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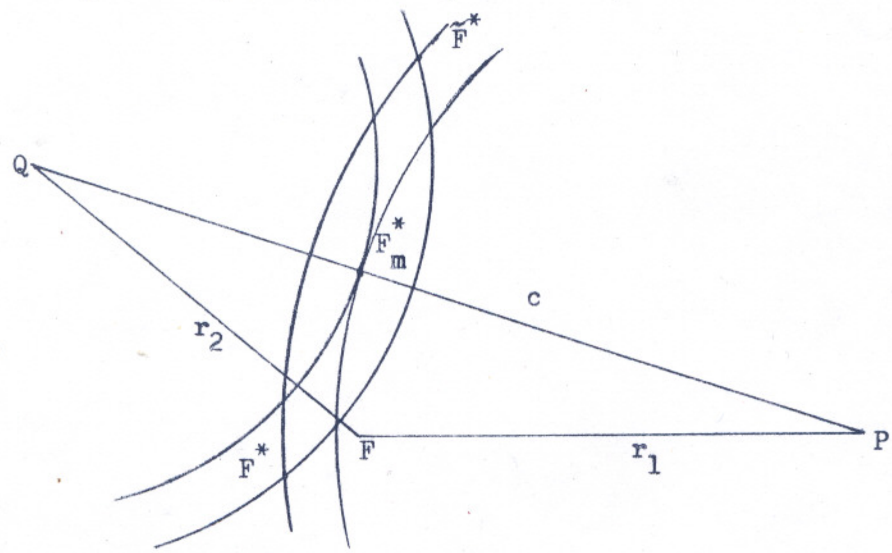
July 11, 1961

AUTHOR: M. Minovich

SUBJECT: An Alternative Method for the Determination of Elliptic and Hyperbolic Trajectories

DISTRIBUTION: Section 312 Engineers, J. F. Scott, W. Scholey

A bound particle moving in the gravitational field of a "stationary" celestial object will have an elliptic trajectory. Its period P is given by $P = 2\pi\sqrt{\frac{a^3}{\mu}}$ where $\mu = GM$; G being the gravitational constant, and M being the mass of the celestial object setting up the field. It is well known that if two points, P and Q, lie on a general conic trajectory, the time required for the particle to traverse the arc $\widehat{PQ}$ is dependent only on the semi-major axis a of the conic, $\overline{FP} + \overline{FQ}$ where F is a focus of the conic and the distance between P and Q. Let us denote $r_1 = \overline{FP}$, $r_2 = \overline{FQ}$ and $c = \overline{PQ}$. Consider the problem of finding an ellipse passing through two specified points, P and Q, and one specified focus F.



Now the definition of an ellipse can be stated as the locus of points the sum of whose distances from two fixed points (called foci) is constant. We may assume without loss of generality that $r_2 > r_1$. Thus if F^* is the other focus, it must satisfy the equations $\overline{PF^*} + r_1 = \overline{QF^*} + r_2 = 2a$. Consequently, if $\overline{PF^*} = 2a - r_1$ and $\overline{QF^*} = 2a - r_2$, F^* will be a second focus. These points are easily obtained by considering families of circles about P and Q with radii $2a - r_1$ and $2a - r_2$, respectively. The intersections of these families determines a set of pairs of points $(F^*, \tilde{F}^*)$ each of which can be the second focus. Consequently, there are two different ellipses which satisfy the conditions of the problem. It is clear that if the radii $2a-r_1$, $2a-r_2$ are too small, the circles will not intersect. Hence, there exists a minimum value of a, say a_m, such that the circles intersect in only one point. Since this intersection must occur on $\overline{PQ}$, we have $2a_m - r_1 + 2a_m - r_2 = c$. Thus letting $\frac{1}{2}(r_1 + r_2 + c) = s$, we have $2a_m = \frac{1}{2}(r_1 + r_2 + c) = s$. Since the kinetic energy of the particle at P of unit mass is $\mu(\frac{1}{r} - \frac{1}{2a})$ where $r = \overline{FP}$, it is clear that this will be minimum if $a = a_m$. Thus the unique ellipse, having a semi-major axis of $a = a_m$, can be called a minimum energy ellipse.

It can be shown (see R. Battin, The Determination of Round-Trip Planetary Reconnaissance Trajectories; ARS Journal/Space Sciences; pages 550-52) that when the vacant foci is at F^*, the time T required for the particle to traverse the elliptic arc $\widehat{PQ}$ is

(1) $$T = \frac{1}{2\pi} \underline{P} \left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$

where $\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$, $\sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$. If the vacant focus is at $\tilde{F}^*$, the time T can be expressed as

(2) $$\tilde{T} = \underline{P} - \frac{1}{2\pi} \underline{P} \left[(\alpha - \sin\alpha) + (\beta - \sin\beta)\right]$$

If we set $x_1 = 1 - \frac{s}{a}$, $x_2 = 1 - \frac{s-c}{a}$ and make use of the trigonometric identities

$$\sin\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1 - \cos\theta)} \quad \text{and} \quad \sin(\cos^{-1}x) = \sqrt{1 - x^2}, \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

equations (1) and (2) can be expressed as

(3) $$T = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1 - x_2^2} + \sin^{-1} x_2 - \sqrt{1 - x_1^2} - \sin^{-1} x_1 \right\}$$

(4) $$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \left\{ \pi + \sqrt{1 - x_2^2} + \sin^{-1} x_2 + \sqrt{1 - x_1^2} + \sin^{-1} x_1 \right\}$$

If $a = a_m$ the two ellipses are coincident and $T = \tilde{T}$. This can be shown analytically by substituting $a = a_m = \frac{1}{2}s$ in (3) and (4) noting that in this case $x_1 = 1 - \frac{s}{\frac{s}{2}} = -1$ and $\sin^{-1}(-1) = -\sin^{-1} 1 = -\frac{\pi}{2}$. Thus it is clear from (3) and (4) that $T \leq \tilde{T}$ where the equality holds only when $a = a_m$. Let the expressions on the right side of equations (3) and (4) be denoted by $f(a)$ and $\tilde{f}(a)$, respectively, so that $T = f(a)$ and $\tilde{T} = \tilde{f}(a)$. Omitting the details, one easily finds

$A = \frac{1-x_1}{1+x_1}$

$B = \frac{1-x_2}{1+x_2}$

(5) $$\frac{df}{da} = \frac{3}{2}\,\frac{f(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} - s\sqrt{\frac{1-x_1}{1+x_1}}\right\}$$

(6) $$\frac{d\tilde{f}}{da} = \frac{3}{2}\,\frac{\tilde{f}(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} + s\sqrt{\frac{1-x_1}{1+x_1}}\right\}$$

(7) $$\frac{d^2f}{da^2} = \frac{3}{4}\,\frac{f(a)}{a} + \frac{1}{\sqrt{a^3\mu}}\left\{(s-c)\sqrt{\frac{1-x_2}{1+x_2}} - s\sqrt{\frac{1-x_1}{1+x_1}} + \frac{s\,A}{\sqrt{1-x_1^2}} - \frac{s-c}{\sqrt{1-x_2^2}}\,B\right\}$$

Now it is clear on physical grounds that $f(a) > 0$ and $\tilde{f}(a) > 0$. From (5) $f'(a) \to -\infty$ as $a \to a_m$ and from (6) $\tilde{f}'(a) \to +\infty$ as $a \to a_m$. Hence the two curves (T, a) and $(\tilde{T} \cdot a)$ are joined at $a = a_m$ such that the total curve C has a well defined tangent line for all values of a where f and $\tilde{f}$ are defined (i.e., $a_m < a$). In order to simplify an analytical investigation of C let us consider values of a in the closed interval $a_m \leqq a \leqq r_1 + r_2$. Since $a_m = \frac{1}{2}s = \frac{1}{4}(r_1 + r_2 + c)$ and $c \leqq r_1 + r_2$, this interval can be expressed as $\frac{1}{2}(r_1 + r_2) \leqq a \leqq r_1 + r_2$. From equation (6) since $s - c = \frac{1}{2}(r_1 + r_2 - c) \geqq 0$ $\tilde{f}'(a) > 0$. Thus this upper half of C increases with a. For the lower half of C where T is given by f(a) it is convenient to consider $\frac{d^2f}{da^2}$. Consider the expression

(8) $$\frac{s-c}{s}\cdot\frac{\sqrt{\frac{1-x_2}{1+x_2}} - \frac{1}{\sqrt{1-x_2^2}}}{\sqrt{\frac{1-x_1}{1+x_1}} - \frac{1}{\sqrt{1-x_1^2}}}$$

Since $x_1 = 1 - \frac{s}{a}$ and $x_2 = 1 - \frac{s-c}{a}$ we obtain $s = a(1 - x_1)$ and $s - c = a(1 - x_2)$. Thus (8) may be written as

$$\frac{1-x_2}{1-x_1}\cdot\frac{\sqrt{\frac{1-x_2}{1+x_2}} - \frac{1}{\sqrt{1-x_2}\,\sqrt{1+x_2}}}{\sqrt{\frac{1-x_1}{1+x_1}} - \frac{1}{\sqrt{1-x_1}\,\sqrt{1+x_1}}} = \frac{1-x_2}{1-x_1}\cdot\frac{\sqrt{1-x_2} - \frac{1}{\sqrt{1-x_2}}}{\sqrt{1-x_1} - \frac{1}{\sqrt{1-x_1}}}\cdot\frac{\sqrt{1+x_1}}{\sqrt{1+x_2}}$$

(9) $$= \frac{1-x_2}{1-x_1}\cdot\frac{1-x_2-1}{1-x_1-1}\cdot\frac{\sqrt{1-x_1}}{\sqrt{1-x_2}}\cdot\frac{1+x_1}{1+x_2} = \frac{x_2}{x_1}\cdot\frac{1-x_2}{1-x_1}\cdot\frac{\sqrt{1-x_1^2}}{\sqrt{1-x_2^2}}$$

Now $1 - x_2^2 = 1 - (1 - \frac{s-c}{a})^2 = 1 - (x_1 + \frac{c}{a})^2 = 1 - x_1^2 - \frac{c}{a}(2x_1 + \frac{c}{a})$.

$2x_1 + \frac{c}{a} = 2 - \frac{2s}{a} + \frac{c}{a} = 2 - \frac{r_1 + r_2 + c}{a} + \frac{c}{a} = 2 - \frac{r_1 + r_2}{a}$. Hence

$\max\,(2x_1 + \frac{c}{a}) = 2 - \frac{r_1 + r_2}{r_1 + r_2} = 1$, $\min\,(2x_1 + \frac{c}{a}) = 2 - \frac{r_1 + r_2}{\frac{r_1 + r_2}{2}} = 0$

Thus $2x_1 + \frac{c}{a} \geqq 0$ and we conclude that

$$(10) \qquad \sqrt{\frac{1 - x_1^2}{1 - x_2^2}} = \sqrt{\frac{1 - x_1^2}{(1-x_1^2) - \frac{c}{a}(2x_1 + \frac{c}{a})}} \geqq 1$$

From $1 \geqq 2 - \frac{1}{a}(r_1 + r_2) \geqq 0$ we write using above results

$$1 \geqq 2 - \frac{1}{a}(2s - c) = 1 - \frac{s-c}{a} + 1 - \frac{s}{a} = x_2 + x_1$$

$$\therefore \quad x_2 - x_1 \geqq (x_2 + x_1)(x_2 - x_1) = x_2^2 - x_1^2$$

$$\therefore \quad x_2 - x_2^2 \geqq x_1 - x_1^2$$

$$\therefore \quad x_2(1 - x_2) \geqq x_1(1 - x_1)$$

$$\therefore \quad \frac{x_2}{x_1} \cdot \frac{1 - x_2}{1 - x_1} \geqq 1$$

With this result and (10) we obtain, since (8) is equal to (9), the important inequality

$$(s - c)\sqrt{\frac{1 - x_2}{1 + x_2}} - (s - c)\frac{1}{\sqrt{1-x_2^2}} \geqq s\,\frac{1-x_1}{1+x_1} - s\,\frac{1}{\sqrt{1 - x_1^2}}$$

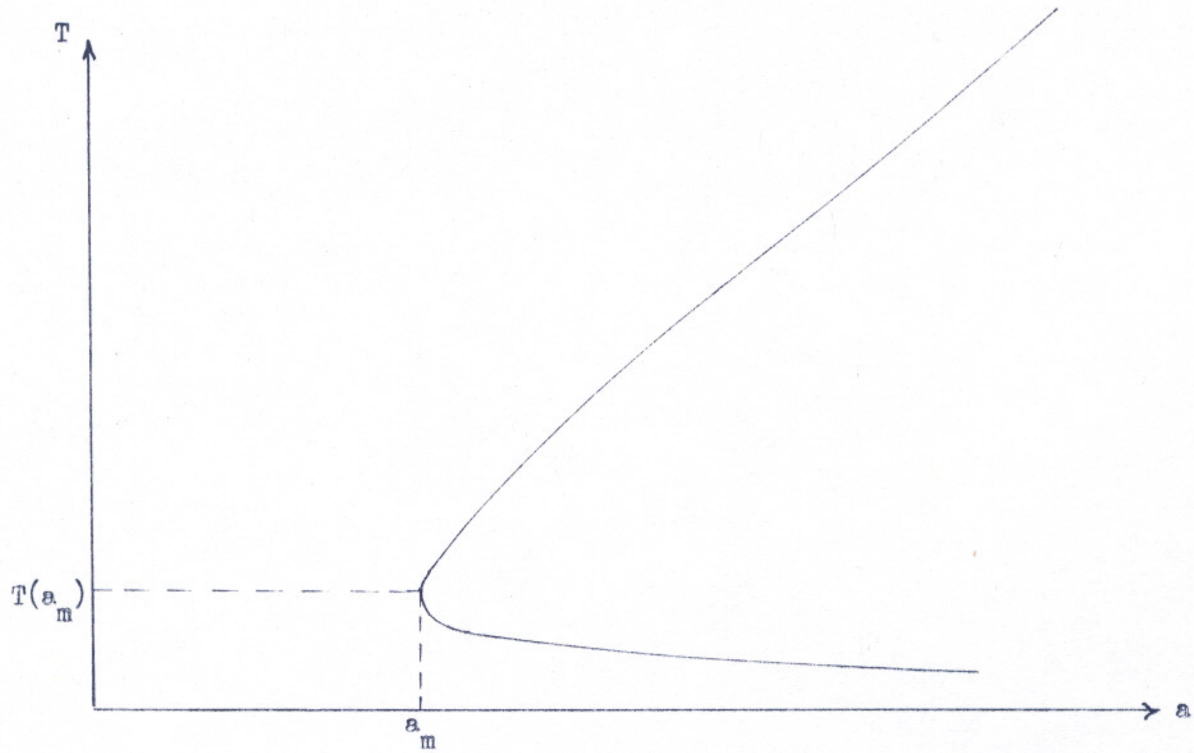
which can be written as

$$\left\{(s - c)\sqrt{\frac{1 - x_2}{1 + x_2}} - s\sqrt{\frac{1 - x_1}{1 + x_1}} + \frac{s}{\sqrt{1-x_1^2}} - \frac{s - c}{\sqrt{1-x_2^2}}\right\} \geqq 0$$

Employing this result in equation (7) we find

$$\frac{d^2f}{da^2} > 0 \qquad a_m \leqq a \leqq r_1 + r_2$$

Since $\frac{df}{da} \rightarrow -\infty$ as $a \rightarrow a_m$ we may now conclude that the lower half of C for values of a in $a_m \leqq a \leqq r_1 + r_2$ will be convex from below. Thus the curve C will take on the general shape of



Suppose a particle is moving in an elliptic trajectory about a gravitating body of mass M (i.e., the particle is in free fall motion but bound in an orbit about the body). If one specifies two points $\vec{r}_1 = \overrightarrow{OP}$ and $\vec{r}_2 = \overrightarrow{OQ}$, which lies on the trajectory, and the time T taken for the particle to pass from $\vec{r}_1$ to $\vec{r}_2$, one--and only one--trajectory exists which satisfies these conditions, (provided T is greater than some minimum value T_o). We consider two possible cases:

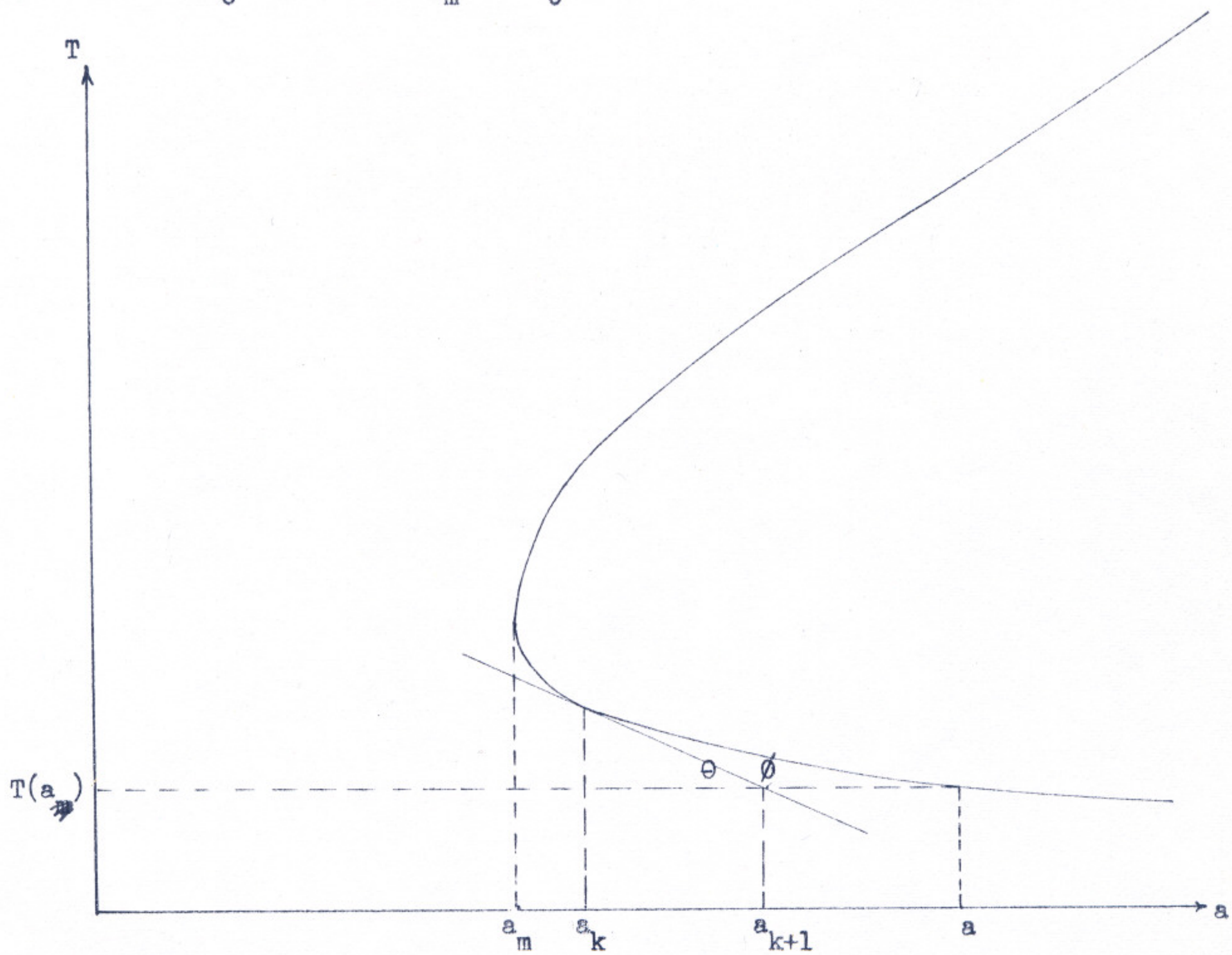
(i) $T < T(a_m)$ (ii) $T > T(a_m)$

If $T < T(a_m)$ we consider two sub-cases:

(a) $T(a_m) > T > T(r_1 + r_2)$

(b) $T(r_1 + r_2) > T$

In practice, it turns out that case (i) is more important since short flight times are desirable. Now as a increases, the kinetic energy of the particle increases, hence the sub-case (b) above will be unlikely. Consequently, we consider a method which, for case (a), will always yield a sequence $\{a_k\}$ converging to the desired value a corresponding to the prescribed flight time T. First, choose an initial value of a, say a_o, such that $T(a_m) > T(a_o) > T$.



By the figure it is evident that

$$\frac{f(a_k) - T}{a_{k+1} - a_k} = \tan \theta = \tan(\pi - \phi) = -\tan \phi = -f'(a_k)$$

Hence

$$\frac{f(a_k) - T}{f'(a_k)} = a_k - a_{k+1}$$

or

(11) $$a_{k+1} = a_k - \frac{f(a_k) - T}{f'(a_k)}$$

The sequence $\{a_k\}$ will necessarily converge to the desired value a because of the convexity of the lower half of C in the interval $a_m \leqq a \leqq r_1 + r_2$. If case (b) is true, one may still apply (11) if it is found by investigating the sign of f" in a neighborhood of a that the method yielding $\{a_k\}$ will be convergent. In a similar manner, it is easy to see that case ii presents no added difficulty and (11) may also be used to calculate the semi-major axis a by appropriate substitutions.

We now consider the error E_{k+1} in the k+1 th iterate. $E_{k+1} = |a - a_{k+1}|$. If $\{a_k\}$ is convergent to a then the following argument holds for either case (i) or (ii). In dealing with case (ii) one replaces f(a) by $\tilde{f}(a)$. Now by equation (11) we have

$$E_{k+1} = |a - a_{k+1}| = \left| a - a_k + \frac{f(a_k) - T}{f'(a_k)} \right|$$

But we may write

$$T = f(a) = f(a_k) + (a-a_k)\, f'(a_k) + \frac{1}{2!}(a-a_k)^2 f''(a_k) + \ldots$$

or

$$T = f(a_k) + (a-a_k)\, f'(a_k) + \tfrac{1}{2}(a-a_k)^2 f''(\zeta_k)$$

where ζ_k lies between a_k and a. Hence

$$\frac{T - f(a_k)}{f'(a_k)} = (a-a_k) + \tfrac{1}{2}(a-a_k)^2 \frac{f''(\zeta_k)}{f'(a_k)}$$

Thus

$$E_{k+1} = \left| (a-a_k) - (a-a_k) - \tfrac{1}{2}(a-a_k)^2 \frac{f''(\zeta_k)}{f'(a_k)} \right|$$

$$= \tfrac{1}{2}(a-a_k)^2 \left| \frac{f''(\zeta_k)}{f'(a_k)} \right|$$

Hence since $a_k \longrightarrow a$

(12) $$E_{k+1} \approx \tfrac{1}{2} E_k^2 \left| \frac{f''(a)}{f'(a)} \right|$$

Since $\frac{1}{2}\frac{f''(a)}{f'(a)}$ is a constant, this shows that the error in a_{k+1} is approximately proportional to the square of the error in a_k. Thus we should expect rapid convergence.

