Higher Transcendental Functions

CALIFORNIA INSTITUTE OF TECHNOLOGY

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Higher Transcendental Functions, 3 volumes.
Tables of Integral Transforms, 2 volumes.

# HIGHER TRANSCENDENTAL FUNCTIONS

## Volume III

Based, in part, on notes left by

Harry Bateman

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and compiled by the

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This work is dedicated to the
memory of

HARRY BATEMAN

as a tribute to the imagination which led him to undertake a project of this magnitude, and the scholarly dedication which inspired him to carry it so far toward completion.

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## FOREWORD

This is the last of the volumes prepared by the staff of the Bateman Manuscript Project, an enterprise whose origin and aims were described in the prefatory material to the first volume. There are altogether three volumes of *Higher Transcendental Functions* supplemented by two volumes of *Tables of Integral Transforms*. The present volume contains chapters on automorphic functions, Lamé and Mathieu functions, spheroidal and ellipsoidal wave functions, functions occurring in number theory and some other functions; and there is also a chapter on generating functions. The volume was prepared after the staff of the Bateman Manuscript Project left Pasadena, but Professor Magnus continued working on Chapters XIV, XVII, XIX after he joined the staff of New York University.

The chapter on automorphic functions contains examples of automorphic functions which can be constructed explicitly. The general theorems given in this chapter serve mainly the purpose of establishing a background for the examples, and the deeper algebraical and number-theoretical aspects of the subject are definitely outside the scope of our book. In the chapter on Lamé functions we neglected somewhat Lamé polynomials (which are discussed adequately in several easily accessible books) and devoted our attention chiefly to the more recent theories of periodic Lamé functions and Lamé-Wangerin functions. Our account of Mathieu functions is largely descriptive and leans heavily on McLachlan's book which is the standard book on the subject. Another book on Mathieu functions, by Meixner and Schäfke, is in preparation and is expected to appear soon. There are in essence two rival systems of notations for Mathieu functions: we adopted the one which is used by all British, most European, and many American mathematicians, even though the most extensive numerical tables of these functions (those prepared by the National Bureau of Standards) use the other notation. In the sections on spheroidal wave functions we neglected much of the older literature (accounts of which are available in other books) and attempted to summarize the results obtained by Bouwkamp, Meixner, and others in the last fifteen years. The forthcoming book by Meixner and Schäfke will cover this field too. The brevity of the sections dealing with ellipsoidal wave functions reflects in some measure the lack of information on this subject. In the chapter on the functions of number theory we attempted to give some of the more important properties of certain arithmetical functions. Here again the more profound aspects of the subject are outside of the scope of our book. Professor Apostol very kindly read this chapter, and he supplied sec. 17.11. We have included brief sections on some of the lesser known special functions, to which several papers have been devoted in recent years. The final chapter, on generating functions, contains an extensive list of generating functions. This is one of several similar chapters planned by the late Professor Bateman. The others were to contain lists of differential equations, power series, $n$th derivative formulas, etc. defining special functions, and it was with regret that we decided to omit them. Chapters XIV and XVII are frankly experimental but we hope that they will prove useful enough to justify their inclusion in a book of this nature.

As in the first two volumes, a list of references is given at the end of each chapter. These lists are by no means complete but they should be sufficient to document the presentation and to enable the reader to find further information about the functions in question. Bibliographies of the various functions are referred to in the text.

At the end of the volume there is a *Subject index* and an *Index of notations*. Some of the notations introduced in the earlier volumes have been included, others (the more common ones) have not been repeated. The system of references is the same as in the first two volumes. In the text, references to literature state the name of the author followed by the year of publication, detailed references being given at the end of the chapter. Equations within the same section are referred to simply by number, equations in other sections by the section number followed by the number of the equation. Chapters are numbered consecutively throughout the book, Chapters I to VI being in vol. I, Chapters VII to XIII in vol. II, and Chapters XIV to XIX in the present volume. Thus, 3.7(27) refers to equation (27) in section 3.7, and will be found on p. 159 of vol. I, while 15.3(2) is on p. 57 of the present volume.

