

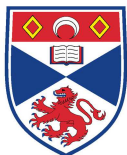
Well quasi-order in combinatorics and algebra

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University
of
St Andrews

Well quasi-orders (WQOs)

Definition

A relation \leq on X is:

- ▶ a **partial order** if it is **Reflexive**, **Anti Symmetric** and **Transitive**;
- ▶ a **quasi-order** if it is **Reflexive** and **Transitive**.

Definition

Well quasi-order (WQO) if in addition there are:

- ▶ no infinite strictly decreasing chains (well founded); and
- ▶ no **infinite antichains**.

Examples

- ▶ WQO: \mathbb{N} , $\mathbb{N} \times \mathbb{N}$, A^* (A finite) under subword ordering.
- ▶ Not WQO: \mathbb{Z} , \mathbb{N} under divisibility, A^* under factor ordering.



WQO: basic properties

Theorem

The following are equivalent for a quasi-order \leq on X :

- (i) \leq is a WQO.*
- (ii) Every infinite sequence contains a non-decreasing subsequence.*
- (iii) Every non-empty subset of X has only finitely many (non-equivalent) minimal elements, and they lie below all other elements of X .*



Order ideals and forbidden containments

Ideal – downward closed set: $x \leq y \in I \Rightarrow x \in I$.

Complements of ideals are **filters** – upward closed sets.

Theorem

Let \leq be a WQO on X . For every ideal I there exists a finite set $B \subseteq X \setminus I$ such that I is the **avoidance set** of B :

$$I = Av(B) = \{x \in X : (\forall b \in B)(b \not\leq x)\}.$$

Remark

B – **basis** – unique if \leq is a partial order.

Corollary

Let (X, \leq) be a countable well founded QO. Then X is WQO iff there are only countably many ideals.



ORDERING BY DIVISIBILITY IN ABSTRACT ALGEBRAS

By GRAHAM HIGMAN

[Received 24 December 1951.—Read 17 January 1952]

1. Introduction

WE shall be concerned in this note with abstract algebras which carry a relation of quasi-order.

THEOREM 1.1. Suppose that (A, M) is a minimal algebra, and that M_r , the set of r -ary operations in M , is a quasi-ordered set with finite basis property for $r = 0, 1, \dots, n$, and is empty for $r > n$. Then A has the finite basis property in any divisibility order of (A, M) compatible with the quasi-orders of M_r .

subgroups of a free group. It is the variety of these applications, rather than any depth in the results obtained, that suggests that the theorems may be interesting.



Higman's Theorem: definitions



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- ▶ Compatibility:

$$\underline{x}' \leq \underline{x}'' \Rightarrow f(\underline{x}') \leq f(\underline{x}'')$$
$$f \leq g \Rightarrow f(\underline{x}) \leq g(\underline{x}).$$

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- ▶ There may be infinitely many basic operations, but their arities are bounded.



Higman's Theorem: informal statement

Intuitively: if

- ▶ generating set is WQO;
- ▶ operations are WQO; and
- ▶ operations and order agree;

then the entire algebra is WQO.



Higman's Theorem: corollaries

The following algebras are WQO:

- ▶ finitely many operations and a WQO generating set;
- ▶ finitely generated and finitely many operations;
- ▶ free monoid A^* (A finite) under subword ordering;
- ▶ A^* where A is WQO under the subword (subsequence) domination ordering.



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- ▶ Infinitely many v_i end with the same x : $v_{i_j} = v'_{i_j}x$.



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- ▶ The sequence $(v_1, \dots, v_{i_1-1}, v'_{i_1}, v'_{i_2}, \dots)$ is bad, a contradiction.



Higman–Aichinger–Mayr–McKenzie ordering on A^*

Definition

$u \leq_E v$ ($u, v \in A^*$) iff v can be obtained from u by inserting letters, following their first occurrence in u .

Example

$ab \leq_E aabb$, but $ab \not\leq_E bab$ even though $ab \leq bab$.

Lemma (Aichinger–Mayr–McKenzie 2014)

A^* is wqo with respect to \leq_E .

WQO in combinatorics

WQO tends to appear when we are concerned with a class of structures which are somehow compared.

Combinatorial structures: graphs, tournaments, trees, permutations, words, ...

Note

Relational structures!

Comparisons: containment, induced containment, minors, subpermutations, subwords, ...

Note

Containment of A in B : an injective mapping $f : A \rightarrow B$ such that:

- ▶ xRy in $A \Rightarrow f(x)Rf(y)$ in B (**plain**)
- ▶ xRy in $A \Leftrightarrow f(x)Rf(y)$ in B (**induced**)

WQO in algebra

- ▶ More a method rather than a topic in its own right: finite basis type considerations.
- ▶ Comparisons of individual algebras, but also lattices of varieties.



