THE CONSTRUCTIVE SECOND NUMBER CLASS*

ALONZO CHURCH

The existence of at least a vague distinction between what I shall call the constructive and the non-constructive ordinals of the second number class, that is, between the ordinals which can in some sense be effectively built up to step by step from below and those for which this cannot be done (although there may be existence proofs), is, I believe, somewhat generally recognized. My purpose here is to propose an exact definition of this distinction and of the related distinction between constructive and non-constructive functions of ordinals in the second number class; where, again to speak vaguely, a function is constructive if there is a rule by which, whenever a value of the independent variable (or a set of values of the independent variables) is effectively given, the corresponding value of the dependent variable can be effectively obtained, effectiveness in the case of ordinals of the second number class being understood to refer to a step by step process of building up to the ordinal from below.

Much of the interest of the proposed definition lies, of course, in its absoluteness, and would be lost if it could be shown that it was in any essential sense relative to a particular scheme of notation or a particular formal system of logic. It is my present belief that the definition is absolute in this way—towards those who do not find this convincing the definition may perhaps be allowed to stand as a challenge, to find either a less inclusive definition which cannot be shown to exclude some ordinal which ought reasonably to be allowed as constructive, or a more inclusive definition which cannot be shown to include some ordinal of the second class which cannot be seen to be constructive.

It is believed that the distinction which it is proposed to develop between constructive and non-constructive ordinals (and functions of ordinals) should be of interest generally in connection with applications of the transfinite ordinals to mathematical problems. The relevance of the distinction is especially clear, however, in the case of applications of the ordinals to certain questions of symbolic logic (for example, various questions more or less closely related to the well known theorem of Gödel on undecidable propositions)†—this is

* An address delivered by invitation of the Program Committee at the Indianapolis meeting of the Society, December 29, 1937.

† Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und*

because of the criterion of effectiveness or "definiteness" which necessarily applies to the rules of procedure of a formal system of logic.* The distinction is also presumably relevant to proposals to classify recursive functions of natural numbers according to ordinals of the second number class†—because the possibility of an effective step by step calculation of the values of the function may reasonably be taken as an essential part of the notion of a recursive function.

For present purposes it will be convenient to make a minor departure from the established terminology, using "second number class" in such a sense that the first number class is included as a part of the second number class (thus avoiding the sometimes awkward phrase, "first and second number classes"). On this basis, the *second number class* may be described as the simply ordered set which results when we take 0 as the first (or least) element of the set and allow the two following processes of generation: (1) given any element of the set, to generate the element which next follows it (the least element greater than it); (2) given any infinite increasing sequence of elements, of the order type of the natural numbers, to generate the element which next follows the sequence (the least element greater than every element of the sequence).‡ The elements of the set are *ordinals*. The ordinal which next follows a given ordinal is the *successor* of that ordinal. The ordinal which next follows a given infinite increasing sequence of ordinals, of the order type of the natural numbers, is the *upper limit* of that sequence. The ordinal which has a given ordinal as successor is the *predecessor* of that ordinal. An infinite increasing sequence of ordinals, of the order type of the natural numbers, which has a given ordinal as upper limit is a *fundamental sequence* of that

verwandter Systeme I, Monatshefte für Mathematik und Physik, vol. 38 (1931), pp. 173–198. See also *On Undecidable Propositions of Formal Mathematical Systems*, mimeographed lecture notes, Princeton, N. J., 1934.

* There is some current tendency to apply the name "logic" to schemes which are similar to accepted systems of symbolic logic but involve one or more rules of procedure which lack this characteristic of effectiveness. Such schemes may perhaps be of interest as abstract definitions of classes of formulas, but they cannot in my opinion be called "logics" except by a drastic (and possibly misleading) change in the usual meaning of that word. For they do not provide an *applicable* criterion as to what constitutes a valid proof.

† Cf. David Hilbert, *Über das Unendliche*, Mathematische Annalen, vol. 95 (1926), pp. 161–190; Wilhelm Ackermann, *Zum Hilbertschen Aufbau der reellen Zahlen*, Mathematische Annalen, vol. 99 (1928), pp. 118–133.

