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UNKNOTTING COMBINATORIAL BALLS

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Introduction

A piecewise linear embedding of a q-ball B^q in a p-ball B^p is called *proper* if the boundary of B^q is contained in the boundary of B^p, and the interior in the interior. We call $p-q$ the *codimension*. We show in Theorem 1 that, if the codimension $\geqq 3$, then any proper embedding is unknotted, i.e., the pair of balls is piecewise linearly homeomorphic to a standard pair. This implies (Theorem 2) the combinatorial unknotting of spheres in spheres, and of spheres in euclidean space, provided that the codimension is $\geqq 3$.

If the codimension $= 2$, then spheres can be knotted. If the codimension $= 1$, the unknotting problem is the same as the combinatorial Schönflies conjecture, and, as far as I know, is still unsolved. The problem is:—Given a piecewise linear embedding of S^{p-1} in S^p, $p \geqq 4$, is the closure of each component of the complement a combinatorial p-ball? The answer is yes, if $p \leqq 3$.

In differential topology the situation is quite different. The combined results of Brown [2] and Smale [6] prove the differential Schönflies theorem for $p \geqq 6$. For higher codimension, Haefliger [3, 4] has shown that S^q differentiably unknots in S^p if $p > 3(q + 1)/2$, but can be differentiably knotted if $q + 2 \leqq p \leqq 3(q + 1)/2$. If $p-q \geqq 3$, this latter knotting is somewhat delicate, because Smale [6] has shown that the closure of the complement of a tubular neighbourhood of S^q in S^p is the same as if S^q were unknotted. The difference between the combinatorial and differential unknotting of spheres is one of the most marked points of divergence between the two theories, a divergence which increases as we pass to the more general problem of isotopies of manifolds in manifolds (see [3, 12, 13]).

In pure topology, knotting can occur with any codimension due to the existence of wild embeddings. But with a hypothesis of topological local unknottedness, Brown [2] has proved the topological Schönflies theorem in all dimensions. With a similar hypothesis, Stallings [7] has proved the topological unknotting of spheres in spheres for codimension $\geqq 3$. Although analogous to the theorem we prove here, his theorem is independent of ours, because his hypothesis and thesis are both topological, whereas our hypothesis and thesis are both combinatorial. Stallings also obtains a criterion for unknottedness in codimension 2; the criterion is that the complement be a homotopy circle.

By a deceptive coincidence, the first attempt [9, 10] at the combinatorial unknotting of spheres was valid only down to the same dimension as in the differential case. An improvement of that technique extended the theorem down to codimension 3, and was announced in [12], and a short proof of the first interesting case S^3 in S^6 was given in [11]. The general proof however was much more complicated than that presented here, and I am indebted to John Stallings for suggesting some of the improvements; in particular the proof of Lemma 7 is his. The crux of the proof occurs in Lemma 9.

There is an interesting difference between Theorem 1, the unknotting of balls in balls, and Theorem 2, the unknotting of spheres in spheres. Theorem 2 is an immediate corollary of Theorem 1, but not *vice versa*; the reason is that we run into trouble near the boundary. Theorem 1 is the more basic because it really combines four separate ideas:

(1) a local unknotting of the smaller ball in the larger;
(2) a global unknotting of the smaller ball in the larger;
(3) a global tubular neighbourhood of the smaller ball in the larger; and
(4) compatible collars to the boundaries of the two balls.

When we try to generalise Theorem 1 from balls to arbitrary bounded manifolds, the difference between the four concepts becomes apparent.

The first can be extended, because it is an immediate corollary to Theorems 1 and 2.

The second cannot be extended without further hypotheses. There are algebraic obstructions to the global unknotting of manifolds in manifolds unless they are sufficiently highly connected; in other words there exist embeddings that are homotopic but not isotopic (see [3, 5, 12, 13]).

The third is unsolved. The extension of (3) to manifolds is a major outstanding problem in the combinatorial theory. The problem is: — Given a proper locally unknotted embedding of M^q in M^p, can a regular neighbourhood of M^q in M^p be fibered by $(p-q)$-balls in a piecewise linear fashion? By Theorems 1 and 2, any proper embedding of codimension $\geqq 3$ is locally unknotted, and so the fibering can be done locally, but the difficulty comes in matching the local fiberings to form a global fibering. In the special case of a ball embedded in a manifold, we can grow the global fibering out from the centre of the ball like a single crystal (as in the proof of Lemma 6 below), but for arbitrary manifolds a new technique is needed.

The fourth can be extended. We show in Theorem 3 that, given two bounded manifolds, one properly embedded in the other so as to be locally unknotted on the boundary (which always happens if the codimension $\geqq 3$), then we can construct compatible collars to their boundaries. Another way of saying this is that we can choose local coordinates so that the smaller manifold is everywhere orthogonal to the boundary of the larger manifold.

We conclude the paper by briefly extending Theorems 1, 2, and 3 from pairs to three or more.

Definitions

We define the *standard* n-simplex Δ^n as follows: let $x_0, x_1, \cdots$ be a sequence of independent points in Hilbert space, and let $\Delta^n = x_0 x_1 \cdots x_n$. Consequently for each n, Δ^n is a face of Δ^{n+1}. Let $\dot{\Delta}^n$ denote the boundary of Δ^n. As usual we define a *combinatorial n-ball* B^n (or a *combinatorial n-sphere* S^n) to be a finite simplicial complex piecewise linearly homeomorphic to Δ^n (or $\dot{\Delta}^{n+1}$). A *combinatorial n-manifold* M^n is a finite simplicial complex whose closed vertex stars are combinatorial n-balls. Let $\dot{M}^n$, $\mathring{M}^n$ denote the boundary, interior of M^n, respectively. If $\dot{M}^n$ is empty we call M^n *closed*; otherwise we call M^n *bounded*. If B^{n-1} is a subcomplex of $\dot{B}^n$ we call B^{n-1} a *face* of B^n.

Throughout this paper we shall only be concerned with finite simplicial complexes. Sometimes we shall revert to polyhedra, in order to avoid excessive subdivision. By a polyhedron, we mean the space underlying a finite simplicial complex; and by a subpolyhedron, we mean the subspace underlying a subcomplex of some rectilinear subdivision. Whenever we say sphere, ball or manifold we shall always mean combinatorial sphere, combinatorial ball, and combinatorial manifold. Whenever we say map, homeomorphism or embedding, we shall always mean (with one exception) piecewise linear map, piecewise linear homeomorphism, and piecewise linear embedding. The one exception is the projective map π used in the proof of Lemma 10.

Pairs

Define a *(p, q)-sphere-pair*, $P = (S^p, S^q)$, $p > q$, to be a pair of spheres such that S^q is a subcomplex of S^p. Define a *(p, q)-ball-pair*, $Q = (B^p, B^q)$, $p > q$, to be a pair of balls such that B^q is a subcomplex of B^p and B^q is properly embedded in B^p, i.e., $\dot{B}^q \subset \dot{B}^p$ and $\mathring{B}^q \subset \mathring{B}^p$. Define the boundary of Q to be the sphere-pair $\dot{Q} = (\dot{B}^p, \dot{B}^q)$. By a *pair* $X = (X^p, X^q)$ we shall mean either a sphere-pair or a ball-pair.

We can perform certain operations on pairs. The *join* of a sphere-pair $P = (S^p, S^q)$ to a sphere S^r is the sphere-pair $PS^r = (S^pS^r, S^qS^r)$. Similarly the join of a sphere-pair to a ball, or the join of a ball-pair to a sphere or a ball, gives a ball-pair. In particular the join of a pair to a point is called a *cone-pair*, and is an example of a ball-pair.

If $X = (X^p, X^q)$, $Y = (Y^r, Y^s)$ are two pairs, we say X *contains* Y, written $X \supset Y$, if Y^r is a subcomplex of X^p, and $Y^s = X^q \cap Y^r$. In particular if $P \supset Q$, where $P = (S^p, S^q)$ and $Q = (B^p, B^q)$, then the closure of the complement

$$P - \mathring{Q} = (S^p - \mathring{B}^p, S^q - \mathring{B}^q)$$

is a ball pair by Alexander [1, Th. 14 : 2]. If Q_1 is a $(p-1, q-1)$-ball-pair contained in the boundary $\dot{Q}$ of a (p, q)-ball-pair Q, we call Q_1 a *face* of Q.