After determining the semi-major axis a with sufficient accuracy, the trajectory will be completely determined by finding the corresponding value of the eccentricity ϵ. This is obtained by making use of the dependence of ϵ on a, set up by the initial condition of requiring F to be a focus and P and Q to lie on the ellipse. It can be shown (see above reference, page 549) that if the second focus if F^* or $\tilde{F}^*$ the corresponding values of the latus rectum are given by

$$l = \left[\frac{4a}{c^2}(s - r_1)(s - r_2)\right] \sin^2 \frac{\alpha - \beta}{2}$$

$$\tilde{l} = \left[\frac{4a}{c^2}(s - r_1)(s - r_2)\right] \sin^2 \frac{\alpha - \beta}{2}$$

respectively. Making use of the relation $l = a(1 - \epsilon^2)$ and introducing x_1 and x_2 defined above, these equations can be written as

(13) $$\epsilon = \left\{1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1 x_2 + \sqrt{1-x_1^2}\sqrt{1-x_2^2}\right\}^{\frac{1}{2}}$$

(14) $$\tilde{\epsilon} = \left\{1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1 x_2 - \sqrt{1-x_1^2}\sqrt{1-x_2^2}\right\}^{\frac{1}{2}}$$

Thus if the given value of T is such that $T > T(a_m)$ then after determining a with sufficient accuracy by (11) with f(a) replaced by $\tilde{f}(a)$, the eccentricity $\tilde{\epsilon}$ of the elliptic orbit is given by (14). If $T < T(a_m)$ one uses (13) after finding a by (11).

Before considering hyperbolic trajectories, it is important to know that a solution of the above problem (having initial conditions F, $\vec{r}_1$, $\vec{r}_2$ and T prescribed) exists. Clearly if T and $\vec{r}_1$, $\vec{r}_2$ are chosen so that $\frac{c}{T}$ is sufficiently large then, since maximum $|\vec{V}| > \frac{T}{c}$ where $\vec{V}$ is the velocity of the particle, the particle may be required to have a kinetic energy such that it cannot be in a bound state. Now since this kinetic energy is given by $\mu(\frac{1}{r} - \frac{1}{2a})$, if $T < T(a_m)$ then as $a \to \infty$ the path $\widehat{PQ}$ is traversed such that $\lim_{a\to\infty} f(a) = T_o$ exists, and that $o < T_o < f(a)$ for all $a_m \leqq a < \infty$. Consequently, if the prescribed value of T is such that $T \leqq T_o$, no elliptic trajectory is possible and the solution of the above problem does not exist. This critical value T_o may be obtained by employing a device known as L'Hospital's rule. This rule for calculating limits states that

$$\lim_{t\to t_o} \frac{F(t)}{G(t)} = \lim_{t\to t_o} \frac{F'(t)}{G'(t)} \quad \text{if } F(t) \to 0 \text{ as } t \to t_o \text{ and } G(t) \to 0 \text{ as } t \to t_o.$$

If we make the change of variable $\frac{1}{t} = a^{\frac{1}{2}}$, f(a) becomes

$$\frac{1}{t^3\sqrt{\mu}}\left\{\sqrt{1 - x_2^2} + \sin^{-1} x_2 - \sqrt{1 - x_1^2} - \sin^{-1} x_1\right\}$$

where $x_1 = 1 - st^2$ and $x_2 = 1 - (s-c)t^2$. Hence as $a \to \infty$, $t \to 0 = t_o$ and we set

$$F(t) = \sqrt{1-x_2^2} + \sin^{-1} x_2 - \sqrt{1-x_1^2} - \sin^{-1} x_1$$

$$G(t) = \sqrt{\mu}\; t^3.$$

Thus $\lim_{a \to \infty} T = \lim_{t \to o} \frac{F'(t)}{G'(t)} = \frac{-2(s-c)\sqrt{\frac{1-x_2}{1+x_2}} + 2s\sqrt{\frac{1-x_1}{1+x_1}}}{3\sqrt{\mu}\; t}$

By a re-application of the rule we obtain

$$\lim_{a \to \infty} T = \lim_{t \to o} \frac{1}{3\sqrt{\mu}}\left\{\frac{s-c}{1+x_2}\frac{dx_2}{dt}\left[\frac{\sqrt{1+x_2}}{\sqrt{1-x_2}} + \frac{\sqrt{1-x_2}}{\sqrt{1+x_2}}\right] - \frac{s}{1+x_1}\frac{dx_1}{dt}\left[\frac{\sqrt{1+x_1}}{\sqrt{1-x_1}} + \frac{\sqrt{1-x_1}}{\sqrt{1+x_1}}\right]\right\}$$

$$= \lim_{t \to o}\left\{\frac{-4(s-c)^2\, t}{3\sqrt{\mu}(1+x_2)\sqrt{1-x_2^2}} + \frac{4s^2\, t}{3\sqrt{\mu}(1+x_1)\sqrt{1-x_1^2}}\right\}$$

Now $\lim_{t \to o} x_1 = \lim_{t \to o} x_2 = 1$ hence

$$\lim_{a \to \infty} T = \frac{-2(s-c)^2}{3\sqrt{\mu}} \cdot \lim_{t \to o} \frac{t}{\sqrt{1-x_2^2}} + \frac{2s^2}{3\sqrt{\mu}} \cdot \lim \frac{t}{\sqrt{1-x_1^2}}$$

Let $L_1 = \lim_{t \to o} \frac{t}{\sqrt{1-x_1^2}}$. Then

$$L_1 = \lim_{t \to o} \frac{1}{\frac{1}{2}(1-x_1^2)^{-\frac{1}{2}}(-2x_1)\frac{dx_1}{dt}} = \lim_{t \to o} \frac{\sqrt{1-x_1^2}}{2x_1 st}$$

$$= \frac{1}{2s} \lim_{t \to o} \frac{\sqrt{1-x_1^2}}{t} = \frac{1}{2sL_1}$$

Thus we obtain $L_1 = \frac{1}{\sqrt{2s}}$. In a similar manner we find letting

$$L_2 = \lim \frac{t}{\sqrt{1-x_2^2}}$$

$$L_2 = \frac{1}{\sqrt{2(s-c)}}$$

Consequently we obtain

$$T_o = \lim_{a \to \infty} T = \frac{2}{3} \cdot \frac{s^2}{\sqrt{\mu}} \cdot \frac{1}{\sqrt{2s}} - \frac{2}{3} \cdot \frac{(s-c)^2}{\sqrt{\mu}} \cdot \frac{1}{\sqrt{2(s-c)}}$$

$$\therefore \quad (15) \quad T_o = \frac{2}{3\sqrt{2\mu}}\left(\sqrt{s^3} - \sqrt{(s-c)^3}\right)$$

Hence if the prescribed T is such that $T \leqq \frac{2}{3\sqrt{2\mu}}\left(\sqrt{s^3} - \sqrt{(s-c)^3}\right)$ an elliptical trajectory will be impossible.

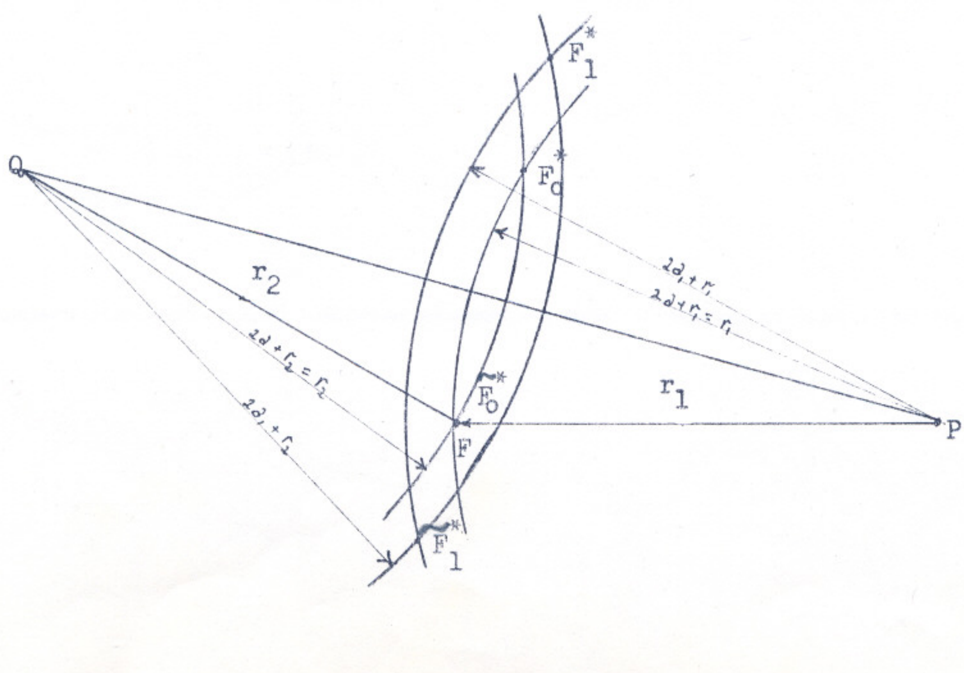
We turn now to the case when the prescribed value of T is such that an elliptic trajectory is impossible. That is to say when $T \leqq T_o$. In this case, we must consider hyperbolic trajectories. The trajectory will be parabolic if $T = T_o$. It will be shown that when the semi-major axis a of a hyperbolic trajectory (with vacant focus F*) increases indefinitely the corresponding time of flight approaches T_o, and the path becomes parabolic. We proceed as before specifying P and Q to lie on the path with $\vec{r}_1 = \overrightarrow{OP}$, $\vec{r}_2 = \overrightarrow{OQ}$ with O the center of an attractive body of mass M which, of course, is at a focus F of the hyperbolic path. Since the field is attractive, P and Q must both lie on the concave branch of the hyperbola with F its nearest focus. Hence if F* is a second focus

$$\overline{PF}^* - r_1 = \overline{QF}^* - r_2 = 2a,$$

according to the definition of a hyperbola which we take to be the locus of points the difference of whose distances from two fixed points (foci) is constant. Thus

$$\overline{PF}^* = 2a + r_1 \qquad \overline{QF}^* = 2a + r_2$$

Hence the vacant foci can be described as the intersection of families of circles about P and Q with radii $2a + r_1$ and $2a + r_2$, respectively.



These circles will intersect in two points (F_1^*, $\tilde{F}_1^*$). Unlike the elliptic case, the minimum value of a = 0. In this case one vacant focus $\tilde{F}_0^*$ coincides with F. The other $\tilde{F}_0^*$ is such that $\overline{PQ}$ bisects $\overline{FF_0^*}$ and hence the path $\widehat{PQ}$ is $\overline{PQ}$ and corresponds to an infinite velocity. The flight time T in this case must, of course, be 0. The path corresponding to $\tilde{F}_0^* = F$ is $\overline{PF}$ to $\overline{FQ}$. These cases, of course, are unrealized. Hence there exists two possible hyperbolic paths having the same semi-major axis a corresponding to the vacant foci F_1^* or $\tilde{F}_1^*$. We observe from the figure that the path with vacant focus at F_1^* has greater eccentricity ϵ than the eccentricity $\tilde{\epsilon}$ of the path with vacant focus at $\tilde{F}^*$. $\epsilon > \tilde{\epsilon}$

The time required to traverse the path $\widehat{PQ}$ when the vacant focus is at $\tilde{F}^*$ or F^* was expressed by Bottin as

$$T = \sqrt{\frac{a^3}{\mu}} \left[(\sinh \alpha - \alpha) - (\sinh \beta - \beta) \right]$$

$$\tilde{T} = \sqrt{\frac{a^2}{\mu}} \left[(\sinh \alpha - \alpha) + (\sinh \beta - \beta) \right]$$

where $\sinh \frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$, $\sinh \frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$, corresponding to paths having vacant focus at F^* or $\tilde{F}^*$, respectively. Thus since $\sqrt{\frac{s}{2a}} \geqq 0$ $\sqrt{\frac{s-c}{2a}} \geqq 0$, α, β, $\geqq 0$. Also since in this case $\sinh \alpha \geqq \alpha$, $\sinh \beta \geqq \beta$, it is clear that $T \leqq \tilde{T}$, which we expect by observing the figure. Employing the identities, $\sinh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x - 1)}$, $\cosh^2 x - \sinh^2 x = 1$ and $\sinh(\cosh^{-1} x) = \sqrt{x^2 - 1}$ for $x > 1$, the above expressions can be written as

$$T = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 - \sqrt{y_2^2 - 1} + \cosh^{-1} y_2 \right]$$

$$\tilde{T} = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 + \sqrt{y_2^2 - 1} - \cosh^{-1} y_2 \right]$$

where $y_1 = 1 + \frac{s}{a}$, $y_2 = 1 + \frac{s-c}{a}$. Let the right-hand sides of these equations be denoted by h(a) and $\tilde{h}(a)$, respectively. Thus

$$(16) \quad h(a) = \sqrt{\frac{a^3}{\mu}} \left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 - \sqrt{y_2^2 - 1} + \cosh^{-1} y_2 \right] = T$$

(17) $$\tilde{h}(a) = \sqrt{\frac{a^3}{\mu}}\left[\sqrt{y_1^2 - 1} - \cosh^{-1} y_1 + \sqrt{y_2^2 - 1} - \cosh^{-1} y_2\right] = \tilde{T}$$

Omitting the details, we find

(18) $$\frac{dh(a)}{da} = h'(a) = \frac{3}{2}\cdot\frac{h(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}$$

(19) $$\frac{d\tilde{h}(a)}{da} = \tilde{h}'(a) = \frac{3}{2}\cdot\frac{\tilde{h}(a)}{a} + \frac{1}{\sqrt{a\mu}}\left\{-(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}$$

We now consider the limits of the equations as $a \to 0$ and $a \to \infty$. In doing this we shall make use of the expansion for $\cosh^{-1}x$,

$$\cosh^{-1}x = \log 2x - \frac{1}{2}\cdot\frac{1}{2x^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4x^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{6x^2} - \dots \quad (x>1)$$

$$\lim_{a\to 0} h(a) = \lim\left\{a\sqrt{a}\cdot\sqrt{\left(1+\frac{s}{a}\right)^2 - 1}\right\} + \lim\left\{a\sqrt{a}\left(\cosh^{-1}y_2 - \cosh^{-1}y_1\right)\right\}$$

$$- \lim\left\{a\sqrt{a}\cdot\sqrt{\left(1+\frac{s-c}{a}\right)^2 - 1}\right\}$$

$$= \lim\left\{\sqrt{a}\cdot\sqrt{(a+s)^2 - a^2}\right\} + \lim\left\{a\sqrt{a}\left\{\log\frac{y_2}{y_1}\right.\right.$$

$$\left.\left. + \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right) + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}\left(\frac{1}{y_1^4} - \frac{1}{y_2^4}\right) + \dots\right\}\right\}$$

$$- \lim\left\{\sqrt{a}\cdot\sqrt{(a+s-c)^2 - a^2}\right\}$$

$$= 0 + \lim\left\{a\sqrt{a}\log\frac{y_2}{y_1}\right\} - 0 \quad \text{since}$$

$$y_1 = 1 + \frac{s}{a} \to \infty \text{ and } y_2 = 1 + \frac{s-c}{a} \to \infty \quad \text{as } a \to 0$$

$$\lim_{a\to 0}\frac{y_2}{y_1} = \lim_{a\to 0}\frac{1+\frac{s-c}{a}}{1+\frac{s}{a}} = \lim\frac{a+s-c}{a+s} = \frac{s-c}{s}. \quad \text{Hence}$$

$$\lim_{a\to 0} h(a) = 0 \quad \text{as expected.}$$

From the above results we may write

$$\lim_{a\to 0}\tilde{h}(a) = -\lim\left\{a\sqrt{a}\left(\log 2y_1 + \log 2y_2\right)\right\}$$

$$= -\lim\left\{a\sqrt{a}\log 4y_1 y_2\right\}$$

$$= -\lim\left\{a\sqrt{a}\log y_1 y_2\right\}$$

Now $\lim y_1 y_2 = \lim\left\{\left(1 + \frac{s}{a}\right)\left(1 + \frac{s-c}{a}\right)\right\} = \lim \frac{s(s-c)}{a^2}$. Hence

$$\lim_{a\to o} \tilde{h}(a) = -\lim\left\{a\sqrt{a}\ \log \frac{s(s-c)}{a^2}\right\} = \lim\left\{a\sqrt{a}\ \log a^2\right\}$$

$$= 2\lim\left\{\sqrt{a}\ \log a^a\right\}$$ But $a^a \to 1$ as $a \to o$. Consequently

$\lim_{a\to o} \tilde{h}(a) = 0$ as expected.

We now find $\lim_{a\to o} h'(a)$

$$\lim_{a\to o} h'(a) = \frac{3}{2}\lim_{a\to o} \frac{h(a)}{a} + \lim\left\{\frac{1}{\sqrt{a\mu}}\left\{(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right\}\right\}$$

$$= \frac{3}{2}\left\{\lim\left[\sqrt{\frac{a}{\mu}}\left[\sqrt{y_1^2-1} - \cosh^{-1} y_1 - \sqrt{y_2^2-1} + \cosh^{-1} y_2\right]\right]\right\}$$

$$+ \lim\left\{\frac{1}{\sqrt{a\mu}}\left[(s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}}\right]\right\}$$

$$= \frac{3}{2}\left[\lim\sqrt{\frac{a}{\mu}}\left(\frac{\sqrt{(a+s)^2 - a^2}}{a} - \frac{\sqrt{(a+s-c)^2 - a^2}}{a}\right)\right]$$

$$+ \lim \frac{1}{\sqrt{a\mu}}\cdot(s - c - s)$$ since

$\lim_{a\to o} \sqrt{a}\ (\cosh^{-1} y_2 - \cosh^{-1} y_1) = 0$ as shown above and $\frac{y_2 - 1}{y_2 + 1} \to 1$,

$\frac{y_1 - 1}{y_2 + 1} \to 1$ as $a \to o$

$$\therefore \quad \lim_{a\to o} h'(a) = \lim\left[\frac{\frac{3}{2}}{\sqrt{a\mu}}\,(s - (s-c)) + \frac{1}{\sqrt{a\mu}}\,(s - c - s)\right]$$

$$= \frac{1}{2}\lim_{a\to o} \frac{c}{\sqrt{a\mu}} = +\infty$$

Since $\lim_{a\to o} \tilde{h}(a) = 0$ and $\tilde{h}(a) \leqq h(a)$, $\lim_{a\to o} h'(a) = +\infty$ implies

$$\lim_{a\to o} \tilde{h}'(a) = +\infty$$

We now compute $\lim_{a\to\infty} h(a)$. Let $\frac{1}{t} = a^{\frac{1}{2}}$. Then

$$\lim_{a\to\infty} h(a) = \lim_{t\to o}\left\{\frac{1}{\sqrt{\mu}}\ t^3\left[\sqrt{y_1^2-1} - \cosh^{-1} y_1 - \sqrt{y_2^2-1} + \cosh^{-1} y_2\right]\right\}$$

Employing L'Hospital's rule with

$$F(t) = \sqrt{y_1^2 - 1} - \cosh^{-1} y_1 - \sqrt{y_2^2 - 1} + \cosh^{-1} y_2$$

$$G(t) = \mu\sqrt{t^3}$$

$$\lim_{a\to\infty} h(a) = \lim_{t\to o} \frac{\frac{dF}{dt}}{\frac{dG}{dt}} = \lim \frac{\frac{1}{2}(y^2{}_1-1)^{-\frac{1}{2}} \cdot 2y_1 \frac{dy_1}{dt} - \sqrt{y^2{}_1-1}\frac{dy_1}{dt} - \frac{1}{2}(y^2{}_2-1)^{-\frac{1}{2}} \cdot 2y_2 \frac{dy_2}{dt} + \frac{1}{\sqrt{y^2{}_2-1}}\frac{dy_2}{dt}}{3\sqrt{\mu}\ t^2}$$

$$= \lim\left(2s\sqrt{\frac{y_1-1}{y_1+1}} - 2(s-c)\sqrt{\frac{y_2-1}{y_2+1}}\right)\frac{1}{3\sqrt{\mu}\ t}$$

$$= \lim \frac{\frac{d}{dt}\left\{2s\sqrt{\frac{y_1-1}{y_1+1}} - 2(s-c)\sqrt{\frac{y_2-1}{y_2+1}}\right\}}{\frac{d}{dt}(3\sqrt{\mu}\ t)}$$

$$= \lim_{t\to o} \frac{\frac{s}{y_1+1}\frac{dy_1}{dt}\left\{\sqrt{\frac{y_1+1}{y_1-1}} - \sqrt{\frac{y_1-1}{y_1+1}}\right\} - \frac{s-c}{y_2+1}\frac{dy_2}{dt}\left\{\sqrt{\frac{y_2+1}{y_2-1}} - \sqrt{\frac{y_2-1}{y_2+1}}\right\}}{3\sqrt{\mu}}$$

$$= \lim_{t\to o} \frac{1}{3\sqrt{\mu}}\left\{\frac{2s^2t}{y_1+1}\left(\frac{2}{\sqrt{y^2{}_1-1}}\right) - \frac{2(s-c)^2t}{y_2+1}\left(\frac{2}{\sqrt{y^2{}_2-1}}\right)\right\}$$

$$= \frac{1}{3\sqrt{\mu}}\left\{2s^2 \lim_{t\to o}\frac{t}{\sqrt{y^2{}_1-1}} - 2(s-c)^2 \lim_{t\to o}\frac{t}{\sqrt{y^2{}_2-1}}\right\}$$

$$= \frac{1}{3\sqrt{\mu}}\left\{2s^2 L_1 - 2(s-c)^2 L_2\right\}$$

where $L_1 = \lim_{a\to o}\frac{t}{\sqrt{y^2{}_1-1}}$, $\quad L_2 = \lim_{a\to o}\frac{t}{\sqrt{y^2{}_2-1}}$

$$\therefore\ L_1 = \lim \frac{1}{\frac{1}{2}(y^2{}_1-1)^{-\frac{1}{2}}\, 2y_1\frac{dy_1}{dt}}$$

$$= \lim \frac{\sqrt{y^2{}_1-1}}{y_1\cdot 2st} = \frac{1}{2s}\lim\frac{\sqrt{y^2{}_1-1}}{t} = \frac{1}{2s}\cdot\frac{1}{L_1}$$

$$\therefore\ L_1 = \frac{1}{\sqrt{2s}}$$

In a similar manner we find $L_2 = \frac{1}{\sqrt{2(s-c)}}$. Hence we obtain

$$\lim_{a\to\infty} h(a) = \frac{1}{3\sqrt{\mu}}\left\{\frac{2s^2}{\sqrt{2s}} - \frac{2(s-c)^2}{\sqrt{2(s-c)}}\right\}$$

$$= \frac{2}{3\sqrt{2\mu}}\left(\sqrt{s^3} - \sqrt{(s-c)^3}\right) \qquad \text{or}$$

$$\lim_{a\to\infty} h(a) = T_o$$

We now calculate $\lim_{a\to\infty} \tilde{h}(a)$. Let us choose $\frac{1}{t} = a^{\frac{1}{2}}$ so that as $a \to \infty$, $t \to o$ and apply L'Hopital's rule with

$$\tilde{F} = \sqrt{y_1^2-1} - \cosh^{-1} y_1 + \sqrt{y_2^2-1} - \cosh^{-1} y_2$$

$$\tilde{G} = \sqrt{\mu}\; t^3$$

Thus

$$\lim_{a\to\infty} \tilde{h}(a) = \lim_{t\to o} \frac{\tilde{F}}{\tilde{G}} = \lim_{t\to o} \frac{\tilde{F}'}{\tilde{G}'}$$

$$= \lim_{t\to o}\left\{\tfrac{1}{2}(y_1^2-1)2y_1 \frac{dy_1}{dt} - \frac{1}{\sqrt{y_1^2-1}}\frac{dy_1}{dt} + \tfrac{1}{2}(y_2^2-1)^{-\frac{1}{2}} 2y_2 \frac{dy_2}{dt} - \frac{1}{\sqrt{y_2^2-1}}\frac{dy_2}{dt}\right\}\frac{1}{3\sqrt{\mu}\, t^2}$$

$$\lim_{a\to\infty} \tilde{h}(a) = \lim_{t\to o}\left\{\frac{\frac{dy_1}{dt}}{\sqrt{y_1^2-1}}(y_1-1) + \frac{\frac{dy_2}{dt}}{\sqrt{y_2^2-1}}(y_2-1)\right\}\frac{1}{3\sqrt{\mu}\, t^2}$$

$$= \lim_{t\to o}\left\{2s\sqrt{\frac{y_1-1}{y_1+1}} + 2(s-c)\sqrt{\frac{y_2-1}{y_2+1}}\right\}\frac{1}{3\sqrt{\mu}\, t}$$

$$= \lim_{t\to o}\left\{s\left[\frac{\sqrt{y_1+1}\,(y_1-1)^{-\frac{1}{2}}\frac{dy_1}{dt} - \sqrt{y_1-1}\,(y_1+1)^{-\frac{1}{2}}\frac{dy_1}{dt}}{y_1+1}\right] + (s-c)\left[\frac{\sqrt{y_2+1}\,(y_2-1)^{-\frac{1}{2}}\frac{dy_2}{dt} - \sqrt{y_2-1}\,(y_2+1)^{-\frac{1}{2}}\frac{dy_2}{dt}}{y_2+1}\right]\right\}\frac{1}{3\sqrt{\mu}}$$

$$= \lim_{t\to o}\left\{\frac{s\frac{dy_1}{dt}}{y_1+1}\left(\sqrt{\frac{y_1+1}{y_1-1}} - \sqrt{\frac{y_1-1}{y_1+1}}\right) + \frac{(s-c)\frac{dy_2}{dt}}{y_2+1}\left(\sqrt{\frac{y_2+1}{y_2-1}} - \sqrt{\frac{y_2-1}{y_2+1}}\right)\right\}\frac{1}{3\sqrt{\mu}}$$

$$= \lim_{t\to o}\left\{\frac{2s^2 t}{y_1+1}\left(\frac{2}{\sqrt{y_1^2-1}}\right) + \frac{2(s-c)^2 t}{y_2+1}\left(\frac{2}{\sqrt{y_2^2-1}}\right)\right\}\frac{1}{3\sqrt{\mu}}$$

