Fractal Geometry Derived from Complex Bases

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Each complex number can be expressed as a single number in positional notation using certain complex bases, just as the positive real numbers can be expressed as decimal expansions. These representations yield some intriguing geometric patterns in the complex plane, whose boundaries are fractal curves. One of these curves is known from the investigation of dragon curves; the others are new examples of fractals.

Bases for Complex Numbers

The positive integers can be represented in any integer base $b > 1$ using digits $0, 1, 2, \ldots, b-1$; the decimal and binary systems are of course the most familiar. *Both* positive *and* negative integers can be represented (without using a sign prefix), in any negative base $b < -1$ using the digits from 0 to $|b| - 1$.

The concept of base, or radix representation, can be extended to the complex numbers. A Gaussian integer, $z = x + iy$ where x and y are real integers, is said to be expressed in the complex base b if it is written in the form $z = \sum_{r=0}^{k} a_r b^r$, where the numbers a_r are called the digits of the representation. We denote such a representation by $(a_k a_{k-1} \ldots a_1 a_0)_b$. The standard algorithm for converting a number into a given integer base can be extended to these complex bases if the allowable digits form a complete residue system modulo the base.

In this paper we will only allow natural numbers as digits since this is the most straightforward generalization of the familiar systems. The number of elements in a complete residue system modulo a complex number $b = n + im$ is $n^2 + m^2$. Now Gauss showed that if n and m are relatively prime then the natural numbers $0, 1, 2, \ldots, n^2 + m^2 - 1$ form a complete residue system modulo $b = n + im$. Moreover, if n and m have a common factor, then any complete residue system modulo b must contain some numbers with nonzero imaginary parts. A further necessary condition for the base $b = n + im$ to represent *all* the Gaussian integers, using natural numbers as digits, is that $m = \pm 1$, since all the powers of the base $(n + im)^r$ have their imaginary parts divisible by m. Therefore we only consider bases of the form $n \pm i$ with digits $0, 1, 2, \ldots, n^2$.

One such example is the base $b = -1 + i$, which provides a binary representation of all the complex numbers. This system has been known for a number of years by computer scientists; see Knuth [6; § 4.1] for its history. For example, $5 - 3i = (101110)_{-1+i}$ since $(-1+i)^5 + (-1+i)^3 + (-1+i)^2 + (-1+i) = 5 - 3i$, and $9 = (111000001)_{-1+i}$.

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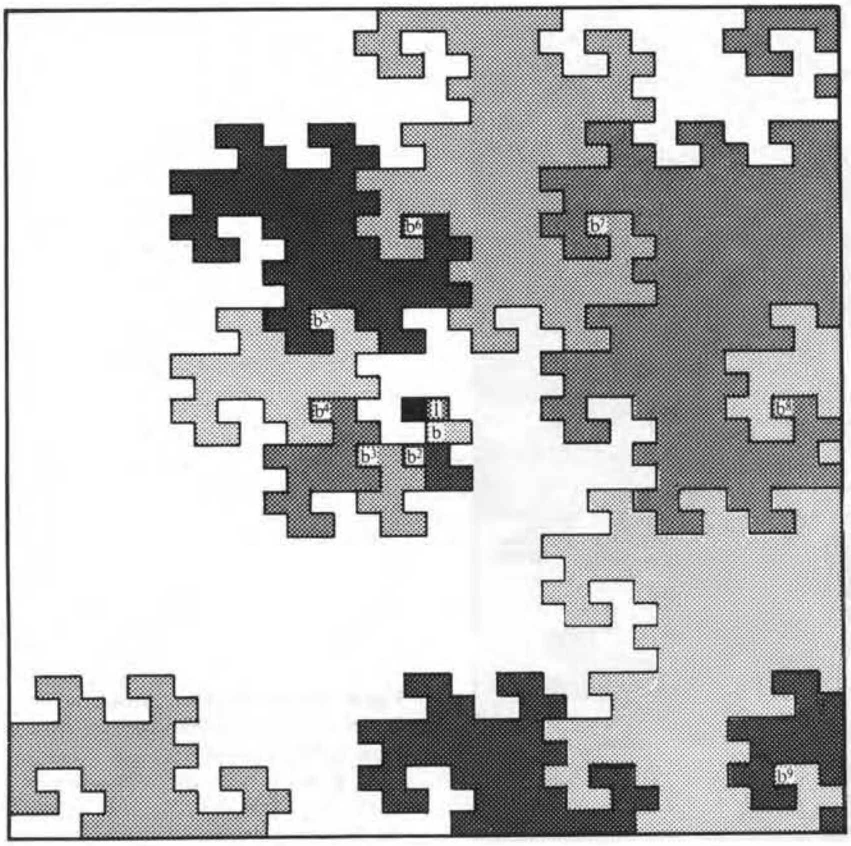

Figure 1. Spiral jigsaw puzzle derived from the base $1 - i$. Each shaded square depicts a Gaussian integer representable in the base $b = 1 - i$. The different shadings correspond to representations of different lengths. Note that the unshaded region is precisely the same shape as the shaded spiral but is turned through one half of a revolution.