If two pairs $X = (X^p, X^q)$, $Y = (Y^p, Y^q)$ are contained in as subcomplexes of a larger complex, then we can define the union and intersection

$$X \cup Y = (X^p \cup Y^p, X^q \cup Y^q), \qquad X \cap Y = (X^p \cap Y^p, X^q \cap Y^q).$$

In general the union and intersection will be neither sphere-pairs nor ball-pairs, but they are in two particular cases of importances :

(i) If two ball-pairs intersect in their common boundary, then their union is a sphere-pair.

(ii) If two ball-pairs intersect in a common face, then their union is again a ball-pair.

Both these statements follow at once from Alexander [1].

Two pairs $X = (X^p, X^q)$, $Y = (Y^p, Y^q)$ are said to be *homeomorphic* if there is a homeomorphism (always piecewise linear of course) of X^p onto Y^p that throws X^q onto Y^q.

Definition of unknottedness

If K is a complex let ΣK denote the suspension of K. The n^{th} suspension is defined inductively, $\Sigma^n K = \Sigma(\Sigma^{n-1}K)$, $\Sigma^1 K = \Sigma K$. Define the *standard* (p, q)-ball-pair to be

$$\Gamma^{p,q} = (\Sigma^{p-q}\Delta^q, \Delta^q),$$

and the standard (p, q)-sphere-pair to be the boundary $\dot{\Gamma}^{p+1, q+1}$ of the standard $(p + 1, q + 1)$-ball-pair. Since we originally defined Δ^q to be an explicit face of Δ^{q+1}, it follows that $\Gamma^{p,q}$ is an explicit face of $\Gamma^{p+1,q+1}$.

Define a ball-pair or a sphere-pair to be *unknotted* if it is homeomorphic to a standard pair. Sometimes it is convenient to use a different form of words to cover the case when S^q is (piecewise linearly) embedded in S^p, rather than being a subcomplex of S^p: we say that *S^q is unknotted in S^p*, if, having chosen a subdivision S^p_1 of S^p that contains a subcomplex S^q_1 covering the image of the embedded S^q, the sphere-pair (S^p_1, S^q_1) is unknotted. Similarly for balls.

THEOREM 1. *If $p - q \geqq 3$ then any (p, q)-ball-pair is unknotted.*
THEOREM 2. *If $p - q \geqq 3$ then any (p, q)-sphere-pair is unknotted.*

The proof of Theorems 1 and 2 is the main burden of this paper. The proof is by induction on p, keeping the codimension fixed. First we show in Lemma 1 that Theorem $1_{p,q}$ implies Theorem $2_{p,q}$. The main part of the proof consists of showing, with the help of the next ten lemmas, that Theorems $1_{p-1,q-1}$ and $2_{p-1,q-1}$ together imply Theorem $1_{p,q}$, provided that $p \geqq q + 3$. For codimension r the induction begins trivially with Theorem $1_{r,0}$, for this is merely the observation that, given a ball B^r with an interior point B^0, then there is a homeomorphism of B^r onto the r-dimensional "octahedron", throwing B^0 onto the centre of the octahedron. Since the inductive steps are the same for all codimensions $\geqq 3$, we can drop the suffix q, and let Theorem 1_p denote the inductive assumption that Theorem 1 is true for all (p', q)-ball-pairs such that $p \geqq p' \geqq q + 3$; let Theorem 2_p denote the analogous statement for spheres.

REMARK. Theorem 2 implies that, provided the codimension $\geqq 3$, spheres unknot in euclidean space in the sense of [10]. In other words, given a (piecewise linear) embedding of S^q in E^p, $p - q \geqq 3$, then there exists a (piecewise linear) homeomorphism of E^p onto itself throwing S^q onto the boundary of a $(q + 1)$-simplex. For embed E^p in S^p piecewise linearly onto the complement of a point x, say. By Theorem 2 there is a homeomorphism

$$h : (S^p, S^q) \to \dot{\Gamma}^{p+1,q+1}$$

onto the standard sphere-pair. Choose a suspension vertex y such that $hx \notin y\dot{\Delta}^{q+1}$. Then $h^{-1}(y\dot{\Delta}^{q+1})$ gives a $(q + 1)$-ball in E^p spanning S^q. Consequently S^q is unknotted in E^p in the sense of [10] by [10, Th. 1(3)].

LEMMA 1. *Theorem 1_p implies Theorem 2_p.*

PROOF. Let $P = (S^p, S^q)$ be a sphere-pair with $p - q \geqq 3$. Choose a vertex $x \in S^q$. Then $P = Q \cup x\dot{Q}$, where Q is the ball-pair $Q =$

$(S^p - \mathrm{st}(x, S^p), \dot{S}^q - \mathrm{st}(x, S^q))$. Let y be the vertex of Δ^{q+1} opposite the face Δ^q. Then $\dot{\Gamma}^{p+1\,q+1} = \Gamma^{p,q} \cup y\dot{\Gamma}^{p,q}$. Any homeomorphism $Q \to \Gamma^{p,q}$ given by Theorem 1_p can be extended by mapping $x \to y$ to a homeomorphism $P \to \dot{\Gamma}^{p+1,q+1}$. Hence P is unknotted. The same argument holds for all $p' \leqq p$.

LEMMA 2. *Let Q_1, Q_2 be two unknotted (p, q)-ball-pairs. Then any homeomorphism $\dot{Q}_1 \to \dot{Q}_2$ can be extended to a homeomorphism $Q_1 \to Q_2$.*

PROOF. Let Γ be the standard (p, q)-ball-pair. Let y be the barycentre of Δ^q; then elementary subdivision gives a homeomorphism $\Gamma \to y\dot{\Gamma}$. For $i = 1, 2$ the pair Q_i is unknotted by hypothesis, and so there is a homeomorphism $Q_i \to \Gamma$; let f_i denote the composition

$$Q_i \longrightarrow \Gamma \longrightarrow y\dot{\Gamma}\ .$$

Let g denote the composition of the homeomorphisms

$$\dot{\Gamma} \xrightarrow{(f_1 | \dot{Q}_1)^{-1}} \dot{Q}_1 \xrightarrow{h} \dot{Q}_2 \xrightarrow{(f_2 | \dot{Q}_2)} \dot{\Gamma}$$

where h is given by the hypothesis. Extend g to g: $y\dot{\Gamma} \to y\dot{\Gamma}$. The composition

$$Q_1 \xrightarrow{f_1} y\dot{\Gamma} \xrightarrow{g} y\dot{\Gamma} \xrightarrow{f_2^{-1}} Q_2$$

is the homeomorphism required to prove the lemma.

LEMMA 3. *Assume Theorem 1_{p-1}, and suppose $p - q \geqq 3$. Then if two unknotted (p, q)-ball-pairs intersect in a common face their union is also unknotted.*

PROOF. Let $Q_1 \cap Q_2 = Q_3$, where Q_3 is a common face of the two unknotted (p, q)-ball-pairs Q_1, Q_2. Let $Q_4 = \dot{Q}_1 - \mathring{Q}_3$ and $Q_5 = \dot{Q}_2 - \mathring{Q}_3$. Then by Theorem 1_{p-1}, Q_3, Q_4, Q_5 are unknotted $(p - 1, q - 1)$-ball-pairs, and $\dot{Q}_3 = \dot{Q}_4 = \dot{Q}_5$. Let $\Sigma\Gamma$ be the suspension of the standard $(p - 1, q - 1)$-ball-pair Γ, and let x_1, x_2 be the two suspension points. Choose a homeomorphism h: $Q_3 \to \Gamma$. By Lemma 2 extend $h | \dot{Q}_4$ to a homeomorphism $Q_4 \to x_1\dot{\Gamma}$. This, together with h, defines a homeomorphism $\dot{Q}_1 \to \Gamma \cup x_1\dot{\Gamma}$, which we can extend, again by Lemma 2, to a homeomorphism $Q_1 \to x_1\Gamma$. Similarly extend h to a homeomorphism $Q_2 \to x_2\Gamma$. These two homeomorphisms combine to give a homeomorphism $Q_1 \cup Q_2 \to \Sigma\Gamma$, which is clearly unknotted.

Simplicial collapsing

Recall Whitehead's collapsing techniques, which he introduced in [8]. Let K be a finite simplicial complex and L a subcomplex. We say there is an *elementary simplicial collapse* from K to L if $K - L$ consists of a principal simplex of K together with one top-dimensional face. We say K *simplicially collapses* to L if there exists a sequence of elementary simplicial collapses going from K to L. If K simplicially collapses to a point, we call K *simplicially collapsible*.

We give next an equivalent polyhedral version of the same idea. It is advantageous to use both in practice, the choice depending upon whether the collapse is part of the hypothesis or thesis of the proof in question; if part of the hypothesis, we generally assume simplicial collapsing (as in Lemma 7); and if part of the thesis, we generally prove polyhedral collapsing (as in Lemma 9).