Now we know $\lim\limits_{t \to o} y_2 = \lim\limits_{t \to o} y_1 = 1$. So, recalling

$L_1 = \lim\limits_{t \to o} \dfrac{t}{\sqrt{y_1^2 - 1}}$ and $L_2 = \lim\limits_{t \to o} \dfrac{t}{\sqrt{y_2^2 - 1}}$ we obtain

$$\lim_{a \to \infty} \tilde{h}(a) = \frac{1}{3\sqrt{\mu}} \left\{ 2s^2 L_1 + 2(s-c)^2 L_2 \right\}$$

$$= \frac{2}{3\sqrt{2\mu}} \left(\sqrt{s^3} + \sqrt{(s-c)^3} \right)$$

Let us define $\tilde{T}_o = \dfrac{2}{3\sqrt{2\mu}} \left(\sqrt{s^3} + \sqrt{(s-c)^3} \right)$ so that

$$\lim_{a \to \infty} \tilde{h}(a) = \tilde{T}_o > T_o$$

Hence we conclude that if a prescribed value of T is such that $T > T_o$ but $T < \tilde{T}_o$, i.e.,

$$T_o < T < \tilde{T}_o$$

two trajectories are possible; an elliptical trajectory and a hyperbolic trajectory. Now clearly as the distance between P and Q approaches zero, (i.e., as $c \to o$),one would expect all flight times to correspondingly approach zero. That this is <u>not</u> true can be seen by observing the expression for $\tilde{T}_o$. We notice that as $c \to o$ $\tilde{T}_o \to \dfrac{4}{3\sqrt{2\mu}} \sqrt{s^3}$. But since $s = \frac{1}{2}(r_1 + r_2 + c)$, $s \to \frac{1}{2}(r_1 + r_1) = r_1$. Hence $\tilde{T}_o \to \dfrac{4}{3\sqrt{2\mu}} \sqrt{r_1^3}$. This should not be too surprising for, by the above figure, we notice that the hyperbolic path with $\tilde{F}^*$ as vacant focus always passes around F so that when $c \to o$ the path approaches the path from P to F and F back to P. This can also be demonstrated analytically by using the expression for the kinetic energy of our unit mass particle: $\frac{1}{2} V^2 = \frac{\mu}{r}$ yielding $V = \sqrt{\frac{2\mu}{r}}$. Now the time T required for the particle to go from P to F is

$$T = \int_o^{r_1} \frac{ds}{v}$$

$$= \int_o^{r_1} \frac{ds}{\sqrt{\frac{2\mu}{r}}} = \frac{1}{\sqrt{2\mu}} \int_o^{r_1} r^{\frac{1}{2}} dr$$

$$= \frac{1}{\sqrt{2\mu}} \frac{2}{3} r^{\frac{3}{2}} \Big|_o^{r_1}$$

$$= \frac{2}{3\sqrt{2\mu}} r^3$$

Thus the time to make the round trip is

$$\frac{4}{3}\frac{\sqrt{r^3}}{\sqrt{2\mu}} = \lim \widetilde{T}_o \quad \text{as} \quad c \to o.$$

Notice that $f(a_m) = T(a_m) = f(\frac{s}{2}) = \frac{1}{2}\sqrt{\frac{s^3}{2\mu}}\left\{\sqrt{1-(1-\frac{s-c}{\frac{s}{2}})^2}\right.$

$$\left. + \sin^{-1}(1-\frac{s-c}{\frac{s}{2}}) - \sqrt{1-(1-\frac{s}{\frac{s}{2}})^2} - \sin^{-1}(1-\frac{2s}{s})\right\}$$

$$T(a_m) = f(a_m) = \frac{1}{2}\sqrt{\frac{s^3}{2\mu}}\left\{\frac{\pi}{2} + \frac{2\sqrt{c}}{s}\sqrt{s-c} + \sin^{-1}(-1+\frac{2c}{s})\right\}$$

Hence

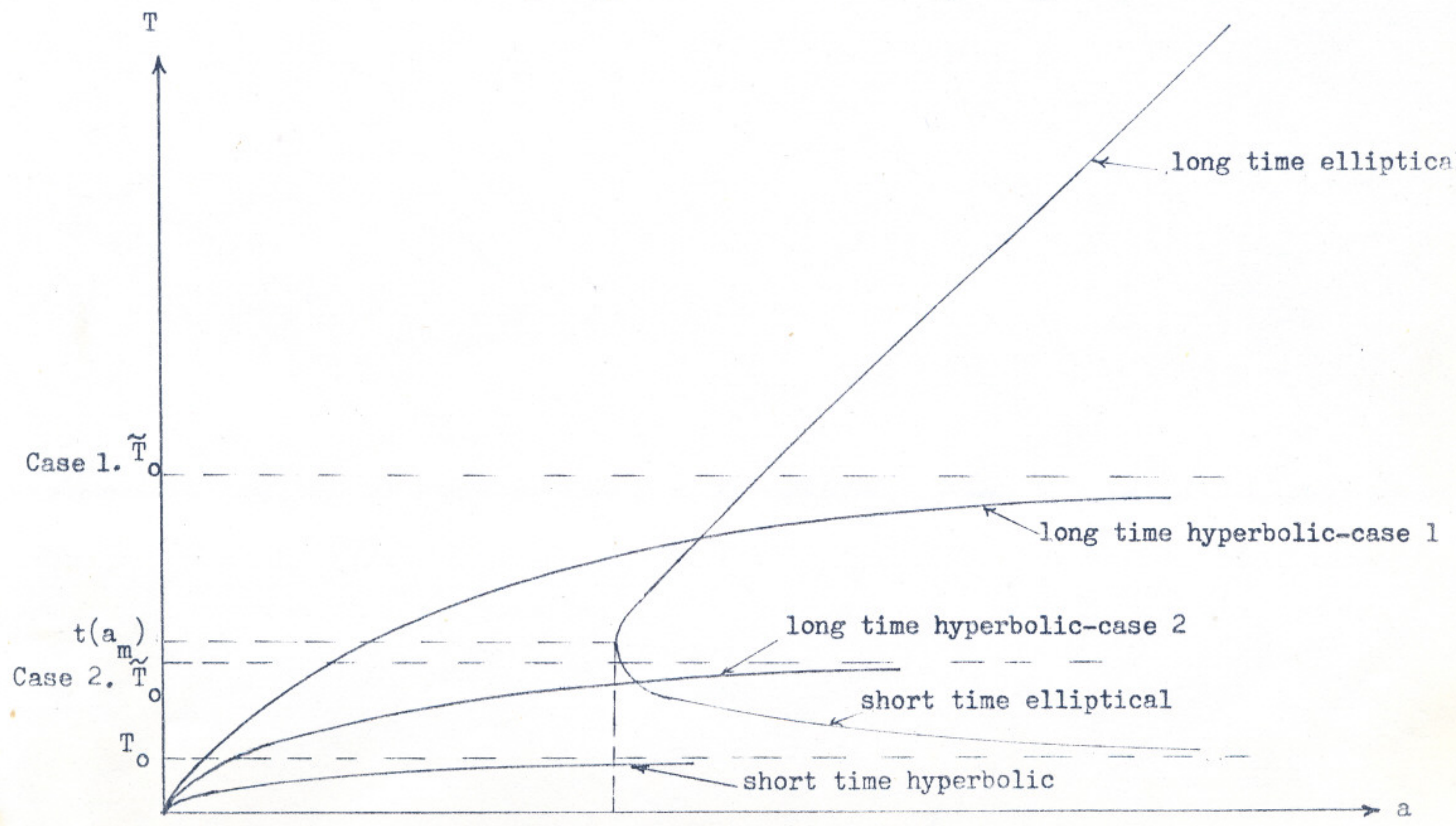
$$T(a_m)\Big|_{c=o} = 0 \quad \text{and} \quad T(a_m)\Big|_{c=s} = \frac{\pi}{2}\sqrt{\frac{s^3}{2\mu}}.$$

Now $\widetilde{T}_o\Big|_{c=o} = \frac{4}{3\sqrt{2\mu}}\sqrt{s^3}$ and $\widetilde{T}_o\Big|_{c=s} = \frac{2}{3\sqrt{2\mu}}\sqrt{s^3}$. Thus we may have $T(a_m) > \widetilde{T}_o$ when c is near max. or $T(a_m) < \widetilde{T}_o$ when c is small.

With this information and information concerning f', f", h', and limits as a approaches limiting values, we can construct general shapes of the graph of T vs. a.

"Graph of T vs. a"

Case 1: $T(a_m) < \widetilde{T}_o$ Case 2: $\widetilde{T}_o < T(a_m)$



The first step in making a detailed study of possible conic trajectories associated with prescribed initial values $\vec{r_1}$, $\vec{r_2}$, T should be determining whether case 1, $f(a_m) < \tilde{T}_o$, or case 2, $\tilde{T}_o < f(a_m)$, is true so that a general graph may be obtained.

To complete our analysis of hyperbolic trajectories we write an iteration method for obtaining a corresponding to $T < \tilde{T}_o$

$$a_{k+1} = a_k - \frac{h(a_k) - T}{h'(a_k)}$$

Since $T_o < \tilde{T}_o$, a second hyperbolic trajectory is possible if $T < T_o$

$$a_{k+1} = a_k - \frac{\tilde{h}(a_k) - T}{\tilde{h}'(a_k)}$$

Thus if $T_o < T < \tilde{T}_o$ two different conic trajectories exist; an elliptical trajectory and a hyperbolic trajectory. For the hyperbolic paths Battin shows that

$$l = \left[\frac{4a}{c^2}(s-r_1)\ (s-r_2)\ \sinh^2 \tfrac{1}{2}(\alpha+\beta)\right]$$

$$\tilde{l} = \left[\frac{4a}{c^2}(s-r_1)\ (s-r_2)\ \sinh^2 \tfrac{1}{2}(\alpha-\beta)\right]$$

corresponding to paths with vacant foci F^* and $\tilde{F}^*$, respectively. Since $l = a(\epsilon^2 - 1)$, these equations yield

$$\epsilon = \left\{1 + \frac{2}{c^2}\ (s-r_1)\ (s-r_2)\ (y_1 y_2 + \sqrt{y_1^2-1}\ \sqrt{y_2^2-1}\ -1\right\}^{\frac{1}{2}}$$

$$\tilde{\epsilon} = \left\{1 + \frac{2}{c^2}\ (s-r_1)\ (s-r_2)\ (y_1 y_2 - \sqrt{y_1^2-1}\ \sqrt{y_2^2-1}\ -1\right\}^{\frac{1}{2}}$$

Summary of results for alternative method of determining possible conic paths associated with prescribed values of $\vec{r_1}$, $\vec{r_2}$, M_1 and T where $\vec{r_1} = \vec{FP}$, $r_2 = \vec{FQ}$, M is the mass of the single gravitating body at the focus F and T is the flight time from P to Q.

(i) Calculate $f(a_m)$ and $\tilde{T}_o$ to determine general graph of T vs. a by: Case 1, $f(a_m) < \tilde{T}_o$, Case 2, $\tilde{T}_o < f(a_m)$ (see graph of T vs. a).

(ii) Calculate T_o to determine whether an elliptical path is possible (if $T < T_o$ an elliptical path is impossible.)

(iii) Determine whether an elliptical path and a hyperbolic path are <u>both</u> possible (i.e., if $T_o < T < \tilde{T}_o$).

(iv) Determine the functions yielding T:

a) if $f(a_m) \leqq T$ use $\tilde{f}(a)$ for elliptic path

b) if $T_o < T \leqq f(a_m)$ use $f(a)$ for elliptic path

c) if $T_o < T \leqq T_o$ then hyperbolic path also exists with T given by $\tilde{h}(a)$

d) if $T < T_o$ only hyperbolic paths exist, $T = h(a)$ for short hyperbolic flight times, $T = h(a)$ for long hyperbolic flight times

(v) Determine a with sufficient accuracy by

$$a_{k+1} = a_k - \frac{F(a) - T}{\frac{dF}{da}} \qquad \left\{ a_k \right\} \to a$$

where F(a) is the function yielding T

error in k'th iterate $= E_k = \left| a - a_k \right|$ has the relation

$$E_{k+1} \approx \frac{1}{2} E_k^2 \left| \frac{F''(a)}{F'(a)} \right|$$

showing rapid convergence to solution $T(a) = T$

(vi) Determine the eccentricity after obtaining good approximation of a by:

$$\epsilon = \left\{ 1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1 x_2 + \sqrt{1-x_1^2}\,\sqrt{1-x_2^2}) \right\}^{\frac{1}{2}}$$

$$\tilde{\epsilon} = \left\{ 1 - \frac{2}{c^2}(s - r_1)(s - r_2)(1 - x_1 x_2 - \sqrt{1-x_1^2}\,\sqrt{1-x_2^2}) \right\}^{\frac{1}{2}}$$

for an elliptical path when T is given by $T = f(a)$, $T = \tilde{f}(a)$, respectively.

$$\epsilon = \left\{ 1 + \frac{2}{c^2}(s - r_1)(s - r_2)(y_1 y_2 + \sqrt{y_1^2-1}\,\sqrt{y_2^2-1} - 1) \right\}^{\frac{1}{2}}$$

$$\tilde{\epsilon} = \left\{ 1 + \frac{2}{c^2}(s - r_1)(s - r_2)(y_1 y_2 - \sqrt{y_1^2-1}\,\sqrt{y_2^2-1} - 1) \right\}^{\frac{1}{2}}$$

for hyperbolic paths when the prescribed time T is given by $T = h(a)$, and $T = \tilde{h}(a)$, respectively.

(vii) Formulas for above expressions:

$$c = \text{distance from P to Q} = \overline{PQ}$$

$$= \sqrt{r_1^2 + r_2^2 - 2\, r_1 r_2 \cos\theta} \qquad \theta = \angle PFQ$$

$$= \sqrt{\vec{r}_1^{\,2} + \vec{r}_2^{\,2} - 2\, \vec{r}_1 \cdot \vec{r}_2}$$

$$s = \frac{r_1 + r_2 + c}{2} \qquad a_m = \frac{s}{2}$$

$$x_1 = 1 - \frac{s}{a} \qquad x_2 = 1 - \frac{s-c}{a}$$

$$y_1 = 1 + \frac{s}{a} \qquad y_2 = 1 + \frac{s-c}{a}$$

$$f(a) = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1-x_2^2} + \sin^{-1} x_2 - \sqrt{1-x_1^2} - \sin^{-1} x_1 \right\}$$

$$\tilde{f}(a) = \sqrt{\frac{a^3}{\mu}} \left\{ \pi + \sqrt{1-x_2^2} + \sin^{-1} x_2 + \sqrt{1-x_1^2} + \sin^{-1} x_1 \right\}$$

$$\frac{df(a)}{da} = f'(a) = \frac{3}{2}\frac{f(a)}{a} + \frac{1}{\sqrt{a\mu}} \left\{ (s-c)\sqrt{\frac{1-x_2}{1+x_2}} - s\sqrt{\frac{1-x_1}{1+x_1}} \right\}$$

$$\frac{d\tilde{f}}{da} = \tilde{f}'(a) = \frac{3}{2}\frac{\tilde{f}(a)}{a} + \frac{1}{\sqrt{a\mu}} \left\{ (s-c)\sqrt{\frac{1-x_2}{1+x_2}} + s\sqrt{\frac{1-x_1}{1+x_1}} \right\}$$

$$h(a) = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{y_1^2-1} - \cosh^{-1} y_1 - \sqrt{y_2^2-1} + \cosh^{-1} y_2 \right\}$$

$$\tilde{h}(a) = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{y_1^2-1} - \cosh^{-1} y_1 + \sqrt{y_2^2-1} - \cosh^{-1} y_2 \right\}$$

$$\frac{dh}{da} = h'(a) = \frac{3}{2}\frac{h(a)}{a} + \frac{1}{\sqrt{a\mu}} \left\{ (s-c)\sqrt{\frac{y_2-1}{y_2+1}} - s\sqrt{\frac{y_1-1}{y_1+1}} \right\}$$

$$\frac{d\tilde{h}}{da} = \tilde{h}'(a) = \frac{3}{2}\frac{\tilde{h}(a)}{a} - \frac{1}{\sqrt{a\mu}} \left\{ (s-c)\sqrt{\frac{y_2-1}{y_2+1}} + s\sqrt{\frac{y_1-1}{y_1+1}} \right\}$$

μ = GM where G is the universal gravitational constant and M is the mass of the body about which the conic trajectory takes place.

It is found convenient to use a year as unit of time and A.U. as unit of distance.

$$T(a_m) = \sqrt{\frac{s^3}{2\mu}} \left\{ \sqrt{\frac{c}{s}\left(1-\frac{c}{s}\right)} + \frac{1}{2}\sin^{-1}\left(\frac{2c}{s}-1\right) + \frac{\pi}{4} \right\}$$

$$T_0 = \frac{1}{3}\sqrt{\frac{2}{\mu}} \left\{ \sqrt{s^3} - \sqrt{(s-c)^3} \right\}$$

MM:ls

JET PROPULSION LABORATORY

TECHNICAL MEMO #312-130

August 23, 1961

TO: Section 312 Engineers, J. F. Scott, W. Scholey

FROM: M. A. Minovich

SUBJECT: A Method For Determining Interplanetary Free-Fall Reconnaissance Trajectories

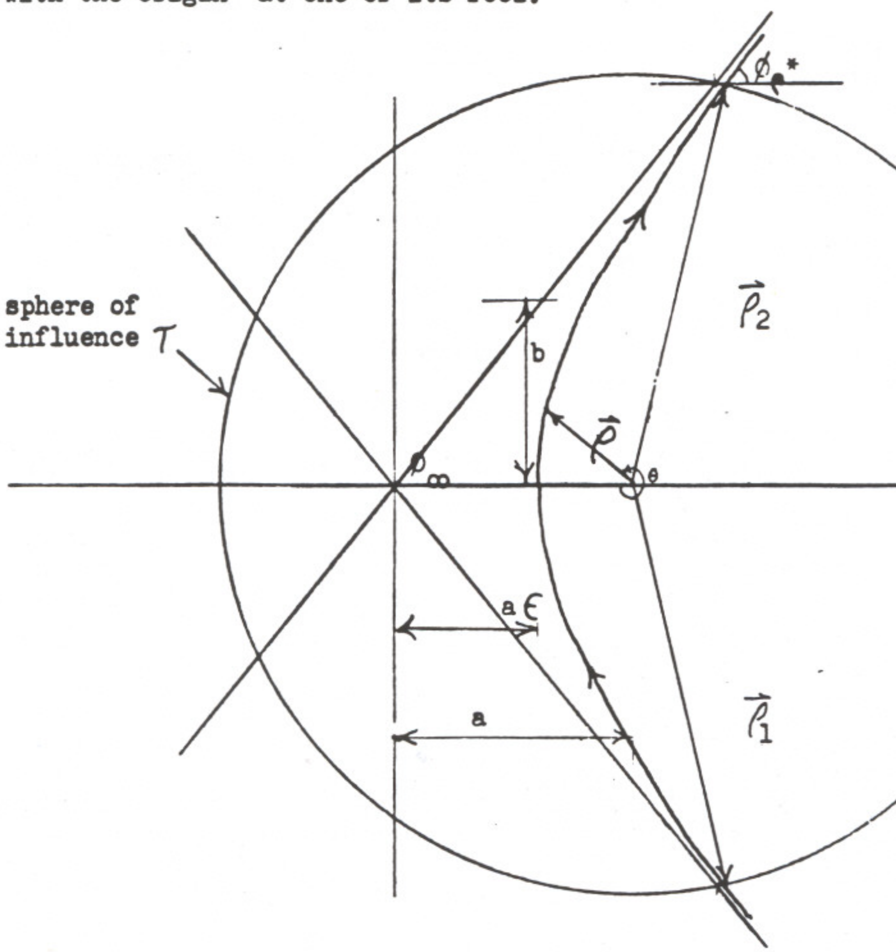
This paper deals with determining round-trip trajectories for reconnaissance vehicles in free-fall motion when certain fundamental assumptions are assumed to hold. After solving the trajectory problem to one planet and back the more general problem of determining a free-fall reconnaissance trajectory to N planets before returning to the launch planet will be solved. No assumptions will be made as to the geometry of the solar system; indeed, it will not matter how eccentric the planets orbits are or how much their planes of motion differ from each other. Vector analysis is employed throughout the paper giving it a somewhat neat mathematical appearance which should offer interesting reading. As far as the author knows the method and results are new.

The problem of finding an exact analytical solution for round-trip,free-fall reconnaissance trajectories is, to say the least, not trivial. Consequently, in papers treating these problems certain simplifying assumptions are very common. In this paper we shall assume only the most basic:

I. When the vehicle (treated as a particle) is inside a "sphere of influence" τ, centered at the center of the target planet, only the field of this body influences its motion. When the vehicle is outside τ only the sun influences its motion.

II. If Δt is the amount of time the vehicle spends in τ, Δt is small so that the planets motion can be assumed to be constant; its velocity being that velocity it has when the vehicle makes its closest approach.

Before stating the last assumption suppose Σ is some inertial cartesian frame of reference with origin at the center of the sun. Let Σ' be a moving frame with origin at the center of the target planet and whose axes are kept parallel to the corresponding axes of Σ. Then our second basic assumption implies that when the vehicle is in τ, Σ' is an inertial frame. Consequently, during the time interval Δt the path of the vehicle with respect to Σ' will be a hyperbolic conic section with the origin at one of its foci.