Gaussian Integer Representations in the Complex Plane

The representations of the Gaussian integers in the base $b = n \pm i$ can be visualized in the complex plane as follows. Divide up the plane into unit squares corresponding to the Gaussian integers and then shade those squares that can be written in the base b using the digits $0, 1, 2, \ldots, n^2$.

The number b will be a "good" base if each Gaussian integer can be uniquely represented using b as base; that is, if every square in the plane is eventually shaded. The uniqueness of the representation follows from the fact that the allowable digits form a complete residue system modulo b.

What are some examples of bases with small norms? The numbers $\pm i$ are not suitable as bases because they have norm 1 and so any complete residue system modulo i contains only one element. Therefore we first consider bases with norm 2.

In Figure 1 all the squares have been shaded which correspond to numbers that have a base $1 - i$ representation using the binary digits 0 and 1. Different shadings have been used to indicate the length of the expansion. The origin is the central black square. The only other number with a base $1 - i$ expansion of length one is $(1)_{1-i} = 1$. There are two numbers with expansions of length two, namely $(10)_{1-i} = 1 - i$ and $(11)_{1-i} = 2 - i$; these are both given the same shading. In general, the 2^r numbers $(1a_{r-1} \ldots a_1a_0)_{1-i}$, with expansions of length $r + 1$, have the same shading. Since $(1a_{r-1} \ldots a_1a_0)_{1-i} = (1 - i)^r + (a_{r-1} \ldots a_1a_0)_{1-i}$, this number can be written as $(1 - i)^r$ plus a number requiring r or fewer digits. Therefore the shaded region consisting of the 2^r squares corresponding to the Gaussian integers requiring $r + 1$ digits can be obtained by translating the union of all the smaller regions along the vector $(1 - i)^r$. In this way we obtain an infinite jigsaw puzzle with one piece corresponding to each power of two and an extra black piece for the origin. If this jigsaw puzzle eventually fills the plane, then every Gaussian integer can be expressed in base $1 - i$.

Figure 1 *suggests* that the jigsaw puzzle will never cover the whole plane but will keep on spiraling outwards. In fact the number -1 is the white region and, applying the standard algorithm for converting a number to a given base:

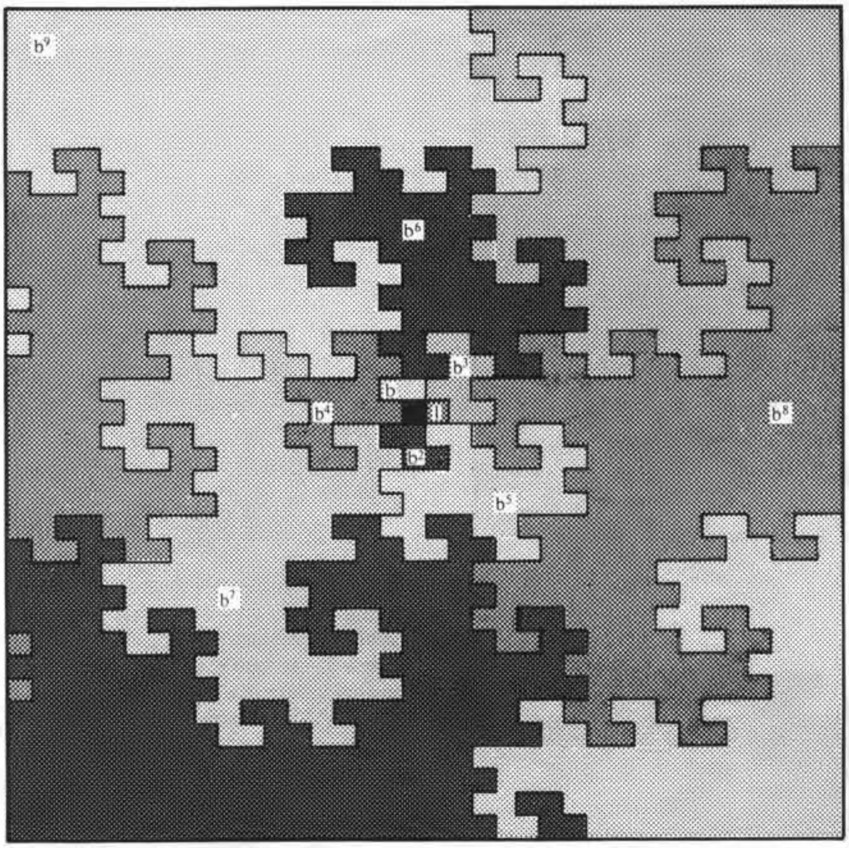

Figure 2. Jigsaw puzzle derived from the base $b = -1 + i$. The pieces are the same as those in the spiral derived from the base $1 - i$. However they exactly fill the plane, which shows that every Gaussian integer can be represented in base $-1 + i$.

$$
\begin{aligned}
-1 &= (-1-i)(1-i)+1\\
-1-i &= (-i)(1-i)+0\\
-i &= (-i)(1-i)+1\\
-i &= (-i)(1-i)+1, \quad \text{etc.}
\end{aligned}
$$

the algorithm never terminates, but cycles indefinitely. Consequently, -1 cannot be represented in the base $1 - i$. In a similar way, the base $1 + i$ yields a spiral jigsaw which is the reflection in the real axis of the jigsaw obtained from $1 - i$. Neither $1 - i$ nor $1 + i$ is a good base.