Thanks of the California Institute of Technology to various organizations and persons contributing in their several ways towards the success of this undertaking have been expressed in the Foreword to vol. I. At the conclusion of the work I should like to express my thanks to my associates and our assistants for the support they have given me in compiling this book, to Miss Rosemarie Stampfel who achieved what would have seemed impossible to many, to Mr. John B. Johnston who read the proofs of this volume and rendered other technical assistance, and last but indeed not least to a large number of correspondents who pointed out misprints and made valuable suggestions for improvement thereby indicating their active interest in the success of this work.

A. ERDÉLYI

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# CHAPTER XIV

## AUTOMORPHIC FUNCTIONS

This chapter contains a number of basic definitions and some easily accessible examples of automorphic functions, in particular of modular functions. The numerous ramifications of the subject involving the theory of groups, various branches of geometry, the theory of numbers, and important aspects of the general theory of complex variables will be left aside. The fundamental ideas of Felix Klein, the painstaking investigations of Fricke, more recent discoveries by Hecke and C.L. Siegel, and the relevance of their results in the theory of numbers, are scarcely mentioned, and the brief remarks on Poincaré's theta series are far from being adequate.

A list of references is given at the end of the chapter. The most important works for the whole of this chapter are Fricke (1901 - 1921), Fricke and Klein (1897, 1912), Fubini (1908), Giraud (1920), Schlesinger (1924), and Ford (1929, with an extensive bibliography). For occasional references to number theory consult Reid (1910), and for algebra van der Waerden (1949).

Specific references for individual sections are

14.1.4 Ford (1929)
14.3 Klein (1884)
14.4 Krazer and Wirtinger (1901 - 1921)
14.6 Klein and Fricke (1890, 1892)
14.6.4 Fricke (1916, 1922).

Other references will be given as they are needed.

### 14.1. Discontinuous groups and homographic transformations

### 14.1.1. Homographic transformations

Let $z$ be a complex variable which will be represented either as a point $z = x + iy$ in the complex plane (completed by the point at infinity), or else as a point $(x_1, x_2, x_3)$ on the sphere

(1) $x_1^2 + x_2^2 + x_3^2 = 1$

in three dimensional space: this sphere will be denoted by $S_0$ and called the Riemann sphere. The correspondence between the points of the complex plane and the points of the Riemann sphere is determined by the equations

(2) $x = \dfrac{x_1}{1 + x_3}, \quad y = \dfrac{x_2}{1 + x_3}, \quad z = x + iy$

(3) $x_1 = \dfrac{2x}{1 + x^2 + y^2}, \quad x_2 = \dfrac{2y}{1 + x^2 + y^2}, \quad x_3 = \dfrac{1 - x^2 - y^2}{1 + x^2 + y^2}$ .

The mapping of $S_0$ on the $z$-plane is conformal and is known as the *stereographic projection*. The circles on the sphere are mapped upon circles or straight lines in the plane. In this chapter straight lines will be regarded as special circles (those passing through $z = \infty$) so that circle will mean circle or straight line, and circular arc will mean a segment of a circle or a segment of a straight line. If a segment of a straight line contains the point at infinity, its representation in the Euclidean plane will contain two components; nevertheless, in the complex plane the segment is a connected set (the two components being joined at infinity).

Let $a$, $b$, $c$, $d$ be any complex numbers such that

(4) $ad - bc = 1$.

The relation

(5) $z' = \dfrac{az + b}{cz + d} = \sigma(z)$

defines a mapping of the $z$-plane, or of $S_0$, onto itself: this mapping is called a *homographic transformation* (or substitution) $\sigma$. In this interpretation $z'$ appears as another point of the complex plane. An alternative interpretation regards $z'$ as a new variable, or as new coordinates, of the same point but in this chapter we shall generally adhere to the first interpretation. The mapping (5) is non-degenerate if $ad - bc \neq 0$, and since (5) is homogeneous in $a$, $b$, $c$, $d$, it is always possible to achieve that (4) holds. Thus, (4), (5) define the most general non-degenerate mapping of the form (5). [For a degenerate mapping (5), $ad - bc = 0$, and the map is either indeterminate or else it consists of a single point.] The relationship between $z$ and $z'$ is one-one. From (4) and (5) we have

(6) $z = \dfrac{dz' - b}{-cz' + a}$.

