Algebras, Operads, Combinads

Jean-Louis Loday

IRMA (Cnrs) et Institut de Mathématique Zinbiel, Strasbourg

HOGT Lille, 23 mars 2012

Plan of the talk

I. Types of algebras, types of operads

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II. An example: planar trees and ns operads

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  a) Feynman diagrams
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    (relationship with quantum groups)

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(thanks to María Ronco and Bruno Vallette)

[JLL-MR] JLL, M.Ronco, Permutads, soumis à J.Algebraic Combinatorics A.
[JLL-BV] JLL and B.Vallette, Algebraic operads, Grundlehren Math.Wiss. 346, Springer, Heidelberg, 2012.

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$$\mathcal{P} : \mathrm{Mod} \to \mathrm{Mod}$$

1. $\mathcal{P}(V) = \bigoplus_n \mathcal{P}_n \otimes V^{\otimes n}$
2. $\mathcal{P}(V) = \bigoplus_n \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$
3. $\Gamma\mathcal{P}(V) = \bigoplus_n (\mathcal{P}(n) \otimes V^{\otimes n})^{\mathbb{S}_n}$ (B.Fresse)

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Given an operad of some type, which type is the Koszul dual?

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The strategy: look for the free object, discard the variable
ex: $T(V)$ gives As

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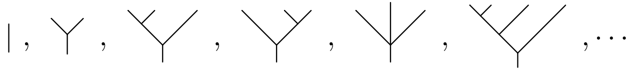

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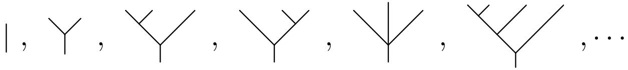

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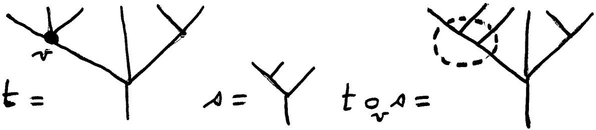

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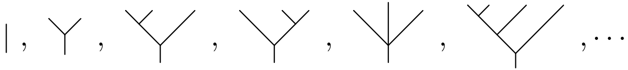

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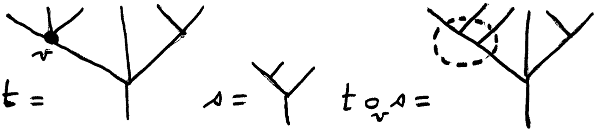

Key point: substitution is associative.

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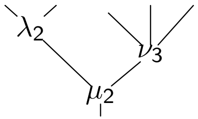
$\mathbb{PT}(M)_n =$ span of the pbr trees with n leaves decorated by M



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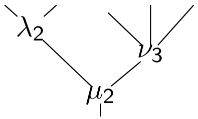

THM The substitution process defines a monoid structure $\Gamma : \mathbb{PT} \circ \mathbb{PT} \to \mathbb{PT}$ on the endofunctor $\mathbb{PT}$, hence $(\mathbb{PT}, \Gamma)$ is a monad on the category of $\mathbb{N}$-modules.

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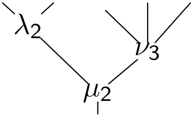

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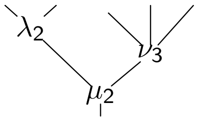

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PROP $\mathbb{PT}(M)$ is the free ns operad on M

Other examples of types of operads

Construction of the free object:

- ns operads use planar rooted trees
- symmetric operads use nonplanar rooted trees
- cyclic operads use nonplanar nonrooted trees
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Question: what is the general framework for all these examples?

III. Combinatorial patterns and combinads (work in progress)

Definition of a **combinatorial pattern** $\mathbb{X}$ over $\mathbb{N}$:
X is a set, whose elements are called *trees* (abuse of terminology)
Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$
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We assume **associativity** of this composition, that is
I. sequential axiom, for $w \in \mathrm{vert}(s)$: $(t \circ_v s) \circ_w r = t \circ_v (s \circ_w r)$
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In the planar case all the sets $\mathrm{leaves}(t), \mathrm{in}(v)$ are completely determined by their cardinality $n \in \mathbb{N}$.