‡ This definition of the second number class is selected as fundamental because it represents the way in which, ordinarily, the ordinals actually arise in applications to mathematical problems—in particular the way in which they arise in connection with those questions of symbolic logic to which reference was just made.

ordinal. Every ordinal except 0 has either a predecessor or a fundamental sequence but not both; in the first case the ordinal is of the *first kind*, in the second case of the *second kind.*

As a definition of the distinction between constructive and nonconstructive ordinals in the second number class might be proposed the following:

An ordinal ξ is constructive if it is possible to devise a system of notation which assigns a unique notation to every ordinal less than or equal to ξ and, associated with the system of notation, three effective processes by which, respectively, (1) given the notation for any ordinal it can be determined whether the ordinal is of the first or the second kind, (2) given the notation for any ordinal of the first kind the notation for the predecessor of the ordinal can be obtained, (3) given the notation for any ordinal of the second kind a fundamental sequence of that ordinal can be obtained, in the sense of an effective process for calculating the notations for the successive terms of the sequence.*

It will be seen that the contemplated system of notation for ordinals less than or equal to ξ will then also admit an effective process by which, given the notations for any two ordinals, it can be determined which ordinal is greater.

Moreover, of course this contemplated system of notation is required (by definition) to yield an effective simultaneous selection, for every ordinal μ of the second kind less than or equal to ξ, of one out of the various fundamental sequences of μ. In fact, such an assignment of a unique fundamental sequence to every ordinal of the second kind less than or equal to ξ is held evidently to be a necessary consequence of the step by step process of building up to ξ from below which was first taken as characterizing constructiveness.

The present definition of constructiveness of ξ (that there exists a system of notation, of the kind described, for the ordinals less than or equal to ξ) is thought to correspond satisfactorily to the vaguer notion with which we began, and also to be satisfactorily exact, except for one thing, the vagueness of the notion of an effective process.

This notion of an effective process occurs frequently in connection with mathematical problems, where it is apparently felt to have a clear meaning, but this meaning is commonly taken for granted

* A similar but essentially different property of an ordinal ξ is employed by S. C. Kleene; see *On notation for ordinal numbers*, abstract in this Bulletin, vol. 43 (1937), p. 41.

without explanation. For our present purpose it is desirable to give an explicit definition.

Perhaps the most convincing form in which this definition of an effective process can be put results from the adoption of an idea of Turing.* A process is effective if it is possible to devise a calculating machine, with a finite number of parts of finite size, which, with the aid of an endless tape, running through the machine, on which symbols are printed, will carry out the process in any particular case—of course only a finite portion of the tape being used in any particular case. (It will be seen that a human calculator, provided with pencil and paper and explicit instructions, can be thought of as a machine of this kind, the paper taking the place of the tape.)

In the case of a process which, applied to a natural number, yields a natural number, another form of the definition of effectiveness is that the process is effective if it corresponds to an arithmetic function which is recursive in the most general sense.† This definition can be extended to processes upon the formulas of an arbitrary system of notation by employing the now familiar device of representing the formulas by Gödel numbers.

Still another form of the definition of an effective process is obtained by replacing the condition of general recursiveness by the condition of λ-definability.‡

The equivalence of these three definitions is established in papers by Kleene and Turing.§

This completes the explanation of the proposed definition of the distinction between constructive and non-constructive ordinals in the second number class. In order, however, to obtain a definition of the related distinction between constructive and non-constructive functions of ordinals, and in order to establish that not every ordinal in the second number class is constructive, it is desirable to extend the notion of λ-definition, which was first introduced for positive in-

* A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, (2), vol. 42 (1936–1937), pp. 230–265.

† I.e., general recursive in the sense of Herbrand and Gödel. Cf. Kurt Gödel, *On Undecidable Propositions of Formal Mathematical Systems*, pp. 26–27; S. C. Kleene, *General recursive functions of natural numbers*, Mathematische Annalen, vol. 112 (1935–1936), pp. 727–742.

‡ Cf. Alonzo Church, *An unsolvable problem of elementary number theory*, American Journal of Mathematics, vol. 58 (1936), pp. 345–363.