Let $\sigma'$ be a second homographic substitution

$$(7) \quad \sigma'(z) = \frac{a'z + b'}{c'z + d'}, \quad a'd' - b'c' = 1.$$

Then

$$(8) \quad \sigma'[\sigma(z)] = \frac{(a'a + b'c)\, z + a'b + b'd}{(c'a + d'c)\, z + c'b + d'd}$$

defines a homographic substitution, since

$$(a'a + b'c)(c'b + d'd) - (a'b + b'd)(c'a + d'c)$$
$$= (ad - bc)(a'd' - b'c') = 1.$$

The substitution (8) is called the *product* of the substitutions $\sigma'$ and $\sigma$ (in this order) and is denoted by $\sigma'\sigma$. The product of any (finite) number of homographic substitutions is defined in a similar manner. In general $\sigma'\sigma$ and $\sigma\sigma'$ are different. The *inverse* of $\sigma$ is the homographic substitution

$$(9) \quad z' = \frac{dz - b}{-cz + a} = \sigma^{-1}(z), \quad ad - bc = 1.$$

and is denoted by $\sigma^{-1}$. If $I$ is the identity substitution, $I(z) = z$, then clearly

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = I$$

or

$$\sigma[\sigma^{-1}(z)] = \sigma^{-1}[\sigma(z)] = z.$$

Any homographic substitution maps any circle of $S_0$ onto a circle, and conversely, any continuous one-one mapping of $S_0$ onto itself which maps circles onto circles, is a homographic transformation.

### 14.1.2. Fixed points. Classification of transformations

The point $\zeta$ is called a *fixed point* of the transformation $\sigma(z)$ if $\sigma(z) = z$. If $c \neq 0$, the fixed points of the transformation $\sigma$ given by (5) are

$$\zeta_1 = \frac{1}{2c}\{a - d + [(a + d)^2 - 4]^{1/2}\}$$

$$\zeta_2 = \frac{1}{2c}\{a - d - [(a + d)^2 - 4]^{1/2}\};$$

and if $c = 0$ and $a \neq d$, the fixed points are

$$\zeta_1 = b/(d - a), \quad \zeta_2 = \infty.$$

If $c = 0$ and $a = d$, both fixed points coincide at infinity, unless also $b = 0$ when every point is a fixed point. In writing down the formulas for $\zeta_1$ and $\zeta_2$ in the general case use was made of (4).

Homographic transformations may be classified according to their fixed points as follows:

(i) *The identity*. Every point is a fixed point,

$$a = d = \pm 1, \quad b = c = 0.$$

(ii) *Parabolic substitutions*. The two fixed points coincide,

$$a + d = \pm 2, \quad \zeta_1 = \zeta_2 = \zeta.$$

The substitution may be put in one of the forms

$$\frac{1}{z' - \zeta} = \frac{1}{z - \zeta} + \delta \qquad \zeta \neq \infty$$

$$z' = z + \delta \qquad \zeta = 0.$$

In the first case

$$(a + d)^2 = 4, \quad \zeta = \frac{a - d}{2c}, \quad \delta = \pm c \neq 0,$$

and in the second case

$$a = d = \pm 1, \quad c = 0, \quad \delta = b/d \neq 0.$$

(iii) Substitutions with two distinct fixed points, $\zeta_1$ and $\zeta_2$. Such substitutions may be put in one of the forms

$$\frac{z' - \zeta_1}{z' - \zeta_2} = \lambda \frac{z - \zeta_1}{z - \zeta_2} \qquad \zeta_1, \zeta_2 \neq \infty$$

$$z' - \zeta_1 = \lambda (z - \xi_1) \qquad \zeta_1 \neq \infty, \quad \zeta_2 = \infty,$$

where

$$\lambda^{1/2} = \frac{1}{2} \{(a + d) - [(a + d)^2 - 4]^{1/2}\} \text{ if } c \neq 0, \quad \lambda = a \text{ if } c = 0,$$

and there are three possibilities

(iii a) $|\lambda| = 1$. *Elliptic substitution*

(iii b) $\lambda$ real. *Hyperbolic substitution*

(iii c) $\lambda$ is not real and $|\lambda| \neq 1$. *Loxodromic substitution.*

For any homographic substitution $\tau$, the substitutions $\sigma$ and $\tau^{-1}\sigma\tau$ are called *similar*. Similar substitutions belong to the same type, i.e., both are elliptic, or both parabolic etc.