Polyhedral collapsing*

Let X be a polyhedron and Y a subpolyhedron. We say there is an *elementary collapse* from X to Y if there exist complexes K, L triangulating X, Y and a ball B^n with a face B^{n-1}, such that $X = Y \cup B^n$ and $B^{n-1} = Y \cap B^n$. We say X *collapses to* Y, written $X \searrow Y$, if there exists a finite sequence of elementary collapses going from X to Y. If X collapses to a point we call X *collapsible*, and write $X \searrow 0$. In particular a ball is collapsible. If L is a subcomplex of K, we write $K \searrow L$ if the underlying polyhedron of K collapses onto that of L. The relation between collapsing and simplicial collapsing is contained in [8, Theorems 6 and 7]:

LEMMA 4 (Whitehead). *If L is a subcomplex of K, then $K \searrow L$ if and only if there is a subdivision αK of K such that αK simplicially collapses onto αL.*

Regular neighbourhoods

Let M be an n-manifold and X a subpolyhedron. A *regular neighbourhood* of X in M is a subpolyhedron of M such that

(i) N is a closed neighbourhood of X in M,
(ii) N is an n-manifold, and
(iii) $N \searrow X$.

LEMMA 5 (Whitehead). *If N_1, N_2 are two regular neighbourhoods of*

* There is a slight technical difference between our definition of collapsing $K \searrow L$ and Whitehead's definition of "geometrical contraction" [8, § 5]. Our definition is easier to use because it is combinatorially invariant and not tied down to any particular triangulation, whereas Whitehead's process is tied down to subdivisions of K, and his proof of the combinatorial invariance of his process [8, Corollary to Th. 7] contains a flaw which is by-passed by the use of our definition.

X in M, then there is a homeomorphism $N_1 \to N_2$ keeping X fixed.

The proof is contained in [8, Th. 23]. Notice that we have stated a slightly stronger result than that stated by Whitehead, inasmuch as we have claimed that the homeomorphism is fixed on X. This stronger result is permissible because we have used a stronger definition of regular neighbourhood than Whitehead, in that we have included the proviso (i), that N also be a topological neighbourhood. Consequently, of the sequence of regular moves passing from N_1 to N_2 that Whitehead uses in [8, Lemmas 10 and 11] to prove the homeomorphism, none intersects X, and so X can be left undisturbed during each move. We are now ready to make the next step towards the proof of Theorems 1 and 2.

LEMMA 6. *Assume Theorems 1_{p-1} and 2_{p-1}. If (B^p, B^q), $p - q \geqq 3$, is a ball-pair such that $B^p \searrow B^q$, then it is unknotted.*

PROOF. A ball is collapsible; therefore choose a subdivision αB^q of B^q that is simplicially collapsible. Extend this to a subdivision αB^p of B^p. Let βB^p be the second derived complex of αB^p. Let N be the closed simplicial neighbourhood of βB^q in βB^p. By [8, Theorem 22] N is a regular neighbourhood of B^q in B^p. But the hypothesis indicates that B^p is itself a regular neighbourhood of B^q in B^p. Therefore Lemma 5 gives a homeomorphism between the ball-pairs

$$(B^p, B^q) \to (N, \beta B^q) .$$

Therefore to prove the lemma, it suffices to show that $(N, \beta B^q)$ is unknotted. Let

$$\alpha B^q = K_k \searrow K_{k-1} \searrow \cdots \searrow K_1 \searrow K_0 = x$$

be the simplicial collapse of αB^q down to a point x. Let Q_i be the ball-pair consisting of the closed simplicial neighbourhoods of K_i in $(\beta B^p, \beta B^q)$. We shall show inductively that Q_i is unknotted.

The induction begins with $Q_0 = xL_x$, the cone-pair on

$$L_x = \big(\operatorname{lk}(x, \beta B^p), \operatorname{lk}(x, \beta B^q)\big) .$$

Now L_x is either a sphere-pair if x is in the interior of αB^q, or a ball-pair if x is on the boundary of αB^q, but in either case is unknotted by the hypothesis, Theorems 1_{p-1} and 2_{p-1}. Hence Q_0 is unknotted.

For the inductive step, suppose Q_{i-1} is unknotted, where $1 \leqq i \leqq k$. Since $K_i \searrow K_{i-1}$ is an elementary simplicial collapse, $K_i - K_{i-1}$ consists of a simplex A with a top-dimensional face C, say. Let a, c denote the barycentres of A, C, respectively. Let $Q_a = aL_a$, the cone-pair on $L_a = (\operatorname{lk}(a, \beta B^p), \operatorname{lk}(a, \beta B^q))$, which is unknotted for the same reason as Q_0 above. Similarly let $Q_c = cL_c$, which is also unknotted. Now

$$Q_i = Q_{i-1} \cup Q_a \cup Q_c .$$

But Q_{i-1} and Q_a intersect in a common face (see [8, p. 295]), and so do $(Q_{i-1} \cup Q_a)$ and Q_c. Hence by applying Lemma 3 twice, we see that Q_i is unknotted. At the end of the induction we have $Q_k = (N, \beta B^q)$ unknotted, which completes the proof of the lemma.

REMARK. With codimension 2 the previous lemma is no longer true; it is possible to have $B^p \searrow B^{p-2}$ but (B^p, B^{p-2}) knotted. For example let (B^4, B^2) be the cone-pair on a knotted (S^3, S^1). Then $B^4 \searrow B^2$ because a cone collapses onto any subcone. Also with codimension 2, it is possible to have a ball-pair (B^p, B^{p-2}) such that B^p does not $\searrow B^{p-2}$, as for example a knotted arc in a 3-ball. The next lemma shows this cannot happen with higher codimension.

LEMMA 7. *If* (B^p, B^q), $p - q \geqq 3$, *is a ball-pair, then* $B^p \searrow B^q$.

Once we have proved Lemma 7 we shall have completed the proof of Theorems 1 and 2, because Lemmas 6 and 7 together provide the inductive step that Theorems 1_{p-1} and 2_{p-1} imply Theorem 1_p. However we shall postpone the proof of Lemma 7 until after that of Lemma 9, because we shall first have to make some geometrical constructions. Indeed we have not yet used any geometry that would suggest the significance of codimension 3.

Shadows

Let I^p be the p-cube. We single out the last coordinate for special reference and write $I^p = I^{p-1} \times I$. Intuitively we think of I^{p-1} as horizontal and I as vertical, and we identify I^{p-1} with $I^{p-1} \times 0$, the base of the cube I^p. Let X be a polyhedron in I^p. Imagine the sun, vertically overhead, causing X to cast a shadow; a point of I^p is said to lie in the shadow of X if it lies vertically below some point of X.

DEFINITION. Let X^* be the closure of the set of points of X that lie in the same vertical line as some other point of X (i.e., the set of points of X that either lie in the shadow of X or else overshadow some other point of X). Then X^* is a subpolyhedron of X.

LEMMA 8. *Let X be a polyhedron in I^p such that* $\dim X = q < p$ *and* $\dim (X \cap \dot{I}^p) < p - 1$. *Then there exists a homeomorphism of I^p onto itself throwing X into a position that satisfies the properties:*

(i) *X does not meet the top or the bottom of the cube;*

(ii) *X meets any vertical line finitely; and*

(iii) $\dim X^* \leqq 2q - p + 1$.

PROOF. Choose a face I^{p-2} of I^{p-1}, so that $I^{p-2} \times I$ is a vertical top-dimensional face of I^p. Since $X \not\supset \dot{I}^p$ there is a homeomorphism of I^p onto itself throwing $X \cap \dot{I}^p$ into the interior of this vertical face, satisfying condition (i). Now triangulate I^p so that X is a subcomplex. Then shift all the vertices of this triangulation by arbitrarily small moves into general position, in such a way that any vertex in the interior on I^p remains in the interior, and any vertex in a face of I^p remains inside that face. If the moves are sufficiently small, the new positions of the vertices determine an isomorphic triangulation, and a homeomorphism of I^p onto itself. The general position ensures that X is thrown onto a polyhedron with the desired properties.