WQO in theoretical computer science

- ▶ WQO and (regular) languages.
- ▶ Rewriting systems.



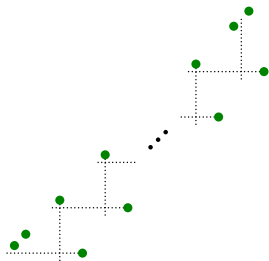
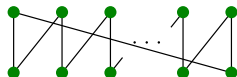
Containment

Usual classes of combinatorial objects

- ▶ graphs
- ▶ tournaments
- ▶ trees
- ▶ permutations
- ▶ posets, etc.

are **not WQO** under containment
(plain or induced).

Exceptions: words (Higman) and
equivalence relations.



WQO subclasses

Theorem (Ding 1992)

The class of all graphs avoiding the path P_n as a subgraph is WQO.

Theorem (Ding 1992)

An ideal \mathcal{F} of graphs under subgraph relation is WQO iff it contains only finitely many cycles C_n and double-ended forks F_n .

Theorem (Damaschke 1990)

P_4 is the unique (maximal) graph G with the property that the class of graphs avoiding G as an induced subgraph is WQO.



WQO subclasses (2)

Conjecture (Korpelainen, Lozin 2012)

For any $k \in \mathbb{N}$ there are only finitely many minimal classes of graphs defined by k forbidden induced subgraphs which are not WQO.

Theorem (Korpelainen, Lozin 2012)

Conjecture is true for $k = 2$.

Theorem (Korpelainen, Lozin 2011)

P_6 -avoiding bipartite graphs are WQO, while P_7 avoiders are not.



Algorithmically deciding WQO

G. Cherlin, Forbidden substructures and combinatorial dichotomies: WQO and universality, *Discrete Math.* 311 (2011), 1534–1584.

General Problem

For a class of combinatorial objects \mathcal{C} , does there exist an algorithm which tests for a finite set F of forbidden objects whether the subclass of \mathcal{C} consisting of all F -avoiders is WQO?

Theorem (Cherlin, Latka 2000)

- (i) *The class of tournaments avoiding a linear tournament is WQO.*
- (ii) *If T is a non-linear tournament of size ≥ 7 then the class of T -free tournaments is not WQO.*

For more than one obstruction, the problem appears to be hard, and remains open for graphs, tournaments, permutations, etc.



Changing the order: minors

Definition

Graph H is a **minor** of G if it can be obtained by successively removing vertices and contracting or removing edges in G .

Theorem (Robertson, Seymour 2004)

The class of graphs is WQO under the minor ordering.

Theorem (Kim 2013)

The class of tournaments is WQO under the minor ordering.



Trees

Definition

Graph H is a **topological minor** of G if an edge-subdivision of H is contained in G .

Theorem (Kruskal 1960)

The class of all trees is WQO under the topological minor ordering.

Theorem (Nešetřil, Ossona de Mendez 2012)

The class of graphs of bounded tree depth is WQO under the induced subgraph relation.

Homomorphisms

Combinatorial objects \longleftrightarrow relational structures.

Homomorphism – mapping respecting basic relations.

Three ‘flavours’ of homomorphisms $\phi : S \rightarrow T$:

- ▶ **Plain:** $xR^S y \Rightarrow \phi(x)R^T \phi(y)$.
- ▶ **Strong:** if $uR^T v$ then there are $x, y \in S$ such that

$$xR^S y, \quad u = \phi(x), \quad v = \phi(y).$$

- ▶ **M-strong:** $xR^S y \Leftrightarrow \phi(x)R^T \phi(y)$.

Remark

Embedding = existence of an injective homomorphism.

Induced embedding = existence of an injective strong (equivalently, M-strong) homomorphism.

Edge contractions = epimorphisms with connected fibres.



Homomorphic image orderings

Theorem

The class of countable linear order types is WQO under:

- ▶ *embedding order (Laver 1971); and*
- ▶ *homomorphic image order (Landraitis 1979).*

S. Huczynska, N. Ruškuc, Homomorphic orders on combinatorial structures, *Order* 32 (2015), 205–226.

Question

Are any of the common classes of combinatorial objects WQO under any of the three homomorphic image orderings?

Rough answer: mostly no (graphs, digraphs, tournaments, posets, permutations, words), but yes for equivalence relations and . . .

Homomorphic image orderings (2)

Theorem

The class of all trees is well quasi-ordered under the plain and strong homomorphic image orderings.