### 14.1.3. Discontinuous groups

A set $G$ of homographic substitutions $\sigma$, $\sigma'$, ... is called a *group* if it has the following properties:

(i) The identity $I$ is in $G$.

(ii) If $\sigma$ is in $G$ then also $\sigma^{-1}$ is in $G$.

(iii) If $\sigma$ and $\sigma'$ are in $G$ then also $\sigma\sigma'$ is in $G$.

The substitutions $\sigma_1$, $\sigma_2$, ... are called *generators* of $G$ if any substitution of $G$ is a product of a finite number of positive or negative powers of some of the substitutions $\sigma_i$.

Two points $P$ and $P'$ (of $S_0$ or of the complex plane) are called *equivalent or congruent* with respect to $G$ if $P \neq P'$, and $G$ contains a substitution which maps $P$ upon $P'$.

Let $D_0$ be a fixed open region (= open connected set of points) on $S_0$ or in the complex plane, and let $G$ be a group of homographic substitutions each of which maps $D_0$ onto itself. Some of the substitutions of $G$ may have fixed points in $D_0$. Let us remove from $D_0$ all points which are either fixed points of some substitution of $G$ (other than $I$) or else are limit points of fixed points; we assume that the remaining set $D_1$ (which is open) is connected and hence a region. For any point $P_1$ of $D_1$ consider the set of all points equivalent to $P_1$ with respect to $G$. If $P_1$ is not a limit point of the set of points equivalent to $P_1$, i.e., if all points equivalent to $P_1$ lie outside some neighborhood of $P_1$, and if this happens for *all* points of $D_1$, then $G$ is called a *discontinuous group* in the region $D_0$. For a simple proof of a criterion for discontinuity of a group of real substitutions see Siegel (1950).

### 14.1.4. Fundamental region

We shall consider a group $G$ of homographic substitutions with which it is possible to associate a closed region (= closed connected set) $F^*$ with the following properties. (i) $F^*$ is bounded by a finite number of circles or arcs of circles (several disjoint arcs of the same circle may occur). We shall denote these circles and circular arcs by $A_1, A_2, \dots, A_n$; a point at which two arcs meet will be called a *vertex*, and the vertices will be denoted by $V_1, V_2, \dots, V_m$. (ii) No two interior points of $F^*$ are equivalent with respect to $G$. (iii) The components $A_1, \dots, A_n$ of the boundary may be arranged in pairs $A_\nu, A_{\nu'}$, $\nu \neq \nu'$ in such a manner that for each $\nu$ there exists exactly one $\sigma_\nu^*$ in $G$ which maps $A_\nu$ onto $A_{\nu'}$. (iv) The substitutions $\sigma_\nu^*$ of (iii) are *generators* of $G$, that is, every substitution of $G$ is a product of (positive or negative) powers of the $\sigma_\nu^*$.

We first remark that no substitution of $G$ (other than $I$) has a fixed point in the interior of $F^*$. If $P$ ( in the interior of $F^*$) were a fixed point of $\sigma$ in $G$ then $\sigma$ maps a neighborhood of $P$ onto some neighborhood of $P$, both may be assumed to lie in $F^*$, and this is excluded by (ii). Consider now the maps of $F^*$ under all substitutions of $G$. The union of all these maps forms a region [the union being a connected set by virtue of (iii) and (iv)]. No point occurs twice as the map of interior points of $F^*$, for if we assume $\sigma(P) = \sigma'(P')$ for two interior points $P$, $P'$, we have $P' = \sigma'^{-1}[\sigma(P)]$, $\sigma'^{-1}\sigma$ belongs to $G$, and this contradicts either (ii), if $P \neq P'$, or the remark about the absence of fixed points, if $P = P'$. On the other hand a point which is a map of a boundary point of $F^*$ certainly occurs several times, for instance $\sigma_\nu^* P_\nu = IP_{\nu'}$, where $P_\nu$ is a point on $A_\nu$, and $P_{\nu'}$, is the corresponding point on $A_{\nu'}$. By removing part of the boundary of $F^*$ we shall construct a region $F$ which is neither open nor closed and whose maps under the substitutions of $G$ cover a region of $S_0$, or of the $z$-plane, *simply*. The region $F$ will be called a *fundamental region* or *fundamental domain* of $G$.