§ S. C. Kleene, *λ-definability and recursiveness*, Duke Mathematical Journal, vol. 2 (1936), pp. 340–353; A. M. Turing, *Computability and λ-definability*, The Journal of Symbolic Logic, vol. 2 (1937), pp. 153–163.

tegers, to the transfinite ordinals by introducing appropriate formulas (of the λ-formalism) to represent the ordinals. For the constructive ordinals of the second number class this is done as follows.

Using an arrow to mean "stands for" or "is an abbreviation for," let

$$0_o \rightarrow \lambda m.m(1),$$
$$S_o \rightarrow \lambda am.m(2, a),$$
$$L \rightarrow \lambda arm.m(3, a, r),$$

where,

$$1 \rightarrow \lambda fx.f(x),$$
$$2 \rightarrow \lambda fx.f(f(x)),$$
$$3 \rightarrow \lambda fx.f(f(f(x))).$$

Then let a class of formulas, to be called *ordinal-formulas*, and a relation $<$ between ordinal-formulas be defined simultaneously by induction as follows (1–6):

1. If $\boldsymbol{a}$ is an ordinal-formula and $\boldsymbol{b}$ conv $\boldsymbol{a}$, then $\boldsymbol{b}$ is an ordinal-formula; further any ordinal-formula which bears the relation $<$ to $\boldsymbol{a}$ bears that relation also to $\boldsymbol{b}$; further $\boldsymbol{b}$ bears the relation $<$ to any ordinal-formula to which $\boldsymbol{a}$ bears that relation.

2. 0_o is an ordinal-formula.

3. If $\boldsymbol{a}$ is an ordinal-formula, $S_o(\boldsymbol{a})$ is an ordinal-formula, and $\boldsymbol{a} < S_o(\boldsymbol{a})$.

4. If $\boldsymbol{r}$ is a well-formed formula and each of the formulas in the infinite list, $\boldsymbol{r}(0_o)$, $\boldsymbol{r}(S_o(0_o))$, $\boldsymbol{r}(S_o(S_o(0_o)))$, $\cdots$, is an ordinal-formula and bears the relation $<$ to the formula which follows it in the list, then $L(0_o, \boldsymbol{r})$ is an ordinal-formula, and each of the formulas in the infinite list bears the relation $<$ to $L(0_o, \boldsymbol{r})$.

5. If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are ordinal-formulas and $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{b} < \boldsymbol{c}$, then $\boldsymbol{a} < \boldsymbol{c}$.

6. The ordinal-formulas are the smallest class of formulas possible consistently with 1–5, and the relation $<$ subsists between two ordinal formulas only when compelled to do so by 1–5.

It will be seen that under the relation $<$ the ordinal-formulas form, not a simply ordered, but a partially ordered set.

Then an assignment of formulas to represent ordinals in the second number class is defined by induction as follows (i–v):

i. If $\boldsymbol{a}$ represents an ordinal a, and $\boldsymbol{b}$ conv $\boldsymbol{a}$, then $\boldsymbol{b}$ also represents a.

ii. 0_o represents the ordinal 0.

iii. If $\boldsymbol{a}$ represents an ordinal a, then $S_o(\boldsymbol{a})$ represents the successor of a.

iv. If $\boldsymbol{r}$ is a well-formed formula, and each of the formulas in the infinite list, $\boldsymbol{r}(0_o)$, $\boldsymbol{r}(S_o(0_o))$, $\boldsymbol{r}(S_o(S_o(0_o)))$, $\cdots$, is an ordinal-formula and bears the relation $<$ to the formula which follows it in the list, and if the formulas in the infinite list represent respectively the ordinals $b_0, b_1, b_2, \cdots$, then $L(0_o, \boldsymbol{r})$ represents the upper limit of the sequence $b_0, b_1, b_2, \cdots$.*

v. A formula represents an ordinal only if compelled to do so by i–iv.

Evidently, every formula which represents an ordinal is an ordinal-formula (as previously defined) and every ordinal-formula represents an ordinal. If the relation $<$ (between ordinal-formulas) holds between two given ordinal-formulas, the relation *less than* (between ordinals) must hold between the ordinals which they represent—but not conversely.