Areas for future investigation:

- ▶ forbidden objects;
- ▶ subclasses;
- ▶ compositions;
- ▶ Fraïse type considerations/inverse limits.



n -wqo

Definition

For a collection of structures \mathcal{C} and $n \in \mathbb{N}$ let $\mathcal{C}^{(n)}$ be the collection obtained by colouring points of members of \mathcal{C} with n colours in all possible ways. Homomorphisms between objects in $\mathcal{C}^{(n)}$ are then required to respect colours.

Remarks

- ▶ $1\text{-wqo} \Leftrightarrow \text{wqo}$.
- ▶ The set of all paths is wqo but not 2-wqo .

Conjecture (Pouzet 1972)

If a class of graphs closed under induced subgraphs is 2-wqo then it is $n\text{-wqo}$ for all n .

Questions

What about classes other than graphs? What about other orderings, e.g. homomorphism orderings?

Words



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- ▶ This is a **regular language**.



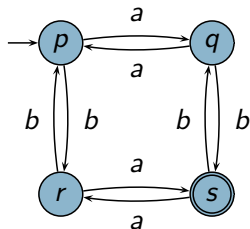
Words (2)

Regular languages are accepted
by finite state automata
(Kleene's Theorem).



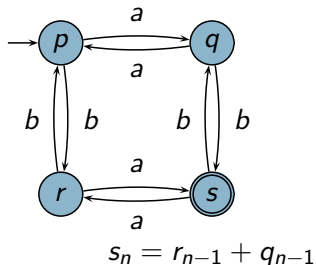
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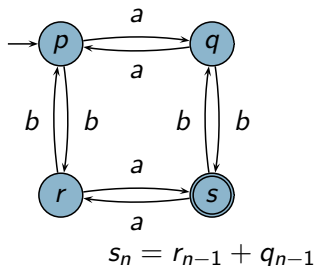
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Definition

Enumeration of X = how many words of length n in X ? =
generating function $\sum_{w \in X} x^{|w|}$.

Theorem (folklore)

Every downward closed set in A^ has a rational generating function and a linear membership test.*

WQO and languages

Theorem (Ehrenfeucht, Haussler, Rozenberg 1983)

A language $L \subseteq A^$ is regular iff it is a down-set with respect to some monotone well quasi-ordering on A^* .*

Remark

Effective regularity: van Leeuwen 1978; Ilie Salomaa 1998.

Theorem (Atminas, Lozin, Moshkov 2013)

The problem of deciding whether a factorial language given by a finite collection of (contiguous) forbidden subwords is wqo can be decided in polynomial time.



WQO and languages: some questions

- ▶ Is wqo question for factorial languages given by a regular or context-free infinite family of forbidden factors decidable?
- ▶ Suppose a relation R is given by a letter-inserting finite transducer. Can it be decided whether R is: (a) quasi-order? (b) partial order? (c) wqo?
- ▶ What can be said about monotone well quasi-orders on A^* ?
- ▶ Can the Ehrenfeucht et al. result be used to shed light on some open problems in formal language theory, e.g. the generalised star height of regular languages?



Permutations: various representations



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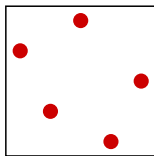
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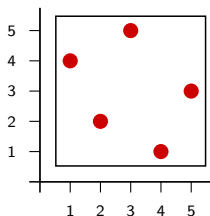


Permutations: various representations

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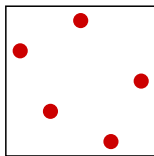
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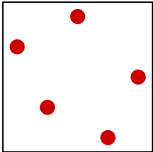


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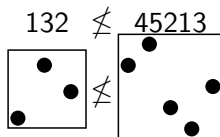
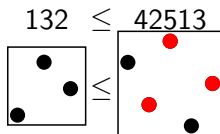


Permutations: various representations

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- Points in plane 
- Relational structure $(\{a, b, c, d, e\}, \leq, \preceq)$
 $a < b < c < d < e$
 $d \prec b \prec e \prec a \prec c$

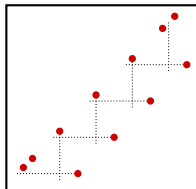
Permutations: pattern containment relation

$\sigma \leq \tau$ if τ contains a subsequence order-isomorphic to σ .



As relational structures: standard containment relation.

Not WQO



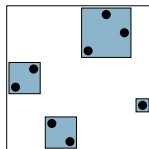
Theorem (folklore)

Up to symmetry, the only WQO classes defined by a single forbidden pattern are $Av(12)$ and $Av(231)$.