The third fundamental assumption is now written as:

III. $\phi_\infty = \phi_{\rho^*}$

The vectors $\vec{\rho}_1$ and $\vec{\rho}_2$ are the position vectors of the vehicle as it enters and leaves τ with respect to the origin of Σ'. The axes of Σ' are not drawn since, in general, they will not be parallel to any of the above lines. It has been found[3] that the best sphere of influence τ has radius ρ^* given by

$$\rho^* = \left(\frac{m}{M}\right)^{\frac{2}{5}} c$$

where m and M are the masses of the target planet and sun, respectively; c is the distance between the target planet and the sun.

By taking I, II and III to be our only assumptions we must deal with the three-dimensional character of the solar system. Consequently, the use of vector analysis is indispensable. Thus at this point, we digress to set up the necessary mathematical apparatus which shall be used throughout this paper.

Newton's law of motion is

$$m \frac{d\vec{V}}{dt} = -G \frac{Mm}{R^2} \hat{R}$$

where m is the mass of particle having velocity $\vec{V}$, M is the mass of a second particle, $\hat{R}$ is the unit vector in the direction from M to m; (Unit vectors shall be denoted by placing $\wedge$ over letter instead of $\rightarrow$), G is the gravitational constant. If M >> m we may assume that M is at rest, ~~hence taking m = 1 and~~ letting MG = μ

(1) $$\frac{d\vec{V}}{dt} = -\mu \frac{\hat{R}}{R^2}$$

Since this implies

$$\frac{d}{dt}(\vec{R} \times \vec{V}) = \frac{d\vec{R}}{dt} \times \vec{V} + \vec{R} \times \frac{d\vec{V}}{dt} = \vec{V} \times \vec{V} - \frac{\mu}{R^2} \vec{R} \times \hat{R} = 0,$$

integrating

$$\frac{d}{dt}(\vec{R} \times \vec{V})$$

yields

(2) $$\vec{R} \times \vec{V} = \vec{h}$$

where $\vec{h}$ is some constant vector of integration and is equal to the vector called angular momentum of m about M. This shows that $\vec{R}$ and $\vec{V}$ must then be perpendicular to $\vec{h}$, hence the motion of m takes place in a fixed plane. Writing

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{d}{dt}(R \hat{R})$$

(2) can be expressed as

$$\vec{h} = \vec{R} \times \vec{V} = R \times \frac{d}{dt}(R\hat{R}) = \vec{R} \times (R\frac{d\hat{R}}{dt} + \frac{dR}{dt}\hat{R}) = R^2 \hat{R} \times \frac{d\hat{R}}{dt}$$

Thus in view of (1)

$$\frac{d\vec{V}}{dt} \times \vec{h} = -\mu \frac{\hat{R}}{R^2} \times R^2(\hat{R} \times \frac{d\hat{R}}{dt})$$

$$= -\mu \left[(\hat{R} \cdot \frac{d\hat{R}}{dt}) \hat{R} - (\hat{R} \cdot \hat{R}) \frac{d\hat{R}}{dt} \right]$$

If θ is the angle between $\vec{R}$ and some arbitrary line in the plane of motion

$$\frac{d\hat{R}}{dt} = \frac{d\hat{R}}{d\theta}\frac{d\theta}{dt}$$

but $\frac{d\hat{R}}{d\theta}$ is perpendicular to $\hat{R}$ hence

$$\hat{R}\cdot\frac{d\hat{R}}{dt} = 0$$

yielding

$$\frac{d\vec{V}}{dt} \times \vec{h} = \mu\frac{d\hat{R}}{dt} = \frac{d}{dt}(\mu\hat{R}) .$$

Now since $\vec{h}$ is a constant vector

$$\frac{d}{dt}(\vec{V} \times \vec{h}) = \frac{d\vec{V}}{dt} \times \vec{h}$$

consequently, we obtain

$$\frac{d}{dt}(\vec{V} \times \vec{h}) = \frac{d}{dt}(\mu\hat{R})$$

whereupon integration yields

(3) $$\vec{V} \times \vec{h} = \mu(\hat{R} + \vec{\epsilon})$$

where $\vec{\epsilon}$ is another constant of integration. Notice that since $\vec{V} \times \vec{h}$ is in the plane of motion so is $\vec{\epsilon}$. We also observe that $\vec{h}$ and $\vec{\epsilon}$ are not independent of each other for if $\vec{R}$, $\vec{V}$ and $\vec{h}$ are known at any time t

(4) $$\vec{\epsilon} = \frac{1}{\mu}\vec{V} \times \vec{h} - \hat{R}$$

Now

$$\vec{h} \times (\vec{V} \times \vec{h}) = (\vec{h}\cdot\vec{h})\,\vec{V} - (\vec{h}\cdot\vec{V})\,\vec{h} = h^2\,\vec{V}$$

Consequently, employing (3) we obtain the important formula

(5) $$\vec{V} = \frac{\mu}{h^2}\vec{h} \times (\hat{R} + \vec{\epsilon}) .$$

Thus if $\vec{h}$ and $\vec{\epsilon}$ are known and $\vec{R}$ is a point on the particles trajectory, its velocity at $\vec{R}$ can be calculated from (5).

Let θ be the angle measured from $\vec{\epsilon}$ in the positive direction (i.e., counterclockwise) to $\vec{R}$. Hence in view of (2) and (3)

$$h^2 = \vec{h}\cdot\vec{h} = \vec{h}\cdot\vec{R} \times \vec{V} = \vec{R}\cdot\vec{V} \times \vec{h} = \vec{R}\cdot\mu(\hat{R} + \vec{\epsilon})$$

$$\therefore \quad \frac{h^2}{\mu} = R + R\,\epsilon \cos\theta = R(1 + \epsilon\cos\theta)$$

(6) $$\therefore \quad R = \frac{\frac{h^2}{\mu}}{1 + \epsilon \cos \theta}$$

But this is the general equation of a conic with eccentricity ϵ and semi-latus rectum

(7) $$\ell = \frac{h^2}{\mu} .$$

Thus we obtain the well known fact that the trajectory is a conic section. Since (6) implies that R is smallest when $\theta = 0$, the direction of $\vec{\epsilon}$ is along the direction of perihelion.

We now establish another important and well known relation. From (5) and (7) we write

$$V^2 = \vec{V}\cdot\vec{V} = \frac{1}{\ell}\vec{V}\cdot\vec{h} \times (\hat{R} + \vec{\epsilon}) = \frac{1}{\ell}(\vec{V}\cdot\vec{h} \times \hat{R} + \vec{V}\cdot\vec{h} \times \vec{\epsilon})$$

$$= \frac{1}{R\ell}\left[\vec{h}\cdot\vec{R} \times \vec{V} + R(\vec{\epsilon}\cdot\vec{V} \times \vec{h})\right]$$

Employing (2) and (3) this becomes

$$V^2 = \frac{1}{R\ell}\left[h^2 + \vec{\epsilon}\cdot\mu(\vec{R} + R\,\vec{\epsilon})\right]$$

$$= \frac{1}{R\ell}\left[h^2 + \mu(R\,\epsilon \cos\theta + R\,\epsilon^2)\right]$$

With the aid of (6) we obtain

$$V^2 = \frac{1}{R\ell}\left[h^2 + \mu(\frac{h^2}{\mu} - R + R\,\epsilon^2)\right]$$

$$= \frac{1}{R\ell}\left[2\,h^2 + \mu R(\epsilon^2 - 1)\right]$$

which becomes, after using (7) a second time,

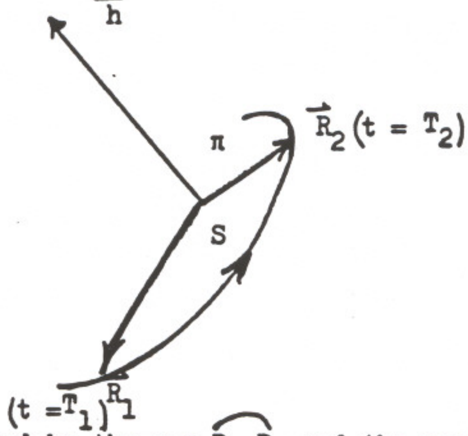
$$V^2 = \mu(\frac{2}{R} + \frac{\epsilon^2 - 1}{\ell})$$

Since $\ell = a(1 - \epsilon^2)$ for ellipses and $\ell = a(\epsilon^2 - 1)$ for hyperbolas where a is the semi-major axis of the conic we obtain

(8) $$V^2 = \mu(\frac{2}{R} \mp \frac{1}{a})$$

where the negative or positive sign is chosen if the conic is an ellipse or hyperbola, respectively.

Let π denote the plane of motion.



Let S denote the area of π bounded by the arc $\overset{\frown}{R_1 R_2}$ and the vectors $\vec{R}_1$ and $\vec{R}_2$. If C denotes this closed curve we obtain by Stokes' theorem

$$\oint_C \vec{f} \cdot d\vec{R} = \iint_S \vec{n} \cdot (\nabla \times \vec{f})\, dS$$

setting $\vec{f} = \vec{\zeta} \times \vec{R}$ where ζ is any arbitrary constant vector

(9) $$\oint_C (\vec{\zeta} \times \vec{R}) \cdot d\vec{R} = \iint_S \hat{h} \cdot \nabla \times (\vec{\zeta} \times \vec{R})\, dS.$$

Now

$$(\vec{\zeta} \times \vec{R}) \cdot d\vec{R} = d\vec{R} \cdot (\vec{\zeta} \times \vec{R}) = \vec{\zeta} \cdot \vec{R} \times d\vec{R}$$

and

$$\nabla \times (\vec{\zeta} \times \vec{R}) = \vec{R} \cdot \nabla \vec{\zeta} - \vec{\zeta} \cdot \nabla \vec{R} + \vec{\zeta}\, \nabla \cdot \vec{R} - \vec{R}\, \nabla \cdot \vec{\zeta}$$

but since $\vec{\zeta}$ is a constant vector the dyadic $\nabla \vec{\zeta}$ and the scalar $\nabla \cdot \vec{\zeta}$ vanish. Since $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ the dyadic $\nabla \vec{R}$ is the idemfactor I

$$\nabla \vec{R} = \frac{\partial \vec{R}}{\partial x}\hat{i} + \frac{\partial \vec{R}}{\partial y}\hat{j} + \frac{\partial \vec{R}}{\partial z}\hat{k} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k} = I$$

Consequently, since $\nabla \cdot \vec{R} = 3$

$$\nabla \times (\vec{\zeta} \times \vec{R}) = -\vec{\zeta} \cdot I + 3\vec{\zeta} = 2\vec{\zeta}$$

Thus since $\vec{\zeta}$ and $\hat{h}$ are constant vectors (9) yields the expression

$$\vec{\zeta} \cdot \oint_C \vec{R} \times d\vec{R} = 2\vec{\zeta} \cdot \hat{h} \iint_S dS = 2\vec{\zeta} \cdot \hat{h}\, S$$

The vector $\vec{\zeta}$ is arbitrary hence we obtain

$$\oint_C \vec{R} \times d\vec{R} = 2\hat{h}S$$

By writing $d\vec{R} = \frac{d\vec{R}}{dt}dt = \vec{V}dt$ this expression can be written as

$$\int_{T_1}^{T_2} \vec{R} \times \vec{V}\,dt = \int_{T_1}^{T_2} \vec{h}\,dt = \vec{h}(T_2 - T_1) = 2\hat{h}S$$

(10) $$\therefore \quad 2S = h(T_2 - T_1)$$

This is equivalent to Kepler's second law. Setting $T_2 - T_1 = \Delta T$, (10) yields

$$h\,\Delta T = 2S = 2\int_{\theta_1}^{\theta_2} \tfrac{1}{2}\rho^2\,d\theta \qquad \text{where}$$

$$\rho = \frac{\ell}{1 + \epsilon \operatorname{Cos}\theta}$$

$$\therefore \quad \operatorname{Cos}\theta = \frac{\ell}{\rho\epsilon} - \frac{1}{\epsilon}$$

$$\therefore \quad -\sin\theta\,d\theta = -\frac{\ell}{\rho^2\epsilon}\,d\rho$$

$$d\theta = \frac{\ell}{\rho^2\epsilon\sin\theta}\,d\rho$$

Hence

$$\Delta T = \frac{1}{h}\int_{R_1}^{R_2} \rho^2 \frac{\ell}{\rho^2\epsilon\sin\theta}\,d\rho$$

If $\vec{R}_1 = (a\epsilon - a)\,\hat{\epsilon} = a(\epsilon - 1)\,\hat{\epsilon}$ and $\vec{R}_2 = \vec{\rho}_1$ (see figure on page 2), ΔT will be one-half of the total time Δt a vehicle spends in $\mathcal{T}$. Thus

$$\Delta t = \frac{2}{h}\int_{R_1 = a(\epsilon-1)}^{R_2 = \rho^*} \frac{\ell}{\epsilon\sin\theta}\,d\rho = \frac{2\ell}{h\epsilon}\int_{a(\epsilon-1)}^{\rho^*} \frac{d\rho}{\sqrt{1 - \left(\frac{\ell}{\rho} - 1\right)^2 \frac{1}{\epsilon^2}}}$$

$$= \frac{2\ell}{h\epsilon}\int_{a(\epsilon-1)}^{\rho^*} \frac{\epsilon\rho\,d\rho}{\sqrt{\epsilon^2\rho^2 - (\ell - \rho)^2}}$$

$$= \frac{2\ell}{h}\int_{a(\epsilon-1)}^{\rho^*} \frac{\rho\,d\rho}{\sqrt{(\epsilon^2 - 1)\rho^2 + 2\ell\rho - \ell^2}}$$

$$= \frac{2\ell}{h}\left\{\frac{1}{\epsilon^2-1}\sqrt{(\epsilon^2-1)\rho^2+2\ell\rho-\ell^2}\,\Bigg|_{a(\epsilon-1)}^{\rho^*} - \frac{\ell}{(\epsilon^2-1)^{\frac{3}{2}}}\log\Big|2.\right.$$

$$\left.\cdot\sqrt{\epsilon^2-1}\cdot\sqrt{(\epsilon^2-1)\rho^2+2\ell\rho-\ell^2}+2(\epsilon^2-1)\rho+2\ell\Big|\,\Bigg|_{a(\epsilon-1)}^{\rho^*}\right\}$$

Since $\ell = a(\epsilon^2-1)$

$$\sqrt{(\epsilon^2-1)\,a^2(\epsilon-1)^2+2\ell a(\epsilon-1)-\ell^2} = a\sqrt{\epsilon^2-1}\cdot\sqrt{(\epsilon-1)^2+2(\epsilon-1)-(\epsilon^2-1)} = 0$$

Thus

$$\Delta t = \frac{2a}{h}\left\{\sqrt{\epsilon^2-1}\sqrt{\rho^{*2}+2a\rho^*-a^2(\epsilon^2-1)}\right.$$
$$-a\sqrt{\epsilon^2-1}\Big[\log\Big|2(\epsilon^2-1)\sqrt{\rho^{*2}+2a\rho^*-a^2(\epsilon^2-1)}+2(\epsilon^2-1)\rho^*$$
$$\left.+2a(\epsilon^2-1)\Big|-\log\Big|2(\epsilon^2-1)\,a(\epsilon-1)+2a(\epsilon^2-1)\Big|\Big]\right\}$$

Substituting $\rho^* = (\frac{m}{M})^{\frac{2}{5}}c$ we obtain

$$\text{(11)}\qquad \Delta t = 2\sqrt{\frac{a}{\mu}}\left\{\sqrt{(\frac{m}{M})^{\frac{4}{5}}c^2+2a(\frac{m}{M})^{\frac{2}{5}}c-a^2(\epsilon^2-1)}\right.$$
$$-a\log\frac{1}{\epsilon a}\left[\sqrt{(\frac{m}{M})^{\frac{4}{5}}c^2+2a(\frac{m}{M})^{\frac{2}{5}}c-a^2(\epsilon^2-1)}\right.$$
$$\left.\left.+(\frac{m}{M})^{\frac{2}{5}}c+a\right]\right\}$$

We now introduce the notation which shall be used to find the trajectory which will take a free-fall vehicle from a certain launch planet P to the vacinity of a certain target planet Q such that its interaction with Q will send it on an interception trajectory with the launch planet P.

(a) $\vec{c}_o$ = position vector of the launch planet with respect to Σ when the vehicle begins its reconnaissance mission at time t_o; this vector will also be taken to be the initial position vector of the vehicle.

(b) $\vec{r}_1$ = position vector of vehicle when it enters the sphere of influence $\mathcal{T}$ of the target planet at time t_1.

(c) $\vec{\rho}_1$ = position vector of vehicle when it enters $\mathcal{T}$ at time t_1 with respect to Σ'.

(d) $\vec{c}_1$ = position vector of target planet at time t_1 when the vehicle enters its sphere of influence $\mathcal{T}$.

(e) $\vec{r}_{cA}$ = position vector of target planet when the vehicle makes its closest approach to its surface at time t_{cA}.

(f) $\vec{r}_2$ = position vector of vehicle when it leaves $\mathcal{T}$ at time t_2.

(g) $\vec{\rho}_2$ = position vector of vehicle when it leaves $\mathcal{T}$ with respect to $\sum'$ at time t_2.

(h) $\vec{c}_2$ = position vector of the target planet when the vehicle leaves its sphere of influence $\mathcal{T}$ at time t_2.

(i) $\vec{c}_3$ = position vector of launch planet at end of reconnaissance mission; this vector is also taken as the final position vector of the vehicle for the mission.

(j) $\vec{h}_1$, $\vec{\epsilon}_1$, a_1, ℓ_1 and $\vec{h}_3$, $\vec{\epsilon}_3$, a_3, ℓ_3 are the vector and scalar parameters corresponding to the departing elliptical trajectory and the returning elliptical trajectory, respectively.

(k) $\vec{h}_2$, $\vec{\epsilon}_2$, a_2, ℓ_2 are the vector and scalar parameters when the trajectory is in $\mathcal{T}$ with respect to $\sum'$ (by the manner in which $\sum'$ was chosen, these vectors given with respect to $\sum'$ have the same coordinate values with respect to $\sum$).

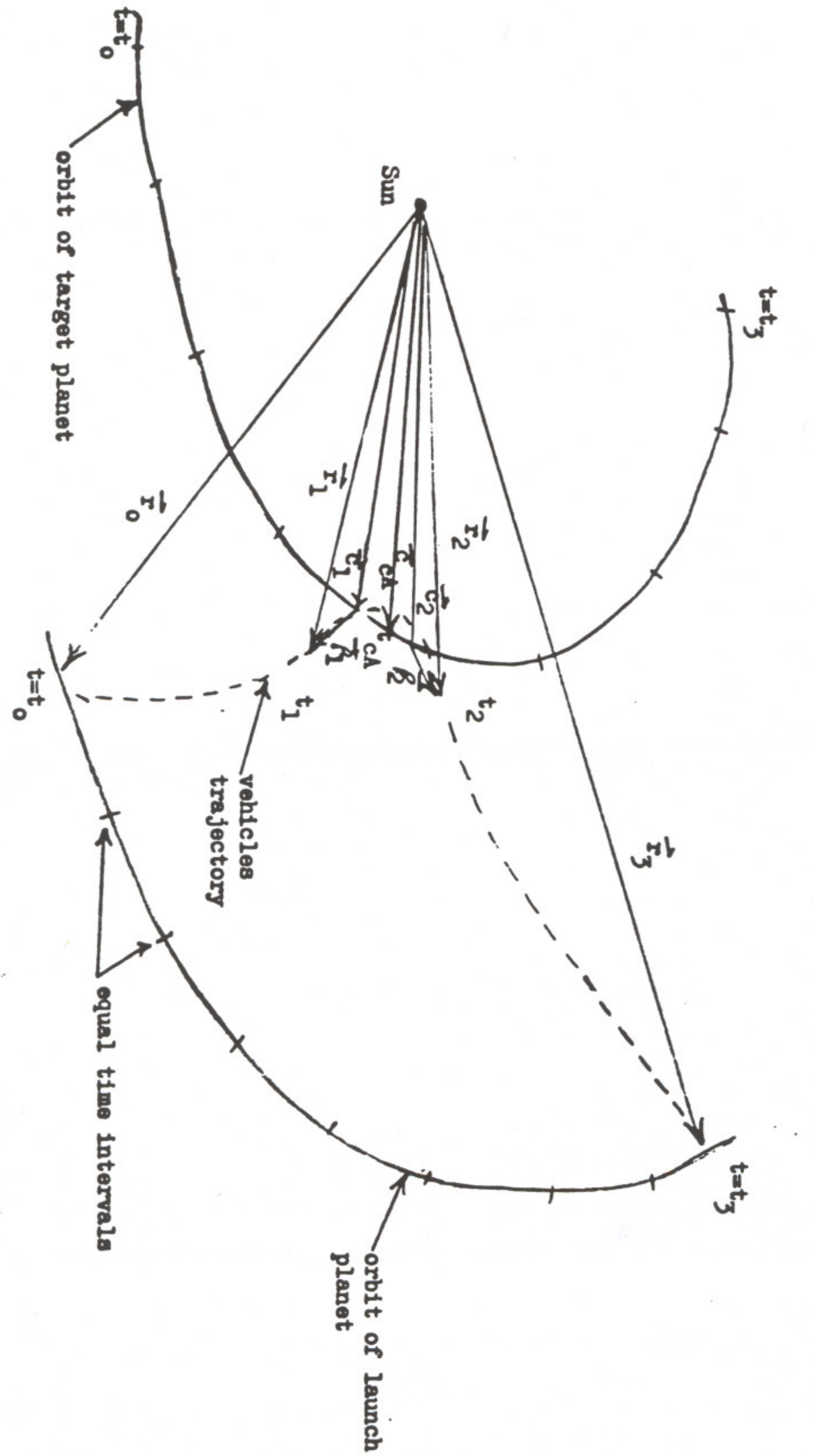
(l) $\vec{P}(t)$ and $\vec{Q}(t)$ denote the position vectors of the launch planet and target planet as functions of time. (These functions are obtain by ephemeris tables.)

(m) $\vec{V}_1$ and $\vec{V}_2$ denote the velocity vectors with respect to $\sum$ as the vehicle enters and leaves $\mathcal{T}$, respectively; the velocity vectors of the vehicle as it enters and leaves $\mathcal{T}$ with respect to $\sum'$ are $\vec{V}_1'$ and $\vec{V}_2'$.

(n) d = distance of closest approach.

(o) R_Q = radius of target planet.

(p) $\widehat{r_i\, r_j}$ = arc of trajectory between $\vec{r}_i$ and $\vec{r}_j$; $\overline{r_i r_j}$ = distance between $\vec{r}_i, \vec{r}_j$.


Sun
orbit of target planet
vehicles trajectory
equal time intervals
orbit of launch planet

The problem of determining a round-trip, free-fall reconnaissance trajectory to one planet shall be formulated as follows:

Assuming that the three basic assumptions hold, find a trajectory of a vehicle launched from the "center" of a given planet at the prescribed time t_o, which makes a closest approach to a given target planet at the prescribed time t_{cA} and returns to the "center" of the launch planet. Notice that after selecting the launch and target planets, only t_o and t_{cA} are prescribed. In theory this problem will always have a solution in Newtonian mechanics; however, if a solution gives a trajectory which comes closer to the center of the target planet than its own surface, it will be physically unrealizable and is said not to exist. For definiteness we shall assume that $T_o \leqq t_{cA} - t_o \leqq T(a_m)$ where T_o is the shortest flight time it takes a vehicle to pass from $\vec{c}_o$ to $\vec{c}_{cA}$ on an elliptical trajectory and $T(a_m)$ is the time taken when the vehicle has least energy (see Technical Memo #312-118).