Figure 2 shows the result of applying the same technique using the base $-1 + i$. The pieces of size 2^r are exactly the same shape as those derived from the base $1 - i$, but now they fit together in a different way to exactly fill the plane. It follows that $-1 + i$ is a good base for the complex numbers and, of course, so is its conjugate $-1 - i$.

The Gaussian integers $2 - i$ and $-2 + i$ have norm 5, so the digits 0, 1, 2, 3 and 4 are required when using these as bases. The numbers representable in base $2 - i$ are shown in Figure 3. In this case there are $4(5)^r$ numbers which have a base $2 - i$ expansion of length $r + 1$. They are of the form $(a_r a_{r-1} \dots a_1 a_0)_{2-i} = a_r(2-i)^r + (a_{r-1} \dots a_1 a_0)_{2-i}$, where $a_r = 1, 2, 3$ or 4. Hence each shaded piece is obtained from the union of all the smaller pieces by translation along the vectors $(2 - i)^r$, $2(2 - i)^r$, $3(2 - i)^r$ and $4(2 - i)^r$. The jigsaws obtained from the bases $2 - i$ and $-2 + i$ show a similar pattern to those obtained from $1 - i$ and $-1 + i$. The one obtained from the base $-2 + i$, shown in Figure 4, is built up from the same pieces as those used in Figure 3. However the jigsaw puzzle derived from the base $2 - i$ forms a large spiral while that derived from $-2 + i$ fills the plane. The illustrations *suggest* that $-2 + i$ is a good base and $2 - i$ is not good.

The natural conjectures one might make from these diagrams have been confirmed by Katai and Szabo [5]. They proved, by algebraic means, that the only Gaussian integers that can be used as a base for all the complex numbers, using natural numbers as digits, are $-n + i$ and $-n - i$, where n is a positive integer. The digits used in these representations are $0, 1, 2, \dots, n^2$. The most common representations for the real numbers in use today are the decimal and binary systems. It is an interesting coincidence that the complex numbers also have decimal and binary representations using the bases $-3 + i$ and $-1 + i$ respectively. For example

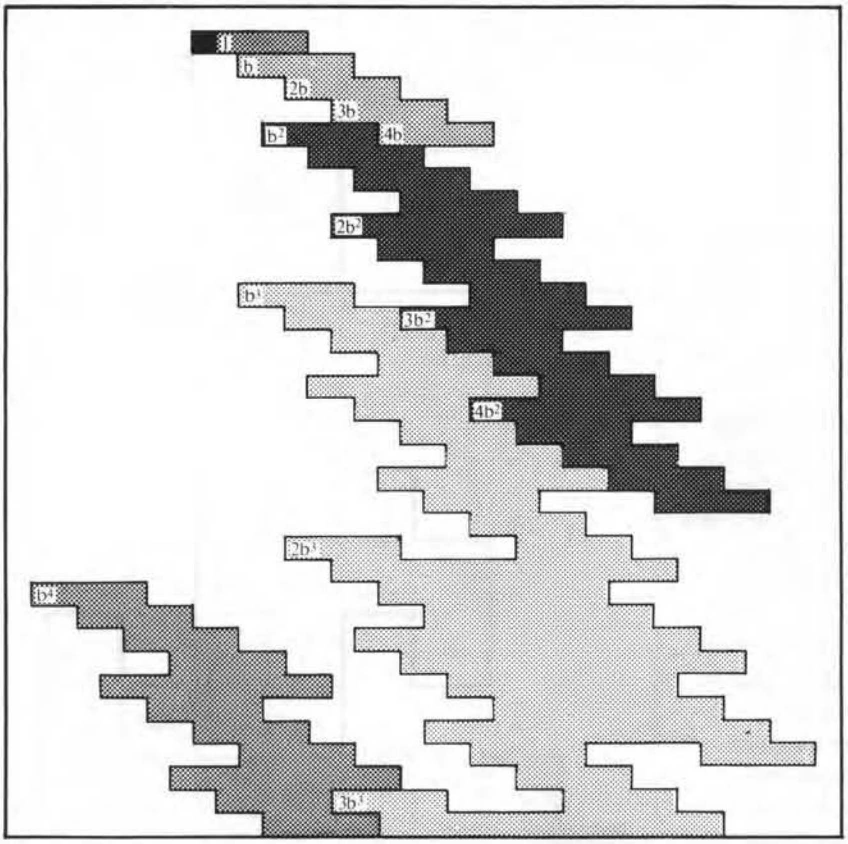

Figure 3. Spiral jigsaw puzzle derived from the base $b = 2 - i$. The origin is the single black square. The other pieces contain 4.5^r squares. Since they do not fill the plane, $2 - i$ is not a good base for the complex numbers.

