* u to u'.

Marriage Systems and Stable Marriages: A *Marriage System* consists of a complete bipartite graph $K_{n,n}$ with bipartition ($men, women$) which is equipped with a system of preferences. We say that a matching M is *stable* if there do not exist edges $mw, m'w' \in M$ with $m, m' \in men$ and $w, w' \in women$ so that m prefers w' to w and w' prefers m to m'. A matching which covers every vertex and is stable is called a *stable marriage.*

Gale-Shapley Algorithm:

input: A marriage system.

output: A stable marriage.

procedure: At each step, every man proposes to the woman he prefers most among those who have not yet rejected him. If every woman receives at most one proposal, stop and output the corresponding matching. Otherwise, every woman who receives more than one proposal says "maybe" to the man who proposes to her whom she most prefers, and rejects the others who proposed.

Theorem 3.4 *The Gale-Shapley Algorithm outputs a stable marriage (as claimed).*

Proof: Note first that this algorithm must terminate, since some man is rejected at each non-final step (and the total number of rejections is no more than n^2). Let M be the marriage resulting from this algorithm, and say that a man m and woman w are *married* if $mw \in M$.

Suppose that the woman w receives proposals from some nonempty set $X \subseteq men$ at some step. Then w says "maybe" to the man m who she prefers most among X, and at the next step, m will again propose to w (since the set of women who have rejected him has not changed). This immediately implies the following claim.

Claim: Every woman w is married to the man m she most prefers among those who propose to her during the algorithm. In particular, if w has at least one proposal, then w is married to some man.

With this claim, we now show that M covers every vertex. Suppose (for a contradiction) that M does not cover some man m. Then m must have been rejected by every woman. But then, by the claim, every woman must be married. Since $|men| = |women|$, this is contradictory.

Next let us show that M is stable. Suppose (for a contradiction) that it is not, and choose $mw, m'w' \in M$ so that m prefers w' to w and w' prefers m to m'. It follows from the definition of the algorithm that m must have proposed to w' at some step (since m will propose to w' before w). Applying the claim to w', we see that w' must be married to m or a person she prefers to m, thus contradicting our assumptions.

It follows that M is a stable marriage, as claimed. $\square$

Fact: Let M be the stable marriage output by the above algorithm and let M' be another stable marriage. Then, for every man m, if $mw \in M$ and $mw' \in M'$, then either $w = w'$ or m prefers w to w'. Similarly, for every woman w, if $wm \in M$ and $wm' \in M'$, then either $m = m'$, or w prefers m' to m. So, among all stable marriages, the Gale-Shapley algorithm produces one which is best possible for every man, and worst possible for every woman.

Covers

Covers: A *vertex cover* of G is a set of vertices $X \subseteq V(G)$ so that every edge is incident with some vertex in X. Simlarly, an *edge cover* of G is a set of edges $S \subseteq V(G)$ so that every vertex is incident with some edge in S.

Independent Set: A subset of vertices $X \subseteq V(G)$ is *independent* if there is no loop with endpoint in X and there is no non-loop with both ends in X.

Matching & Cover Parameters: For every graph G, we define the following parameters

$\alpha(G)$	maximum size of an independent set
$\alpha'(G)$	maximum size of a matching
$\beta(G)$	minimum size of a vertex cover
$\beta'(G)$	minimum size of an edge cover

Observation 3.5 $\alpha(G) + \beta(G) = |V(G)|$ *for every simple graph* G.

Proof: A set $X \subseteq V(G)$ is independent if and only if $V(G) \setminus X$ is a vertex cover. Thus, the complement of an independent set of maximum size is a vertex cover of minimum size. $\square$

Theorem 3.6 (König, Egerváry) *If G is bipartite, then $\alpha'(G) = \beta(G)$.*

Proof: It is immediate that $\beta(G) \geq \alpha'(G)$ since for a maximum matching M, any vertex cover must contain at least one endpoint of each edge in M.

Next we shall show that $\beta(G) \leq \alpha'(G)$. Let (A, B) be a bipartition of G, let X be a vertex cover of minimum size, and define two bipartite subgraphs H_1 and H_2 so that H_1 has bipartition $(A \cap X, B \setminus X)$, H_2 has bipartition $(A \setminus X, B \cap X)$, and both H_1 and H_2 have all edges with both ends in their vertex sets.

