BOOK REVIEWS

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Number theory, an approach through history from Hammurapi to Legendre, by André Weil, Birkhäuser, Boston, Inc., Cambridge, Mass., 1984, xxi + 375 pp., $19.95. ISBN 3-7643-3141-0

A recent book contained the dedication

Hommage à André Weil
pour sa Leçon: goût,
rigueur et pénétration.

The author thus expressed his appreciation for Weil's refined mathematical taste, his rigor in exposition, and the depth of his work. The present book displays once more all these qualities. It is written in a prose which is precise, with a pleasant rhythm, very agreeable to read.

To state that the subject matter has been very well researched and the author has found the relevant documents—is obvious, but insufficient to express the lifelong familiarity of Weil with the historical development of number theory. Nourished in the mathematics of the past, Weil propelled the future. In number theory and algebraic geometry his well-known discoveries and conjectures have their roots in genuinely classical work.

Weil has chosen to develop his book around four mathematicians among past giants, Fermat, Euler, Lagrange and Legendre—the period to be covered excluded a priori their successors Gauss, Dirichlet, Kummer, Riemann, and others.

In reviewing this book, I have decided that, rather than paraphrase what is already so well written, I'd quote directly from the text—a good "collage" should be worthier than a bad painting.

A protohistory precedes the main chapters, alluding to some significant developments of number theory since antiquity.

"It is not prehistory, since it depends on written sources; protohistory seems more appropriate."

"The modern theory of numbers, like the god Bacchus...seems to have been twice-born." The first birth is ascribed to the period when Fermat studied the book of Diophantus, translated into Latin and published by Bachet in 1621—"the same one, no doubt, into whose margins [alas, too narrow] he was later to jot down some of his best discoveries." The rebirth took place when "on the first of December 1729, Goldbach asked Euler for his views about Fermat's statement that all integers $2^{2^n} + 1$ are primes" and on the next "fourth of June, when Euler reports that he has...been greatly impressed by Fermat's assertion that every integer is a sum of four squares (and also of 3 triangular numbers, of 5 pentagonal numbers, etc....)."

The chapter on protohistory deals first with the work treated in Euclid's books VII, VIII and IX; "it is generally agreed upon that much, if not all of the contents of those books is of earlier origin." The Euclidean algorithm and the irrationality of $\sqrt{2}$ are discussed; "if Aristotle...hints at one proof for [the irrationality of] $\sqrt{2}$, this hardly gives us a right to credit it to some hypothetical 'Pythagoreans'." It is a bit surprising to read: "Even in Euclid, we fail to find a general statement about the uniqueness of the factorization of an integer into primes," and "...the proof for the existence of infinitely many primes represents undoubtedly a major advance, but there is no compelling reason either for attributing it to Euclid or for dating it back to earlier times".

Next, in this chapter, there is a brief mention of Euclid's theorem that $2^n(2^{n+1} - 1)$ is perfect if the second factor is a prime.

Indeterminate equations of the first degree, to be solved in integers, were considered in China; and the general method of solution is essentially identical to the Euclidean algorithm for finding the greatest common divisor of two numbers. Aside from China, "the first explicit description of the general solution occurs in the mathematical portion of the Sanskrit astronomical work *Āryabhatīya* of the fifth–sixth century A.D." "In 1621, Bachet, blissfully unaware (of course) of his Indian predecessors, but also of the connection with the seventh book of Euclid, claimed the same method emphatically as his own in his comments in Diophantus."

Results about sums of arithmetical or geometric progressions or sums of squares of successive integers, as well as results about "pythagorean" triangles "must have become fairly universally known at a comparatively early date; invoking the name of Pythagoras adds little to our understanding of the matter". Indeed, the old Babylonian tablet, Plimpton 322 (reproduced as an illustration in this book), shows fifteen pythagorean triples; it is dated by Neugebauer and Sachs to between 1900 and 1600 B.C.

Pythagorean triples appeared in Euclid's book X, and were very familiar to Diophantus; they emerge again in Bombelli's *Algebra* (1572) and in Viète's books.

The identity $(x^2 + y^2)(z^2 + t^2) = (xz \pm yt)^2 + (xt \mp yz)^2$, useful for constructing numbers which can be written as sums of two squares in more than one way, appears "with an elaborate proof [...] in Fibonacci's *Liber Quadratorum* of 1225; Fibonacci claims no credit for it..." A copy of this book, not like the popular *Liber Abaci*, was located and published in 1856 by the prince Boncompagni. The identity "must have been familiar to Diophantus". Viète applied the identity for the construction of two new rectangular triangles from two given ones.