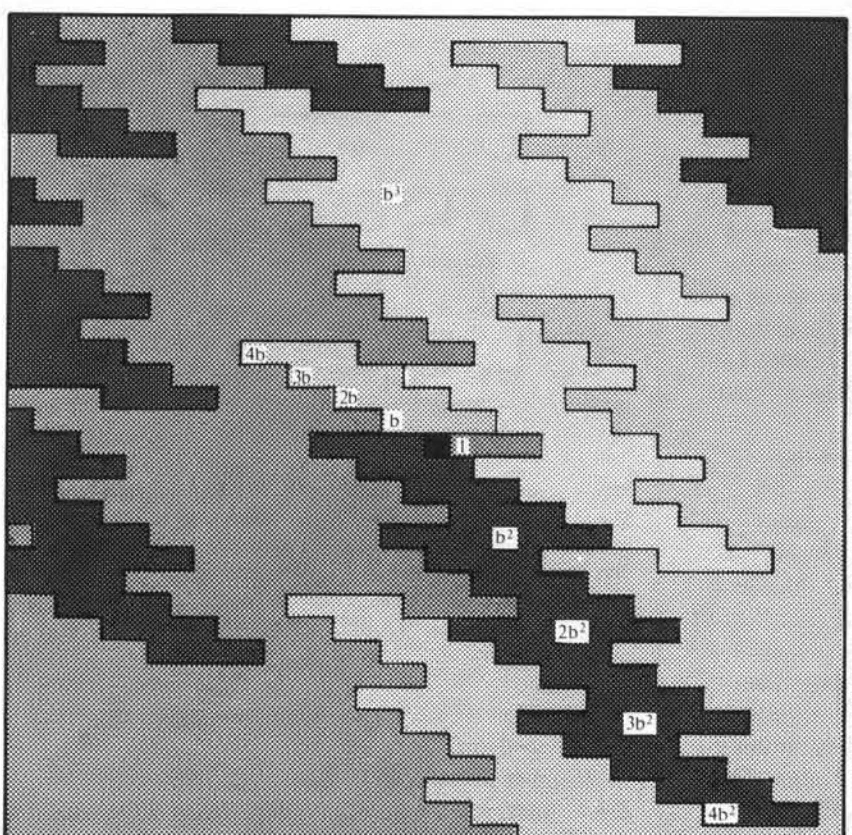

Figure 4. Jigsaw puzzle derived from the good base $b = -2 + i$. The pieces are the same as those derived from the base $2 - i$, but this time they fill the plane.

$(326)_{-3+i}$ represents $3(-3+i)^2 + 2(-3+i) + 6 = 24 - 16i$ while $(1543)_{-3+i}$ represents $(-3+i)^3 + 5(-3+i)^2 + 4(-3+i) + 3 = 13$.

The Representation of All Complex Numbers

These representations of the Gaussian integers using base $-n+i$ (or $-n-i$) can be extended to cover all the complex numbers in the same way that the decimal or binary systems can be extended to all positive real numbers. Geometrically these bases lead to some bizarre patterns in the plane.

We say that a complex number is represented in base b when it is written in the form $\sum_{r=-\infty}^{k} a_r b^r$, with each a_r an allowable digit for the base b. We denote this infinite expansion by $(a_k a_{k-1} \ldots a_1 a_0 \cdot a_{-1} a_{-2} \ldots)_b$. The digits to the left of the radix point define a Gaussian integer $(a_k a_{k-1} \ldots a_1 a_0)_b$, called the integer part of the expansion. For example, if $b = -1+i$, then $b^{-1} = (-1-i)/2$ and $b^{-2} = i/2$ so $1/2 = 1 + b^{-1} + b^{-2} = (1.11)_{-1+i}$, with integer part one. As in the decimal system, most numbers cannot be written using a terminating expansion.

The number $1-i$ is not good base for the Gaussian integers but the set of all the complex numbers representable in the base $1-i$ takes on an interesting form. (We investigate this base before $-1+i$ because the base $1-i$ exposes the underlying geometric structure more clearly.) Take Figure 1 and subdivide each integer square into four squares; these smaller squares can be coordinatized by the complex numbers $x+iy$, where x and y are multiples of $1/2$. Shading those squares corresponding to numbers that can be represented in base $1-i$ produces a similar spiral pattern to Figure 1, but which has shrunk to one half its size and has turned counter-clockwise through one quarter of a revolution. Subdivide these smaller squares into four and repeat the process. Figure 5 shows the first three stages. At the rth stage, the complex plane is divided into squares of side 2^{-r}. If a coordinate $x+iy$, with x and y multiples of 2^{-r}, has a base $1-i$ representation, then it will be a terminating expansion with at most $2r$ digits to the right of the radix point.

