Network dynamics: advanced models

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Advanced models

Modifications of the Barabasi-Albert model

- Liu et al's hybrid model.
- Bianconi-Barabási hybrid model.
- Dorogovtsev-Mendes model (accelerated growth).

Liu et al's hybrid model

- Motivation: modelling the mixture of power-law and exponential behavior of real degree distributions.
- A hybrid attachment rule: preferential + (degree-independent) random attachment.

$$\pi(k_i) = \frac{(1-p)k_i + p}{\sum_j [(1-p)k_j + p]}$$

- The model is reminiscent of the mean field approach adopted for $\partial k_i / \partial t$ in the copying model (previous session).
- $0 \leq p \leq 1$

Degree distribution for $p = 0$ and for $p = 1$?

The degree distribution of Liu et al's model I

$$p(k) \sim \left(\frac{\frac{k}{m_0} + b}{1 + b} \right)^{-\gamma}$$

where

- $$\gamma = 3 + b$$
- $$b = \frac{p}{m_0(1-p)}$$

A mean-field proof as that of the Barabási-Albert model is not difficult [Liu et al., 2002].

The degree distribution of Liu et al's model II

Limit distributions

- If $p = 0$ then $p(k) \sim k^{-3}$
- If $p \to 1$, exponential $p(k) \sim e^{-k/m}$. Easy proof: [Barabási et al., 1999]
  - Impose $p = 1$ which gives $\pi(k_i) = p$.
  - Derive $p(k)$ from $\partial k_i / \partial_t = m_0 \pi(k_i) = m_0 p$ (mean-field non-rigorous proof).

The degree distribution of Liu et al's model III

$$\gamma = 3 + b$$

with

$$b = \frac{p}{m_0(1-p)}$$

.

- What is range of variation of γ?
- Warning: the higher the value of γ the less valid the power-law
- A serious problem (recall that exponents are close to two in the majority of cases).

An example of a consistent degree distribution

Word thesaurus network [Motter et al., 2002]

- Thesaurus: list of entries. Entry: word + list of related words.
- friend: Maecenas, acquaintance, adherent, advocate, ally, alter ego,amigo, angel, associate, baby, backer, beau, bedfellow, benefactor, best friend, bird, boon, companion, bosom, buddy, bosom friend, boyfriend, chum, co-worker, cocker, cohort, colleague, compatriot, compeer, comrade, concubine, confederate, confidant, confidante, confrere,consociate,crony, doxy, escort, familiar, fellow,financier,girl,intimate, investor, lover, man, mate, mistress, moll, pal, partner, patron, playmate, roomie, soul ,mate, squeeze, supporter, sweetheart, twist, woman, person, individual, someone, somebody, mortal, human, soul, protagonist, champion, admirer, booster, advocator, proponent, exponent, Friend, Quaker, Christian

Thesaurus network

- Two words are connected if one is in the entry of the other.
- Two regimes: 1st regime exponential and 2nd regime power-law with $\gamma \approx 3.5$.
- Was the Moby thesaurus build at random?

Bianconi and Barabási hybrid model

- Barabási-Albert model: growth + preferential attachment
- Bianconi-Barabási model: growth + preferential attachment + fitness [Bianconi and Barabási, 2001]
  - Every vertex has a fitness. η_i is the fitness of the *i*-th vertex ("etha").
  - Every vertex is assigned a random fitness when added to the network. The random fitness is obtained with a probability density function $\rho(\eta)$
  - New attachment probability:

$$\pi(k_i, \eta_i) = \frac{\eta_i k_i}{\sum_1^n \eta_j k_j}$$

The degree distribution of the Bianconi-Barabási model

- The degree distribution of the model depends on $\rho(\eta)$.
- If $\rho(\eta)$ is uniform ($\rho(\eta)$ constant)

$$p(k) \sim \frac{k^{-(1+C^*)}}{\log k}$$

with $C^* \approx 1.255$.

The model reproduces degree correlations (disassortative mixing) of the Internet autonomus systems [Vázquez et al., 2002] (vertices are Autonomous systems, autonomous systems are partitions of Internet).

Dorogovtsev and Mendes model

Growth + preferential attachment + accelerated edge growth [Dorogovtsev and Mendes, 2001]
The evolution of an undirected network over time t.

1. $t = 0$, a disconnected set of n_0 vertices (no edges). **Assume** $n_0 = 1$ **here**.
2. At time $t > 0$,
   2.1 Add a new vertex with m_0 edges. **Assume** $m_0 = 1$ **here**.
      - The new vertex connects to the i-th vertex with probability

$$\pi(k_i) = \frac{k_i}{\sum_j k_j}$$

   2.2 **Add** ct **new edges (**c **is a parameter of the model).**
      - The probability that the i-th and the j-th vertex are connected is proportional to $k_i k_j$.

Accelerated growth

Thus

$$n = n_0 + t$$

(as for the Barabási-Albert model)

$$m \approx m_0 t + c \sum_{t'=1}^{t} t'$$

Assuming $m_0 = 1$,

$$m \approx t + ct(t+1)/2 = \left(\frac{c}{2} + 1\right) t + \frac{c}{2} t^2$$

(accelerated growth!)

Degree distribution

$$p(k) \sim \begin{cases} k^{-3} & \text{for } k \geq k^* \\ k^{-3/2} & \text{for } k \leq k^* \end{cases}$$

$$k^* \approx \sqrt{ct}(2+ct)^{3/2}$$

More ingredients for modelling

- Vertex growth → edge growth (edges added without adding new vertices)
- Vertex growth → ageing (vertex death)
- Edge removal
- ...

We have focused on the degree distribution: clustering, geodesic distances, degree correlations,...are important aspects to determe the best model for a real network.

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