Sunny collapsing

Suppose we are given polyhedra $Y \subset X \subset I^p$. If there is an elementary collapse from X to Y, define this collapse to be *sunny* if no point of $X - Y$ lies in the shadow of X. We say there is a *sunny collapse* $X \searrow Y$ if there exists a finite sequence of elementary sunny collapses going from X to Y. If there is a sunny collapse $X \searrow 0$, then X is called *sunny collapsible*. Similarly we can define sunny simplicial collapses between complexes in I^p, and deduce:

COROLLARY TO LEMMA 4. *X is sunny collapsible if and only if some triangulation of X is sunny simplicially collapsible.*

The proof of the corollary follows from [8, Th. 6], because each elementary sunny collapse can be factored into a sequence of elementary simplicial collapses, each of which will be sunny.

LEMMA 9. *Suppose (I^p, X) is homeomorphic to a (p, q)-ball-pair, $p - q \geqq 3$, and suppose X satisfies the three properties of Lemma 8. Then X is sunny collapsible.*

REMARK. Lemma 9 fails with codimension 2. The classical example of a knotted arc in I^3 gives a good intuitive feeling for the obstruction to a sunny collapse: looking down from above, it is possible to start collapsing away until we hit underpasses, which are in shadow and so prevent any further progress.

PROOF OF LEMMA 9. The proof is long, by a complicated induction. Let $Y_0 = \dot{\Delta}^p$ and $Z_0 = X$. We shall construct inductively two descending sequences of subpolyhedra

$$Y_0 \supset Y_1 \supset \cdots \supset Y_i \supset \cdots \supset Y_q$$
$$Z_0 \supset Z_1 \supset \cdots \supset Z_i \supset \cdots \supset Z_q ,$$

and for each i, $0 \leqq i \leqq q$, a homeomorphism

$$f_i \colon C_i \to Z_i$$

from C_i onto Z_i, where C_i is the cone on Y_i, such that the following four properties are satisfied:

(1) Y_i is everywhere $(q - i - 1)$-dimensional;

(2) $\dim Z_i^* \leqq q - i - 2$;

(3) $f_i^{-1}(Z_i^*)$ does not contain the vertex of the cone C_i, and meets each generator of the cone finitely; and

(4) there exists a sunny collapse $Z_{i-1} \searrow Z_i$.

The induction begins with $Z_0 = X$ and finishes with Z_q being a point (Y_q being empty). Therefore the lemma will follow from Property (4), because the sequence

$$X = Z_0 \searrow Z_1 \searrow \cdots \searrow Z_q$$

shows that X is sunny collapsible.

Beginning of the induction

We have defined $Y_0 = \dot{\Delta}^q$, $Z_0 = X$. Therefore Property (1) is trivial, because $\dot{\Delta}^q$ is everywhere $(q - 1)$-dimensional. Property (2) follows from Lemma 8 (iii), because (and this is the point where codimension 3 enters)

$$\dim X^* \leqq 2q - p + 1 \leqq q - 2 .$$

Property (4) is vacuous because Z_{-1} is not defined. There remains to define the homeomorphism f_0 so as to satisfy Property (3).

Choose a homeomorphism $f \colon \Delta^q \to X$ onto the q-ball $X = Z_0$. Choose a vertex v in the interior of Δ^q in general position with respect to $f^{-1}(X^*)$. General position means that $v \notin f^{-1}(X^*)$, and that any straight line in Δ^q through v meets $f^{-1}(X^*)$ finitely. Subdividing Δ^q at v gives a complex isomorphic to the cone C_0 on $Y_0 = \dot{\Delta}^q$. Define $f_0 = f \colon C_0 \to Z_0$, and then Property (3) is satisfied by construction.

The inductive step

Fix i, $1 \leqq i \leqq q$. Suppose we are given the polyhedra Y_{i-1}, Z_{i-1} and the homeomorphism $f_{i-1} \colon C_{i-1} \to Z_{i-1}$ satisfying the four inductive properties. We have to define subpolyhedra Y_i, Z_i and a homeomorphism $f_i \colon C_i \to Z_i$, and prove the four properties for them. Since the inductive step is complicated, let us drop the suffix $i - 1$ and retain the suffix i. That is to say we are given $Y = Y_{i-1}$, $Z = Z_{i-1}$, $C =$ the cone on Y, and $f \colon C \to Z$; and we shall eventually define Y_i, Z_i and f_i.

Let v be the vertex of the cone C, and let $W = f^{-1}(Z^*)$. Then W is a subpolyhedron of C of dimension $\leqq q-i-1$ (by the inductive Property (2)) that does not contain v, and meets each generator of the cone finitely (by the inductive Property (3)). Let π: $W \to Y$ be the map defined by projecting from the vertex v onto the base Y of the cone C. Now in general π is not piecewise linear, and so it is impossible to find triangulations of W, Y with respect to which π is simplicial; however π is projective, so that we can do the next best thing.

LEMMA 10. *There exist triangulations K, L of W, Y such that for each simplex $A \in K$, πA is a simplex of L of the same dimension.*

PROOF. Choose some triangulation K_0 of W. For each simplex $A_0 \in K_0$, πA_0 is a simplex contained in Y. The dimension of πA_0 is the same as that of A_0 because of the inductive Property (3). As A_0 runs over the simplexes in K_0, the set of image simplexes πA_0 may criss-cross each other in Y, but, nevertheless, it is possible to find a triangulation L of Y, such that every πA_0 is covered by a subcomplex of L. Lift these subcomplexes under π to form a subdivision K of K_0. Then K, L satisfy the requirements of the lemma.

Definition of Y_i

We are now in a position to define Y_i. Let Y_i be the polyhedron underlying the $(q-i-1)$-skeleton of L. By the inductive Property (1), Y is everywhere $(q-i)$-dimensional. Therefore every principal simplex of L is $(q-i)$-dimensional, and Y_i is everywhere $(q-i-1)$-dimensional. Hence Property (1) holds for Y_i.

The cone $C_i = vY_i$ is a subcone of C. However it is no good trying to define $f_i = f \mid C_i$, because then we should have to have $Z_i = fC_i \supset fW = Z^*$, and so Z_i^* would in general be of dimension $q-i-1$, which is too high for Property (2). In fact this is the crux of the matter: we must arrange some device for collapsing away the top-dimensional shadows of Z^*.

The first thing to observe is that the triangulation K of W is in no way related to the embedding of $fW = Z^*$ in the cube I^p. The images in I^p of the simplexes of K may link around and overshadow each other in an unpredictable fashion. Our next task is to take a subdivision K' of K that remedies this confusion. Let g: $K \to I^{p-1}$ denote the composition of f followed by vertical projection onto the base of the cube:

$$\underbrace{K \subset C \xrightarrow{f} Z \subset X \subset I^p \longrightarrow I^{p-1}}_{g} .$$

Since g is piecewise linear, we can find subdivisions K', M of K, I^{p-1} such that $g: K' \to M$ is simplicial.

Recall that $\dim K' = \dim W \leqq q - i - 1$. Let $A_1, A_2, \cdots, A_m$ be the $(q - i - 1)$-simplexes of K'. Each A_j is mapped non-degenerately by g, by Lemma 8 (ii). For each pair $A_j, A_k, j \neq k$, the interiors $\mathring{A}_j, \mathring{A}_k$ are mapped disjointly by f, and are either mapped disjointly or identified by g. If $gA_j \neq gA_k$ then no point of $f\mathring{A}_j$ overshadows any point of $f\mathring{A}_k$ and *vice versa*. If $gA_j = gA_k$, then vertical projection establishes a homeomorphism between $f\mathring{A}_j$ and $f\mathring{A}_k$, so that either $f\mathring{A}_j$ overshadows $f\mathring{A}_k$ or *vice versa*. Consequently overshadowing induces a partial ordering between the A's, and we choose the ordering $A_1, A_2, \cdots, A_m$ to be compatible with this partial ordering. We state this in the form of a lemma:

LEMMA 11. *All the points of X that overshadow $f\mathring{A}_k$ are contained in* $\bigcup_{1 \leqq j < k} f\mathring{A}_j$.

Construction of the blisters

The next step is to construct a little $(q - i + 1)$-dimensional blister J_j about each A_j in the cone C. The blisters are the device that enable us to make the sunny collapse, and the fact that there is just sufficient room to construct them is an indication of why codimension 3 is a necessary and sufficient condition for unknotting.

Choose $\varepsilon > 0$ and sufficiently small (the criterion for sufficiently small will appear at the end of the construction). There are two cases depending on whether or not A_j happens to lie in Y.