In the limit this process yields a fascinating region, shown in Figure 6, that I call a snowflake spiral. The boundary of the figure is an example of a snowflake curve. The distance along the boundary between any two points is always infinite. The snowflake spiral consists of a large spiral with smaller spirals coming off it. Each small spiral has smaller spirals growing on its back and so on ad infinitum. This snowflake spiral has the property that if it is rotated counter-clockwise about the origin through one quarter of a revolution and simultaneously shrunk to one half its size then it remains unchanged! Notice that the white area in Figure 6 is identical to the black area when rotated through one half of a revolution. Hence

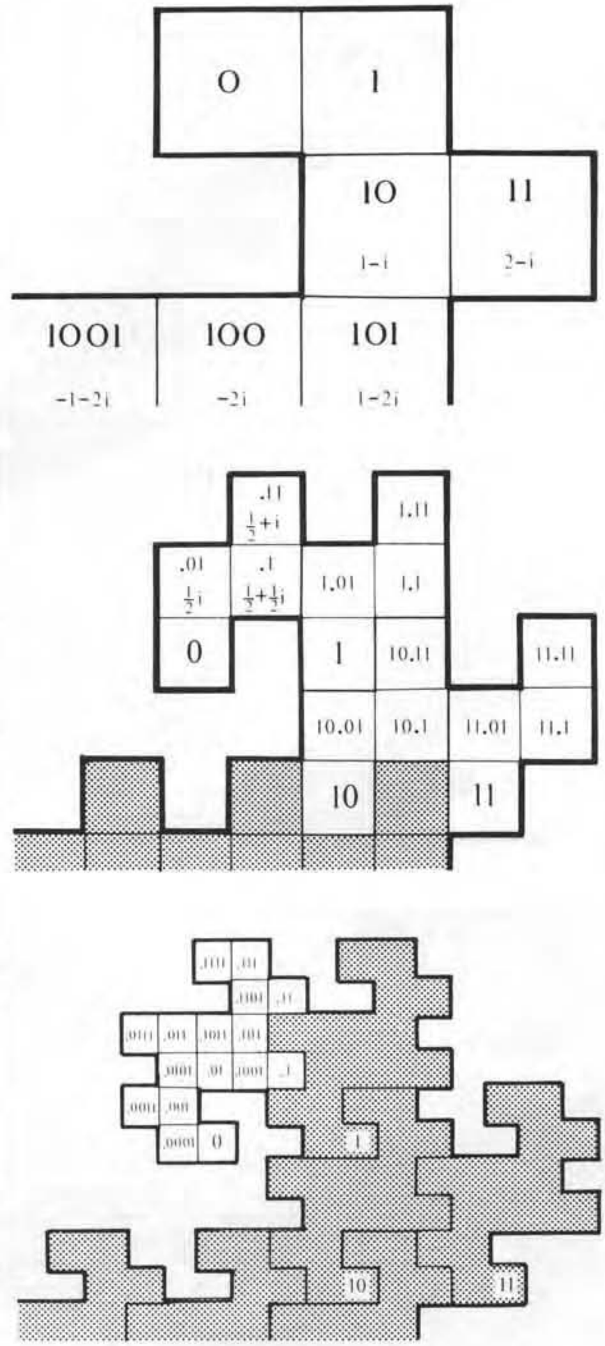

Figure 5. Approximations in the construction of a snowflake spiral. The squares in these three approximations are of size one, one half and one quarter, respectively, and they depict complex numbers expressible in the base $1-i$ using at most zero, two and four negative powers of the base.

"one half" of all the complex numbers can be written in the base $1-i$.

A similar process can be applied to the base $-1+i$. Even though every complex number can be written in binary form using the base $-1+i$, snowflake curves emerge

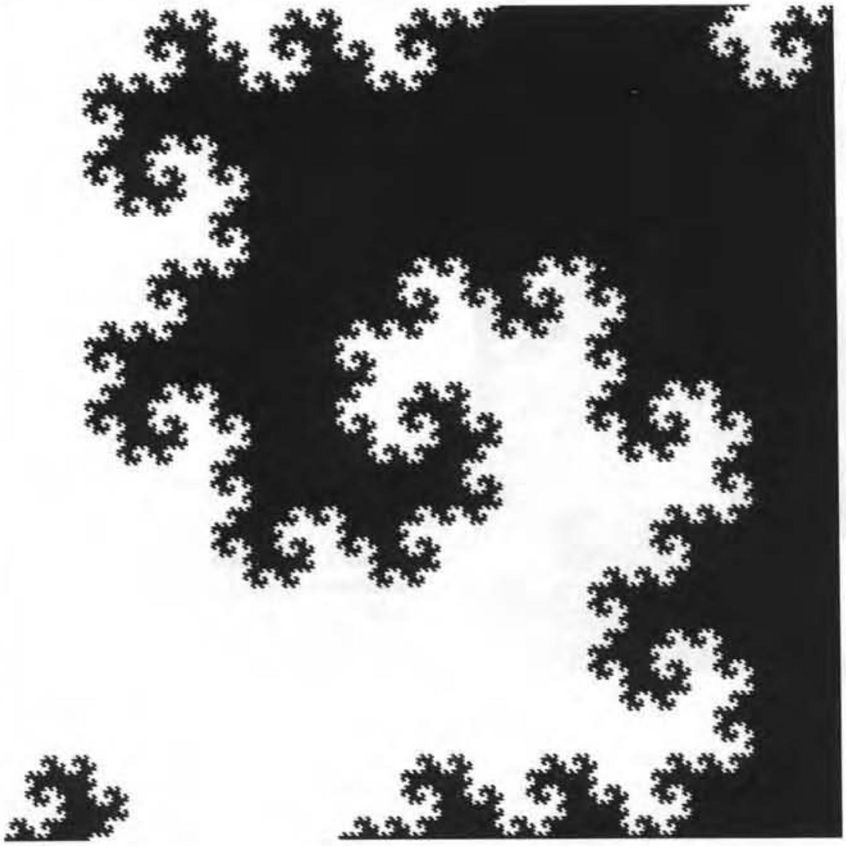

Figure 6. The snowflake spiral consisting of all the complex numbers expressible in binary form using the base $1 - i$. This is the limiting region of the approximations in Figure 5. The boundary is an example of a fractal curve.