In the *Liber Quadratorum* Fibonacci considered the problem of finding three squares in arithmetic progression. An interesting discussion revolves around whether Leonardo could have known a Byzantine manuscript of the eleventh or twelfth century, where there occurs the related problem: To find a pythagorean triangle of area $5m^2$.

"One is on somewhat firmer ground in assuming that problems of the type of $x^2 - Ny^2 = \pm m$, for given positive integers N and m, must have occurred rather early in Greek mathematics, presumably in connection with the problem of obtaining good rational approximations for $\sqrt{N}$, when N is not a square." In particular, this method is used by Archimedes to give the approximations $265:153$ and $1351:780$ for $\sqrt{3}$.

And how about the interesting Greek "epigram" in 22 distichs, discovered by Lessing (1773) in a manuscript of the famous Wolfenbüttel library? It states a problem in eight unknown integers involving linear algebra, but also squares and triangular numbers. The author says that he who can solve the problem wins the prize for supreme wisdom. "He may well say so; it can be shown that the smallest solution is of the order of magnitude of 10^{103275}." "There is... every reason to accept the attribution [of this epigram] to Archimedes, and none for putting it in doubt."

Viète, and now I quote Fermat (as quoted by Weil—a second degree collage, if you like): "'Viète, by extending Diophantus's work to continuous quantities, has made it clear that it does not really belong to number theory.' For us, Viète is an algebraist, both in notations and in contents; his *Zetetica*...in our views pertains to algebraic geometry." The problems by Diophantus and Viète involve the two main cases of (in our modern terminology) curves of genus 0 and curves of genus 1, with "visible" rational points or pairs (to begin the search for rational solutions). Plane cubics fall within the scope of the methods.

So goes the protohistory. Despite my great pleasure in reviewing what Weil wrote, I have to refrain from describing the section on Xylander (alias Holzmann, hellenized) Bombelli, and later Bachet, translators of Diophantus. It is not out of place to quote the recently reprinted book by Heath, *Diophantus of Alexandria*, which is a source of reliable information.

Fermat, my favorite, is treated well by Weil—how could it be otherwise? It is well known that the proof of only one of Fermat's theorems in number theory has been preserved for posterity. It is the famous proof, by the method of infinite descent, that the area of a pythagorean triangle cannot be the square of an integer. This is deduced from the fact that the equation $x^4 - y^4 = z^2$ cannot have solutions in nonzero integers. And how about the other penetrating statements made by Fermat, for which he claimed to have proofs? Had he indeed? By which methods? From Weil's careful analysis of Fermat's correspondence, and in the light of what was known at the time, it is possible to imagine how Fermat might have proceeded. Such speculations, which may be risky if not well founded, are intriguing. In a situation where documents are lacking, they constitute a valid procedure to evaluate the methods which were used.

Fermat's earlier attention in number theory was directed to binomial coefficients, like the triangular and pyramidal numbers; he succeeded also in finding theorems on sums of powers of consecutive integers—"the same approach was rediscovered by Jacob Bernoulli (and is described in his posthumous *Ars*

Conjectandi of 1713), leading him to the definition of the 'Bernoulli numbers' and 'Bernoulli polynomials' whose importance for number theory did not begin to appear until later at the hands of Euler". An interesting discussion of §III concerns the proofs by induction and contains Fermat's statement, "The essence of a proof is that it compels belief". So Weil writes: "In view of the above quotation, when Fermat asserts that he has a proof for some statement, such a claim has to be taken seriously."

"It is difficult to take magic squares seriously, in spite of Fermat's professed enthusiasm for it and of E. Lucas's intriguing suggestion that they may have led to the discovery of the fundamental identity for sums of four squares".

Not only was Fermat attracted by magic squares, but he also studied perfect numbers. "Actually, not a little ingenuity is required in order to obtain all the perfect, amicable and submultiple numbers which turn up in the letters exchanged at that time between Fermat, Mersenne, Frenicle and Descartes." These investigations led to the study, among others, of Mersenne numbers $2^p - 1$ (p a prime), and Fermat was able to show that $2^{37} - 1$ is not a prime. His factorization method involved what is now called the "little Fermat theorem". Even though it may be safely assumed that Fermat had actually proved it, his proof is not available, and we had to wait for Euler.