To construct $F$, take the bounding circles and (open) arcs of $F^*$, arranged in pairs $A_\nu$, $A_{\nu'}$ as above: from each pair remove one arc, retaining the other. Remove also those vertices at which an infinite number of maps of $F^*$ meet, and divide the remaining vertices in classes of equivalent vertices, retaining one of each class, and removing all the others. The set of all remaining points (and this includes all interior points of $F^*$) is a fundamental region $F$ of $G$: it contains no two equivalent points.

Let $\sigma_1$, $\sigma_2$, ... be the substitutions of $G$, $\sigma_1$ being the identity. The substitution $\sigma_r$ maps $F$ onto $F_r$, and $F_1 = F$. The union of all the $F_r$ forms a region $D_1$ (which in general is neither open nor closed), and the interior, $D_0$, of $D_1$ is the open region which was discussed in sec. 14.1.3.

Let $z$ be any point of $F$, and set $z_r = \sigma_r(z)$. A limit-point of the sequence $\{z_r\}$ is called a *limit point* of $G$ ($\infty$ may occur as a limit point). The set of all limit points is mapped upon itself by any substitution of $G$, and may be used to define the *boundary* of $D_0$ or $D_1$.

A given group $G$ does not determine a unique fundamental region $F$, and it may be proved (see Fricke and Klein 1897, Chapter 2, p. 128) that $F$ can always be chosen in such a manner that none of its vertices is a fixed point of a hyperbolic or of a loxodromic substitution. At an elliptic vertex $V$ of $F^*$, the angle between two arcs meeting at $V$ is of the form $2\pi/l$ where $l$ is a positive integer. If $V$ is a fixed point of the elliptic substitution $\sigma$ of $G$, then $\sigma^l$ is the identity; $l$ is called the *order* of $\sigma$ or of $V$. A vertex of $F^*$ which is a fixed point of a parabolic substitution of $G$ is called a *parabolic cusp*.

Two groups, $G$ and $G'$, of homographic substitutions are called *similar* or *equivalent* if there exists a fixed substitution $\tau$ such that $G' = \tau^{-1} G\tau$, i.e., such that for each $\sigma$ in $G$, the substitution $\sigma' = \tau^{-1}\sigma\tau$ is in $G'$, and such that every $\sigma'$ in $G'$ may be obtained in this manner from a substitution $\sigma$ in $G$. If $F$ is a fundamental region of $G$, then $\tau^{-1}F = F'$ is a fundamental region of $G'$ [$\tau^{-1}F$ being the set of all points $\tau^{-1}(z)$ when $z$ ranges over $F$].

Our definition of a fundamental region is needlessly restrictive, and has been adopted here for the sake of simplicity; for a more general discussion see Fricke and Klein (1897). It is not at all essential for the fundamental region to be bounded by a finite number of circles and circular arcs; it is essential that the fundamental region should form a complete set of non-equivalent points, that it should be connected, and that it should have a reasonably regular shape. The first two of these requirements is fairly easy to formulate but it is very difficult to express the third condition in a manner which is both simple and precise, and sufficiently general. The assumption of a finite number of vertices of $F$ implies certain restrictions on the automorphic functions to be considered, and these restrictions lead to a comparatively simple formulation of certain general theorems.

For the definition of fundamental regions of automorphic functions of several variables see the literature quoted in sections 14.11, 14.12.