from this representation as boundaries of the regions of complex numbers with a given integer part. An approximation to those regions is shown in Figure 7; the same shading has been given to those complex numbers that have the same integer part using eight places of binary expansion in the base $-1 + i$. In the limit, the boundaries of these shaded regions have infinite length but still enclose a unit area.

All points on the boundary of any of these regions will have two base $-1 + i$ expansions with different integer parts. For example, the point $(-1 + 2i)/5$ lies on the boundaries of the regions with integer parts 0 and $i = (11)_{-1+i}$; in fact, the two expansions are $(-1 + 2i)/5 = (0.\overline{01})_{-1+i} = (11.\overline{10})_{-1+i}$, where the digits under the bar are repeated indefinitely. This shows that the representation of all the complex numbers in base $-1 + i$ is *not* unique, even though it *is* unique for the Gaussian integers.

Because the plane is two dimensional, there must be some points on the boundary of *three* of these regions. These will correspond to points with three base $-1 + i$ expansions with different integer parts. For example, $(1 + 3i)/5 = (0.\overline{010})_{-1+i} = (11.\overline{001})_{-1+i} = (1110.\overline{100})_{-1+i}$ lies at the intersection of the regions with integer parts 0, i and $1 + i$. These three periodic expansions of length three can be checked by evaluating them by the standard method of multiplying them by the cube of the base and then substracting the original expansion.

Fractal Curves

The boundary of the snowflake spiral in Figure 6 has been studied by Mandelbrot in his book on Fractals [7]. A subset of a Euclidean space is called a *fractal* if its Hausdorff dimension is strictly larger than its topological dimension. This Hausdorff dimension, or fractal dimension as Mandelbrot calls it, is a real metric invariant that can take nonintegral values. This dimension agrees with the standard topological dimension for most usual sets and measures the jaggedness of more pathological sets. A standard curve would have fractal dimension one while a space-filling curve would have dimension two. The boundary of the snowflake spiral derived from the base $1 - i$ lies between these extremes and Mandelbrot has calculated its fractal dimension to be approximately 1.5236 ([7; p. 313]; see [3] for details). The limit of the approximation shown in Figure 7, consisting of regions of complex numbers with a fixed integer part in base $-1 + i$, has boundaries which are locally the same as that of Figure 6 and hence have the same fractal dimension.

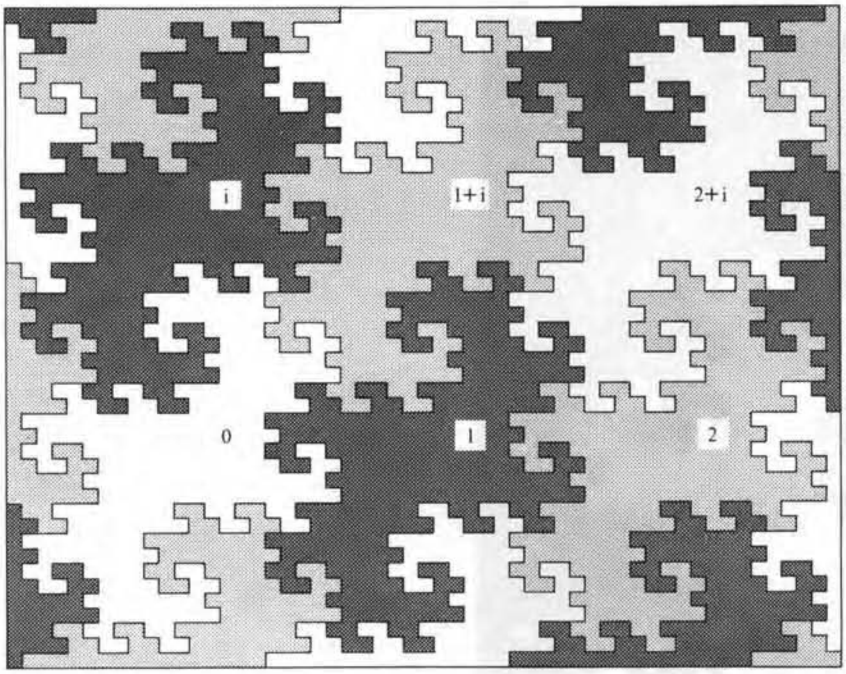

Figure 7. The integer parts of the complex numbers expressed in the base $-1 + i$. Each region is an approximation, using eight binary places, of the numbers with the indicated integer part in base $-1 + i$.