Substitutions and simple permutations

Substitution $\sigma[\tau_1, \dots, \tau_m]$:

$$3142[12, 21, 132, 1] = 45216873 =$$



- ▶ **Simple** = no non-trivial decomposition.
- ▶ $\langle X \rangle$ = substitution closure of X .
- ▶ Substitution into a permutation σ of length m can be regarded as an m -ary operation on $\langle X \rangle$.

Theorem (Albert, Atkinson 2005)

A pattern class of permutations P with only finitely many simple permutations is WQO and has an algebraic generating function $(\sum_{\sigma \in P} x^{|\sigma|})$.

Geometric grid classes

M.H. Albert, M.D. Atkinson, M. Bouvel, N. Ruškuc, V. Vatter,
Geometric grid classes of permutations, Trans. Amer. Math. Soc.
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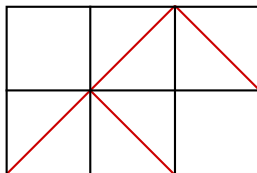
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0	1	-1
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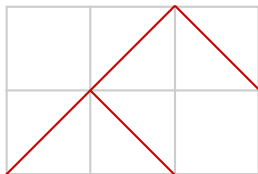
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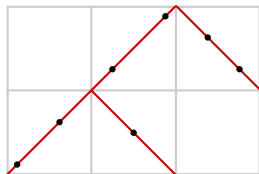
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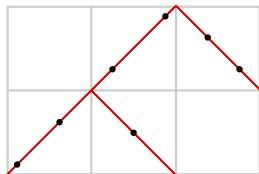
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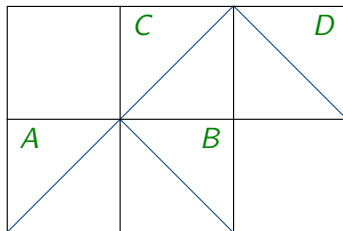
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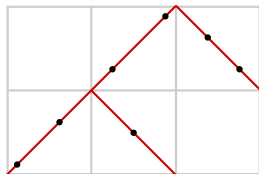
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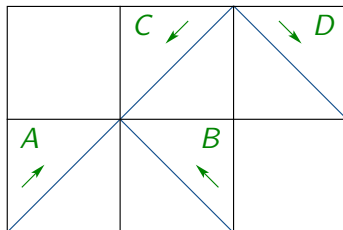
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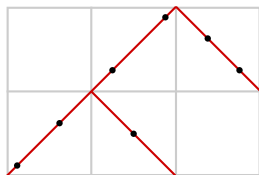
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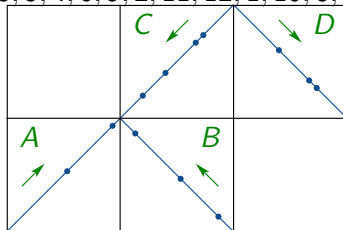
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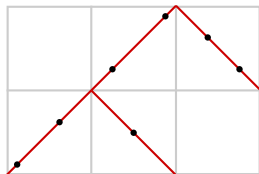
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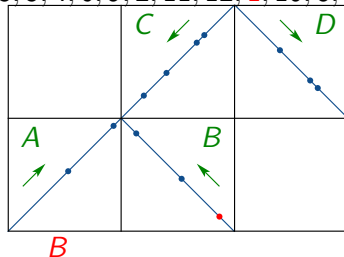
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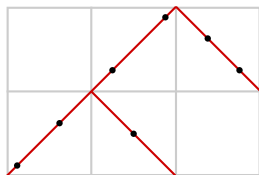
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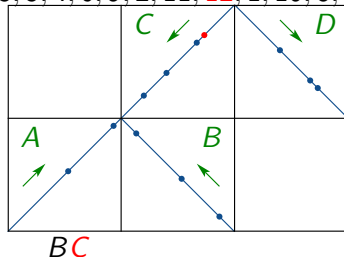
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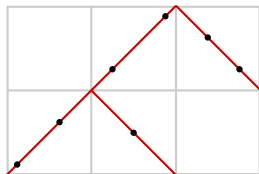
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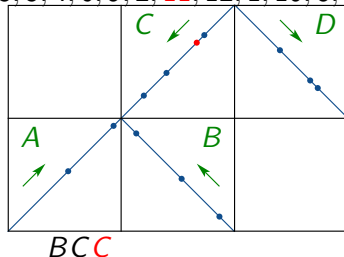
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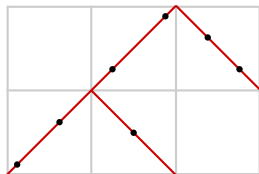
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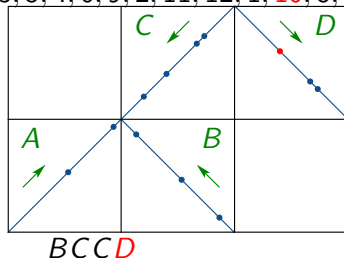
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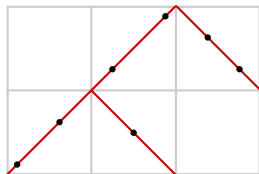
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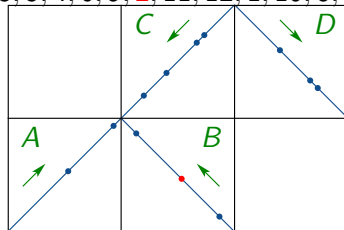
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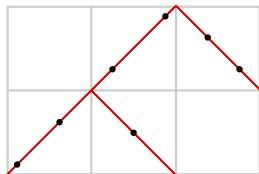