Moreover, under this assignment of formulas to represent ordinals in the second number class, every formula which represents an ordinal has a normal form, and no formula represents more than one ordinal, as has been proved by Church and Kleene.† In general, however, the same ordinal is represented by an infinite number of non-interconvertible formulas.

Let us call an ordinal in the second number class λ-*definable* if there is at least one formula which represents it under the foregoing scheme. Then the class of λ-definable ordinals coincides with the class of constructive ordinals previously defined.

For the ordinal-formulas in principal normal form‡ which bear the relation $<$ to a given ordinal-formula form a simply ordered set un-

* This constitutes a minor but essential correction to the assignment of formulas to represent ordinals which is proposed by Alonzo Church and S. C. Kleene, *Formal definitions in the theory of ordinal numbers*, Fundamenta Mathematicae, vol. 28 (1937), pp. 11–21. The correction is minor in the sense that no further changes in the cited paper are necessitated by it.

The correction is regarded as essential from the point of view of intuitive justification. Since this address was delivered, however, I have seen a proof by Kleene that the two definitions of λ-definability of ordinals in the second number class (the definition given here and our previous definition) are equivalent.

† In the paper cited in the preceding footnote.

‡ For definition of the terms *normal form* and *principal normal form* reference may be made to *An unsolvable problem of elementary number theory*. The normal form of a formula is ambiguous only to the extent of possible alphabetical changes of the bound variables which appear, but it is sometimes convenient to remove even this ambiguity by adopting a device due to Kleene by which a particular one of the normal forms of a formula is designated as the principal normal form.

der the relation $<$, as can be proved by transfinite induction. If ξ is a λ-definable ordinal, and $\boldsymbol{\xi}$ is a formula in principal normal form which represents ξ, then $\boldsymbol{\xi}$ together with the ordinal-formulas in principal normal form which bear the relation $<$ to $\boldsymbol{\xi}$ constitute a system of notation for the ordinals less than or equal to ξ which is of the kind required by the definition of constructiveness. Hence every λ-definable ordinal is constructive.

Moreover, if ξ is a constructive ordinal, the system of notation for ordinals less than or equal to ξ which is referred to in the definition of constructiveness can be replaced by an assignment of a positive integer to each ordinal less than or equal to ξ, the notation for each ordinal being replaced by the Gödel number of the notation. And then the positive integer assigned to each ordinal may in turn be replaced by the formula (of the λ-formalism) which represents that positive integer.* Thus there is correlated to each ordinal less than or equal to ξ a formula (of the λ-formalism), and in fact one of the formulas which represent the positive integers. In view of the definition of the constructiveness of ξ, and since every effective function of positive integers is λ-definable, there will be three formulas $\boldsymbol{K}$, $\boldsymbol{P}$, $\boldsymbol{f}$, having the following properties: if $\boldsymbol{m}$ is any formula (of the λ-formalism) which is correlated (as just described) to an ordinal μ less than or equal to ξ, then $\boldsymbol{K}(\boldsymbol{m})$ conv 1 if μ is of the first kind, $\boldsymbol{K}(\boldsymbol{m})$ conv 2 if μ is of the second kind, $\boldsymbol{K}(\boldsymbol{m})$ conv 3 if μ is 0_o; also if μ is of the first kind, $\boldsymbol{P}(\boldsymbol{m})$ conv the formula correlated to the predecessor of μ; and if μ is of the second kind and $\boldsymbol{n}$ is a formula which represents a finite ordinal n,† then $\boldsymbol{f}(\boldsymbol{m}, \boldsymbol{n})$ conv the formula correlated to the $(1+n)$th term of that fundamental sequence of μ which is given by the effective process (3) referred to in the definition of the constructiveness of ξ. Using a theorem of Kleene,‡ one may obtain a formula $\boldsymbol{T}$ having the following conversion properties, where $\boldsymbol{m}$ is any formula which represents a positive integer:

$$\boldsymbol{T}(\boldsymbol{m}) \text{ conv } S_o(\boldsymbol{T}(\boldsymbol{P}(\boldsymbol{m}))), \text{ if } \boldsymbol{K}(\boldsymbol{m}) \text{ conv } 1;$$

$$\boldsymbol{T}(\boldsymbol{m}) \text{ conv } L(0_o, \lambda n \boldsymbol{T}(\boldsymbol{f}(\boldsymbol{m}, n))), \text{ if } \boldsymbol{K}(\boldsymbol{m}) \text{ conv } 2;$$

$$\boldsymbol{T}(\boldsymbol{m}) \text{ conv } 0_o, \text{ if } \boldsymbol{K}(\boldsymbol{m}) \text{ conv } 3.$$