Definition of a combinad

A *combinad* is a monad on $\mathbb{N}$-modules $(\mathcal{X}, \Gamma)$

$$\mathcal{X} : \mathbb{N}\text{-}mod \to \mathbb{N}\text{-}mod \quad , \quad \Gamma : \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$$

where $\mathcal{X}$ is induced by a given combinatorial pattern $\mathbb{X}$

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Related to rewriting of polygraphs (work in progress with Ph.Malbos and Y.Guiraud)

Algebras	Operads	Combinads
(associative alg) $T(V)$, $U(\mathfrak{g})$, $S(V)$ …	(ns operads) As	
(dendriform alg) $PBT(V)$, $T^{sh}(V)$, …	$Dend$	$\mathbb{PT}$
(whatever alg) …	…	
(Lie alg) $Lie(V)$, $\mathfrak{g}$, sl_n …	(symm operads) Lie	
(commutative alg) $S(V)$, $K[x,y]/\sim$, …	Com	$\mathbb{T}$
(associative alg) $T(V)$, ⋯	Ass	
…	…	
…	(permutads) $PermAs$	
…	q-$permAs$	$\mathbb{SM}$
…	…	
…	(shuffle operads) Ass	
…	Com	
…	q-$permAs$	$Sh\mathbb{T}$
…	$ShAs$	

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$\mathbb{X}_n =$ surjective maps $t : \underline{n} \to \underline{k}$
vertices of t: the elements of $\underline{k}$ (shown as $\times$)
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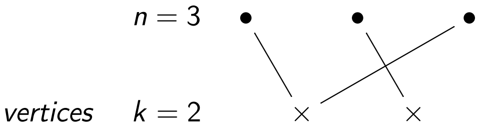

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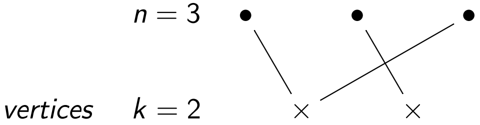

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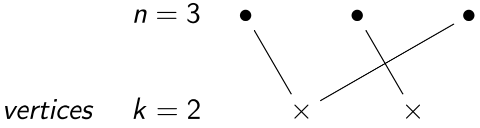

Substitution: given by composition of surjective maps

An algebra over this combinad is called a *permutad*.
This is the *combinatorial definition* of a permutad.

Permutad vs shuffle algebra

Prop A permutad is an $\mathbb{N}$-module $\mathcal{P}$ equipped with linear maps

$$\bullet_\gamma : \mathcal{P}_{n+1} \otimes \mathcal{P}_{m+1} \to \mathcal{P}_{n+m+1}, \text{ for } \gamma \in Sh(n, m),$$

verifying:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda$ in $Sh(n, m, r)$.

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This is the *partial definition* of a permutad. This is essentially the definition given by M. Ronco of a *shuffle algebra*, which are in fact *colored algebras*.

`M. Ronco, Shuffle bialgebras, Ann.Inst.Fourier 61 (2011), 799-850.`

PermAs and the permutohedron

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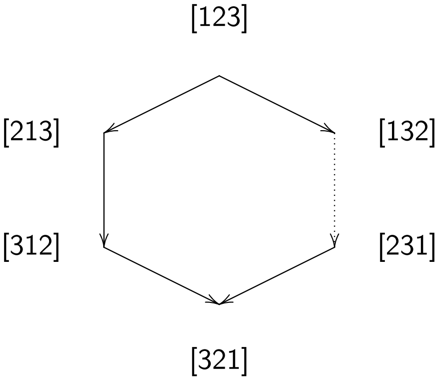
Key lemma: connectedness of a subgraph of the weak Bruhat order graph:



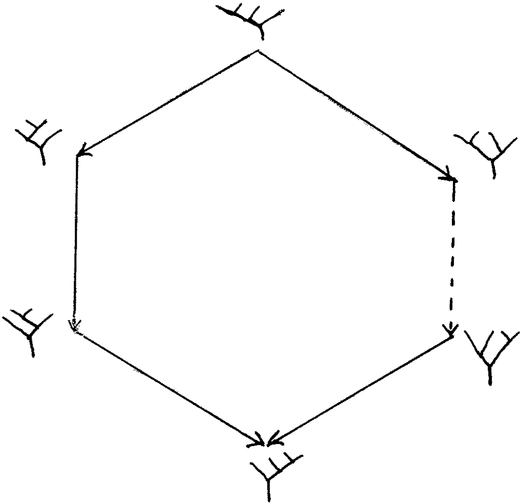
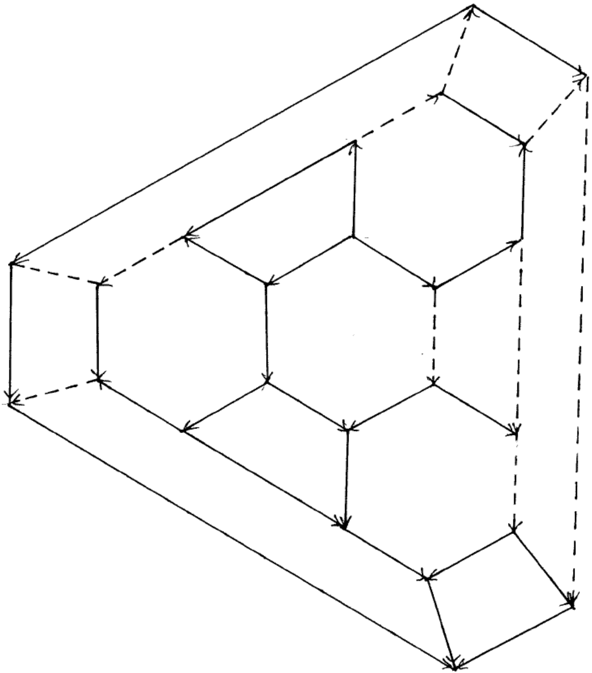