Suppose (for a contradiction) that there does not exist a matching in H_1 which covers $A \cap X$. Then, by Hall's theorem, there is a subset $Y \subseteq A \cap X$ so that $|N_{H_1}(Y)| < |Y|$. Now, we claim that the set $X' = (X \setminus Y) \cup N_{H_1}(Y)$ is a vertex cover. Let $e \in E(G)$. If e has one end in $B \cap X$, then e is covered by X'. If e has no end in $B \cap X$, then (since X is a vertex cover) e must have one end in $A \cap X$ and the other in $B \setminus X$, so $e \in E(H_1)$. If e does not have an end in Y, then e is covered by $X \setminus Y \subseteq X'$. Otherwise, e is an edge in H_1 with one end in Y, so its other end is in $N_{H_1}(Y)$ and we again find that e is covered. But then X' is a vertex cover with $|X'| = |X| - |Y| + |N_{H_1}(Y)| < |X|$, giving us a contradiction.

Thus, H_1 has a matching M_1, which covers $A \cap X$. By a similar argument, H_2 has a matching, M_2, which covers $B \cap X$. Since these subgraphs have disjoint vertex sets, $M = M_1 \cup M_2$ is a matching of G. Furthermore, $\alpha'(G) \geq |M| = |X| = \beta(G)$. This completes the proof. $\square$

Theorem 3.7 (Gallai) *If G is a simple connected graph with at least two vertices, then $\alpha'(G) + \beta'(G) = |V(G)|$.*

Proof: First, let M be a maximum matching (so $|M| = \alpha'(G)$). Now, we form an edge cover L from M as follows: For every vertex v not covered by M, choose an edge e incident with v and add e to L. Then L is an edge cover, so $\beta'(G) \leq |L| = |M| + |V(G)| - 2|M| = |V(G)| - \alpha'(G)$.

Next, let L be a minimum edge cover (so $|L| = \beta'(G)$) and consider the subgraph H consisting of all the vertices, and those edges in L. Since L is a minimum edge cover, it follows that $L \setminus \{e\}$ is not an edge cover for every $e \in L$. Thus, every edge $e \in E(H)$

must have one endpoint of degree 1 in H. It follows from this that every component of H is isomorphic to a star (a graph of the form $K_{1,m}$ for some positive integer m). Choose a matching $M \subseteq L$ by selecting one edge from each component of H. Then we have $\alpha'(G) \geq |M| = comp(H) = |V(G)| - |L| = |V(G)| - \beta'(G)$.

Combining the two inequalities yields $\alpha'(G) + \beta'(G) = |V(G)|$, as required. □

Corollary 3.8 *If G is a connected bipartite graph with at least two vertices, then $\alpha(G) = \beta'(G)$.*

Proof: By Observation 11.1 and Theorem 11.3 we have $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$. Now, subtracting the relation $\beta(G) = \alpha'(G)$ proved in 11.2 we have $\alpha(G) = \beta'(G)$ as desired. □

Tutte's Theorem

Odd Components: For every graph G, we let $odd(G)$ denote the number of components of G which have an odd number of vertices.

Identification: If $X \subseteq V(G)$, we may form a new graph from G by merging all vertices in X to a single new vertex. If an edge has an endpoint in X, then this edge will have the new vertex as its new endpoint. We say this graph is obtained from G by *identifying* X.

Theorem 3.9 (Tutte) *G has a perfect matching if and only if $odd(G-X) \leq |X|$ for every $X \subseteq V(G)$*

Proof: The "only if" is immediate: if G has a set $X \subseteq V(G)$ with $odd(G - X) > |X|$, then G cannot have a perfect matching.

We prove the "if" direction by induction on $|V(G)|$. As a base, observe that this is trivial when $|V(G)| \leq 2$. For the inductive step, let G be a graph for which $odd(G - X) \leq |X|$ for every $X \subseteq V(G)$ and assume the theorem holds for all graphs with fewer vertices. Call a set $X \subseteq V(G)$ *critical* if $odd(G - X) \geq |X| - 1$. We shall establish the theorem in steps.

(1) $|V(G)|$ is even

This follows from $odd(G - \emptyset) \leq |\emptyset| = 0$.

(2) If X is critical, then $odd(G - X) = |X|$.

This follows from the observation that $|X| + odd(G - X) \cong |V(G)|$ (modulo 2).

(3) There is a critical set.

For instance, $\emptyset$ is critical.

Based on (3), we may now choose a maximal critical set X. Let $|X| = k$ and let the odd components of $G - X$ be $G_1, \ldots, G_k$.

(4) $G - X$ has no even components.