Case (i). Suppose $A_j \subset Y$; then the blister will lie at the bottom of the cone. Let a_j be the barycentre of A_j. Let b_j be the point on the line va_j (v is the vertex of the cone) a distance ε from a_j. Since

$$\dim A_j = \dim K' = q - i - 1 ,$$

A_j is contained in a $(q - i - 1)$-simplex D_j of K. Since $A_j \subset Y$, we also have $D_j \subset Y$, and so by Lemma 10, $D_j = \pi D_j \in L$. By the inductive Property (1), Y is everywhere $(q - i)$-dimensional, and so there is at least one $(q - i)$-simplex $E_j \in L$ having D_j as a face. Let a'_j be the point of the join of a_j to the barycentre of E_j, a distance ε from a_j. Define

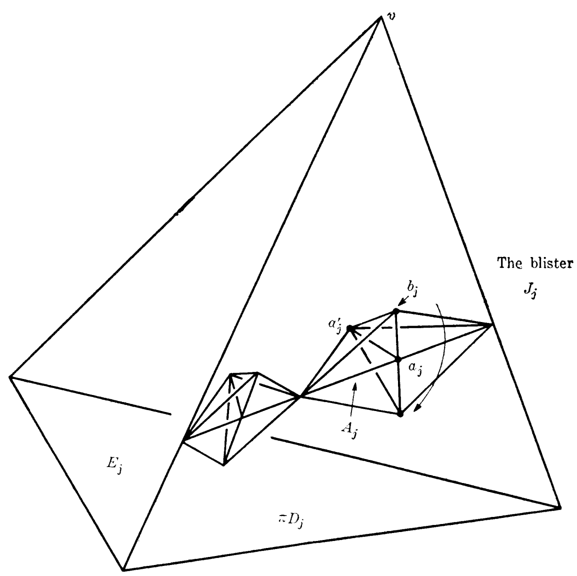
$$J_j = a_j a'_j b_j \dot{A}_j .$$

Case (ii). Suppose $A_j \not\subset Y$; then the blister will lie in the middle of the cone. Again let a_j be the barycentre of A_j; then $a_j \notin Y$. By the inductive Property (3) the generator of the cone va_j through a_j does not meet A_j again. Let b_j denote the pair of points on this generator a distance ε either side of a_j. As before, A_j is contained in a $(q - i - 1)$-simplex

D_j of K, only this time $D_j \not\subset Y$. By Lemma 10, $\pi D_j \in L$, and again we can choose a $(q - i)$-simplex $E_j \in L$ having πD_j as a face. Let a'_j be the point on the line joining a_j to the barycentre of vE_j a distance ε from a_j. Again define J_j by the same formula

$$J_j = a_j a'_j b_j \dot{A}_j \ .$$

Finally choose ε sufficiently small for all the blisters to be well defined, and so that no two overlap more than necessary, i.e., $J_j \cap J_k = \dot{A}_j \cap \dot{A}_k$ for each pair j, k.



Definition of Z_i and f_i

We are now in a position to complete the inductive definitions. Recall that we defined Y_i to be the $(q - i - 1)$-skeleton of L. Hence $C_i = vY_i$, the cone on Y_i, is a subcone of C. Therefore, for each j,

$$C_i \cap J_j = a_j b_j \dot{A}_j \ .$$

Define an embedding $e: C_i \to C$ as follows: let e be the identity outside all the blisters, and inside the j^{th} blister map $a_j b_j \dot{A}_j$ linearly onto $a'_j b_j \dot{A}_j$, for each j. In other words to obtain the embedding $e: C_i \to C$ from the inclusion $C_i \subset C$, we merely push up all the blisters. Define

$$f_i = fe: C_i \to Z \ ,$$

and define

$$Z_i = f_i C_i \ .$$

This completes the inductive definitions. We have already verified Property (1); there remains to verify Properties (2), (3) and (4).

To verify Property (2), observe that

$$\begin{aligned} Z_i^* &\subset Z_i \cap Z^* \\ &= feC_i \cap fW \\ &= f(eC_i \cap W) \ , \end{aligned}$$

since f is a homeomorphism. But the whole point of our construction of the embedding e was to push C_i away from the interior of the top-dimensional simplexes of K'. Hence $eC_i \cap W$ is contained in the $(q - i - 2)$-skeleton of K'. Therefore $\dim Z_i^* \leqq q - i - 2$.

The same observation suffices to verify Property (3), because

$$f_i^{-1}(Z_i^*) \subset \text{the } (q - i - 2)\text{-skeleton of } K' \subset f^{-1}(Z^*) \ .$$

By induction Property (3) holds for $f^{-1}(Z^*)$ and so it also holds for $f_i^{-1}(Z_i^*)$.

The sunny collapse

Finally we come to Property (4), which is the heart of the matter. Let $J = \bigcup_{1 \leq j \leq m} J_j$, the union of all the blisters. Given a $(q - 1)$-simplex $E \in L$, let E_1, E_2 denote respectively the closures of $vE - J$, $E - J$. Now E_1 can be obtained from the $(q - i + 1)$-simplex vE by removing one by one any blisters that happen to protrude into vE; therefore E_1, suitably triangulated, is a $(q - i + 1)$-ball by [1, Corollary 14: 5b]. Similarly E_2 is a $(q - i)$-ball, and a face of E_1. Removing the interiors of E_1 and E_2 defines an elementary collapse of C. Doing this successively for all the $(q - i)$-simplexes in L defines a collapse

$$C \searrow C_i \cup J \ .$$

But $C_i \cup J = eC_i \cup J$ and $f(eC_i \cup J) = Z_i \cup fJ$. Therefore the image under the homeomorphism f of this collapse determines a sunny collapse

$$Z \searrow Z_i \cup fJ \ ,$$

sunny because we have not yet removed any point of Z^*.

We now collapse $eC_i \cup J \searrow eC_i$ by collapsing each blister in turn, $j = 1, 2, \cdots, m$, as follows. The blister J_j is a $(q - i + 1)$-ball, and its intersection with eC_i (and all the other blisters) is the $(q - i)$-face $a_j' b_j \dot{A}_j$. Therefore we may collapse J_j onto this face. The images under f of these collapses determine a sequence of elementary collapses

$$Z_i \cup \textstyle\bigcup_{j=1}^m fJ_j \searrow Z_i \cup \bigcup_{j=2}^m fJ_j \searrow \cdots \searrow Z_i \ .$$

Each of these elementary collapses is sunny by Lemma 11, and by virtue of our choice of the ordering $j = 1, 2, \cdots, m$; because, by the time we come to collapse fJ_k, say, the only points that might have been in shadow are those in the interior $f\mathring{A}_k$, but these are sunny for we have already removed everything that overshadows them. Hence we have demonstrated a sunny collapse

$$Z \searrow Z_i \cup fJ \searrow Z_i \, .$$

The proof of Lemma 9 is complete.

Proof of Lemma 7

We can now return to the proof of Lemma 7, which will conclude the proof of Theorems 1 and 2. We are given a ball-pair (B^p, B^q), $p - q \geqq 3$, and we have to show that $B^p \searrow B^q$. By Lemmas 8 and 9, we can choose a homeomorphism $B^p \to I^p$ such that B^q is thrown onto a sunny collapsible polyhedron X satisfying the three properties of Lemma 8. It suffices to show that $I^p \searrow X$.

DEFINITION. If F is a complex or polyhedron in I^p, let $F^\sharp$ denote the polyhedron consisting of F together with all points of I^p lying in the shadow of F. ($F^\sharp$ is quite different from the construction F^* used in the proofs of Lemmas 8 and 9.) Recall that I^{p-1} denotes the base of the cube. Let

$$M = I^{p-1} \cup X^\sharp \, .$$

First we verify that $I^p \searrow M$, as follows. The vertical projection $X \to I^{p-1}$ is piecewise linear, and so we can choose triangulations of X, I^{p-1} with respect to which it is simplicial. Let L denote the triangulation of I^{p-1}. For each simplex $D \in L$, let $D \times I$ denote the prism lying vertically above D. If the interior of the prism meets X, then by Lemma 8 (ii) it meets it in a finite number of simplexes, each of the same dimension as D and lying vertically above D. Let D_1 be the topmost of these; D_1 does not meet the top or bottom of the prism by Lemma 8 (i). Then M contains the subprism bounded above by D_1, and contains no points above $\mathring{D}_1$. Let D' denote the subprism bounded below by D_1. If, on the other hand, the interior of $D \times I$ does not meet X, let $D' = D \times I$. Consider the elementary collapse of D' from the top onto the walls and base. Now enumerate the simplexes of L in order of decreasing dimension, and the corresponding sequence of elementary collapses determines a collapse

$$I^p \searrow M \, .$$

Next we make use of Lemma 9 and the Corollary to Lemma 4. Let K be a triangulation of X that is simplicially sunny collapsible by the sequence, say, of elementary simplicial sunny collapses