Figure: P^2 and trees



Minimal model of *PermAs*

- Minimal model of the ns operad *As* is A_∞ where

$$(A_\infty)_n = C_*(\text{associahedron})$$

- Minimal model of the permutad *PermAs* is $PermAs_\infty$ where

$$(PermAs_\infty)_n = C_*(\text{permutohedron})$$

Minimal model of *PermAs*

- Minimal model of the ns operad *As* is A_∞ where

$$(A_\infty)_n = C_*(\text{associahedron})$$

- Minimal model of the permutad *PermAs* is $PermAs_\infty$ where

$$(PermAs_\infty)_n = C_*(\text{permutohedron})$$

PROP The permutad with one binary operation and relation

$$(xy)z = q\ x(yz)$$

is Koszul for any q.
(In the operad case, only for $q = 0, 1, \infty$)

V. Further research (work in progress):

a) Feynman diagrams

One can construct a combinad from finite graphs with various decorations **stable by substitution**,
for instance Feynman graphs (QED, φ^4)
(work in progress JLL-N.Nikolov related to vertex algebras)

V. Further research (work in progress):

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Symmetry choice in a combinatorial pattern at vertices with $n+1$ flags:

- planar rooted $\Rightarrow$ symmetry group $= \{1\}$,
- planar nonrooted $\Rightarrow$ symmetry group $= C_{n+1}$ (cyclic group),
- nonplanar rooted $\Rightarrow$ symmetry group $= \mathbb{S}_n$,
- nonplanar nonrooted $\Rightarrow$ symmetry group $= \mathbb{S}_{n+1}$,
- p inputs, q outputs $\Rightarrow$ symmetry group $= \mathbb{S}_p \times \mathbb{S}_q$.

More general combinatorial patterns

Let $\mathbb{Y}$ be a combinatorial pattern, for instance $\bullet$, $\mathbb{N}$ (ladders). A combinatorial pattern $\mathbb{X}$ over $\mathbb{Y}$ is

a set X of elements such that each $t \in X$ has an underlying set $|t| \in Y$ (replaces n)

Any $t \in X$ comes with its set of *vertices* $v \in \mathrm{vert}(t)$ and its set of *leaves* $\mathrm{leav}(t) \in Y$

Any vertex v comes with its set of *inputs* $\mathrm{in}(v) \in Y$

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For any trees t, s and $v \in \mathrm{vert}(t)$ and an isomorphism $\mathrm{in}(v) \cong \mathrm{leav}(s)$ there is given a new tree denoted $t \circ_v s$ such that $|t \circ_v s| = |t|$, $\mathrm{vert}(t \circ_v s) = (\mathrm{vert}(t) \backslash \{v\}) \cup \mathrm{vert}(s)$

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We assume **associativity** of this composition, that is

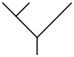
I. sequential axiom, for $w \in \mathrm{vert}(s)$: $(t \circ_v s) \circ_w r = t \circ_v (s \circ_w r)$

II. parallell axiom, for $w \in \mathrm{vert}(t)$: $(t \circ_v s) \circ_w r = (t \circ_w r) \circ_v s$

Combinatorics

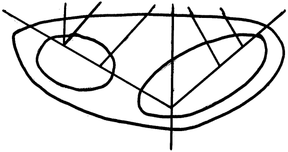
ladder = Dynkin diagram A_n

How to continue the sequence:

$k = 0$	$k = 1$	$k = 2$	$k = 3$
Mod	$\mathbb{N}$-mod	$\mathbb{PT}$-mod	$\mathbb{CPT}$-mod
•			(see picture later)
	ladder	circled ladder = tree	circled tree

Free combinad

The combinatorial pattern is made of **circled trees**



leaves:	elements	in	Algebras
vertices:	operations	in	Operads
circles:	compositions	in	Combinads

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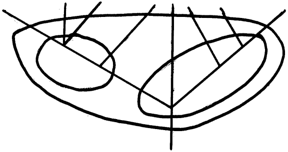

leaves:	elements	in	Algebras
vertices:	operations	in	Operads
circles:	compositions	in	Combinads