Instead of finding an exact solution to the problem (which, in lieu of the three basic assumptions, will not be a true solution) a solution shall be found which will be very close to an exact solution. This solution yielding the trajectory vectors $\vec{h}_1$, $\vec{\epsilon}_1$, $\vec{h}_2$, $\vec{\epsilon}_2$, $\vec{h}_3$, $\vec{\epsilon}_3$ should suffice for a preliminary analysis (e.g. distance of closest approach), but for an actual mission where more accuracy may be desired an iteration method is given in the appendix that will enable one to obtain a solution which is arbitrarily close to the exact solution.

It can be shown (see above mentioned reference) that if $T_o \leqq T \leqq T(a_m)$ where T is the time required for a vehicle to pass from the point $\vec{R}_1$ to the point $\vec{R}_2$ under the gravitational influence of one stationary body

$$(12) \qquad T = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1-x^2{}_2} + \sin^{-1} x_2 - \sqrt{1-x^2{}_1} - \sin^{-1} x_1 \right\}$$

where a is the semi-major axis of the elliptical path, $\mu = GM$ and x_1, x_2 are given by

$$x_1 = 1 - \frac{S}{a} \qquad\qquad x_2 = 1 - \frac{S - \overline{R_1R_2}}{a}$$

$$S = \tfrac{1}{2}(R_1 + R_2 + \overline{R_1R_2})$$

The eccentricity is then given by

(13) $$\epsilon = \left\{1 - \frac{2}{R_1 R_2{}^2} (S-R_1)(S-R_2)(1-x_1x_2 + \sqrt{1-x^2{}_1} \cdot \sqrt{1-x^2{}_2})\right\}^{\frac{1}{2}}$$

If we now substitute $T = t_{CA} - t_o$, $\vec{R}_1 = \vec{c}_o$ and $\vec{R}_2 = \vec{c}_{CA}$ into (12) a_1 can be calculated. Employing this in (13) the corresponding eccentricity ϵ_1 can be calculated. Consequently, since $\ell_1 = a_1(1- \epsilon^2{}_1)$ the semi-latus rectum ℓ_1 is obtained. These calculated values of a_1, ϵ_1 and ℓ_1 will clearly be very close to the exact values. Now by energy considerations a vehicle passing from $\vec{c}_o$ to $\vec{c}_{CA}$ on an elliptical trajectory will have an angular momentum $\vec{h}_1$ given by

(14) $$\vec{h}_1 = \pm \frac{\vec{c}_o \times \vec{c}_{CA}}{|\vec{c}_o \times \vec{c}_{CA}|}\sqrt{\mu_s \ell_1}$$

where the positive or negative sign is chosen so that

$$\vec{h}_1 \cdot \vec{h}_p > 0$$

where $\vec{h}_p$ is the angular momentum of the launch planet about the sun. This can be easily seen by (2) with the aid of (7).

Consider the problem of calculating the vector $\vec{\epsilon}$ corresponding to an elliptical path if ϵ and two points $\vec{R}_1$ and $\vec{R}_2$ on the path are known. The property which distinguishes a long-time elliptical path from a short-time path is that in the former case the chord joining $\vec{R}_1$ and $\vec{R}_2$ intersects the line segment $\overline{F\vec{F}^*}$ joining the two foci F and $\vec{F}^*$. For each of these cases we consider all the possible situations. Notice that (6) and (7) imply

(15) $$\cos \theta = \left(\frac{\ell}{R} - 1\right) \frac{1}{\epsilon}$$

Thus writing

$$\cos \theta_1 = \left(\frac{\ell}{R_1} - 1\right) \frac{1}{\epsilon} \qquad \cos \theta_2 = \left(\frac{\ell}{R_2} - 1\right) \frac{1}{\epsilon}$$

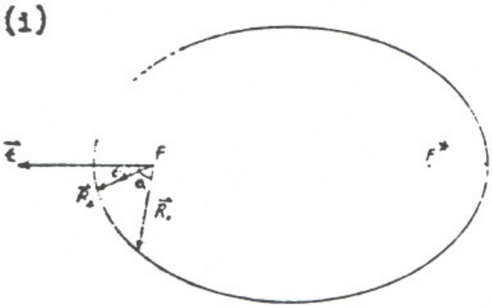
we consider

$$\left.\begin{array}{lll}
(1.1) & \cos\theta_1 \geqq 0 & \cos\theta_2 \geqq 0 \\
(1.2) & \cos\theta_1 \geqq 0 & \cos\theta_2 < 0 \\
(1.3) & \cos\theta_1 < 0 & \cos\theta_2 \geqq 0 \\
(1.4) & \cos\theta_1 < 0 & \cos\theta_2 < 0
\end{array}\right\} R_1 > R_2$$

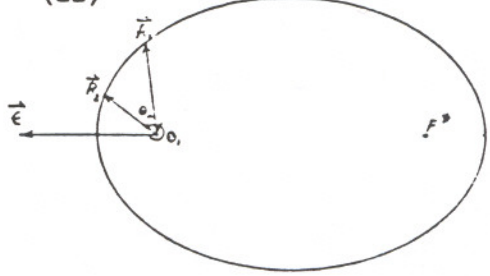
$$\left.\begin{array}{lll}
(2.1) & \cos\theta_1 \geqq 0 & \cos\theta_2 \geqq 0 \\
(2.2) & \cos\theta_1 \geqq 0 & \cos\theta_2 < 0 \\
(2.3) & \cos\theta_1 < 0 & \cos\theta_2 \geqq 0 \\
(2.4) & \cos\theta_1 < 0 & \cos\theta_2 < 0
\end{array}\right\} R_1 < R_2$$

Let us first take up the case of "short-time elliptical trajectories" and assume that (1.1) - (2.4) correspond to this case. For "long-time elliptical paths" the above eight situations again exhaust all the possibilities and, for convenience, we assume they are numbered (3.1) - (4.4), respectively. Thus it is clear that cases (1.2), (2.3), (3.1), (3.2), (4.1) and (4.3) are impossible. (In all cases we assume that the vehicle passes from $\vec{R}_1$ to $\vec{R}_2$ in counter-clockwise sense.) Now for case (1.1) we may have the following sub-cases:

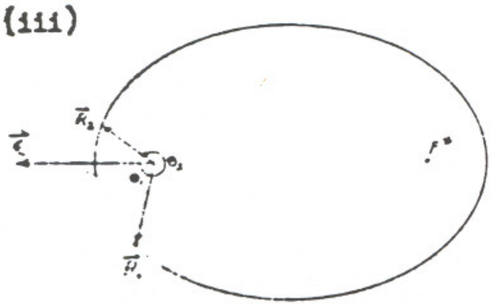
(i)



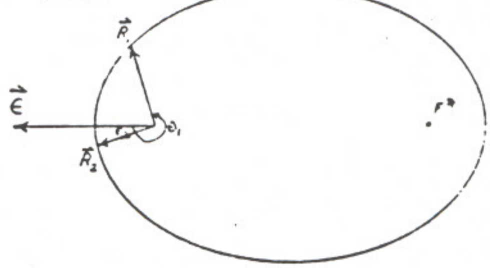
(ii)



(iii)



(iv)



By elementary trigonometry it is easy to see that the first two cases yield

$$\vec{\epsilon} = \frac{\sqrt{1 - \cos^2 \theta_1}\ \hat{R}_2 - \sqrt{1 - \cos^2 \theta_2}\ \hat{R}_1}{\left|\sqrt{1 - \cos^2 \theta_1}\ \hat{R}_2 - \sqrt{1 - \cos^2 \theta_2}\ \hat{R}_1\right|}\ \epsilon$$

or

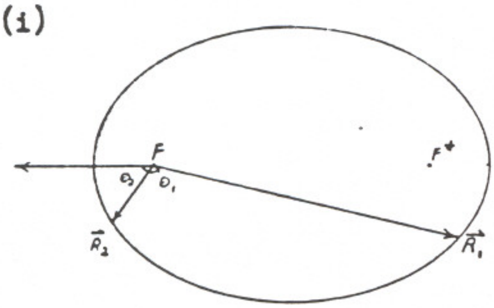
$$(1.11) \quad \vec{\epsilon} = \frac{\sqrt{R_1^2\epsilon^2 - (\ell - R_1)^2}\;\vec{R}_2 - \sqrt{R_2^2\epsilon^2 - (\ell - R_2)^2}\;\vec{R}_1}{\left|\sqrt{R_1^2\epsilon^2 - (\ell - R_1)^2}\;\vec{R}_2 - \sqrt{R_2^2\epsilon^2 - (\ell - R_2)^2}\;\vec{R}_1\right|}\,\epsilon$$

For the last two sub-cases (iii) and (iv)

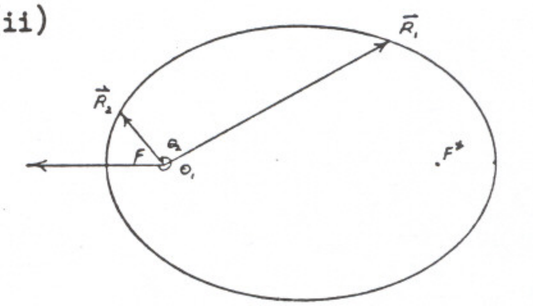
$$(1.12) \quad \vec{\epsilon} = \frac{\sqrt{R_1^2\epsilon^2 - (\ell - R_1)^2}\;\vec{R}_2 + \sqrt{R_2^2\epsilon^2 - (\ell - R_2)^2}\;\vec{R}_1}{\left|\sqrt{R_1^2\epsilon^2 - (\ell - R_1)^2}\;\vec{R}_2 + \sqrt{R_2^2\epsilon^2 - (\ell - R_2)^2}\;\vec{R}_1\right|}\,\epsilon$$

In case (1.3) is true the following sub-cases are possible:

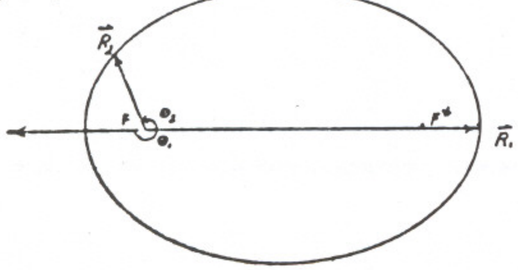
(i)



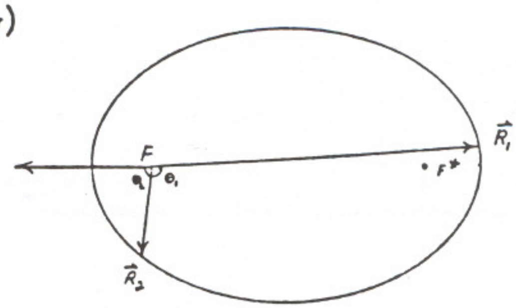
(ii)



(iii)



(iv)



Notice that in cases (iii) and (iv), $\vec{R}_1$ and $\vec{R}_2$ are on opposite sides of $\overline{F\,F^*}$ but $\overline{R_1R_2}$ does not intersect $\overrightarrow{FF^*}$.

The sub-cases (i) and (ii) yield

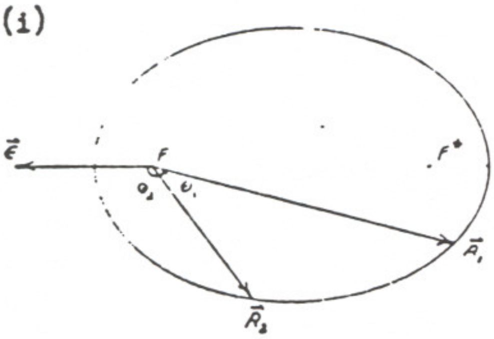
$$(1.31)\quad \vec{\epsilon} = \frac{\sqrt{R^2_1\,\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\,\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1}{\left|\sqrt{R^2_1\,\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\,\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1\right|}\ \epsilon$$

and the sub-cases (iii), (iv) yield

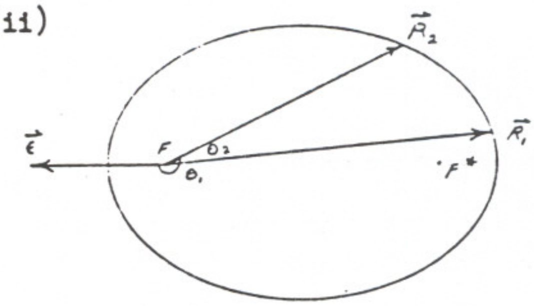
$$(1.32)\quad \vec{\epsilon} = \frac{-\sqrt{R^2_1\,\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\,\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1}{\left|-\sqrt{R^2_1\,\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\,\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1\right|}\ \epsilon$$

If the case (1.4) for the "short-time elliptical path" is true each of the following sub-cases may be true:

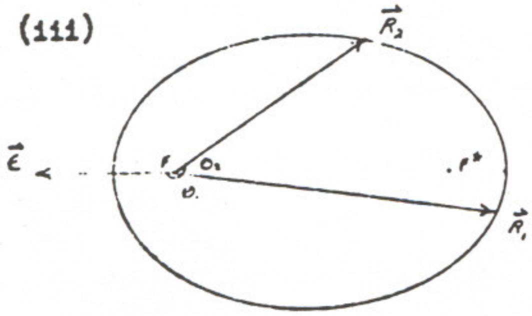
(i)



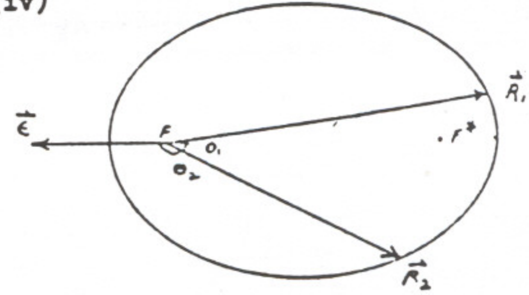
(ii)



(iii)



(iv)



The sub-cases (i) and (ii) yield the same formula for $\vec{\epsilon}$ given by

$$(1.41) \qquad \epsilon = \frac{\sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1}{\left|\sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1\right|}\ \epsilon$$

and for (iii) and (iv)

$$(1.42) \qquad \vec{\epsilon} = \frac{-\sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1}{\left|-\sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1\right|}\ \epsilon$$

When $R_1 < R_2$ for the cases (2.1) - (2.4), $\vec{R}_1$, θ_1 and $\vec{R}_2$, θ_2 in the cases (1.1-1.4) are simply reversed. Hence

$$(2.11) \qquad \vec{\epsilon} = \frac{\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2\right|}\ \epsilon$$

$$(2.12) \qquad \vec{\epsilon} = \frac{\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 + \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 + \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2\right|}\ \epsilon$$

$$(2.21) \qquad \vec{\epsilon} = \frac{\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2\right|}\ \epsilon$$

$$(2.22) \qquad \vec{\epsilon} = \frac{-\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|-\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2\right|}\ \epsilon$$

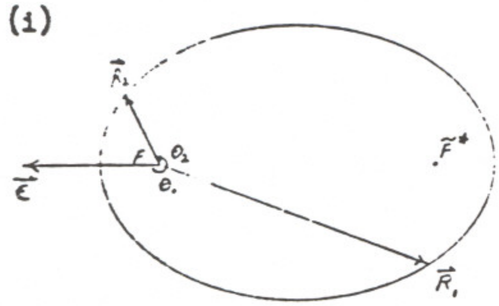
$$(2.41) \qquad \vec{\epsilon} = \frac{\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|\sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2_1\ \ ^2 - (\ \ - R_1)^2}\ \vec{R}_2\right|}\ \epsilon$$

$$(2.42)\qquad \vec{\epsilon} = \frac{-\sqrt{R_2^2\epsilon^2-(\ell-R_2)^2}\;\vec{R}_1-\sqrt{R_1^2\epsilon^2-(\ell-R_1)^2}\;\vec{R}_2}{\left|-\sqrt{R_2^2\epsilon^2-(\ell-R_2)^2}\;\vec{R}_1-\sqrt{R_1^2\epsilon^2-(\ell-R_1)^2}\;\vec{R}_2\right|}\;\epsilon$$

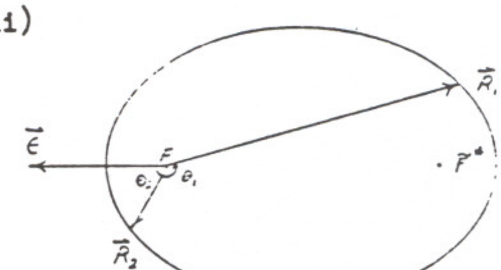
where (2.11) and (2.12) correspond to (i), (ii) and (iii), (iv) of case (1.1) with $\vec{R}_1$ and $\vec{R}_2$ reversed; (2.21) and (2.22) correspond to (i), (ii) and (iii), (iv) of case (1.3), respectively, with $\vec{R}_1$ and $\vec{R}_2$ reversed; (2.41) and (2.42) correspond to (i), (ii) and (iii), (iv), respectively, of case (1.4) with R_1 and R_2 reversed.

In the case of "long-time elliptical paths" the chord joining $\vec{R}_1$ and $\vec{R}_2$ intersects the line segment $\overline{F\tilde{F}^*}$ joining the two foci. Consequently, there are only two sub-cases to be considered for each of the cases (3.3), (3.4), (4.2) and (4.4). In the case of (3.3)

(i)



(ii)

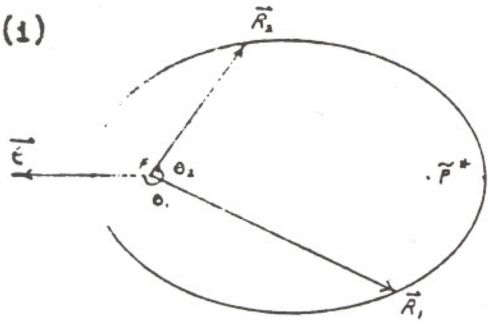


Both of these sub-cases yield

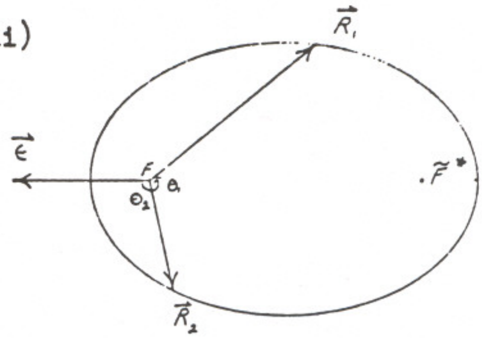
$$(3.31)\qquad \vec{\epsilon} = \frac{-\sqrt{R_1^2\epsilon^2-(\ell-R_1)^2}\;\vec{R}_2-\sqrt{R_2^2\epsilon^2-(\ell-R_2)^2}\;\vec{R}_1}{\left|-\sqrt{R_1^2\epsilon^2-(\ell-R_1)^2}\;\vec{R}_2-\sqrt{R_2^2\epsilon^2-(\ell-R_2)^2}\;\vec{R}\right|}\;\epsilon$$

For the case (3.4)

(i)



(ii)



and in both cases

$$(3.41) \qquad \vec{\epsilon} = \frac{-\sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1}{\left|-\sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1\right|}\ \epsilon$$

and is exactly the same as (3.31). For the cases (4.2) and (4.4) we simply interchange $\vec{R}_1$ and $\vec{R}_2$ yielding

(4.21),(4.41)

$$\vec{\epsilon} = \frac{-\sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|-\sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2\right|}\ \epsilon$$

which is the same as (3.41). Now for the case of "short-time elliptical paths" we find by observing the above figures that the sub-cases (i) and (ii) are more desirable than (iii) and (iv). Consequently, we write

$$(16) \qquad \vec{\epsilon} = \frac{\sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1}{\left|\sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2 - \sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1\right|}\ \epsilon \quad \text{if } R_1 > R_2$$

$$(17) \qquad \vec{\epsilon} = \frac{\sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2}{\left|\sqrt{R^2{}_2\epsilon^2 - (\ell - R_2)^2}\ \vec{R}_1 - \sqrt{R^2{}_1\epsilon^2 - (\ell - R_1)^2}\ \vec{R}_2\right|}\ \epsilon \quad \text{if } R_1 < R_2$$

In the case of long-time elliptical paths is given by

$$(18) \quad \vec{\epsilon} = \frac{-\sqrt{R_2^2\epsilon^2 - (\ell - R_2)^2}\,\vec{R}_1 - \sqrt{R_1^2\epsilon^2 - (\ell - R_1)^2}\,\vec{R}_2}{\left|-\sqrt{R_2^2\epsilon^2 - (\ell - R_2)^2}\,\vec{R}_1 - \sqrt{R_1^2\epsilon^2 - (\ell - R_1)^2}\,\vec{R}_2\right|}\,\epsilon$$

for both cases $R_1 > R_2$ and $R_1 < R_2$. If the formulas (16) and (17) do not yield desired solutions to the round-trip problem, one may try replacing (16) with: (1.12) if $\frac{\ell}{R_1} - 1$ and $\frac{\ell}{R_2} - 1$ are both positive; (1.32) if $\frac{\ell}{R_1} - 1 < 0$; and replace (17) with (2.12) if $\frac{\ell}{R_1} - 1 \geq 0$ and $\frac{\ell}{R_2} - 1 \geq 0$, (2.22) if $\frac{\ell}{R_2} - 1 < 0$.

Returning to the problem of finding a round-trip trajectory to one target planet we may now calculate the vector $\vec{\epsilon}_1$. Since the departing elliptical trajectory is assumed to be a short-time elliptical path, $\vec{\epsilon}_1$ is calculated by (16) or (17) if $c_o > c_{CA}$ or $c_o < c_{CA}$, respectively, by substituting $\vec{R}_1 = \vec{c}_o$, $\vec{R}_2 = \vec{c}_{CA}$, $\ell = \ell_1$, and $\epsilon = \epsilon_1$.

Now as the vehicle passes the target planet it will interact with the target planet's gravitational field (i.e., with the target planet's motion) and, consequently, energy will be exchanged. Thus, in general, the total energy E_1 of the vehicle with respect to Σ before entering $\mathcal{T}$ will be different from the total energy E_2 of the vehicle after leaving $\mathcal{T}$. Hence, since total energy = potential energy + kinetic energy we write by employing (8)

$$E_1 = -\frac{\mu_s}{r_1} + \tfrac{1}{2} V_1^2 \approx -\frac{\mu_s}{c_{CA}} + \mu_s\left(\frac{1}{c_{CA}} - \frac{1}{2a_1}\right)$$

$$E_2 = -\frac{\mu_s}{r_2} + \tfrac{1}{2} V_2^2 \approx -\frac{\mu_s}{c_{CA}} + \mu_s\left(\frac{1}{c_{CA}} - \frac{1}{2a_3}\right)$$

Hence, in general, $a_1 \neq a_3$. In some cases, if $T = t_{CA} - t_o$ is near T_o, the effect of the target planet may increase the vehicle's total energy such that it passes out of $\mathcal{T}$ on a hyperbolic trajectory relative to the sun. This situation will not be considered since high initial energies would have to be imparted to the vehicle at the beginning of its journey. If, perhaps, this is desired one may still solve the problem by using different formulae, all of which appear in this paper. Thus we assume the vehicle returns to the launch planet on an elliptical path. This returning trajectory may be either a short-time or a long-time elliptical path. The vehicle may also make one or more circuits of the sun on its returning trajectory before intercepting its launch planet. The time T, which the vehicle requires to pass from $\overline{R_1}$ to $\overline{R_2}$ after first making k complete circuits of the sun, corresponding to these situations is given by

(19) $$T = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1-x^2{}_2} + \sin^{-1}x_2 - \sqrt{1-x^2{}_1} - \sin^{-1}x_1 \right\} + kP$$

and

(20) $$\widetilde{T} = \sqrt{\frac{a^3}{\mu}} \left\{ \pi + \sqrt{1-x^2{}_2} + \sin^{-1}x_2 + \sqrt{1-x^2{}_1} + \sin^{-1}x_1 \right\} + kP$$

respectively, where a is its semi-major axis, x_1 x_2 are as given in (12) and $P = 2\pi \sqrt{\frac{a^3}{\mu}}$ is the time the vehicle takes to complete one circuit of the sun. The eccentricity corresponding to (19) is given by (13) and corresponding to the case of long-time elliptical paths it is given by

(21) $$\widetilde{\epsilon} = \left\{ 1 - \frac{2}{R_1R_2} 2\,(S-R_1)\,(S-R_2)\,(1-x_1x_2 - \sqrt{1-x_1{}^2}\,\sqrt{1-x^2{}_2}) \right\}^{\frac{1}{2}}$$

It should be borne in mind that at this point we are free in theory to make the following statements:

The vehicle returns on a short-time elliptical path with k = o or k = 1 or any arbitrary positive integer.