Fermat also considered the numbers of the form $2^{2^n} + 1$, which he believed to be always primes. It is again Euler who showed that $2^{2^5} + 1$ admits 641 as a factor. In this respect, Fermat turned out to be wrong, because apart from the first few Fermat numbers, no other is known to be prime.

Between 1636 and 1640, Fermat looked more closely at diophantine equations and sums of squares—questions like: If an integer is a sum of two (respectively three) rational squares is it also a sum of the same number of integral squares? In a letter to Mersenne on 15 July, 1636, Fermat implies that he thinks he had proved it; in a second letter of September 2, he only asserts that he is working at it. Did he ever prove that the answer to the problem is the affirmative? An elementary proof, which Fermat would have understood, was published by L. Aubry in 1912. "It is idle, of course, to ask whether he could have found it; had he done so, occasions would not have been lacking for him to mention it to his correspondents, but the matter never turns up again in his letters."

In a letter of 1638 to Mersenne, Fermat made "the celebrated statement about every integer being the sum of three triangular numbers, of four squares, of five pentagonal numbers, and so on... Also in a letter of 1638, Fermat challenges Sainte-Croix to find two cubes whose sum is a cube, or two fourth powers whose sum is a fourth power, with the implication that he already knew or suspected that there are none. We shall never know for sure when, or whether, Fermat proved all these results."

It should be noted that even though Fermat also wrote, in his famous margin annotation to his copy of Diophantus, that a sum of two nth powers is not a nth power for any integer $n \geqslant 3$, he never actually asked any of his correspondents to prove it, except for $n = 3$ or 4.

In subsequent sections, Weil describes Fermat's work on quadratic residues, anticipating the work of Euler, as well as his discovery about sums of two squares.

"Writing to Mersenne on Christmas day 1640, Fermat told him that every prime $p = 4n + 1$ is, in one and only one way, a sum of two squares... Fortunately, he gave us a faint indication of his method in his communication of 1659 to Huygens. There, he says, he had used his method of descent, showing that if it were not so for some prime, it would also not be true for some smaller prime, and so on, until you reach 5." "Euler, in the years between 1742 and 1747, constructed a proof precisely of that kind; it is such that we may with some verisimilitude attribute its substance to Fermat."

"Fermat does not stop at the question whether $N = x^2 + y^2$ has a solution; he asks for the number of those solutions and for a way to find them."

In §XI, Weil summarizes the main results of Fermat concerning quadratic forms, whether under the guise of triangular numbers; or pythagorean triangles; or the quadratic forms $x^2 + 2y^2$, $x^2 + 3y^2$, etc.; or the diophantine equation $x^2 - Ny^2 = \pm 1$ (N a positive square-free integer), wrongly called Pell's equation by Euler; or "simple and double equations" leading to curves of genus 1.

Several sections are devoted to a detailed consideration of these matters. It is rewarding to follow Weil's discussion and to learn how Fermat challenged Wallis and Brouncker on the equation $x^2 - Ny^2 = 1$.

I am also especially impressed by Fermat's solution of the following problem of Frenicle: to find a pythagorean triangle with sides a, b, c such that $a + b$ and c are squares. Fermat gave the numbers

$$a = 4565486027761$$
$$b = 1061652293520$$
$$c = 4687298610289$$

"What is more...he makes bold to assert confidently that this solution is the smallest possible one." This was indeed verified by Lagrange in 1777, with an elaborate application of Fermat's infinite descent method.

Regretfully, I must depart from Fermat in order to continue the review of this book.

"If Huygens was undoubtedly more capable than most of his contemporaries of appreciating and of criticizing arithmetical work, he was not prepared to take up the torch proffered by Fermat. Only once did he come at all close to number theory; this occurred in his *Descriptio Automati Planetarii*, first printed posthumously in 1703 but probably composed between 1680 and 1687. Here, in connection with practical problems about automata with dented wheels, one finds a thoroughly original and masterly treatment of the best approximation of real numbers by fractions, based on the continued fraction algorithm." Lagrange called it "une des principales découvertes de ce grand géomètre"; "however, Huygens himself never bothered to get it published."