* For the representation of positive integers by formulas (of the λ-formalism) see, for example, *An unsolvable problem of elementary number theory.*

† The finite ordinals are here taken as distinct from the corresponding non-negative integers, as is done in the cited paper of Church and Kleene. In view of the formula $\mathfrak{J}$ derived in that paper (page 19), there is no difficulty caused here by taking n to be a finite ordinal rather than a positive integer.

‡ *λ-definability and recursiveness*, Theorem 19 and footnote 17.

Then if $\boldsymbol{m}$ is the formula which is correlated to an ordinal μ less than or equal to ξ, the formula $\boldsymbol{T}(\boldsymbol{m})$ will represent the ordinal μ (proof by transfinite induction). In particular if $\mathbf{x}$ is the formula which is correlated to ξ, then $\boldsymbol{T}(\mathbf{x})$ will represent ξ. Hence every constructive ordinal is λ-definable.

Now let a function $F(x_1, x_2, \cdots, x_r)$ of ordinals in the second number class be called λ-*definable* if there is a formula $\boldsymbol{F}$ such that whenever the formulas $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r$ represent the ordinals $x_1, x_2, \cdots, x_r$ respectively, the formula $\boldsymbol{F}(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_r)$ will represent the ordinal $F(x_1, x_2, \cdots, x_r)$. As a definition of the notion of a constructive function of ordinals in the second number class, it is proposed simply to identify this notion with that of a λ-definable function of ordinals in the second number class. This is rendered plausible by the known properties of the λ-formalism, and no definition with a more direct appeal suggests itself.

It has been proved by Church and Kleene* that a large class of functions of ordinals are λ-definable, including addition, multiplication, exponentiation, the function ϵ_x, a predecessor function of ordinals, and others. The function $\phi(x, y)$ of ordinals in the second number class whose value is the ordinal 0 when x is less than y, and 1 when x is equal to y, and 2 when x is greater than y is, however, demonstrably not λ-definable.† This is taken to mean that from a strictly finitary point of view the second number class as a simply ordered set is inadmissible and must be replaced by a partially ordered set which has the structure of the set of ordinal-formulas under the relation $<$. Many of the classical theorems about the second number class may, however, be represented by corresponding finitary theorems about this partially ordered set.

In order to establish the existence of non-constructive ordinals in the second number class it is sufficient to observe that the class of all formulas (of the λ-formalism) is enumerable, and hence (by a non-constructive argument) that the class of all λ-definable ordinals is enumerable, whereas the class of all ordinals in the second number class is non-enumerable.

Let ω_1 be the least non-constructive ordinal in the second number

* Loc. cit.

† This is not surprising. It is, for instance, not difficult to give examples of pairs of constructive definitions of ordinals such that the question whether the ordinals defined are equal, or which of the two is greater, depends on this or that unsolved problem of number theory; and indeed this may be done without employing any ordinal greater than ω^2.

class. Then evidently every ordinal greater than ω_1 in the second number class is also non-constructive.

It is readily proved that ω_1 cannot be of the first kind. Thus by an indirect argument the existence of a fundamental sequence of ω_1 is established. But demonstrably there cannot be an effective process of calculating the successive terms of a fundamental sequence of ω_1.*

From a finitary point of view, ω_1 belongs to the third number class but not the second.

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* If it be required that the process of calculating the successive terms of the sequence shall not only provide a step by step process of building up to each term from below, but shall also effectively exhibit the increasing character of the sequence by including the step by step process of building up to any term as a part of the corresponding process for each subsequent term, the impossibility is an immediate consequence of the definition of ω_1. If, however, the condition be omitted that the process effectively exhibit the increasing character of the sequence in this way, the proof of impossibility depends on the theorem of Kleene previously referred to concerning the equivalence of the two definitions of λ-definability of ordinals in the second number class.