The vehicle returns on a long-time elliptical path with k = o or k = 1 or any arbitrary positive integer.

These statements are possible because of the unique way of stating the initial conditions. The determining factor will be the distance of closest approach corresponding to each of the above choices.

In most cases short flight times will be desired hence we set k = o and assume that the vehicle returns on a short-time elliptical path. Since $\vec{r}_2$ is very close to $\vec{c}_{CA}$ we substitute $T = t_3 - t_{CA}$, $a = a_3$, $\vec{R}_1 = \vec{c}_{CA}$, $\vec{R}_2 = \vec{c}_3 = \vec{P}(t_3)$ and $\overline{R_1R_2} = u(t_3)$ into (19) where $\vec{P}(t_3)$ and $u(t_3)$ are known vector and scalar functions of the variable t_3. Consequently, we have an equation relating a_3 and t_3:

$$(22)\quad t_3 - t_{CA} = \sqrt{\frac{a_3^3}{\mu_a}}\left\{\sqrt{1-\left[1-\frac{c_{CA}+P(t_3)-u(t_3)}{2a_3}\right]^2} + \sin^{-1}\left[1-\frac{c_{CA}+P(t_3)-u(t_3)}{2a_3}\right] - \sqrt{1-\left[1-\frac{c_{CA}+P(t_3)+u(t_3)}{2a_3}\right]^2} - \sin^{-1}\left[1-\frac{c_{CA}+P(t_3)+u(t_3)}{2a_3}\right]\right\}$$

Since reconnaissance trajectory problems will employ large digital computers, the functional relation between a_3 and the variable t_3, $a_3 = a_3(t_3)$ expressed by the above equation, shall be taken to represent a large table of numerical values of a_3 corresponding to a set of reasonable values of the variable t_3. These values of a_3 corresponding to various values of t_3 can easily be calculated by the method given in the above reference. Henceforth, if $f(x)$ and $\vec{F}(x)$ are any known scalar and vector functions of a variable x we shall think of $f(x)$ and $\vec{F}(x)$ as tables of f and $\vec{F}$ versus x calculated over some set $\{x_i\}$ of the variable and stored in the computer.

Consequently, substituting the tables $a = a_{3i}$, $\vec{R}_2 = \vec{P}(t_{3i})$, $\overline{R_1R_2} = \overline{c_{CA}P(t_{3i})}$ into (13), $\epsilon_3(t_3)$ is obtained. We then calculate $\vec{\epsilon}_3(t_3)$ by (16) or (17) depending on whether $c_{CA} > c_3$ or $c_{CA} < c_3$, respectively, and $\ell = \ell_3(t_3) = a_3(t_3)(1-\epsilon^2_3(t_3))$, $\vec{R}_1 = \vec{c}_{CA}$, $\vec{R}_2 = \vec{P}(t_3)$, $\epsilon = \epsilon_3(t_3)$ and calculate $\vec{h}_3(t_3)$ by (14) with $\ell_3(t_3) = a_3(t_3)(1-\epsilon^2_3(t_3))$ replacing ℓ_1, $\vec{c}_{CA}$ replacing $\vec{c}_0$ and $\vec{P}(t_3)$ replacing $\vec{c}_{CA}$.

Now if the vehicle is in $\mathcal{T}$

$$(23)\qquad \vec{r} = \vec{c} + \vec{\rho}$$

where $\vec{r}$ is the position vector of the vehicle and $\vec{c}$ is the position vector of the target planet with respect to Σ. The vector $\vec{\rho}$ is the position vector of the vehicle with respect to Σ'. If $\vec{V}_Q$ denotes the velocity vector of the target planet with respect to Σ at time t_{CA}, then according to the second basic assumption, differentiating the above equation yields

$$(24)\qquad \vec{V} = \vec{V}_Q + \vec{V}'$$

where $\vec{V}$ and $\vec{V}'$ are the velocity vectors of the vehicle with respect to Σ and Σ', respectively. Consequently, (24) yields

$$\vec{V}_1 = \vec{V}_Q + \vec{V}'_1 \qquad\qquad \vec{V}_2 = \vec{V}_Q + \vec{V}'_2$$

$$\therefore \quad V^2_1 = \vec{V}_1 \cdot \vec{V}_1 = V_Q^2 + 2\vec{V}_Q \cdot \vec{V}'_1 + V_1'^2$$

$$V^2_2 = \vec{V}_2 \cdot \vec{V}_2 = V^2_Q + 2\vec{V}_Q \cdot \vec{V}'_2 + V'^2_2$$

In view of (8)

$$V_1'^2 = \mu_Q\left(\frac{2}{\rho_1} + \frac{1}{a_2}\right) \qquad V'^2_2 = \mu_Q\left(\frac{2}{\rho_2} + \frac{1}{a_2}\right)$$

Hence since $\rho_1 = \rho_2$,

$$V_1'^2 = V_2'^2$$

Thus

$$V^2_2 - V^2_1 = 2\vec{V}_Q \cdot (\vec{V}'_2 - \vec{V}'_1)$$

Now from (24) $\vec{V}_2 - \vec{V}_1 = \vec{V}'_2 - \vec{V}'_1$

(25) $$\therefore \quad V^2_2 - V^2_1 = 2\vec{V}_Q \cdot (\vec{V}_2 - \vec{V}_1)$$

By (5) the vector $\vec{V}_1$ can be calculated by

$$\vec{V}_1 = \frac{1}{\ell_1} \vec{h}_1 \times (\hat{c}_{CA} + \vec{\epsilon}_1)$$

since $\vec{r}_1$ is very close to $\vec{c}_{CA}$. The table $\vec{V}_2(t_3)$ may also be calculated by

$$\vec{V}_2(t_3) = \frac{1}{\ell_3(t_3)} \vec{h}_3(t_3) \times (\hat{c}_{CA} + \vec{\epsilon}_3(t_3))$$

Making use of (8) we write

$$V^2_1 = \mu_s\left(\frac{2}{c_{CA}} - \frac{1}{a_1}\right) \qquad\qquad V^2_2(t_3) = \mu_s\left(\frac{2}{c_{CA}} - \frac{1}{a_3(t_3)}\right)$$

Substituting the above results into (25) we obtain

(26) $$\mu_s\left(\frac{1}{a_1} - \frac{1}{a_3(t_3)}\right) = 2\vec{V}_Q \cdot \left[\frac{1}{\ell_3(t_3)} \vec{h}_3(t_3) \times \left(\hat{c}_{CA} + \vec{\epsilon}_3(t_3)\right) - \frac{1}{\ell_1} \vec{h}_1 \times (\hat{c}_{CA} + \vec{\epsilon}_1)\right].$$

The solution of this important equation is obtained by comparing the table

$$2\vec{V}_Q \cdot (\vec{V}_2(t_3) - \vec{V}_1)$$

with the table

$$\mu_s\left(\frac{1}{a_1} - \frac{1}{a_3(t_3)}\right)$$

and finding that value of t_3 which gives the corresponding entries in the two tables identical (or nearly identical) values.

After obtaining a solution t_3 of (26) the quantities

$$a_3 = a_3(t_3) \quad \vec{\epsilon}_3 = \vec{\epsilon}_3(t_3) \quad \vec{\ell}_3 = \vec{\ell}_3(t_3) \quad \vec{h}_3 = \vec{h}_3(t_3)$$

are calculated. Hence by (24) and (5)

(27) $$\vec{V}'_1 = \vec{V}_1 - \vec{V}_Q = \frac{1}{\ell_1} \vec{h}_1 \times (\hat{c}_{CA} + \vec{\epsilon}_1) - \vec{V}_Q$$

(28) $$\vec{V}'_2 = \vec{V}_2 - \vec{V}_Q = \frac{1}{\ell_3} \vec{h}_3 \times (\vec{c}_{CA} + \vec{\epsilon}_3) - \vec{V}_Q$$

and ${V'_1}^2$, ${V'_2}^2$ are calculated. Since in theory $V'_1 = V'_2$ we compute the average

(29) $$\bar{V}'^2 = \tfrac{1}{2} ({V'_1}^2 + {V'_2}^2)$$

and employ (8) to obtain

(30) $$a_2 = \frac{\rho^* \mu_Q}{\bar{V}'^2 \rho^* - 2 \mu_Q} \approx \frac{\mu_Q}{\bar{V}'^2}$$

For hyperbolic conic sections the eccentricity ϵ is given by

$$\epsilon = \sqrt{1 + (\tfrac{b}{a})^2}$$

where according to the figure on page 2

$$\tan \phi_\infty = \frac{b}{a}$$

Consequently by the third basic assumption

$$\epsilon_2^2 = 1 + \tan^2 \phi_{\rho^*} = \sec^2 \phi_{\rho^*}$$

$$\therefore \cos \phi_{\rho^*} = \frac{1}{\epsilon_2}$$

Now from the figure we write

$$\vec{V}'_1 \cdot \vec{V}'_2 = V'_1 V'_2 \cos 2 (\tfrac{\pi}{2} - \phi_{\rho^*})$$
$$= - V'_1 V'_2 \cos 2 \phi_{\rho^*}$$
$$= V'_1 V'_2 (1 - 2 \cos^2 \phi_{\rho^*})$$

$$\therefore \cos^2 \rho^* = \frac{V_1' \cdot V_2' - \vec{V}_1' \cdot \vec{V}_2'}{2V_1' V_2'}$$

from which we obtain

$$\epsilon_2 = \sqrt{\frac{2V_1' \cdot V_2'}{V_1' \cdot V_2' - \vec{V}_1' \cdot \vec{V}_2'}} \tag{31}$$

After calculating a_2 and ϵ_2 by (30) and (31) the distance of closest approach d may be calculated by

$$d = a_2 (\epsilon_2 - 1) - R_Q$$

which is evident from the figure of the hyperbola.

If the distance of closest approach turns out to be negative we conclude that the vehicle cannot return to the launch planet on a short-time elliptical path without first making at least one complete circuit of the sun. Consequently, returning to the choices given on page 20, we may let the vehicle make one circuit of the sun before it intercepts the launch planet thus changing $k = 0$ to $k = 1$ which will simply result in the addition of the term

$$2\pi \sqrt{\frac{a_3^3}{\mu_s}}$$

to the right hand side of (22) and repeat the calculations using the same formulas. On the other hand if we assume that the vehicle returns on a long-time elliptical path without making a complete circuit of the sun, then if the resulting distance of closest approach turns out to be positive a shorter flight time will be possible. Hence, we replace (22) by

$$(32) \quad t_3 - t_{CA} = \sqrt{\frac{a_3^3}{\mu_S}} \left\{ \pi + \sqrt{1 - \left[1 - \frac{c_{CA} + P(t_3) + u(t_3)}{2a_3}\right]^2} + \sin^{-1}\left[1 - \frac{c_{CA} + P(t_3) - u(t_3)}{2a_3}\right] + \sqrt{1 - \left[1 - \frac{c_{CA} + P(t_3) + u(t_3)}{2a_3}\right]^2} + \sin^{-1}\left[1 - \frac{c_{CA} + P(t_3) + u(t_3)}{2a_3}\right] \right\}$$

The machine shall then proceed by calculating a new table $a_3 = a_3(t_3)$ corresponding to the set $\{t_{3i}\}$ of expected values of t_3 by solving for a_3 in (32) for each t_{3i} of $\{t_{3i}\}$. Substituting this table (i.e., each entry of the table corresponding to each t_{3i}) into (21) we obtain the table $\epsilon_3(t_3)$. The table of vectors $\vec{\epsilon}_3(t_3)$ is calculated by (18) with $\ell = \ell_3(t_3) = a_3(t_3)(1 - \epsilon_3^2(t_3))$, $\hat{R}_1 = \hat{c}_{CA}$, $\hat{R}_2 = \vec{P}(t_3)$ and $\epsilon = \epsilon_3(t_3)$. After obtaining the table $\vec{\epsilon}_3(t_3)$ the above steps are repeated using the same formulas. If the resulting value of d remains negative and the initial conditions t_o, t_{CA} remain unchanged the vehicle must make at least one circuit of the sun before it can return to the launch planet after leaving the vicinity of the target planet. Consequently one is forced to change the initial values of t_o, t_{CA} and perhaps consider long-time departing trajectories.

Suppose one of the above calculations of d yields a reasonable value for the distance of closest approach. Then since $\ell_2 = a_2(\epsilon_2^2 - 1)$, an application of (7) yields

$$(33) \qquad h_2 = \sqrt{a_2(\epsilon_2^2 - 1)\,\mu_Q}$$

Since $\vec{\epsilon}_2$ is along the direction of perihelion with respect to Σ' and since $V_1' = V_2'$ the vector $\vec{\epsilon}_2$ may easily be calculated by the formula

$$(34) \qquad \vec{\epsilon}_2 = \frac{\vec{V}_1' - \vec{V}_2'}{\left|\vec{V}_1' - \vec{V}_2'\right|} \quad \epsilon_2$$

where $\vec{V}_1'$ and $\vec{V}_2'$ are given by (27) and (28). Also since $\hat{h}_2$ is perpendicular to the plane of motion in Υ and passes from $\vec{\rho}_1$ to $\vec{\rho}_2$ with respect to $\sum'$ the vector $\vec{h}_2$ may be obtained by the formula

(35)
$$\vec{h}_2 = \frac{\vec{V}_1' \times \vec{V}_2'}{|\vec{V}_1' \times \vec{V}_2'|} \quad h_2$$

Employing (3) we obtain

(36)
$$\vec{\rho}_1 = \left(\frac{1}{\mu_Q} \vec{V}_1' \times \vec{h}_2 - \vec{\epsilon}_2\right)\left(\frac{m}{M}\right)^{\frac{2}{5}} \quad c_1$$

(37)
$$\vec{\rho}_2 = \left(\frac{1}{\mu_Q} \vec{V}_2' \times \vec{h}_2 - \vec{\epsilon}_2\right)\left(\frac{m}{M}\right)^{\frac{2}{5}} \quad c_2$$

$(c_1 \approx c_2 \approx c_{CA})$

The amount of time Δt the vehicle spends in Υ can be calculated by (11) substituting $a = a_2$, $\epsilon = \epsilon_2$, $\mu = \mu_Q$, $c = c_{CA}$ with M and m equal to the mass of the sun and target planet respectively consequently

(38)
$$t_1 = t_{CA} - \tfrac{1}{2}\Delta t \qquad\qquad t_2 = t_{CA} + \tfrac{1}{2}\Delta t$$

from which $\vec{c}_1$ and $\vec{c}_2$ can be obtained

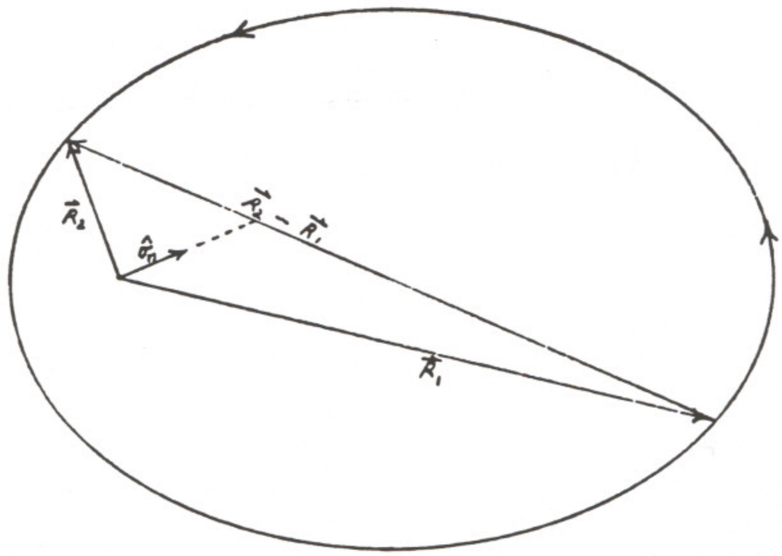
$$\vec{c}_1 = \vec{Q}(t_1) \qquad\qquad \vec{c}_2 = \vec{Q}(t_2)$$

Thus by (23)

(39)
$$\vec{r}_1 = \vec{c}_1 + \vec{\rho}_1 \qquad\qquad \vec{r}_2 = \vec{c}_2 + \vec{\rho}_2$$

The solution will be complete when the position vector and corresponding velocity vector of the vehicle are found as functions of time. Since this will involve a great amount of computation the quantities $\vec{\epsilon}_1$, $\vec{h}_1$, $\vec{\epsilon}_2$, $\vec{h}_2$, $\vec{\epsilon}_3$, $\vec{h}_3$ and $\vec{r}_1$, $\vec{r}_2$, $\vec{\rho}_1$, $\vec{\rho}_2$, should be refined. This can be easily accomplished by a method of successive approximations given in the appendix.

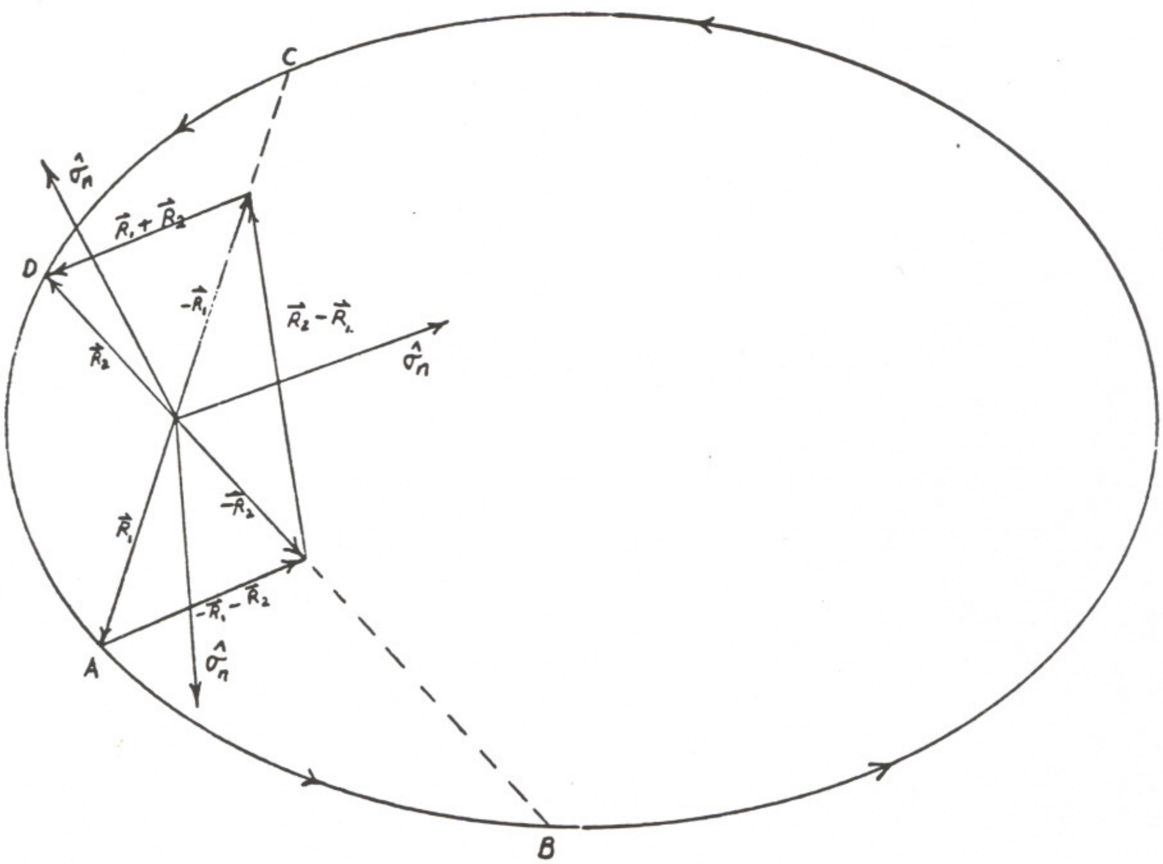
We now develop some important general formulas from which the complete solution may be calculated. Consider any elliptical trajectory which takes a vehicle from $\vec{R}_1$ to $\vec{R}_2$ such that $\measuredangle \vec{R}_1\vec{R}_2 < 180^\circ$.



Thus from this figure it is clear that the vectors

(40)
$$\hat{\sigma}_n = \frac{\vec{R}_1 + \frac{n}{N}(\vec{R}_2 - \vec{R}_1)}{\left|\vec{R}_1 + \frac{n}{N}(\vec{R}_2 - \vec{R}_1)\right|} \qquad (n = 0, 1, \ldots N)$$

represent a set $\{\hat{\sigma}_n\}$ of unit position vectors of the vehicle as it passes from $\vec{R}_1$ to $\vec{R}_2$. Notice that if $n = 0$, $\hat{\sigma}_o = \hat{R}_1$ and $n = N$ yields $\hat{\sigma}_N = \hat{R}_2$. If $\measuredangle \vec{R}_1\vec{R}_2 > 180^\circ$ the set $\{\hat{\sigma}_n\}$ $(n = 0, 1, 2, \ldots N)$ is obtained by constructing three subsets $\{\sigma_n\}$ $(n = 0, 1, \ldots N_1 - 1)$, $\{\sigma_n\}$ $(n = N_1, N_1 + 1, \ldots N_1 + N_2 - 1)$ and $\{\sigma_n\}$ $(n = N_1 + N_2, N_1 + N_2 + 1, \ldots N_1 + N_2 + N_3 = N)$.