BCCDB

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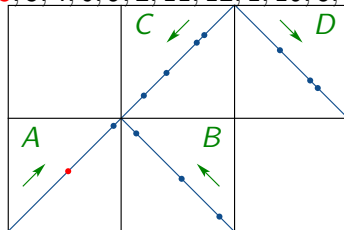
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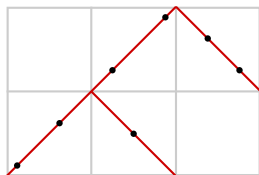


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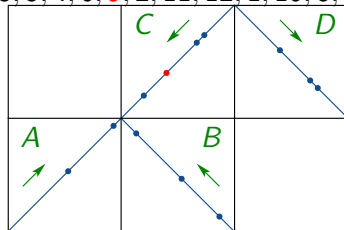
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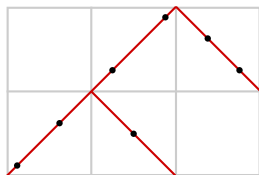


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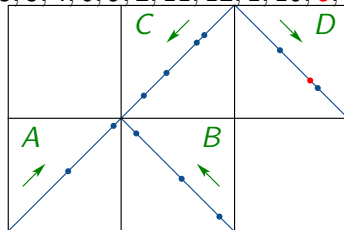
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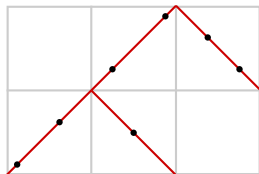


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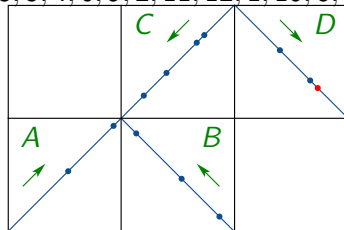
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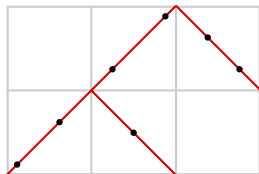


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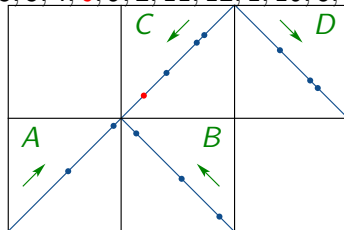
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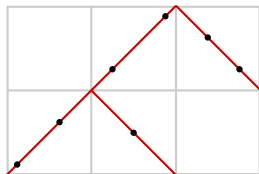


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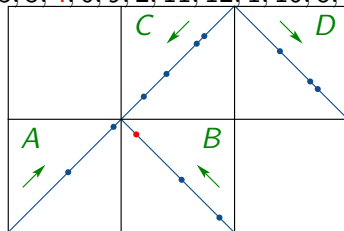
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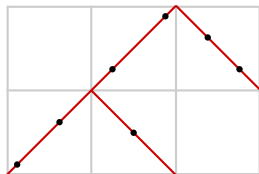


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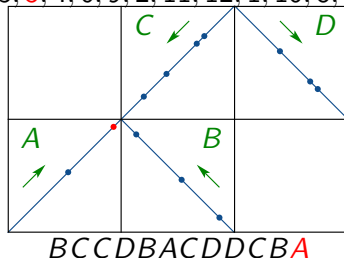
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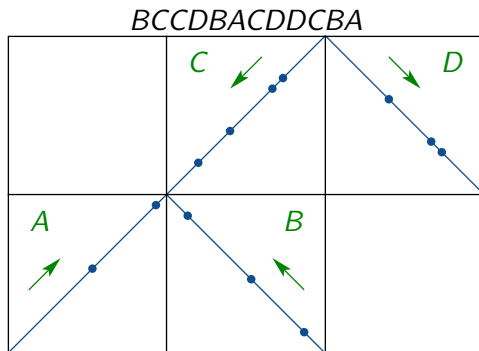
GGC encoding: properties

- ▶ Every word represents a permutation in $\text{Geom}(M)$ in a **length-** and **order-preserving** manner.
- ▶ This correspondence is (finitely) **many-to-one**.
- ▶ To utilise the enumerative power of regular languages we **need a bijection!**
- ▶ What are the obstacles to uniqueness of encoding?