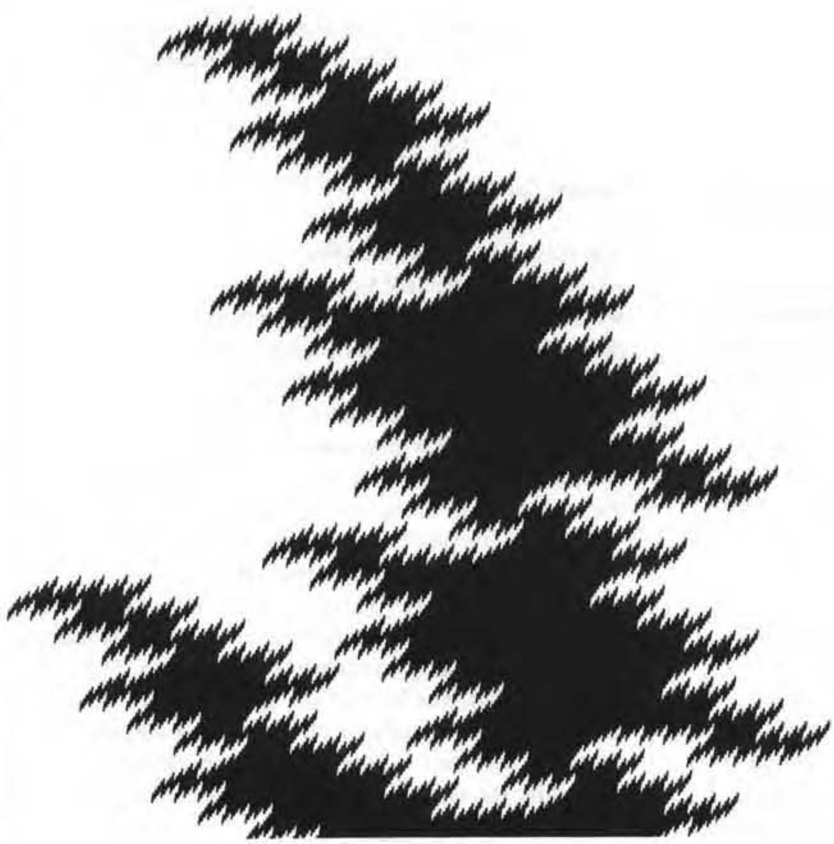

Figure 8. The fractal curve derived from the base $2 - i$. The dark region represents all the complex numbers expressible in the base $2 - i$ using digits 0, 1, 2, 3 and 4. The origin is at the tip of the top left peninsula.

The other representations of the complex numbers using the bases $n - i$ and $-n + i$, for n larger than one, yield new examples of snowflake curves. The complex numbers that can be written in base $n - i$ form a spiral snowflake region with the property that when rotated counter-clockwise through an angle arctan $(1/n)$ and shrunk by a linear factor $(n^2 + 1)^{1/2}$ it remains unchanged.

One of the chapters in Mandelbrot's book [7] is entitled "How long is the coast of Britain? " and in it he estimates the fractal dimension of the coast to be roughly between 1.2 and 1.3. Part of the boundary of the region derived from the base $2 - i$, shown in Figure 8, has a form remarkably like the British coast, but a calculation shows that its fractal dimension is approximately 1.6087 [3]. This means that it is much too jagged to be a good model of the coastline.

Dragon Curves

The snowflake spiral obtained from the base $1 - i$ has another interesting interpretation as the limit of twin space-filling dragon curves. The remarkable dragon curves were introduced in Martin Gardner's Mathematical Games column of the Scientific American [2]. One way of constructing these curves is to repeatedly fold a sheet of paper in half in the same direction, and then unfold it so that all the creases are right angles. A dragon curve is obtained by looking along the edge of the paper. A dragon curve is said to be of order r if its construction requires r folds. The curve of order $r + 2$ can be obtained from the order r curve as follows. Start at one end and systematically replace each L-shaped piece by the order 3 curve, whose sides are one half the length of the original. This process can be continued indefinitely and the limit yields an example of a space-filling curve. These dragon curves have been studied and coordinatized using complex bases by Chandler Davis and Donald Knuth [1].

Two dragon curves of the same size and order can be connected by joining the head of each to the tail of the other. Two such twin dragons of order r lie naturally inside the jigsaw piece containing 2^r squares derived from the base $-1 + i$. The process of replacing the twin dragons of order r by ones of order $r + 2$, shown in Figure 9, agrees with the construction of the jigsaw piece containing 2^{r+2} smaller squares by increasing the base $-1 + i$ expansion by two binary places. The limit of these twin dragon curves placed head to tail is precisely a jigsaw piece consisting of those complex numbers that can be written in the base $-1 + i$ using numbers whose integer parts have a given fixed length.

It is easily seen from the folding construction that a dragon curve of order $r + 1$ consists of two copies of order r curves. Therefore, by increasing r, it is possible to define a dragon curve of infinite order whose initial 2^r segments form the curve of order r. Donald Knuth has proved that four of these infinite order dragons joined at their tails form an infinite grid that covers the plane and that, as the length of the segments approaches zero, the four resulting space-filling dragons fill the entire plane. Figure 10 shows two such infinite dragon curves joined at their tails, one dragon being rotated through 90°. They lie naturally inside the jigsaw derived from the base $1 - i$. In the limit, as the length of the segment approaches zero, these twin space-filling dragons will yield the snowflake spiral of Figure 6.