Adieu, Fermat. Be assured that you "cast your shadow well into the present century and perhaps the next one." In 1702, "with prophetic insight", Leibniz was trying to say (in modern terms) that "the study and classification of algebraic differentials and their integrals depended upon the methods of algebraic geometry for which, in his days, Diophantus, Viète and Fermat offered the only existing models." "When the same threads are later picked up, first by Euler and then by Lagrange, with brilliant success, the close connection between 'Diophantine algebra' (as Leibniz called it) and elliptic integrals begins to appear below the surface."

"Fermat's torch had indeed been long extinguished when Euler, in 1730, picked it up, kindled it anew, and kept it burning brightly for the next half-century."

The chapter on Euler begins with the observation that "until the latter part of the seventeenth century, mathematics had sometimes bestowed high reputation upon its adepts but had seldom provided them with the means to social advancement and honorable employment." It goes on with an interesting list of the various bread-earning activities of the mathematicians and scientists of the period. Everybody knows that Fermat was a magistrate, Copernicus an ecclesiastical dignitary, Galileo a professor in Padova, but I was amused to read that "Kepler plied his trade as an astrologer and maker of horoscopes", while Barrow relinquished his position of Lucasian professor in Cambridge "to become preacher to Charles II and achieve high reputation as a divine." And Descartes "felt himself, by the grace of God, above the need of gainful employment."

"By the time of Euler's birth in 1707, a radical change had taken place. The first academies were created. *L'Académie des Sciences* in 1666, the *Royal Society* the year before. These institutions began scientific publications; thus the Philosophical Transactions of the Royal Society, started in 1665, have been continued down to the present day." "Soon, universities and academies were competing for scientific talent and sparing neither effort nor expense in order to attract it." "Scientific life, by the turn of the century, had acquired a structure not too different from what we witness today."

"At the time of Euler's birth, Jacob Bernoulli was dead, and Johann had succeeded him. Euler became a close friend of Johann's two sons, Nicolas and Daniel, and he was Johann's favorite disciple." It is touching to learn how Euler, "in his old age, liked to recall how he had visited his teacher every Saturday and laid before him the difficulties he had encountered during the week, and how hard he had worked so as not to bother him with unnecessary questions." Needless to say, this behavior is now out of fashion...

"Three monarchs came to play a decisive role in Euler's career. Peter the Great (a truly great czar perhaps), Frederic the Great and the Great Catherine." And, of course, their greatness demanded academies of sciences...

"In those days, academies were well-endowed research institutions, provided with ample funds and good libraries. Their members enjoyed considerable freedoms; their primary duty was to contribute substantially to the academy's publications and keep high its prestige in the international scientific world."

Euler was lured to the new Petersburg Academy when he was not quite twenty years old. And there, his productivity exceeded all expectations. Later, in 1741, Euler moved to Berlin to the new Academy founded by Frederic. "Euler's Berlin period were twenty-five years of prodigious activity. More than 100 memoirs sent to Petersburg, 127 published in Berlin on all possible topics in pure and applied mathematics...side by side with major treatises on analysis, but also on artillery, ship-building, lunar theory,...and prize-winning essays sent to the Paris academy, to which one has to add the *Letters to a*

German Princess (one of the most popular books on science ever written)"...!! Once Catherine II "seized power after ridding herself and Russia of her husband...Euler was back in Petersburg, after a triumphal journey through Poland, where Catherine's former lover, King Stanislas, treated him almost like a fellow sovereign." Hundreds of memoirs were written in the last decade of Euler's life. "Enough, as he had predicted, to fill up the academy publications for many years to come."

"No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century." To his old teacher Johann Bernoulli, he was "mathematicorum princeps", to d'Alembert, "ce diable d'homme", "on finding himself anticipated by Euler in some results which he had felt rather proud of."

Of more than seventy volumes in his complete works, only four are concerned with number theory—but "this work alone would have earned him a distinguished place in the history of mathematics." Besides the published papers, the most valuable information about Euler's ideas on number theory, from the years 1730 to 1756, is to be found in his letters to Goldbach; this correspondence has been published with annotations by Juškevič and Winter.

Number theory makes its appearance in Goldbach's reply to Euler's very first letter. "Is Fermat's observation known to you that all numbers $2^{2^n} + 1$ are primes?" wrote Goldbach. Already in June 1730, Euler was studying Fermat's works, with the aim of providing proofs for his statements. He was particularly bewildered by Fermat's assertion that every natural number is a sum of four squares—but the proof of this theorem was to be Lagrange's.