GGC encoding: ambiguities

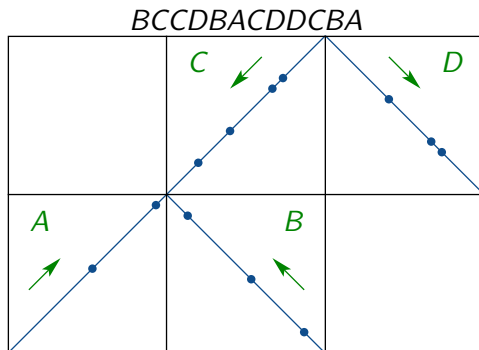
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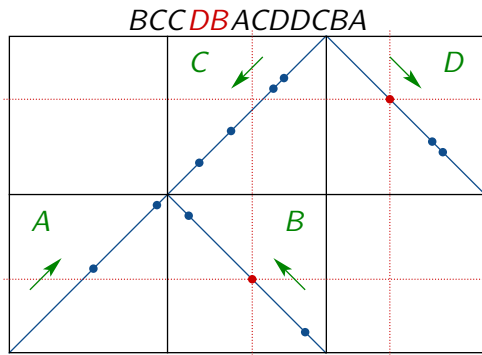
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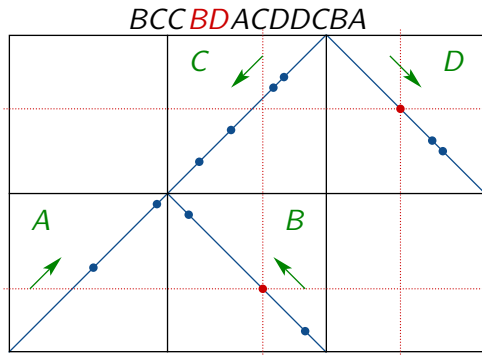
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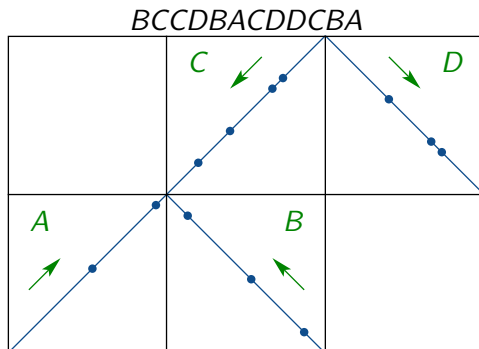
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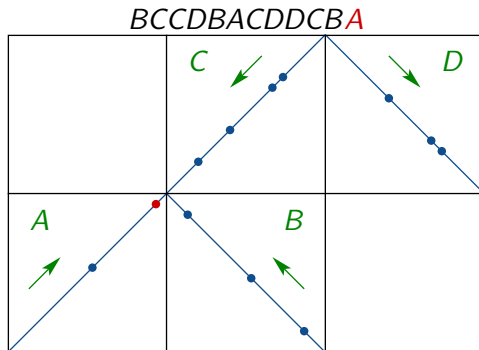
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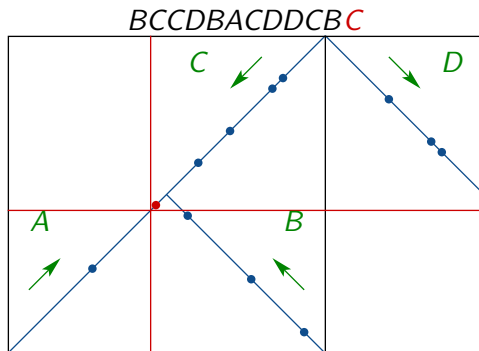
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GGC encoding: resolving ambiguities

Theorem (Anisimov & Knuth; Diekert)

The *trace monoid*, defined by the presentation

$$\langle A \mid ab = ba \ ((a, b) \in P) \rangle$$

where $P \subseteq A \times A$ is arbitrary, admits a regular set of normal forms with uniqueness.