$$K = K_0 \searrow K_1 \searrow \cdots \searrow K_n = \text{a point} .$$

Let

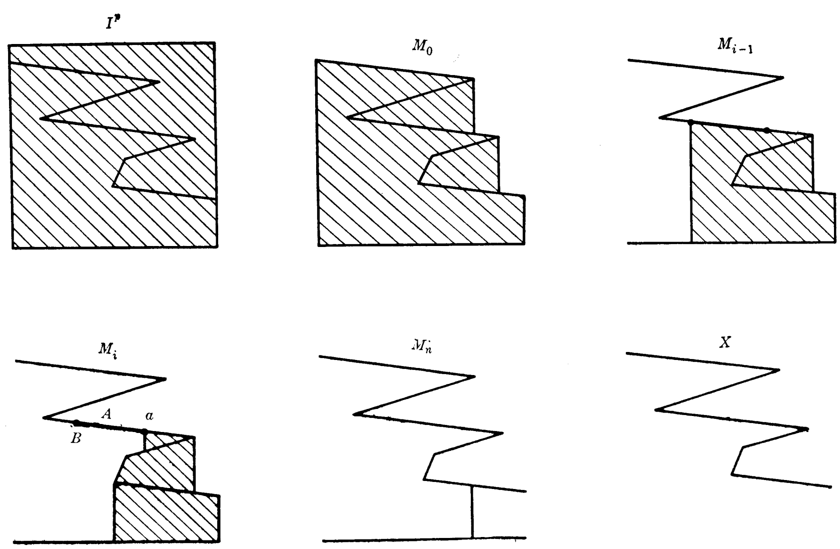
$$M_i = I^{p-1} \cup X \cup K_i^{\#} .$$

In particular $M_0 = M$. We shall complete the proof of Lemma 7 by showing that

$$I^p \searrow M_0 \searrow M_1 \searrow M_2 \searrow \cdots \searrow M_n \searrow X .$$

The first step $I^p \searrow M_0$ we have already demonstrated above. The last step is easy, because $I^{p-1} \cap X = \emptyset$ by Lemma 8 (i). Therefore M_n consists of I^{p-1} and X connected by a single arc $K_n^{\#}$. Therefore collapse $M_n \searrow X$ by collapsing I^{p-1} onto the bottom point of the arc, and then collapsing the arc.

There remain the intermediate steps $M_{i-1} \searrow M_i$, $1 \leqq i \leqq n$.



We are given a sunny elementary simplicial collapse $K_{i-1} \searrow K_i$. Suppose that the collapse is across the simplex $A \in K_{i-1}$, from the face B. Let a be the vertex of A opposite B. Therefore

$$K_i \cup A = K_{i-1} , \qquad K_i \cap A = a\dot{B} .$$

Let a_1, A_1, B_1 be the vertical projections of a, A, B on the base I^{p-1} of the cube; A_1, B_1 are simplexes of the same dimension as A, B by Lemma 8 (ii). Let $A_1 \times I$ denote the prism lying above A_1. Let

$$U = (K^{\#}_{i-1} \cup I^{p-1}) \cap (A_1 \times I) ,$$
$$V = (K^{\#}_{i} \cup I^{p-1}) \cap (A_1 \times I) .$$

Then $M_{i-1} - M_i = U - (V \cup A)$. Therefore to show $M_{i-1} \searrow M_i$ it suffices to prove that $U \searrow V \cup A$.

Let us examine U. To begin with U contains the subprism lying between A and A_1 (A does not meet A_1 by Lemma 8 (i)). Since the collapse $K_{i-1} \searrow K_i$ is sunny, U contains no points above $\mathring{A} \cup \mathring{B}$. However U may contain material above $a\dot{B}$ in the walls $a_1\dot{B}_1 \times I$ of the prism $A_1 \times I$.

Now examine V. To begin with V agrees with U in the walls $a_1\dot{B}_1 \times I$ of the prism. However V does not contain A, and in fact

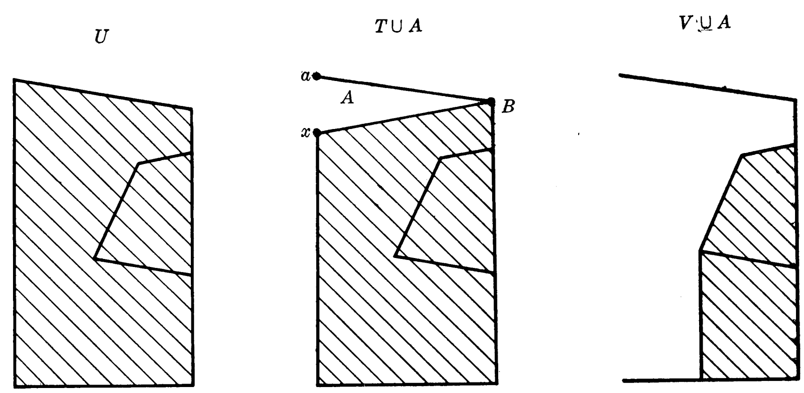
$$A \cap V = a\dot{B} .$$

Since V is a polyhedron we can find a point x vertically below the barycentre of B, such that

$$xA \cap V = a\dot{B} .$$

Let T be the closure of $U - xA$. Then $U \supset T \cup A \supset V \cup A$. We shall show that we can collapse

$$U \searrow T \cup A \searrow V \cup A .$$



The first step is an elementary collapse across xA from the free face xB. For the second step, since $(T \cup A) - (V \cup A) = T - V$, it suffices to show that $T \searrow V$. We use the same device that we used at the beginning of the proof of this lemma, as follows. Choose triangulations of T, V, A_1 such that the vertical projections $T \to A_1$ and $V \to A_1$ are simplicial. Let L be the triangulation of A_1, and let L_0 denote the subcomplex covering $a_1\dot{B}_1$.

Let D be a simplex of $L - L_0$, and let $E = D \cap L_0$ (possibly E is empty). The intersection of T with the prism $D \times I$ above D consists of a non-degenerate subprism bounded above by a simplex, D_1 say, contained in $xa\dot{B}$, together with possibly some material above E. Similarly the intersection of V with $D \times I$ is a possibly degenerate subprism bounded above by a simplex, D_2 say, contained in $K_i \cup I^{p-1}$, together with the same material above E. Now $D_1 \neq D_2$ by our choice of x, and $D_1 \cap D_2$ is a common face, perhaps empty, above E. Therefore there is a non-degenerate subprism, D' say, bounded above by D_1 and below by D_2. Consider the elementary collapse of D' from the top D_1 onto the walls and base D_2. Now enumerate the simplexes of $L - L_0$ in order of decreasing dimension, and the corresponding sequence of elementary collapses determine the required collapse

$$T \searrow V .$$

Therefore we have shown $U \searrow V \cup A$, and hence $M_{i-1} \searrow M_i$, and hence $I^p \searrow X$. This completes the proof of Lemma 7 and Theorems 1 and 2.

Manifold-pairs

One feature of the unknotting of balls is the straightening up of the boundaries. The second half of the paper is concerned with generalising this particular feature to arbitrary manifolds.

Define a *manifold-pair* $V = (M^p, M^q)$, $p > q$, to be a pair of manifolds such that M^q is a subcomplex of M^p, and M^q is properly embedded in M^p (i.e., $\dot{M}^q \subset \dot{M}^p$ and $\mathring{M}^q \subset \mathring{M}^p$). Each of the manifolds may or may not be connected. If both manifolds are closed, we call V closed. If both manifolds are bounded we call V bounded, and define the boundary $\dot{V} = (\dot{M}^p, \dot{M}^q)$. The third possibility is that M^p is bounded and M^q closed, in which case the boundary $\dot{V} = \dot{M}^p$, a single manifold rather than a manifold-pair.

Local knotting

If A is a simplex of M^q, the link of A in V,

$$\text{lk}(A, V) = \big(\text{lk}(A, M^p), \text{lk}(A, M^q)\big) ,$$

is either a ball-pair or a sphere-pair, according as to whether A lies in the boundary or the interior of M^q. If the links of all the vertices of M^q are unknotted we say V is *locally unknotted*. If just the links of the vertices of $\dot{M}^q$ (or $\mathring{M}^q$) are unknotted, we say V is *locally unknotted on the boundary* (or *in the interior*).

COROLLARY 1 TO THEOREMS 1 AND 2. *If $p - q \geqq 3$, then V is locally unknotted.*

In codimension 2, local knotting can occur, as is shown by the following examples.