If $G - X$ has an even component G', then choose $v \in V(G')$. Now $X \cup \{v\}$ is critical, contradicting the choice of X.

(5) For every $1 \leq i \leq k$ and $v \in V(G_i)$, the graph $G_i - v$ has a perfect matching.

If not, then by induction there exists $Y \subseteq V(G_i - v)$ so that $(G_i - v) - Y$ has $> |Y|$ odd components. But then $G \setminus (X \cup Y \cup \{v\})$ has $\geq |X| + |Y|$ odd components, so it is critical - again contradicting the maximality of X.

(6) G has a matching M with $|M| = k$ so that M covers X and every G_i has exactly one vertex covered by M.

Construct a graph H from G by identifying $V(G_i)$ to a new vertex y_i for every $1 \leq i \leq k$ and then deleting every loop and every edge with both ends in X. Now, H is bipartite with bipartition (X, Y) where $Y = \{y_1, \ldots, y_k\}$. Suppose (for a contradiction) that H does not have a perfect matching. Then by Hall's Theorem there exists $Y' \subseteq Y$ with $|N_H(Y')| < |Y'|$. Let $X' = N_H(Y')$. Now the graph $G - X'$ has $\geq |Y'| > |X'|$ odd components, giving us a contradiction. So, H has a perfect matching, which proves (6).

It follows from (5) and (6) that G has a perfect matching, as desired. $\square$

Theorem 3.10 (Tutte-Berge Formula)

$$\alpha'(G) = \frac{1}{2}\left(|V(G)| - \max_{X \subseteq V(G)} (odd(G - X) - |X|)\right)$$

Proof: Let $k = \max_{X \subseteq V(G)}(odd(G - X) - |X|)$ and choose $X \subseteq V(G)$ so that $k = odd(G - X) - |X|$. Note that $k = odd(G - X) - |X| \cong odd(G - X) + |X| \cong |V(G)|$ (modulo 2). By considering X and $odd(G - X)$ we find that every matching of G must not cover $\geq k$ vertices, so $\alpha'(G) \leq \frac{1}{2}(|V(G)| - k)$.

To prove the other inequality, we construct a new graph G' from G by adding a set Y of k new vertices to G each adjacent to every other vertex. Let $Z' \subseteq V(G')$. We claim that $odd(G' - Z') \leq |Z'|$. If $Z' = \emptyset$, then this follows from the observation that $k \cong |V(G)|$ (modulo 2). If $Y \not\subseteq Z'$, then $G' - Z'$ is connected, so $|Z'| \geq 1 \geq odd(G' - Z')$. Finally, if $Y \subseteq Z'$, then we have

$$odd(G' - Z') = odd(G - Z) \leq k + |Z| = |Y| + |Z| = |Z'|.$$

Since Z' was arbitrary, Tutte's Theorem shows that G' has a perfect matching, and it follows that G has a matching covering all but k vertices, so $\alpha'(G) \geq \frac{1}{2}(|V(G)| - k)$ as required. $\square$

Theorem 3.11 (Petersen) *If every vertex of G has degree 3 and G has no cut-edge, then G has a perfect matching.*

Proof: We shall show that G satisfies the condition for Tutte's Theorem. Let $X \subseteq V(G)$, let $G_1, \ldots, G_k$ be the odd components of $G - X$, and for every $1 \leq i \leq k$ let S_i be the set of edges with one end in X and the other in $V(G_i)$. Now, for every $1 \leq i \leq G_i$, we have $3|V(G_i)| = \sum_{v \in V(G_i)} deg_G(v) = |S_i| + 2|E(G_i)|$. Since $|V(G_i)|$ is odd, it follows that $|S_i|$ must also be odd. By our assumptions, $|S_i| \neq 1$, so we conclude that $|S_i| \geq 3$.

Now, form a new graph H from G by deleting every vertex in every even component of $G - X$, then identifying every G_i to a single new vertex y_i, and then deleting every loop and every edge with both ends in X. This graph H is bipartite with bipartition (X, Y) where $Y = \{y_1, \ldots, y_k\}$. Furthermore, by our assumptions, every vertex in Y has degree ≥ 3 and every vertex in X has degree ≤ 3. Thus, we have

$$3|X| \geq \sum_{x \in X} deg_H(x) = |E(H)| = \sum_{y \in Y} deg_H(y) \geq 3|Y|.$$

So, $|X| \geq |Y| = k = odd(G - X)$. Since X was arbitrary, it follows from Theorem 3.9 that G has a perfect matching. $\square$