To resolve multiple griddings:

- ▶ Define a notion of **minimal** (or **greedy**) griddings.
- ▶ Show that the set of encodings of such griddings is a down set of A^* .
- ▶ Use Higman.
- ▶ Beware: loss of effectiveness!



Geometric grid classes: conclusions

Theorem

Every geometric grid class $\text{Geom}(M)$ is

- ▶ *WQO,*
- ▶ *finitely based, and*
- ▶ *has a rational generating function.*

Furthermore, every subclass of $\text{Geom}(M)$ is a finite union of geometric grid classes, and has the same properties.



Geometric grid classes: questions

Questions

- ▶ *Are the finite basis and rational generating function effectively computable from the matrix M ?*
- ▶ *Can it be effectively decided from a finite basis whether the class it defines is a GGC (or a subclass)?*

Application: small classes – definition

Theorem (Markus, Tardos 2004; Stanley-Wilf Conjecture)

For every proper pattern class of permutations \mathcal{C} there exists $q \in \mathbb{R}$ such that $|\mathcal{C}_n| \leq q^n$ for all n .

Growth rate of a permutation class: $\limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{X}_n|}$.

Theorem (Vatter 2011)

There are only countably many pattern classes with growth rate $< \kappa$ (unique +ve root of $x^3 - 2x^2 - 1$), and uncountably many with growth rate $= \kappa$.

Definition

X is **small** if its growth rate is $< \kappa$.



Application: small classes, strong rationality and substitutions

M.H. Albert, N. Ruškuc, V. Vatter, Inflations of geometric grid classes of permutations, Israel J. Math. 205 (2015), 73–108.

- ▶ A **strongly rational** class X : all subclasses have rational generating functions.
- ▶ GGCs are strongly rational.
- ▶ Strongly rational classes are WQO.

Theorems

- ▶ *If C is a GGC and D is strongly rational then $C[D]$ (substitutions of members of D into the members of C) is strongly rational.*
- ▶ *Every small pattern class is contained in an iterated substitution of a GGC into itself.*



Application: enumeration in small classes

Corollary (Albert, NR, Vatter)

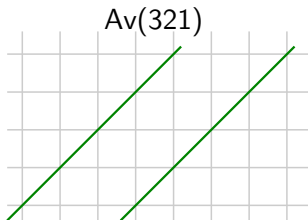
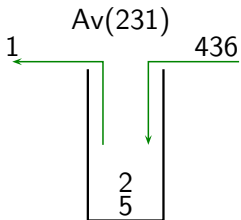
Every small pattern class has a rational generating function.



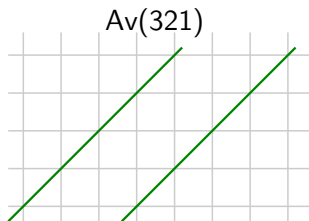
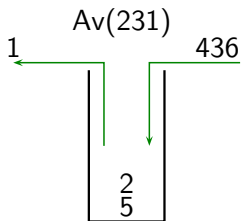
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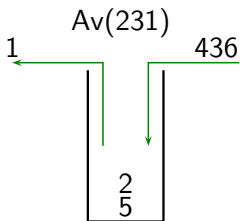


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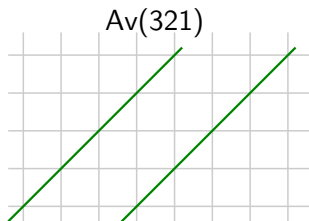


- ▶ enumerated by Catalan numbers;

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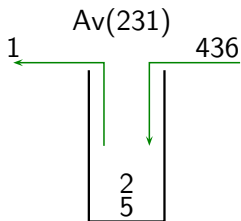


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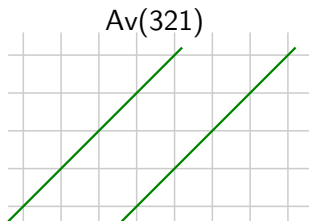


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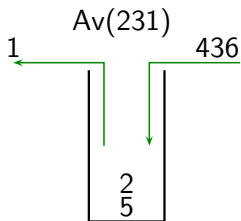


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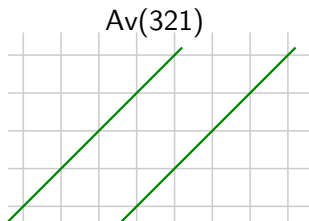


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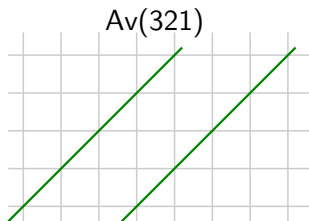
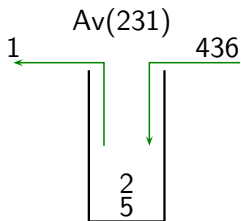