(i) The cone-pair on a knotted (S^3, S^1) is locally knotted in the interior at the vertex of the cone, but is locally unknotted on the boundary.

(ii) The suspension V of a knotted (B^3, B^1) is locally unknotted in the interior, but locally knotted on the boundary at the suspension points, although the boundary $\dot{V}$ itself is both unknotted and locally unknotted.

In codimension 1 it is an open question whether or not local knotting can occur, owing to the unsolved state of the combinatorial Schönflies conjecture. The phenomenon of local knotting is therefore restricted to codimension 2 and possibly codimension 1, and our subsequent remarks on the subject will refer only to these two cases.

Using the methods of [1], it is straightforward to show that an unknotted sphere- or ball-pair is also locally unknotted. Hence if V is locally unknotted, the link of every simplex (as well as every vertex) of M^q is unknotted; if V is locally unknotted in the interior, then the link of every simplex in the interior of M^q is unknotted; and if V is locally unknotted on the boundary, then the link of every simplex of M^q that meets the boundary is unknotted. It follows that local unknottedness, local unknottedness in the interior, and local unknottedness on the boundary, are all combinatorial invariants of V (i.e., independent of the triangulation).

Collared manifolds

If M is a manifold, define the *collared manifold* M^+ of M to be the mapping cylinder of the inclusion $\dot{M} \subset M$. If M is closed then $M^+ = M$. If M is bounded, then M^+ is formed from $\dot{M} \times I \cup M$ by identifying $x \times 1 = x$ for each $x \in \dot{M}$. We call $\dot{M} \times I$ the *collar*.

In this paper, whenever we say manifold we mean combinatorial manifold, and so to define M^+ as a manifold, it is necessary to specify the triangulation of M^+. Now there is no natural triangulation of M^+, but since M^+ is the mapping cylinder of an embedding, there is a natural combinatorial structure*, and we choose a convenient triangulation in

* A combinatorial structure is a family of piecewise linearly related triangulations. The mapping cylinder C of a simplicial map $f: K \to L$ between finite simplicial complexes does not possess a *natural* combinatorial structure unless f is an embedding, because, although we can use K, L to construct a triangulation of C, we do not in general obtain a piecewise linearly related triangulation if we start from piecewise linearly related triangulations of K, L.

this structure, as follows.

Take the first barycentric derived complex M' of M. The boundary $\dot{M}'$ is identified with, and therefore triangulates, the bottom $\dot{M} \times 1$ of the collar. Triangulate the top $\dot{M} \times 0$ of the collar isomorphically; the top of the collar is the same as the boundary of M^+ and so we have defined $\dot{M}^+ \cong \dot{M}'$. For each simplex D in $\dot{M}'$, $D \times I$ is a prism in the collar. Let $D^0, \in M^+$, denote the top of the prism; the bottom of the prism is identified with D. Order the vertices $x_0, x_1, \cdots, x_n$ of D so that the induced ordering of the simplexes of $\dot{M}$, of which they are the barycentres, is the order of increasing dimension. By our notation

$$D \times 0 = D^0 = x_0^0 x_1^0 \cdots x_n^0 , \qquad D \times 1 = D = x_0 x_1 \cdots x_n .$$

Triangulate the prism $D \times I$ by the simplexes

$$x_0^0 x_1^0 \cdots x_{j-1}^0 x_j^0 x_j x_{j+1} \cdots x_n , \qquad j = 0, 1, \cdots, n ,$$

and their faces. Do this for each simplex of $\dot{M}'$. The process is compatible, in the sense that if E is a face of D, then the triangulation of $E \times I$ is the same as that induced from the triangulation of $D \times I$. Therefore we have defined a triangulation, call it $(\dot{M} \times I)'$, of the collar, which agrees with $\dot{M}'$ on the bottom of the collar. Consequently we can define

$$M^+ = (\dot{M} \times I)' \cup M' .$$

It follows from [8, Th. 40] that M^+ is homeomorphic to M. We can in fact say more: let ρ: $M^+ \to M$ be the retraction that shrinks the collar, i.e., $\rho \mid \dot{M} \times I$ is the projection onto the first factor, and $\rho \mid M$ is the identity. In particular ρ maps the boundary $\dot{M}^+$ of M^+ homeomorphically onto the boundary $\dot{M}$ of M, and isomorphically onto $\dot{M}'$. We can choose the homeomorphism h: $M^+ \to M$ so as to agree with ρ on the boundary and outside any given neighbourhood of the collar. In other words h also maps $\dot{M}^+$ isomorphically onto $\dot{M}'$, and keeps fixed every point of M outside the given neighbourhood of $\dot{M}$. Our purpose is to do the same for manifold-pairs.

If $V = (M^p, M^q)$ is a manifold-pair, the *collared pair* V^+ is defined to be the mapping cylinder of the inclusion $\dot{V} \subset V$, and triangulated by the manifold-pair

$$V^+ = (M^{p+}, M^{q+}) .$$

As before let ρ: $V^+ \to V$ be the retraction that shrinks the collar.

THEOREM 3. *Let V be a manifold pair either of codimension $\geqq 3$ or locally unknotted on the boundary. Then there exists a homeomorphism h: $V^+ \to V$ that agrees with ρ on the boundary and outside a neighbour-*

hood of the collar.

Alternatively we can state the theorem in an equivalent form:

COROLLARY. *Let $j: M^q \subset M^p$ be a proper embedding between bounded manifolds. Suppose that either $p - q \geqq 3$, or that (M^p, M^q) is locally unknotted on the boundary. Then there exists a commutative diagram*

$$\begin{array}{ccc} \dot{M}^q \times I & \xrightarrow{k^q} & M^q \\ \Big\downarrow {\scriptstyle (j\dot{M}^q)\times 1} & & \Big\downarrow {\scriptstyle j} \\ \dot{M}^p \times I & \xrightarrow{k^p} & M^p \end{array}$$

where k^p, k^q are homeomorphisms into, such that $k^p(x \times 0) = x$, $x \in \dot{M}^p$, and $k^q(x \times 0) = x$, $x \in \dot{M}^q$.

It is easy to verify the equivalence, for given the theorem, the restriction of h to the collar provides the k's of the corollary. Conversely, given the corollary, then by stretching the collar twice as long, we obtain a manifold pair homeomorphic to both V^+ and V.

REMARK. Theorem 3 fails if V is of codimension 2 and locally knotted on the boundary. For, consider Example (ii) above, the suspension of a knotted arc in a 3-ball. Here V is locally unknotted in the interior but locally knotted on the boundary, whereas V^+ is the other way round. Therefore V^+ cannot be homeomorphic to V

LEMMA 12. *Let Q be an unknotted ball-pair, aQ the cone-pair on Q, and $ba\dot{Q}$ the cone-pair on $a\dot{Q}$. Then there exists a homeomorphism*

$$f: ba\dot{Q} \cup aQ \to aQ ,$$

that maps b to a, is the identity on Q, and maps $b\dot{Q}$ linearly onto $a\dot{Q}$.

PROOF. For the standard ball-pair the proof is obvious. An unknotting homeomorphism from Q onto the standard ball-pair induces homeomorphisms from the given set-up onto the standard set-up, and the desired homeomorphism is obtained by composition.

Proof of Theorem 3

We are given a manifold-pair $V = (M^p, M^q)$. Let $A_1, A_2, \cdots, A_s$ denote the simplexes of $\dot{M}^q$, arranged in order of increasing dimension, and let $A_{s+1}, A_{s+2}, \cdots, A_t$ denote the simplexes of $\dot{M}^p - \dot{M}^q$, arranged likewise. Let $r_i = \dim A_i$. For each i, $1 \leqq i \leqq t$, the first derived complex A_i' of A_i is an r_i-ball in $\dot{M}^{p\prime}$, and $A_i' \times 0$ is an isomorphic ball in $\dot{M}^{p+}$. Let $D_i^{p-r_i-1}$ be the dual cell to A_i in $\dot{M}^p$, which is a $(p - r_i - 1)$-ball in $\dot{M}^{p\prime}$. By our definition of the triangulation of a collared manifold, the join

$$B_i^p = (A_i' \times 0)D_i^{p-r_i-1}$$

is a p-ball contained in M^{p+}. Let

$$M_i^p = M^{p\prime} \cup \bigcup_{j=1}^{i} B_j^p \ .$$

We have defined an ascending sequence of subcomplexes of M^{p+}:

$$M^{p\prime} = M_0^p \subset M_1^p \subset \cdots \subset M_t^p = M^{p+} \ .$$

The last equality $M_t^p = M^{p+}$ holds because every point of the collar is contained in some simplex

$$x_0^0 x_1^0 \cdots x_j^0 x_j x_{j+1} \cdots x_n$$

in the collar, where $x_0 x_1 \cdots x_n \in \dot{M}'$, and this simplex is contained in the ball B_i^p, where A_i is the simplex having barycentre x_j.