C
D
A
B
$\hat{\sigma}_n$
$\vec{R}_1 + \vec{R}_2$
$-\vec{R}_1$
$\vec{R}_2 - \vec{R}_1$
$\hat{\sigma}_n$
$\vec{R}_2$
$\vec{R}_1$
$-\vec{R}_2$
$-\vec{R}_1 - \vec{R}_2$
$\hat{\sigma}_n$

(41.1) $$\hat{\sigma}_n = \frac{\vec{R}_1 + \frac{n}{N'}(-\vec{R}_2 - \vec{R}_1)}{\left|\vec{R}_1 + \frac{n}{N'}(-\vec{R}_2 - \vec{R})\right|} \qquad (n = 0,1, \ldots N' - 1)$$

(41.2) $$\hat{\sigma}_n = \frac{-\vec{R}_2 + \frac{n-N'}{N''}(\vec{R}_2 - \vec{R}_1)}{\left|-\vec{R}_2 + \frac{n-N'}{N''}(\vec{R}_2 - \vec{R}_1)\right|} \qquad (n = N', N' + 1, \ldots N' + N'' - 1)$$

(41.3) $$\hat{\sigma}_n = \frac{-\vec{R}_1 + \frac{n-N'-N''}{N'''}(\vec{R}_1 + \vec{R}_2)}{\left|-\vec{R}_1 + \frac{n-N'-N''}{N'''}(\vec{R}_1 + \vec{R}_2)\right|} \qquad (n = N'+N'', N' + N'' + 1, \ldots N' + N'' + N''' = N)$$

Notice that $\hat{\sigma}_o = \hat{R}_1$, $\hat{\sigma}_{N'} = -\hat{R}_2$, $\hat{\sigma}_{N'+N''} = -\hat{R}_1$ and $\hat{\sigma}_{N'+N''+N'''} = \hat{\sigma}_N = \hat{R}_2$. Thus this set $\{\hat{\sigma}_n\}$ $(n = 0,1, \ldots N)$ also represents a set of unit position vectors of the vehicle as it passes from $\vec{R}_1$ to $\vec{R}_2$ such that $\measuredangle\, \vec{R}_1\vec{R}_2 > 180^\circ$. Employing (5) the set $\{\vec{V}_n\}$ of velocity vectors corresponding to the vehicles set $\{\hat{\sigma}_n\}$ of unit position vectors can be calculated

(42) $$\vec{V}_n = \frac{1}{\ell}\vec{h} \times (\hat{\sigma}_n + \vec{\epsilon}). \qquad (n = 0,1, \ldots N)$$

From (6) the magnitude of the vehicles position vectors can be calculated. Thus

(43) $$\vec{\sigma}_n = \sigma_n\hat{\sigma}_n = \frac{\ell}{1+\hat{\sigma}_n \cdot \vec{\epsilon}}\,\hat{\sigma}_n .$$

The set $\{t_n\}$ corresponding to the time when the vehicle is at $\vec{\sigma}_n$ can be easily calculated by (19) for short-time trajectories or (20) for long-time elliptical paths by setting $k = 0$ and

$$x_1 = x_{n1} = 1 - \frac{S}{a} = 1 - \frac{R_1 + \sigma_n + \overline{\sigma_n R_1}}{2a}$$

$$x_{n1} = 1 - \frac{R_1 + \sigma_n + \sqrt{R_1^2 + \sigma_n^2 - 2\,\vec{R}_1 \cdot \vec{\sigma}_n}}{2a} -$$

$$x_2 = x_{n2} = 1 - \frac{S - \overline{R_1 \sigma_n}}{a} = 1 - \frac{R_1 + \sigma_n - \overline{\sigma_n R_1}}{2a}$$

$$x_{n2} = 1 - \frac{R_1 + \sigma_n - \sqrt{R_1^2 + \sigma_n^2 - 2\vec{R}_1 \cdot \vec{\sigma}_n}}{2a}$$

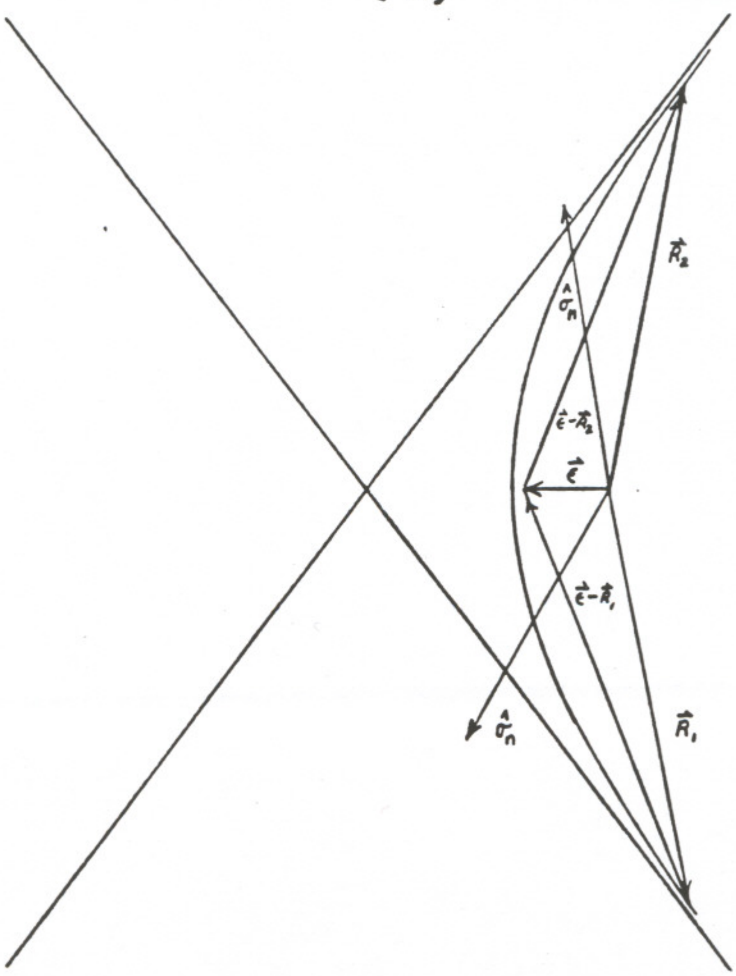
and $T = t_n - t_o$ where t_o is the known time when the vehicle is at $\vec{\sigma}_o = \vec{R}_1$

(44) $$\therefore t_n = t_o + \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{1 - x_{n2}^2} + \sin^{-1} x_{n2} - \sqrt{1 - x_{n1}^2} - \sin^{-1} x_{n1} \right\}$$

or

(45) $$t_n = t_o + \sqrt{\frac{a^3}{\mu}} \left\{ \pi + \sqrt{1 - x_{n2}^2} + \sin^{-1} x_{n2} + \sqrt{1 - x_{n1}^2} + \sin^{-1} x_{n1} \right\}$$

Consider the determination of $\left\{ \hat{\sigma}_n \right\}$ for hyperbolic trajectories.



Thus by the above figure we obtain the formulas

$$(46.1) \qquad \hat{\sigma}_n = \frac{\vec{R}_1 + \frac{n}{N'}(\vec{\epsilon} - \vec{R}_1)}{|\vec{R}_1 + \frac{n}{N'}(\vec{\epsilon} - \vec{R}_1)|} \qquad (n = 0, 1, \ldots, N' - 1)$$

$$(46.2) \qquad \hat{\sigma}_n = \frac{\vec{\epsilon} + \frac{n-N'}{N'}(\vec{R}_2 - \vec{\epsilon})}{|\vec{\epsilon} + \frac{n-N'}{N'}(\vec{R}_2 - \vec{\epsilon})|} \qquad (n = N', N' + 1, \ldots, 2N' = N)$$

The corresponding set of velocity vectors $\{\vec{V}_n\}$ can be calculated from (42) noting that in this case $\ell = a(\epsilon^2 - 1)$. The magnitude of the position vectors can be also obtained from (6) and hence by (43) the set $\{\hat{\sigma}_n\}$ can be calculated. The time the vehicle takes to pass from $\vec{R}_1$ to $\vec{R}_2$ on hyperbolic trajectories can be expressed as

$$(47) \qquad T = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{y_1^2 - 1} - \cosh^{-1} y_1 - \sqrt{y_2^2 - 1} + \cosh^{-1} y_2 \right\}$$

$$(48) \qquad \tilde{T} = \sqrt{\frac{a^3}{\mu}} \left\{ \sqrt{y_1^2 - 1} - \cosh^{-1} y_1 + \sqrt{y_2^2 - 1} - \cosh^{-1} y_2 \right\}$$

where

$$y_1 = 1 + \frac{S}{a} \qquad y_2 = 1 + \frac{S - \overline{R_1 R_2}}{a}$$

corresponding to short-time and long-time hyperbolic path respectively (see technical memo 312-118). If this flight time is known the type of hyperbolic path can be determined by substituting into (47) and (48) the values of y_1 and y_2 corresponding to $\vec{R}_1$ and $\vec{R}_2$ and observing which formula yields the correct flight time.

With the aid of these formulas the solution can be readily calculated. Let us now denote the times when the vehicle enters and leaves the sphere of influence of the target planet by t_{N_1} and t_{N_2} respectively. Let the time when the vehicle completes its journey be denoted by t_{N_3} instead of t_3.

Since we assume in the figures drawn for elliptic trajectories that the vehicle passes from $\vec{R}_1$ to $\vec{R}_2$ in a counter clockwise direction the angular momentum vector $\vec{h}$ always points up out of the paper. Thus the angle $\measuredangle \vec{c}_o \vec{R}_1 \leq 180^\circ$ if and only if

$$\frac{\vec{c}_o \times \vec{r}_1}{|\vec{c}_o \times \vec{r}_1|} - \hat{h}_1 = 0 \tag{49}$$

Consequently the first set of position and velocity vectors $\left\{\vec{\sigma}_n\right\}, \left\{\vec{V}_n\right\}$ at times t_n of the vehicle on its departing elliptical trajectory can be easily calculated by (40) or (41.1, 41.2, 41.3) depending on whether (49) is or is not satisfied respectively; employing the results in (42) and (43) to obtain $\left\{\vec{V}_n\right\}$ and $\left\{\vec{\sigma}_n\right\}$ and substituting $\left\{\vec{\sigma}_n\right\}$ into (44) (since for definiteness we have assumed a short-time elliptical deporting trajection) we obtain the corresponding set of times $\left\{t_n\right\}$. Thus if

$$\frac{\vec{c}_o \times \vec{r}_1}{|\vec{c}_o \times \vec{r}_1|} \approx h_1$$

$$\hat{\sigma}_n = \frac{\vec{c}_o + \frac{n}{N_1}(\vec{r}_1 - \vec{c}_o)}{|\vec{c}_o + \frac{n}{N_1}(\vec{r}_1 - \vec{c}_o)|} \qquad (n = 0, 1, \ldots N_1)$$

if

$$\frac{\vec{c}_o \times \vec{r}_1}{|\vec{c}_o \times \vec{r}_1|} \not\approx \hat{h}_1$$

$$\hat{\sigma}_n = \frac{\vec{c}_o - \frac{n}{N_1'}(\vec{r}_1 + \vec{c}_o)}{\left|\vec{c}_o - \frac{n}{N_1'}(\vec{r}_1 + \vec{c}_o)\right|} \qquad (n = 0, 1, \ldots N_1' - 1)$$

$$\hat{\sigma}_n = \frac{-\vec{r}_1 + \frac{n - N_1'}{N_1''}(\vec{r}_1 - \vec{c}_o)}{\left|-\vec{r}_1 + \frac{n - N_1'}{N_1''}(\vec{r}_1 - \vec{c}_o)\right|} \qquad (n = N_1', N_1' + 1, \ldots N_1' + N_1'' - 1)$$

$$\hat{\sigma}_n = \frac{-\vec{c}_o + \frac{n - N_1' - N_1''}{N_1'''}(\vec{c}_o + \vec{r}_1)}{\left|-\vec{c}_o + \frac{n - N_1' - N_1''}{N_1'''}(\vec{c}_o + \vec{r}_1)\right|} \qquad (n = N_1' + N_1'', N_1' + N_1'' + 1, \ldots, N_1' + N_1'' + N''' = N_1)$$

where N_1 is the total number of observations one wishes to carry out while the vehicle is on its departing trajectory. Referring to the figure on Page 27a the numbers N_1', N_1'' and N_1''' are the number of observations one wishes to perform when the vehicle is on the arcs $\widehat{AB}$, $\widehat{BC}$ and $\widehat{CD}$ respectively such that $N_1' + N_1'' + N_1''' = N_1$. After calculating this set the results are substituted into (42) to obtain a corresponding set $\{\vec{V}_n\}$ $(n = 0, 1, \ldots N_1)$ of velocity vectors.

$$\vec{V}_n = \frac{1}{\ell_1} \vec{h}_1 \times (\hat{\sigma}_n + \vec{\epsilon}_1) \qquad (n = 0, 1, \ldots N_1)$$

The set of position vectors $\{\vec{\sigma}_n\}$ is calculated by

$$\vec{\sigma}_n = \frac{\ell_1}{1 + \hat{\sigma}_n \cdot \vec{\epsilon}_1} \hat{\sigma}_n \qquad (n = 0, 1, \ldots N_1)$$

The corresponding set $\{t_n\}$ is calculated by

$$t_n = t_o + \sqrt{\frac{a_1^3}{\mu_S}} \left\{ \sqrt{1 - x_{n2}^2} + \sin^{-1} x_{n2} - \sqrt{1 - x_{n1}^2} - \sin^{-1} x_{n1} \right\}$$

where

$$x_{n1} = 1 - \frac{c_o + \sigma_n + \sqrt{c_o^2 + \sigma_n^2 - 2\,\vec{c}_o \cdot \vec{\sigma}_n}}{2a_1}$$

$$x_{n2} = 1 - \frac{c_o + \sigma_n - \sqrt{c_o^2 + \sigma_n^2 - 2\,\vec{c}_o \cdot \vec{\sigma}_n}}{2a_1}$$

$$(n = 0, 1, \ldots N_1)$$

When the vehicle enters the sphere of influence of the target planet its position and velocity vectors $\left\{\vec{\rho}_n\right\}$, $\left\{\vec{V}'_n\right\}$ along with $\left\{t_n\right\}$ can be obtained by first calculating $\left\{\hat{\rho}_n\right\}$ by (46.1) and (46.2). Changing the notation of $\vec{\rho}_1$ and $\vec{\rho}_2$ to $\vec{\rho}_{N_1}$ and $\vec{\rho}_{N_2}$ respectively

$$\hat{\rho}_n = \frac{\vec{\rho}_{N_1} + \frac{n-N_1}{N'_2}(\vec{\epsilon}_2 - \vec{\rho}_{N_1})}{\left|\vec{\rho}_{N_1} + \frac{n-N_1}{N'_2}(\vec{\epsilon}_2 - \vec{\rho}_{N_1})\right|} \qquad (n = N_1, N_1 + 1, \ldots, N_1 + N'_2 - 1)$$

$$\hat{\rho}_n = \frac{\vec{\epsilon}_2 + \frac{n-N_1-N'_2}{N'_2}(\vec{\rho}_{N_2} - \vec{\epsilon}_2)}{\left|\vec{\epsilon}_2 + \frac{n-N_1-N'_2}{N'_2}(\vec{\rho}_{N_2} - \vec{\epsilon}_2)\right|} \qquad (n=N_1+N'_2, N_1+N'_2+1, \ldots, N_1+2N'_2 = N_2)$$

where $2N'_2$ is the total number of observations one wishes to perform when the vehicle is in the vicinity of the target planet (i.e., in its sphere of influence). The corresponding set $\left\{\vec{V}'_n\right\}$ of the vehicles velocity vectors with respect to $\sum'$ is calculated by

$$\vec{V}'_n = \frac{1}{\ell_2}\,\vec{h}_2 \times (\hat{\rho}_n + \vec{\epsilon}_2) \qquad (n = N_1, N_1 + 1, \ldots, N_2)$$

The position vectors $\left\{\vec{\rho}_n\right\}$ are calculated by

$$\vec{\rho}_n = \frac{\ell_2}{1 + \hat{\rho}_n \cdot \vec{\epsilon}_2}\,\hat{\rho}_n \,. \qquad (n = N_1, N_1 + 1, \ldots, N_2)$$

By substituting $\vec{\rho}_{N_1}$ and $\vec{\rho}_{N_2}$ for $\vec{R}_1$ and $\vec{R}_2$ in (47) and (48) and comparing the results with $t_{N_2} - t_{N_1} = \Delta t$ one can determine whether the hyperbolic path in τ with respect to $\sum'$ is of the short-time or long-time type. Consequently

$$t_n = t_{N_1} + \sqrt{\frac{a_2^3}{\mu_Q}} \left\{ \sqrt{y_{n1}^2 - 1} - \cosh^{-1} y_{n1} - \sqrt{y_{n2}^2 - 1} + \cosh^{-1} y_{n2} \right\}$$

or

$$t_n = t_{N_1} + \sqrt{\frac{a_2^3}{\mu_Q}} \left\{ \sqrt{y_{n1}^2 - 1} - \cosh^{-1} y_{n1} + \sqrt{y_{n2}^2 - 1} - \cosh^{-1} y_{n2} \right\}$$

for short and long time types respectively and where

$$y_{n1} = 1 + \frac{\rho_{N_1} + \rho_n + \sqrt{\rho_{N_1}^2 + \rho_n^2 - 2\vec{\rho}_{N_1} \cdot \vec{\rho}_n}}{2a_2}$$

$$y_{n2} = 1 + \frac{\rho_{N_1} + \rho_n - \sqrt{\rho_{N_1}^2 + \rho_n^2 - 2\vec{\rho}_{N_1} \cdot \vec{\rho}_n}}{2a_2}$$

and $n = N_1, N_1 + 1, \ldots, N_2$. This set may also be obtained by employing (11) which holds for both short and long time cases.

$$t_n = \tfrac{1}{2}(t_{N_1} + t_{N_2}) - \sqrt{\frac{a_2}{\mu_Q}} \Bigg\{ \sqrt{\rho_n^2 + 2a_2\rho_n - a_2^2(\epsilon_2^2 - 1)}$$
$$- a_2 \log \frac{1}{\epsilon_2 a_2} \left(\sqrt{\rho_n^2 + 2a_2 \rho_n - a_2^2 (\epsilon_2^2 - 1)} + \rho_n + a_2 \right) \Bigg\}$$

for $n = N_1, N_1 + 1, \ldots, N_1 + N_2'$ such that $t_{N_1 + N_2'} = t_{N_1} + \tfrac{1}{2}\Delta t = \tfrac{1}{2}(t_{N_1} + t_{N_2})$.

$$t_n = t_{N_1 + N_2' + j} = t_{N_1} + t_{N_2} - t_{N_1 + N_2' - j} \quad (j = 1,2,\ldots N_2')$$

so that n in this formula ranges from $n = N_1 + N_2' + 1$ to $n = N_1 + N_2' + N_2' = N_2$. The values of $t_{N_1 + N_2' - j}$ are first computed from the previous formula. One may use both methods for the computation of $\left\{ t_n \right\} (n = N_1, N_1 + 1, \ldots, N_2)$ to check one against the other. With respect to $\sum$

$$\vec{\sigma}_n = \vec{Q}(t_n) + \vec{\rho}_n \qquad (n = N_1, N_1 + 1, \ldots, N_2)$$
$$\vec{V}_n = \vec{V}_Q + \vec{V}_n'$$

where $\vec{V}_Q$ is taken as the velocity of the target planet when the vehicle makes its closest approach at time $t_{CA} = t_{N_1 + N_2'}$.

The solution corresponding to the returning elliptical trajectory proceeds by calculating

$$\hat{\sigma}_n = \frac{\vec{r}_2 + \frac{n - N_2}{N'_3}(\vec{c}_3 - \vec{r}_2)}{\vec{r}_2 + \frac{n - N_2}{N'_3}(\vec{c}_3 - \vec{r}_2)} \qquad (n = N_2, N_2 + 1, \cdots N_2 + N'_3 = N_3)$$

if

$$\frac{\vec{r}_2 \times \vec{c}_3}{|\vec{r}_2 \times \vec{c}_3|} \approx \hat{h}_3$$

where N'_3 is the total number of observations made of the vehicle on its returning ellipse If $\frac{\vec{r}_2 \times \vec{c}_3}{|\vec{r}_2 \times \vec{c}_3|} \approx -\hat{h}_3$ these vectors are calculated by

$$\hat{\sigma}_n = \frac{\vec{r}_2 + \frac{n - N_2}{N'_3}(-\vec{c}_3 - \vec{r}_2)}{\left|\vec{r}_2 + \frac{n - N_2}{N'_3}(-\vec{c}_3 - \vec{r}_2)\right|} \qquad (n = N_2, N_2 + 1, \cdots, N_2 + N'_3 - 1)$$

$$\hat{\sigma}_n = \frac{-c_3 + \frac{n - N_2 - N'_3}{N''_3}(\vec{c}_3 - \vec{r}_2)}{\left|-\vec{c}_3 + \frac{n - N_2 - N'_3}{N''_3}(\vec{c}_3 - \vec{r}_2)\right|} \qquad (n = N_2 + N'_3, N_2 + N'_3 + 1, \cdots, N_2 + N'_3 + N''_3 - 1)$$

$$\hat{\sigma}_n = \frac{-\vec{r}_2 + \frac{n - N_2 - N'_3 - N''_3}{N_3^{!''}}(\vec{r}_2 + \vec{c}_3)}{\left|-\vec{r}_2 + \frac{n - N_2 - N'_3 - N_3^{!'}}{N_3^{!''}}(\vec{r}_2 + \vec{c}_3)\right|} \qquad (n = N_2 + N_3^{!} + N_3^{!'}, N_2 + N_3^{!} + N_3^{!'} + 1, \cdots N_2 + N_3^{!} + N_3^{!'} + N_3^{!''})$$

where

$N_2 + N_3^{!} + N_3^{!'} + N_3^{!''} = N_3$ and by observing the figure on page 27a, $N_3^{!}$ is the number of observations made when the vehicle is on $\widehat{AB}$ $N_3^{!'}$ is the number of observations made when the vehicle is on $\widehat{BC}$ and $N_3^{!''}$ is the number of observations made when the vehicle is on $\widehat{CD}$. The total number of observations performed on the vehicle on its returning trajectory is $N_3^{!} + N_3^{!'} + N_3^{!''}$.

The corresponding velocity vectors are calculated by

$$\vec{V}_n = \frac{1}{\ell_3} \vec{h}_3 \times (\hat{\sigma}_n + \vec{\epsilon}_3) \qquad (n = N_2, N_2 + 1, \cdots, N_3)$$

The position vectors are obtained by

$$\vec{\sigma}_n = \frac{\ell_3}{1 + \hat{\sigma}_n \cdot \vec{\epsilon}_3} \hat{\sigma}_n \qquad (n = N_2, N_2 + 1, \cdots N_3)$$

The time t_n when the vehicle is at $\vec{\sigma}_n$ with velocity $\vec{V}_n$ can be calculated by

$$t_n = t_{N_3} - \sqrt{\frac{a_3^3}{\mu_s}} \left\{ \sqrt{1 - x_{n2}^2} + \sin^{-1} x_{n2} - \sqrt{1 - x_{n1}^2} - \sin^{-1} x_{n1} \right\}$$

or

$$t_n = t_{N_3} - \sqrt{\frac{a_3^3}{\mu s}} \left\{ \pi + \sqrt{1 - x_{n2}^2} + \sin^{-1} x_{n2} + \sqrt{1 - x_{n1}^2} + \sin^{-1} x_{n1} \right\}$$

depending on whether the vehicle returns on a short or long time elliptical trajectory respectively where

$$x_{1n} = 1 - \frac{c_3 + \sigma_n + \sqrt{c_3^2 + \sigma_n^2 - 2 \vec{c}_3 \cdot \vec{\sigma}_n}}{2a_3}$$

$$x_{2n} = 1 - \frac{c_3 + \sigma_n - \sqrt{c_3^2 + \sigma_n^2 - 2 \vec{c}_3 \cdot \vec{\sigma}_n}}{2a_3}$$

with $n = N_2, N_2 + 1, \cdots N_3$

The above calculations represent a complete solution for the problem of determining a reconnaissance trajectory for a vehicle launched at t_o from a certain launch planet, making a closest approach to a certain target planet at t_{cA} and returning to the launch planet. Suppose these initial conditions yield the value $a = a_1$ for the departing elliptical trajectories which is of the short time type. Then by observing the formula for long-time elliptical paths if

$$\tilde{T} = t'_{cA} - t_o = \sqrt{\frac{a_1^3}{\mu_s}} \left\{ \pi + \sqrt{1 - x_2^2} + \sin^{-1} x_2 + \sqrt{1 - x_1^2} + \sin^{-1} x_1 \right\}$$

where

$$x_1 = 1 - \frac{c_o + Q(t'_{cA}) + \overline{c_o\, Q(t'_{cA})}}{2a_1}$$

$$x_2 = 1 - \frac{c_o + Q(t'_{cA}) - \overline{c_o Q(t'_{cA})}}{2a_1}$$

yields a solution for t'_{cA}, there will in most cases be four distinct possible reconnaissance trajectories with $k = 0$ taking the vehicle from the launch planet at the same launch time t_o and the same energy.