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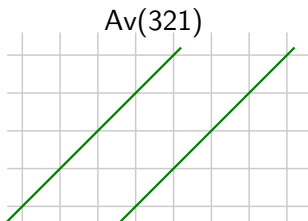
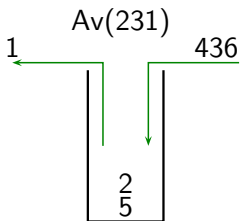
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- ▶ enumerated by Catalan numbers;
- ▶ not WQO (antichain \mathcal{A});
- ▶ every **finitely based** proper subclass and every **WQO** subclass have rational generating functions (Albert, Brignall, NR, Vatter; in prep).

WQO in algebra: embeddings, quotients and divisibility

Theorem

The collection of countable boolean algebras is wqo with respect to the embedding order.

Remarks

- ▶ Follows from the Laver's result on countable linear orders.
- ▶ Bonnet and Rubin 1991: elementary embeddability.

Theorem (Almeida 1987)

The only pseudovarieties of finite commutative semigroups wqo under the embedding, quotient or divisibility orderings are

- ▶ *zero semigroups ($xy = 0$);*
- ▶ *a pseudovariety of groups with only finitely many primes dividing orders of elements.*



WQO in algebra: varieties

In **group theory** wqo has been widely used to show certain varieties hereditarily finitely based.

Theorem (Krasilnikov, Shmelkin 1981; Quick 2003)

Every variety of (pointed) groups which are nilpotent by abelian-of-finite-exponent is finitely based.

Semigroups:

- ▶ conditions for hbf (Pollák 1986);
- ▶ lattices of varieties and pseudovarieties of commutative semigroups (Almeida 1986; Grech 2008).

WQO in algebra: Malcev clones

Theorem (Aichinger, Mayr, McKenzie 2014)

Every clone on a finite set A that contains a Malcev operation (or, more generally, a k -edge operation) is finitely related, i.e. it is precisely the set of operations on A which preserve some finitary relation R on A . Therefore there are at most countably many Malcev clones on a finite set A .



Towards some conclusions



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- ▶ But ultimately, most questions reduce to whether a poset is wqo or not.
- ▶ Theoretical computer science has kept closest to Higman's Theorem via theory of languages.



Towards some conclusions

Disclaimer: what follows are mostly personal views/musings/preferences.

- ▶ The modes of deployment of wqo have been rather different in combinatorics, theoretical computer science and in algebra.
- ▶ Most diverse/advanced in combinatorics: variety of structures, possible orders, and questions.
- ▶ But ultimately, most questions reduce to whether a poset is wqo or not.
- ▶ Theoretical computer science has kept closest to Higman's Theorem via theory of languages.
- ▶ Some promising new developments at the interface of combinatorics and theoretical computer science, leading to more structural and computational considerations in the former.



Towards some conclusions (2)

- ▶ In algebra, attempts at studying 'involvement' of (algebraic) structures in terms of wqo have typically hit problems early on.



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- ▶ . . . and finite relatedness properties of individual structures.



Some possible future directions

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- ▶ In combinatorics: extend the structural and computational point of view to graphs and other structures (linear clique-width, letter graphs).
- ▶ Possible influence from algebra/model theory on combinatorics: integrate homomorphisms more tightly into the landscape.
- ▶ Why is Higman's Theorem nearly always applied in its special instantiation for words? Be on the look-out for new applications in its full, algebraic form.



Some possible future directions (2)

- ▶ Which algebraic structures would be more amenable to wqo comparisons than groups and semigroups?



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- ▶ What are the natural routes via which regularity links from language theory enter into algebraic aspects of wqo?
- ▶ How about a theory of wqo lattices of clones on a finite set?



The Theory of Well-Quasi-Ordering: A Frequently Discovered Concept

JOSEPH B. KRUSKAL

Bell Telephone Laboratories, Incorporated Murray Hill, New Jersey

Communicated by the late Theodore S. Motzkin

Received November 4, 1970

Results from the rich and well-developed theory of well-quasi-ordering have often been rediscovered and republished. The purpose of this paper is to describe this intriguing subject. To illustrate the theory, here are two definitions and an elementary result. A partially ordered set is called well-partially-ordered if every subset has at least one, but only a finite number, of minimal elements. For sequences s and t , we define $s < t$ if some subsequence of t majorizes s term by term. Then the space of all finite sequences over a well-partially-ordered set is itself well-partially-ordered.



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Thank you!