Substitution: as usual

(related to the dendroidal sets of Ieke Moerdijk and Ittay Weiss)

b) higher operads: opetopes

Alg=0-op	Op=1-op	Comb=2-op	$\cdots$
			$\cdots$
		$\mathbb{CPT}(W)$	$\mathbb{CPT}$
	$Mag(M) = \mathbb{PT}(M)$	$\mathbb{PT}$	
$T(V) = As(V)$	$\mathrm{As}=\{\mathrm{As}_n\}_{n\geq 1}$		

$V = \bullet$-module $\quad \bullet$

$M = \mathbb{N}$-module $\quad \mathbb{N}$  comb. pattern over $\bullet$

$W = \mathbb{PT}$-module $\quad \mathbb{PT}$ comb. pattern over $\mathbb{N}$

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etc. comb. pattern over $\mathbb{CPT}$

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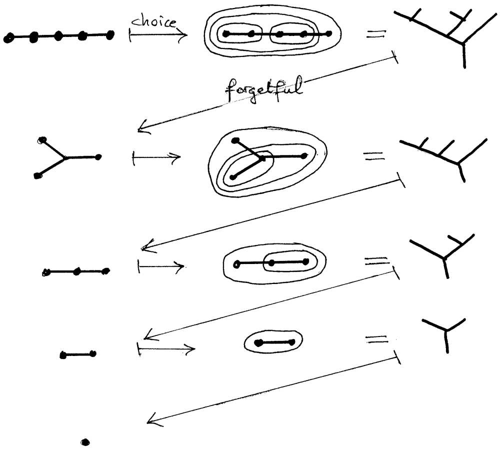
etc.   comb. pattern over $\mathbb{CPT}$

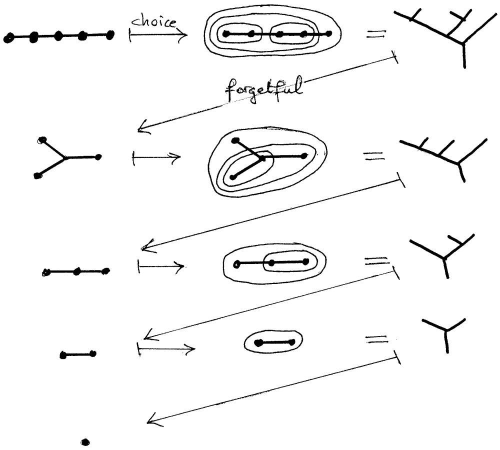
$T(V), Mag(M), \mathbb{CPT}(W)$ are free objects,
As, $\mathbb{PT}$ are associative objects

A tower of binary trees



graphs circled graphs rooted trees

A tower of binary trees = binary opetope



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Opetopes

Similar objects already appeared in the literature in the work:

`Baez, J.C.; Dolan, J., Higher-dimensional algebra. III. n-categories and the algebra of opetopes. Adv. Math. 135 (1998)`

under the name **opetopes**.

Opetopes

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under the name **opetopes**.
Not surprizing since the philosophy is the same, except that Baez, Dolan (and then Joyal, Kock, Batanin, etc.) work in the set environment, while we are working in the linear environment.
Main differences: a set is a co-object, because of the diagonal (duplication is possible),
in the linear case there is more freedom (*Lie* is not set-theoretic).

Quantum groups

Number of binary opetopes $a(n)$, for A_n, :

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144		

$$10 = 5 \times 2, \quad 144 = (12 \times 5 + 2 \times 6) \times 2$$

Quantum groups

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n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144		

This is, up to $n = 5$, the number of regions of linearity for Lusztig's piecewise-linear function in type A_{n-1} appearing in the study the canonical basis of

$$U_q^-(sl_n)$$

Canonical bases for quantized enveloping algebras were introduced by George Lusztig and Masaki Kashiwara in the nineties.

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CONJECTURE The number of regions of linearity for A_6 is

$$1\,044\,736$$

Quantum groups

Number of binary opetopes $a(n)$, for A_n, $d(n)$, for D_n, etc.:

n	1	2	3	4	5	6	7
a(n)	1	1	2	10	144	6 608	1 044 736
d(n)	1	1	2	12	184	8 704	1 395 456

This is, up to $n = 6$, the number of regions of linearity for Lusztig's piecewise-linear function in type A_{n-1} appearing in the study the canonical basis of

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More: a helpful lemma in order to compute $a(n)$ and $d(n)$ is the following. A *partition* of the graph G is: I and J nonempty connected subgraphs of G such that each vertex is either in I or in J. Define $\varphi(G) =$ sum of "derived graphs" (forgetful $\circ$ choice)

Lemma For any graph G we have

$$\varphi(G) = \sum_{(I,J)} \varphi(I) \vee \varphi(J) \ ,$$

where the sum is over all partitions of G.

This decomposition is close to Carter's method to construct regions of linearity in

`Carter, R.W. Canonical bases, reduced words, and Lusztig's piecewise-linear function. ...(1997).`

Thanks to Richard Green and Robert Marsh for their help.

MERCI !