### 14.2. Definition of automorphic functions

Let $G$ be a group of homographic substitutions

$$(1)\quad z_r = \sigma_r(z) = \frac{a_r z + b_r}{c_r z + d_r} \qquad a_r d_r - b_r c_r = 1, \quad r = 0, 1, 2, \dots,$$

$\sigma_0$ being the identity,

$$a_0 = d_0 = \pm 1, \quad b_0 = c_0 = 0.$$

Let $G$ be discontinuous in a region $D_0$, and let $F$ be a fundamental region of $G$. We shall consider *automorphic functions*, $\phi(z) = \phi(z;\, G)$ which satisfy the identity

$$(2)\quad \phi(z_r) = \phi[\sigma_r(z)] = \phi(z) \qquad r = 0, 1, 2, \dots .$$

The behavior of these functions in the neighborhood of a singularity $z_0$ will be described in terms of a *uniformizing variable* $t$ in the form

$$(3)\quad \phi(z) = t^m (a_0 + a_1 t + a_2 t^2 + \cdots),$$

where $m$ is an integer, and the uniformizing variable is defined with reference to $G$ as follows.

If $z_0$ is not a fixed point of a substitution of $G$ we put

(4) $t = z - z_0$ $\qquad z_0 \neq \infty$

(5) $t = z^{-1}$ $\qquad z_0 = \infty$.

If $z_0$ is the fixed point of a parabolic substitution

(6) $$\frac{1}{z' - z_0} = \frac{1}{z - z_0} + \delta \qquad z_0 \neq \infty$$

we put

(7) $$t = \exp\left(\pm \frac{2\pi i}{\delta} \frac{1}{z - z_0}\right),$$

choosing the sign so that $t \to 0$ as $z \to z_0$ in $F$, and if $z_0 = \infty$ is the fixed point of a parabolic substitution

(8) $z' = z + \delta$

we put

(9) $$t = \exp\left(\pm \frac{2\pi i}{\delta} z\right),$$

again choosing the sign so as to make $t \to 0$ as $z \to \infty$ in $F$. If $z_0$ is a fixed point of an elliptic substitution of order $l$, and $z_0'$ is the other fixed point of that substitution we put

(10) $$t = \left(\frac{z - z_0}{z - z_0'}\right)^l \qquad z_0 \neq \infty, \quad z_0' \neq \infty$$

(11) $t = z^{-l}$ $\qquad z_0 = \infty, \quad z_0' \neq \infty$

(12) $t = (z - z_0)^l$ $\qquad z_0 \neq \infty, \quad z_0' = \infty$.

With the foregoing notations and definitions, $\phi(z) = \phi(z; G)$ will be called an *automorphic function* of $G$ (or belonging to $G$) if it satisfies the following conditions:

(i) $\phi(z)$ is analytic and single-valued in $F$ with the possible exception of a finite number of points.

(ii) If $\phi(z)$ is analytic at $z_0$ in $F$, then it may be continued analytically, within $D_0$, to $z_r = \sigma_r(z_0)$, all possible analytic continuations (within $D_0$) lead to the same value $\phi(z_r)$, and $\phi(z_r) = \phi(z_0)$.

(iii) In the neighborhood of a singularity $z_0$, $\phi(z)$ may be represented in the form (3).

(iv) $\phi(z)$ is not a constant.

It has already been mentioned that our definition of automorphic functions (and fundamental regions) is not the most general definition. The class of functions defined above leads to a simple formulation and general validity of the theorems of sec. 14.7; Ford (1929, sec. 86) calls automorphic functions of the kind discussed here *simple* automorphic functions.

The most characteristic property of automorphic functions is their invariance under substitutions of $G$: this property is expressed by (2). More generally, the term automorphic function may be applied when a function of one or several variables is invariant under a group of transformations of the variable or variables. Some instances of such generalizations will occur in sections 14.11, 14.12.