Similarly for each i, $1 \leqq i \leqq s$, let $D_i^{q-r_i-1}$ be the dual cell to A_i in $\dot{M}^q$, and let B_i^q be the join

$$B_i^q = (A_i' \times 0)D_i^{q-r_i-1} \ ,$$

which is a q-ball in M^{q+}. Let

$$M_i^q = M^{q\prime} \cup \bigcup_{j=1}^{i} B_j^q \ .$$

We have an ascending sequence of subcomplexes of M^{q+}:

$$M^{q\prime} = M_0^q \subset M_1^q \subset \cdots \subset M_s^q = M_{s+1}^q = \cdots = M_t^q = M^{q+} \ .$$

Let

$$V_i = (M_i^p, M_i^q) \ , \qquad i = 1, 2, \cdots, t \ .$$

We shall show inductively that there exists a homeomorphism

$$h_i \colon V_i \longrightarrow V$$

that agrees with ρ on $\dot{V}_i$. Since $V_0 = V'$, the first derived of V, the induction begins trivially with h_0 being the identity map. The induction ends at $i = t$ with the statement of the theorem.

To prove the inductive step, assume that $h_{i-1} \colon V_{i-1} \longrightarrow V$ has been defined. There are two cases, according as to whether or not $A_i \in \dot{M}^q$.

Case (i). Suppose $1 \leqq i \leqq s$, so that $A_i \in \dot{M}^q$. Let R denote the (p, q)-ball-pair $R = (B_i^p, B_i^q)$. Then

$$V_i = R \cup V_{i-1} \ .$$

Let a be the barycentre of A_i, and let Q denote the $(p-1, q-1)$-ball-pair

$$Q = \mathrm{lk}(a, V_{i-1}) = \big(\mathrm{lk}(a, M_{i-1}^p), \mathrm{lk}(a, M_{i-1}^q)\big) \ ,$$

which is unknotted because V_{i-1} is homeomorphic to V by induction, and

V is locally unknotted on the boundary by hypothesis. Then, since V_{i-1} already contains the ball-pairs corresponding to all the faces of A_i,

$$R \cap V_{i-1} = a\dot{Q} = (\dot{A}_i' \times 0)(D_i^{p-r_i-1}, D_i^{q-r_i-1}) ,$$

and

$$R = a^0 a\dot{Q} .$$

By Lemma 12, there is a homeomorphism $a^0 a\dot{Q} \cup aQ \to aQ$ that keeps Q fixed, and maps $a^0\dot{Q}$ linearly onto $a\dot{Q}$. Extend this by the identity on $V_{i-1} - aQ$ to a homeomorphism

$$f\colon a^0 a\dot{Q} \cup V_{i-1} \to V_{i-1} .$$

But $a^0 a\dot{Q} \cup V_{i-1} = R \cup V_{i-1} = V_i$. Define h_i to be the composite homeomorphism

$$V_i \xrightarrow{\ f\ } V_{i-1} \xrightarrow{\ h_{i-1}\ } V .$$

We have to check that h_i agrees with ρ on $\dot{V}_i$. For points not in $a^0\dot{Q}$ this follows by induction, because $\dot{V}_i - a^0\dot{Q}$ is kept fixed by f. For points in $a^0\dot{Q}$ we have a commutative diagram

$$\begin{array}{ccc} a^0\dot{Q} & \xrightarrow{\quad f \quad} & a\dot{Q} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle h_{i-1} = \rho} & \\ & a(\rho\dot{Q}) & \end{array} .$$

Hence $h_i = h_{i-1}f$ agrees with ρ on $a^0\dot{Q}$.

Case (ii). Suppose $s < i \leqq t$, so that $A_i \notin \dot{M}^q$. Then

$$M_i^p = B_i^p \cup M_{i-1}^p , \qquad M_i^q = M_{i-1}^q .$$

As before, let a be the barycentre of A_i, and let $B = \mathrm{lk}(a, M_{i-1}^p)$. Then $B_i^p \cap B_{i-1}^p = a\dot{B}$, and $B_i^p = a^0 a\dot{B}$. By Lemma 12 (ignoring the smaller ball of the pair) there is a homeomorphism $a^0 a\dot{B} \cup aB \to aB$, keeping B fixed, and mapping $a^0\dot{B}$ linearly onto $a\dot{B}$. Extend this by the identity on $M_{i-1}^p - aB$ to a homeomorphism

$$f\colon M_i^p \to M_{i-1}^p .$$

Since $a \notin M_{i-1}^q$, M_{i-1}^q is kept fixed under f. Hence f is a homeomorphism $V_i \to V_{i-1}$. As in *Case* (i), define $h_i = h_{i-1}f$, and verify that h_i agrees with ρ on $\dot{V}_i$.

There only remains to confirm the last remark in the statement of Theorem 3, that h can be made to agree with ρ outside an arbitrary given neighbourhood N of the collar. Let us subdivide the interior of V before we start, so that the simplicial neighbourhood of the collar in V^+ is contained in N. Then during the homeomorphism f of the inductive step, and hence during each h_i, no point outside N is moved. Hence h agrees with ρ outside N. The proof of Theorem 3 is complete.

Manifold-flags

We conclude the paper by extending Theorems 1, 2 and 3 from pairs to triples or more. Define a *manifold-flag of length* t to be a sequence of manifolds

$$V = (M^{p_1}, M^{p_2}, \cdots, M^{p_t}) , \qquad p_1 > p_2 > \cdots > p_t ,$$

such that each is a subcomplex properly embedded in its predecessor. If $p_i - p_{i+1} = r$, then $(M^{p_i}, M^{p_{i+1}})$ is called a *pair of neighbours of codimension* r. Each manifold may or may not be connected, and may be bounded or closed, but if any manifold is closed, then all its successors must also be closed in order that the embeddings be proper. Suppose that the first s manifolds are bounded and the rest closed. The *boundary*

$$\dot{V} = (\dot{M}^{p_1}, \dot{M}^{p_2}, \cdots, \dot{M}^{p_s})$$

is a flag of length s. As before, we define the *collared flag* V^+ to be the mapping cylinder of the inclusion $\dot{V} \subset V$, and denote by ρ the retraction ρ: $V^+ \to V$ that shrinks the collar. If all the manifolds are balls we have a *ball-flag*, with boundary a *sphere-flag*. A ball-flag (or sphere-flag) is *unknotted* if it is homeomorphic to a standard flag, defined by a sequence of suspensions of a simplex (or its boundary).

COROLLARY 2 TO THEOREMS 1 AND 2. *A ball- or sphere-flag is unknotted if and only if each pair of neighbours of codimension* 1 *or* 2 *is unknotted.*

PROOF. The result for sphere-flags follows from that for ball-flags by taking the boundaries of cone-flags. The proof for ball-flags is by induction on t, the length of the flag. By the hypothesis and Theorem 1, every pair of neighbours is unknotted, and the corollary is true for $t = 2$. Assume the corollary for $t - 1$, $t \geqq 3$. Suppose we are given a ball-flag of length t,

$$(B^p, \cdots, B^q, B^r) .$$

By induction there is a homeomorphism of the subflag of length $t - 1$ onto an unknotted flag:

$$f: (B^p, \cdots, B^q) \to (\Sigma^{p-r}\Delta^r, \cdots, \Sigma^{q-r}\Delta^r) .$$

By hypothesis the last pair of neighbours is also unknotted; therefore the image of this pair under f is unknotted, and so there is a homeomorphism

$$g\colon (\Sigma^{q-r}\Delta^r, fB^r) \to (\Sigma^{q-r}\Delta^r, \Delta^r) .$$

The composition of f followed by the $(p-q)$-fold suspension of g gives the unknotting we want.

ADDENDUM TO THEOREM 3. *Let V be a manifold-flag such that all pairs of neighbours of codimension 1 or 2 are locally unknotted on the boundary. Then there exists a homeomorphism $h\colon V^+ \to V$ that agrees with ρ on the boundary and outside a neighbourhood of the collar.*

We call this an addendum rather than a corollary to Theorem 3, because the proof is a duplication of the proof of Theorem 3, using flags instead of pairs, rather than a consequence of the statement of Theorem 3. We leave the duplication to the reader.

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