In §V, Weil summarizes Euler's work on numbers, before embarking on a detailed description in subsequent sections. Ten main headings are needed to classify these discoveries:

(a) Fermat's theorem, the multiplicative group of integers modulo N and the beginning of group theory;

(b) sums of squares and "elementary" quadratic forms;

(c) diophantine equations of degree 2;

(d) diophantine equations of genus 1, and others;

(e) elliptic integrals;

(f) continued fractions, Pell's equation and recurrent sequences;

(g) summation of $\zeta(2\nu)$ and related series;

(h) "partitio numerorum" and formal power series;

(i) prime divisors of quadratic forms;

(j) large primes.

Fascinating topics, in which Euler's clairvoyance invariably led to complete solutions or to the developments of new theories.

Weil's masterful writing is not the usual dry listing of theorems. It contains, of course, all that is needed to understand the problems and the achievements, but it is also lively with an excellent documentation of the relationships and influences between the mathematicians involved. It renders the atmosphere. The readers will delight. For myself, I plan to return to the reading of this chapter, for there is much still for me to ponder. Take, for example, the topic of elliptic integrals.

"As Euler knew, Leibniz and Joh. Bernoulli had already asked whether the differential

$$w = \frac{dx}{\sqrt{1 - x^4}}$$

can be integrated by means of logarithms or inverse trigonometric functions and had guessed that it could not. When Euler asked the same question in 1730 and then, in 1738, gave a proof of Fermat's theorem about the diophantine equation $x^4 - y^4 = z^2$, it must surely have occurred to him that any substitution transforming w into a rational differential might well supply rational solutions for $z^2 = 1 - x^4$, and hence integral solutions for Fermat's equation." "In 1751, Fagnano's *Produzioni Matematiche*, reaching him in Berlin, opened his eyes to the fruitful field of investigation that lay unexplored there." Weil goes on to tell how one month later Euler presented his first memoir on the subject, then gave "a proof of the addition and multiplication theorems for integrals of the form

$$\int \frac{F(x)\,dx}{\sqrt{P(x)}},$$

where P is a polynomial of degree 4 and F an arbitrary polynomial or even a rational function."

Alas, I have to curtail my review of this chapter on Euler and move to the last chapter about Lagrange and Legendre—an age of transition to the glorious time of Gauss, Riemann, Kummer and Dirichlet.

This chapter is to be considered more as a postscript to the core of the book. As such, the subject matter is more outlined than developed. To be sure, nothing of importance is omitted, but I would like to continue at the same pace to witness the unveiling of the development of classical number theory.

"Le célèbre Lagrange, le premier des géomètres", according to Lavoisier, and the successor to Euler at the Berlin Academy. So it should be, since it was while reading and "meditating assiduously" about Euler's work that Lagrange produced his first important results on the theory of maxima and minima, as it applies to curves—in effect, he created the classical calculus of variations, sketched in a letter to Euler, on 12 August 1755. And the reply was "not only prompt and generous; it was enthusiastic: 'You seem to have brought the theory of maxima and minima to almost its highest degree of perfection; my admiration for your penetration knows no bounds.'" Yet nothing could alter Lagrange's innate modesty. Of him, Clairaut said: "a young man, no less remarkable for his talents than for his modesty; his temperament is mild and melancholic; he knows no other pleasure than study."

Much and perhaps the greater part of Lagrange's best work is directly inspired by that of Euler... "which, already as a young man, he knew by heart down to the minutest details". His writings on arithmetical topics consist of the following:

(a) papers on Pell's equation, proving Fermat's assertion on the existence of infinitely many solutions, and an algorithm to obtain them, using continued fractions;

(b) the proof that all quadratic irrationals have periodic continued fractions;

(c) solution in integers of the equations of degree 2 in two unknowns, as well as of the equations $z^2 = Ax^2 + By^2$;

(d) when Euler's *Algebra* was translated into French, Lagrange wrote a long supplement containing his own results;

(e) the proof of Fermat's theorem on sums of four squares;

(f) the proof of Wilson's theorem;

(g) a long study, called *Recherches d'Arithmétique*, about the theory of binary quadratic forms;

(h) the study of the diophantine equation $x^4 - 2y^4 = \pm z^2$, which originated with a problem of Fermat on pythagorean triangles, "a carefully done exercise on Fermat's method of infinite descent, but applied for the first time to an equation of genus 1 and rank > 0 (i.e., with infinitely many solutions)."