(i) short-time departing elliptical trajectory making closest approach to target planet at t_{cA}; short time returning elliptical trajectory with $k = 0$.

(ii) short-time departing elliptical trajectory making closest approach to target planet at t_{cA}; long-time returning elliptical trajectory with $k = 0$.

(iii) long-time departing elliptical trajectory making closest approach to target planet at t'_{cA}; short-time returning trajectory, $k = 0$.

(iv) long-time departing elliptical trajectory making closest approach to target planet at t'_{cA}; long-time returning trajectory, $k = 0$.

It is conceivable that a reconnaissance vehicle may be required to visit more than one planet before returning to its launch planet. This, of course, would require a very accurate guidance system. Such systems will no doubt be developed, hence, we are motivated to consider such reconnaissance missions. The statement of the problem shall be formulated as follows:

Assuming that the basic assumptions I, II and III hold, find a trajectory of a vehicle launched from the "center" of a given planet at the prescribed time t_{o2}, which makes a closest approach to the first planet to be observed at the prescribed time t_{1cA} and continue on a journey of visiting N-1 more planets in a prescribed order and return to the launch planet.

An example of such a reconnaissance mission may be the following: at t_{o2} the vehicle leaves the "center" of the earth and makes a closest approach to the first planet Venus at time t_{1cA}. It then proceeds to visit the remaining N-1 planets in the following order:

Mars
Earth
Saturn
Pluto
Jupiter
Earth

In this problem we shall make use of the following notation:

(a) $\sum_j'$ = moving frame of reference centered at center of j'th planet whose axes are kept parallel to the axes of a primary inertial frame $\sum$ having origin fixed at center of sun ($j = 1,2,\cdots N$)

(b) $\mathcal{T}_j$ = sphere of influence of j'th planet ($j = 1,\cdots, N$)

(c) $\vec{c}_{o2}$ = position vector of launch planet and initial position vector of vehicle at beginning of mission at time t_{o2}

(d) $\vec{r}_{j,1}$ $\vec{r}_{j,2}$ = position vectors of vehicle as it enters and leaves $\mathcal{T}_j$ respectively at time t_{j1} and t_{j2} ($j = 1,2,\cdots, N$)

(e) $\vec{\rho}_{j1}, \vec{\rho}_{j2}$ = position vectors of vehicle as it enters and leaves $\mathcal{T}_j$ respectively with respect to $\sum_j'$

(f) $\vec{c}_{j1}, \vec{c}_{jcA}, \vec{c}_{j2}$ = position vectors of j'th planet when vehicle enters $\mathcal{T}_j$, makes its closest approach, and leaves $\mathcal{T}_j$ respectively at time t_{j1}, t_{jcA}, and t_{j2}

(g) $\vec{c}_{N+1,1}$ = position vector of launch planet and vehicle at end of mission at time $t_{N+1,1}$

(h) $\vec{P}_j(t)$ = known position vector of j'th planet expressed as a vector function of time ($j = 0,1,\cdots N+1$) $j = 0$ and $j = N+1$ corresponds to the launch planet

(i) $\widehat{r_{ij}\, r_{k\ell}}$ = arc of trajectory from $\vec{r}_{ij}$ to $\vec{r}_{k\ell}$

(j) $\overline{r_{ij}\, r_{k\ell}}$ = distance between $\vec{r}_{ij}$ and $\vec{r}_{k\ell}$

(k) $\vec{h}_{j,j+1}, \vec{\epsilon}_{j,j+1}$ = vector trajectory parameters corresponding to arc $\widehat{r_{j,2}\, r_{j+1,1}}$ ($j = 0, 1,\cdots N$)

(l) $\vec{h}_j$ $\vec{\epsilon}_j$ = vector trajectory parameters corresponding to arc $\widehat{r_{j1}\, r_{j2}}$ ($j = 1,2,\cdots N$)

(m) $\vec{V}_{jcA}$ = velocity of j'th planet when vehicle makes its closest approach at time t_{jcA}

(n) $\vec{V}_{j1}, \vec{V}_{j2}$ = velocity of vehicle as it enters and leaves $\mathcal{T}_j$ with respect to $\sum$

(o) $\vec{V}'_{j1}, \vec{V}'_{j_2}$ = velocity of vehicle as it enters and leaves $\mathcal{T}_j$ with respect to $\sum'_j$

(p) d_j = distance of closest approach to j'th planets surface

($j = 1,2,\cdots, N$)

(q) R_j = radius of j'th planet

(r) $\mu_j = Gm_j$ where m_j is mass of j'th planet

(s) $\vec{h}_{pj}$ = angular momentum of j'th planet.

A solution (first approximation but still very close to exact solution) then proceeds by the following steps:

(i) assume $\widehat{c_{o2}\, r_{11}}$ is a short time elliptical path

(ii) calculate a_{o1} by (12) with $\vec{R}_1 = \vec{c}_{o2}$, $\vec{R}_2 = \vec{c}_{1CA}$, $T = t_{1CA} - t_{o2}$, and $\mu = \mu_s$

(iii) calculate ϵ_{o1} by (13) with $\vec{R}_1 = \vec{c}_{o2}$, $\vec{R}_2 = \vec{c}_{1CA}$ and $a = a_{o1}$ obtained from (ii)

(iv) calculate $\ell_{o1} = a_{o1}(1 - \epsilon_{o1}^{2})$

(v) calculate $\vec{h}_{o1}$ by (14) with $\vec{c}_o$, $\vec{c}_{CA}$ replaced by $\vec{c}_{o2}$, $\vec{c}_{1CA}$ and ℓ_1 by ℓ_{o1} obtained from (iv) where the sign is chosen so that $\vec{h}_{o1} \cdot \vec{h}_{Po} > 0$

(vi) calculate $\vec{\epsilon}_{o1}$ by (15) ~~(16) if $c_{o2} > c_{1CA}$ or (17) if $c_{o2} < c_{CA}$ with~~ $\vec{R}_1 = \vec{c}_{o2}$, $\vec{R}_2 = \vec{c}_{1CA}$, $\ell = \ell_{o1}$ $\epsilon = \epsilon_{o1}$

(vii) assume $\widehat{r_{12}\, r_{21}}$ is a short-time elliptical path which makes $k = 0$ circuits of the sun

(viii) calculate $a_{12}(t_{2CA})$ by (19) with $k = 0$, $T = t_{2CA} - t_{1CA}$, $\vec{R}_1 = \vec{c}_{1CA}$, $\vec{R}_2 = \vec{c}_{2CA} = \vec{P}_2(t_{2CA})$ $\overline{R_1R_2} = \overline{c_{1CA}P(t_{2CA})}$ (known function of t_{2CA})

(ix) calculate $\epsilon_{12}(t_{2CA})$ by (13) with $a = a_{12}(t_{2CA})$, $\vec{R}_1 = \vec{c}_{1CA}$, $\vec{R}_2 = \vec{P}_2(t_{2CA})$, $\overline{R_1R_2} = \overline{c_{1CA}P_2(t_{2CA})}$

(x) calculate $\ell_{12}(t_{2CA}) = a_{12}(t_{2CA})\,(1 - \epsilon^{2}_{12}(t_{2CA}))$

(xi) calculate $\vec{\epsilon}_{12}(t_{2CA})$ by (16) if $c_{1CA} > c_{2CA}$ or (17) if $c_{1CA} < c_{2CA}$ with $\vec{R}_1 = \vec{c}_{1CA}$, $\vec{R}_2 = \vec{P}_2(t_{2CA})$, $\ell = \ell_{12}(t_{2CA})$, $\epsilon = \epsilon_{12}(t_{2CA})$

(xii) calculate $\vec{h}_{12}(t_{2CA})$ by (14) with $\vec{c}_o$, $\vec{c}_{CA}$ replaced by $\vec{c}_{1CA}$, $\vec{c}_{2CA}$ and ℓ_1 by $\ell_{12}(t_{3CA})$ where the sign is chosen so that $\vec{h}_{12} \cdot \vec{h}_{P1} > 0$

(xiii) calculate t_{2CA} explicitly by (26) with $a_1 = a_{o1}$, $a_3(t_3) = a_{12}(t_{2CA})$, $\ell_1 = \ell_{o1}$, $\ell_3(t_3) = \ell_{12}(t_{2CA})$ $\vec{h}_1 = \vec{h}_{o1}$, $\vec{\epsilon}_1 = \vec{\epsilon}_{o1}$, $\vec{h}_3(t_3) = \vec{h}_{12}(t_{2CA})$ $\vec{\epsilon}_3(t_3) = \vec{\epsilon}_{12}(t_{2CA})$ $\vec{V}_o = \vec{V}_{1CA}$ $\hat{c}_{CA} = \hat{c}_{1CA}$,

(xiv) if above solution for t_{2CA} does not exist (i.e., not reasonable) assume $\widehat{r_{12}\, r_{21}}$ is a long-time elliptical path which makes $k = 0$ circuits of the sun; if t_{2CA} has reasonable value, proceed to (xxviii).

(xv) replace (19) of (viii) by (20) and calculate $a_{12}(t_{2CA})$

(xvi) replace (13) of (ix) by (21) and calculate $\epsilon_{12}(t_{2CA})$

(xvii) calculate new values for $\ell_{12}(t_{2CA}) = a_{12}(t_{2CA})\ (1 - \epsilon^2_{12}(t_{2CA}))$

(xviii) calculate $\vec{\epsilon}_{12}(t_{2CA})$ by step (xi) replacing (16) or (17) by (18)

(xix) repeat (xii) using the new table for $\ell_{12}(t_{2CA})$

(xx) calculate new value for t_{2CA} by repeating step (xiii) using the new value for $a_{12}(t_{2CA})$ $\ell_{12}(t_{2CA})$, $\vec{h}_{12}(t_{2CA})$, $\vec{\epsilon}_{12}(t_{2CA})$

(xxi) if new solution for t_{2CA} is still not reasonable, assume $\widehat{c_{o2}\ r_{11}}$ is a long-time elliptical path; if new value for t_{2CA} is reasonable, proceed to (xxviii)

(xxii) repeat (ii) replacing (12) by (20) with $k = 0$

(xxiii) repeat (iii) replacing (13) by (21)

(xxiv) repeat (iv) and (v)

(xxv) repeat (vi) by replacing (16) or (17) with (18)

(xxvi) repeat (vii) through (xx)

(xxvii) if resulting values of t_{2CA} are still not reasonable repeat (i) through (xxvi), (if necessary) replacing $k = 0$ with $k = 1,2,...$ except in step (xxii)

(xxviii) calculate a_{12}, ℓ_{12}, $\vec{\epsilon}_{12}$ and $\vec{h}_{12}$ by substituting in $a_{12}(t_{2CA})$, $\ell_{12}(t_{2CA})$ $\vec{\epsilon}_{12}(t_{2CA})$ and $\vec{h}_{12}(t_{2CA})$ the first reasonable solution for t_{2CA}

(xxix) calculate $\vec{V}_{11}$ and $\vec{V}_{12}$ by (5) with $\mu = \mu_s$, $\hat{R} = \hat{c}_{1CA}$, $\vec{h} = \vec{h}_{o1}$, $\vec{\epsilon} = \vec{\epsilon}_{o1}$, and $\vec{h} = \vec{h}_{12}$, $\vec{\epsilon} = \vec{\epsilon}_{12}$, respectively

(xxx) calculate $\vec{V}'_{11}$ and $\vec{V}'_{12}$ by (24) with $\vec{V}_Q = \vec{V}_{1CA}$ $\vec{V} = \vec{V}_{11}$ and $\vec{V} = \vec{V}_{12}$, respectively

(xxxi) calculate a_1 by (29) and (30) with $V_1'^2 = V_{11}'^2$ $V_2'^2 = V_{12}'^2$, $\mu_Q = \mu_1$

(xxxii) calculate ϵ_1 by (31) with $\vec{V}'_1 = \vec{V}'_{11}$, $\vec{V}'_2 = \vec{V}'_{12}$

(xxxiii) calculate $d_1 = a_1(\epsilon_1 - 1) - R_1$

If $d_1 > 0$, repeating (vii) through (xxxiii) with $j = 1$ replaced by $j = 2$ will yield d_2. If $d_2 > 0$ (vii) through (xxxiii) are repeated replacing $j = 2$ by $j = 3$ yielding d_3. This process is repeated until either all $d_j > 0$ or stops when the first $d_i < 0$, in which case, the next best value of t_{iCA} is calculated and the process is continued. When all $d_j > 0$ the calculation continues by calculating $\vec{\epsilon}_j, h_j$

$(j = 1,2,\ldots N)$ by (34) and (35) by replacing $\vec{V}_1', \vec{V}_2'$ by $\vec{V}_{j1}', \vec{V}_{j2}'$ and

$\epsilon_2 = \epsilon_j, \; h_2 = h_j = \sqrt{\ell_j \mu_j} = \sqrt{a_j(\epsilon_j^2 - 1)\mu_j}$

calculate the amount of time $(\Delta t)_j$ the vehicle spends in τ_j by (11) with $\Delta t = (\Delta t)_j \quad a = a_j, \; \mu = \mu_j, \; m = m_j$ = mass of j'th planet $\epsilon = \epsilon_j$

calculate t_{j1} and t_{j2} by (38) with $t_1 = t_{j1}, \; t_2 = t_{j2}, \; t_{CA} = t_{jCA}$

$\Delta t = (\Delta t)_j \quad (j = 1,2,\ldots,N)$

calculate $\vec{\rho}_{j1}$ and $\vec{\rho}_{j2}$ by (36) and (37) with $\mu_Q = \mu_j, \; \vec{V}_1' = \vec{V}_{11}', \; m = m_j, \; \vec{V}_2' = \vec{V}_{12}'$

calculate $\vec{r}_{j1}$ and $\vec{r}_{j2}$ by (39) with $\vec{r}_1 = \vec{r}_{j1}, \; \vec{c}_1 = \vec{P}_j(t_{j1}), \; \vec{c}_2 = \vec{P}_j(t_{j2})$, $\vec{\rho}_1 = \vec{\rho}_{j1}, \; \vec{\rho}_2 = \vec{\rho}_{j2}$

The complete solution $\left\{\vec{\sigma}_n\right\}, \left\{\vec{V}_n\right\}, \left\{t_n\right\}$ can now be calculated by employing the formulas developed on pages 27-30 in the same fashion as was done for the case $N = 1$. Before this calculation begins $\vec{\epsilon}_{j,j+1}, \vec{h}_{j,j+1}, a_{j,j+1}, \ell_{j,j+1} (j=0,1,\ldots N)$ and $\vec{\epsilon}_j, \vec{h}_j \; a_j, \ell_j (j=1,2,\ldots N)$ and $\vec{r}_{j1}, \vec{r}_{j2}, \vec{\rho}_{j1} \; \vec{\rho}_{j2}$ should be refined.

In conclusion, we notice the remarkable fact that if E is the total heliocentric energy of a departing free-fall reconnaissance vehicle to one planet and back, it may be possible to send the vehicle on a trajectory which will take it to N-1 more planets before returning to its launch planet without any appreciable change in E.

APPENDIX

One may proceed by the following method of successive approximations to obtain values of $\vec{\epsilon}_1$, $\vec{h}_1$ $\vec{\epsilon}_2$ $\vec{h}_2$, $\vec{\epsilon}_3$ $\vec{h}_3$ and $\vec{r}_1$ $\vec{r}_2$ $\vec{\rho}_1$ $\vec{\rho}_2$ corresponding to a round-trip reconnaissance trajectory to one planet and back which are arbitrarily close to the exact values of these quantities. In cases where $N > 1$ the process is very similar to that given below and hence will not be explicitly written out.

Suppose $\vec{\epsilon}_1^{(k)}$ $\vec{h}_1^{(k)}$ $a_1^{(k)}$, ... $\ell_3^{(k)}$ corresponds to the k'th approximation of these quantities. The k+1'th approximation can be calculated as follows:

calculate $a_1^{(k+1)}$ by (12) with $\vec{R}_1 = \vec{c}_o$, $\vec{R}_2 = \vec{r}_1^{(k)}$, $T = t_1^{(k)} - t_o$

calculate $\epsilon_1^{(k+1)}$ by (13) with $\vec{R}_1 = \vec{c}_o$, $\vec{R}_2 = \vec{r}_1^{(k)}$, $a = a_1^{(k+1)}$

calculate $\vec{\epsilon}_1^{(k+1)}$ by same formula used to calculate $\vec{\epsilon}_1^{(k)}$ with $\ell = \ell_1^{(k+1)} = a_1^{(k+1)}(1 - \epsilon_1^{(k+1)})$ $\vec{R}_1 = \vec{c}_o$, $\vec{R}_2 = \vec{r}_1^{(k)}$

calculate $\vec{h}_1^{(k+1)}$ by same formula used to calculate $\vec{h}^{(k)}$ with $\vec{R}_1 = \vec{c}_o$, $\vec{R}_2 = \vec{r}_1^{(k)}$ $\ell = \ell_1^{(k+1)}$

calculate $a_3^{(k+1)}(t_3^{(k+1)})$ by same formula used to calculate $a_3^{(k)}(t_3^{(k)})$ with $\vec{R}_1 = \vec{r}_2^{(k)}$, $\vec{R}_2 = \vec{P}(t_3^{(k+1)})$ $T = t_3^{(k+1)} - t_2^{(k)}$, $a = a_3^{(k+1)}(t_3^{(k+1)})$

calculate $\epsilon_3^{(k+1)}(t_3^{(k+1)})$ by same formula used to calculate $\epsilon_3^{(k)}(t_3^{(k)})$ with $\vec{R}_1 = \vec{r}_2^{(k)}$ $\vec{R}_2 = \vec{P}(t_3^{(k+1)})$ $a = a_3^{(k+1)}(t_3^{(k+1)})$

calculate $\ell_3^{(k+1)}(t_3^{(k+1)}) = a_3^{(k+1)}(t_3^{(k+1)}) \left[1 - \epsilon_3^{(k+1)^2}(t_3^{(k+1)})\right]$

calculate $\vec{\epsilon}_3^{(k+1)}(t_3^{(k+1)})$ by same formula used to calculate $\vec{\epsilon}_3^{(k)}(t_3^{(k)})$ with $\ell = \ell_3^{(k+1)}(t_3^{(k+1)})$, $\vec{R}_1 = r_2^{(k)}$ $\vec{R}_2 = \vec{P}(t_3^{(k+1)})$

calculate $h_3^{(k+1)}(t_3^{(k+1)})$ by same formula used to calculate $h_3^{(k)}(t_3^{(k)})$ with $\ell = \ell_3^{(k+1)}(t_3^{(k+1)})$

calculate $\vec{h}_3^{(k+1)}(t_3^{(k+1)})$ by same formula used to calculate $\vec{h}_3^{(k)}(t_3^{(k)})$ with $\vec{R}_1 = \vec{r}_2^{(k)}$, $\vec{R}_2 = \vec{P}(t_3^{(k+1)})$ $h = h_3^{(k+1)}(t_3^{(k+1)})$

calculate $t_3^{(k+1)}$ explicitly by same formula used to calculate $t_3^{(k)}$ with
$a_1 = a_1^{(k+1)}$ $a_3 = a_3^{(k+1)}(t_3^{(k+1)})$, $\ell_3 = \ell_3^{(k+1)}(t_3^{(k+1)})$, $\vec{h}_3 = \vec{h}_3^{(k+1)}(t_3^{(k+1)})$
$\hat{r}_1^{(k-1)}$ and $\hat{r}_2^{k-1}$ replaced by $\hat{r}_1^{(k)}$ and $\hat{r}_2^{(k)}$ $\vec{\epsilon}_3 = \vec{\epsilon}_3^{(k+1)}(t_3^{(k+1)})$
$\ell_1 = \ell_1^{(k+1)}$, $\vec{h}_1 = \vec{h}_1^{(k+1)}$, $\vec{\epsilon}_1 = \vec{\epsilon}_1^{(k+1)}$

calculate $a_3^{(k+1)}$ $\vec{\epsilon}_3^{(k+1)}$ and $\vec{h}_3^{(k+1)}$ explicitly by substituting the value of $t_3^{(k+1)}$ into $a_3^{(k+1)}(t_3^{(k+1)})$, and $h_3^{(k+1)}(t_3^{(k+1)})$

calculate $a_2^{(k+1)}$ by same formula used to caluclate $a_2^{(k)}$ with $\vec{V}_1^{(k-1)'}$ and $\vec{V}_2^{(k-1)'}$ replaced by $\vec{V}_1^{(k)'}$ and $\vec{V}_2^{(k)'}$

calculate $\epsilon_2^{(k+1)}$ by same formula used to calculate $\epsilon_2^{(k)}$ with $\vec{V}_1^{(k-1)'}$ and $\vec{V}_2^{(k-1)'}$ replaced by $\vec{V}_1^{(k)'}$ and $\vec{V}_2^{(k)'}$

calculate $d^{(k+1)}$ by same formula used to calculate $d^{(k)}$ with $\epsilon_2^{(k)}$ and $a_2^{(k)}$ replaced by $\epsilon_2^{(k+1)}$ and $a_2^{(k+1)}$

calculate $(\Delta t)^{(k+1)}$ by same formula used to calculate $(\Delta t)^k$ with $\epsilon_2^{(k)}$, $a_2^{(k)}$ replaced by $\epsilon_2^{(k+1)}$ and $a_2^{(k+1)}$

calculate $t_1^{(k+1)}$, $t_2^{(k+1)}$ by (38) with Δt replaced by $(\Delta t)^{(k+1)}$

calculate $\vec{c}_1^{(k+1)}$ and $\vec{c}_2^{(k+1)}$ by substituting $t_1^{(k+1)}$ and $t_2^{(k+1)}$ into $\vec{Q}(t)$

calculate $\vec{\rho}_1^{(k+1)}$ and $\vec{\rho}_2^{(k+1)}$ by (36) and (37) with $\vec{V}_1' = \vec{V}_1^{(k)'}$, $\vec{V}_2' = \vec{V}_2^{(k)'}$,
$\vec{h}_2 = \vec{h}_2^{(k+1)}$ $\vec{\epsilon}_2 = \vec{\epsilon}_2^{(k+1)}$

calculate $\vec{r}_1^{(k+1)}$ and $\vec{r}_2^{(k+1)}$ by (39) with $\vec{c}_1 = \vec{c}_1^{(k+1)}$, $\vec{c}_2 = \vec{c}_2^{(k+1)}$,
$\vec{\rho}_1 = \vec{\rho}_1^{(k+1)}$, $\vec{\rho}_2 = \vec{\rho}_2^{k+1}$

calculate $\vec{V}_1^{(k+1)}$, $\vec{V}_2^{(k+1)}$ same formula used to calculate $\vec{V}_1^{(k)}$ and $\vec{V}_2^{(k)}$ replacing k by k+1

A continuation of the above calculations letting $k = 1,2, \ldots$ will yield values of $\vec{\epsilon}_1$ $\vec{h}_2$, ... $\vec{h}_3$, which will be arbitrarily close to the exact values of these of these quantities.

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