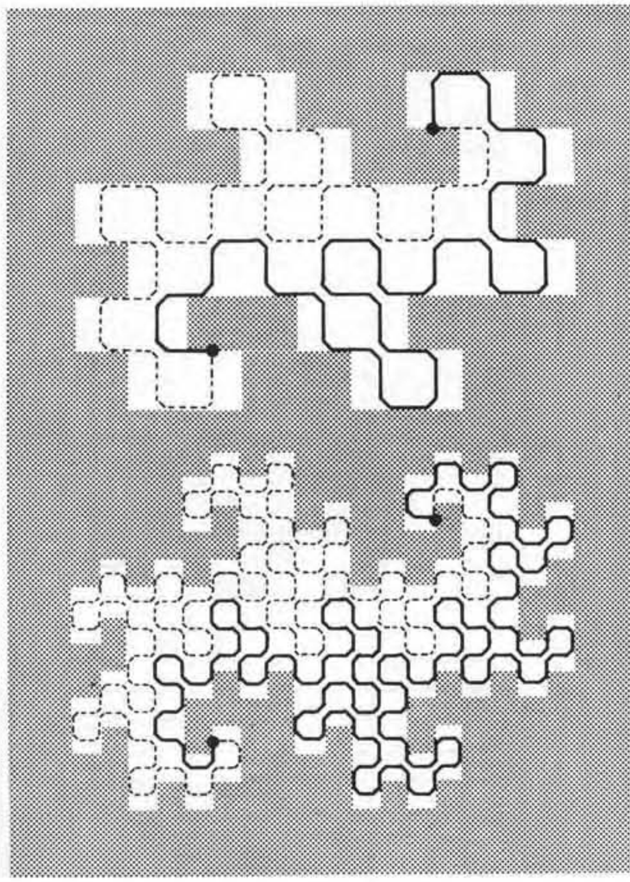

Figure 9. Twin dragons joined head to tail. Two dragon curves of order r lie naturally inside the jigsaw piece containing 2^r squares derived from the base $-1 + i$.

Conclusion

It is possible to investigate bases for other quadratic extensions of the rationals and for number fields of higher degree [4]. These bases yield further examples of fractal curves in the plane as well as higher dimensional fractal surfaces.

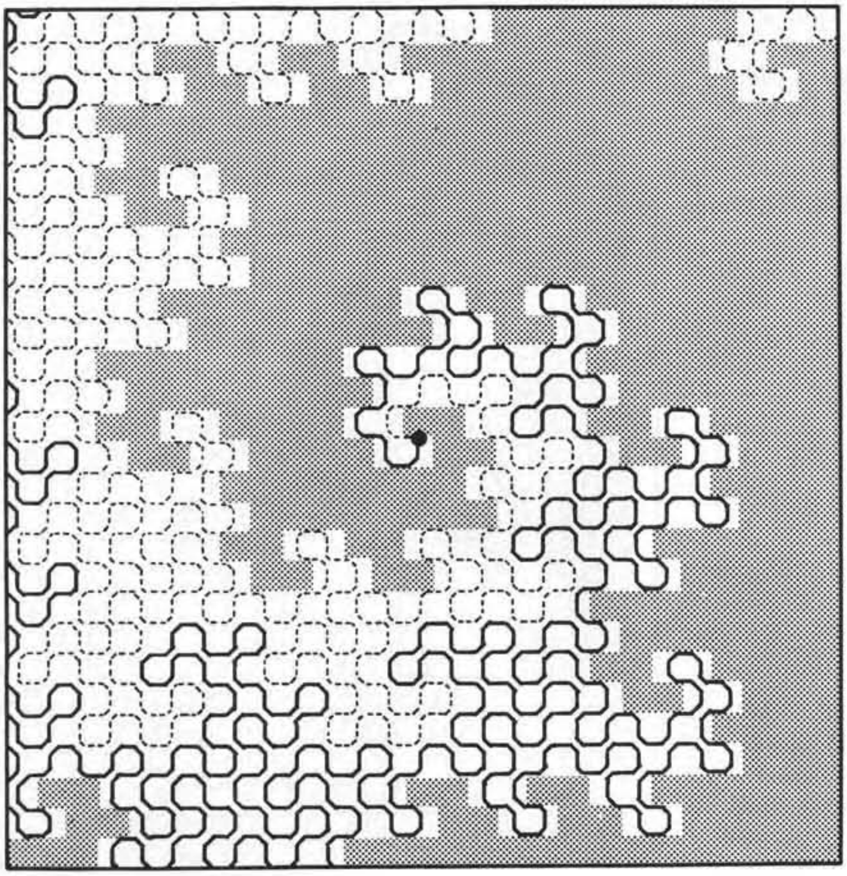

Figure 10. Twin dragons joined tail to tail lie naturally inside the spiral jigsaw derived from the base $1 - i$.

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