Legendre, sixteen years younger than Lagrange, lived until 1833, well into the time of Gauss. A remarkably successful and long career, despite the troubled times of the French revolution. Besides his favorite subject of elliptic functions, Legendre made also many important contributions to number theory. Yet, he suffers in comparison with Gauss, who was able to bring to fruition many of Legendre's ideas. So it was with the quadratic reciprocity law, which Legendre discovered but couldn't prove, despite various attempts. As is well known, it was the feat of young Gauss to arrive at several proofs of this fundamental theorem. The main block to Legendre's proof amounted to establishing the existence of infinitely many primes in arithmetic progressions, with difference and first term relatively prime. Again, this theorem would be proved by Dirichlet in 1837, "by a wholy original method which remains as one of his major achievements; by the same method, he also proved that every quadratic form $ax^2 + bxy + cy^2$ represents infinitely many primes, provided a, b, c have no common divisor, as had been announced by Legendre." "On another matter of great importance, Legendre also appears as a forerunner of Gauss, and more so perhaps than Gauss was willing to concede." It is the elementary construction which is at the origin of the Gaussian concept of composition of binary quadratic forms. "No doubt the Gaussian theory, as Gauss chose to describe it, is far more elaborate; so much so, indeed, that it remained a stumbling-block for all readers of the *Disquisitiones* until Dirichlet restored its simplicity by going back very nearly to Legendre's original construction."

"One last satisfaction was granted Legendre in his old age. ...Fermat's so-called 'last theorem' for exponents greater than 4 had remained as a challenge for all arithmeticians... Interest in the problem was revived in Paris...particularly after the Paris Academy, in 1816, made it the subject of their annual prize-competition for 1818." It is interesting to mention Gauss's reaction: "with luck, its solution might perhaps turn up as a by-product of a wide extension of the higher arithmetic..." One cannot say it better today.

"In the meanwhile, Sophie Germain, whose talent had early attracted the notice of Lagrange, Legendre and Gauss, had started working on Fermat's theorem, obtaining some valuable results based on ingenious congruence arguments."

Then follows the interesting story of how Dirichlet, a young student in Paris in 1825, tried his hand on Fermat's equation with exponent 5. He used an identity known since Euler's days and the infinite descent to show that the equation has no solution where one of the numbers is a multiple of 10. "This is where Legendre, then well over 70 years old, stepped in. After presenting Dirichlet's paper to the Academy in July 1825, it took him only a few weeks to deal with the remaining case." "As to Dirichlet, he was soon to take his flight and soar to heights undreamt of by Legendre.."

One of Legendre's influential contributions was the treatise on numbers which he prepared for more than thirty years. "He sought to give a comprehensive account of number theory, as he saw it at the time, including, besides his own research, all the main discoveries of Euler and Lagrange, as well as numerical evidence (in the form of extensive tables) for many results whose proofs he felt to be shaky."

The *Théorie des Nombres*, published in 1830, is the final form given to two previous editions, appropriately called *Essais*. Yet, "by then, as his younger contemporaries well knew, Gauss's *Disquisitiones* had made it almost wholly obsolete."

An indispensable part of Weil's book is the long series of appendices attached to the three main chapters. Their purpose is to show, from a modern point of view, how to consider certain classical questions, to indicate developments of importance originated in the ideas of that period, but sometimes also to give proofs of results described in the main text. Thus, we may read an illuminating appendix under the title "The Descent and Mordell's Theorem", another about "The Addition Theorem for Elliptic Curves", or also "Hasse's Principle for Ternary Quadratic Forms", etc.

Here I reach the point when it is appropriate to refer to the physical characteristics of the book. Should I say that it is a medium-sized volume, well bound and pleasantly printed, with large size type, greatly facilitating the reading? Should I add that it is well organized, has good indices, and no misprints? I just want to say that the hand holds it well, and does not wish to let it go.

Professor Weil, hear as a distant echo from younger days: Rico é o seu livro que nos revela uma gloriosa exploração intelectual pelos verdadeiros heróis.

PAULO RIBENBOIM

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Fundamentals of generalized recursion theory, by M. C. Fitting, Studies in Logic and The Foundations of Mathematics, Volume 105, North-Holland Publishing Company, Amsterdam, 1981, xx + 302 pp., $63.75 U.S./Dfl. 150.00.

0. The time seems ripe for a broad look at generalized recursion theory (g.r.t.) against the background of ordinary recursion theory (o.r.t.), with