## 14.3. The icosahedral group

In general, the group $G$ occurring in the definition of automorphic functions is an infinite group (i.e., consists of an infinite number of substitutions). In this section we shall discuss automorphic functions of a *finite* group (consisting of a finite number of substitutions). This example will show some of the essential principles involved in the construction of automorphic functions without the complications inherent in the general case. The group in question is the symmetry group of the *icosahedron* (the regular solid consisting of twenty equilateral triangles). The group may be envisaged as the group of rotations of an icosahedron into itself and is known as the *icosahedral group*. It is identical with the symmetry group of the dodecahedron (the regular solid consisting of twelve regular pentagons) and is sometimes also called the *dodecahedral group*. Now, in Euclid's construction, the dodecahedron is derived from a cube, each edge of the cube being a diagonal of a face of the dodecahedron. Altogether five distinct cubes may be inscribed in such a manner in any dodecahedron, any rotation of the dodecahedron effects a permutation of these cubes, and thus our group may be identified as a group of permutations of five elements which turns out to be the *alternating group* (consisting of all the even permutations).

Let an icosahedron be inscribed in the sphere $S_0$ of 14.1 (1), and let the edges of the icosahedron be projected on $S_0$, the center of the sphere being the center of projection. We thus obtain a pattern of 20 congruent equilaterial spherical triangles covering $S_0$. There are 60 rotations of the sphere which leave this pattern invariant, for any centroid of a triangle may be brought into any of 20 positions, and in each position there will be 3 rotations (by $2\pi/3$) which leave the pattern invariant.

If we map the sphere $S_0$ onto the complex $z$-plane by a stereographic projection 14.1(2), we obtain a net of 20 curvilinear triangles (in the $z$-plane) bounded by arcs of circles (in the sense of sec. 14.1.1 so that some of the "arcs of circles" may be segments of straight lines). The 60 rotations of the sphere induce 60 homographic transformations, and these form a group $G_{60}$, a realization of the icosahedral group. Taking the origin of the $z$-plane at one of the vertices, and the real $z$-axis as an axis of symmetry of the fundamental region, it turns out that $G$ contains the three substitutions

$$(1)\quad U(z) = -\frac{1}{z}$$

$$(2)\quad S(z) = \epsilon z = \frac{\epsilon^3 z}{\epsilon^2} \qquad \epsilon = e^{2\pi i/5}$$

$$(3)\quad T(z) = \frac{(1+\epsilon)z + \epsilon^3}{\epsilon^3 z - (1+\epsilon)} = \frac{(\epsilon - \epsilon^4)\,5^{-\frac{1}{2}}\, z - (\epsilon^2 - \epsilon^3)\,5^{-\frac{1}{2}}}{-(\epsilon^2 - \epsilon^3)\,5^{-\frac{1}{2}}\, z - (\epsilon - \epsilon^4)\,5^{-\frac{1}{2}}}.$$

In the cases of $S$ and $T$ the first of the two forms is the simplest form of the substitution, and the last is the form satisfying 14.1(4). The special substitutions $U$, $S$, $T$ are generators of $G_{60}$. More precisely, the 60 substitutions of $G_{60}$ are given by

$$(4)\quad S^\kappa, \quad S^\kappa T S^\lambda, \quad U S^\kappa, \quad U S^\kappa T S^\lambda$$

where $\kappa$, $\lambda = 0, 1, 2, 3, 4$. The identity is $S^0$ in this representation.

The group $G_{60}$ is discontinuous, and $D_0$ is the whole plane. The fundamental region $F$ has vertices at the points $z_0 = 0$, $z_1$, and $\bar{z}_1$, where

$$(5)\quad z_1 = \epsilon^2 \left[ \tfrac{3}{4} + \tfrac{1}{4}\sqrt{5} - \left( \tfrac{15}{8} + \tfrac{3}{8}\sqrt{5} \right)^{\frac{1}{2}} \right],$$

and $z_1$, $\bar{z}_1$ are conjugate complex. The boundary of $F$ consists of the segments $A_1$, $A_2$ of straight lines joining $z_0$ to $z_1$ and $\bar{z}_1$, and the circular arc $A_3$ joining $z_1$ and $\bar{z}_1$ and intersecting the real axis at

$$(6)\quad z_2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5} + \left( \tfrac{5}{2} + \tfrac{1}{2}\sqrt{5} \right)^{\frac{1}{2}}.$$

All substitutions of $G_{60}$ are elliptic, $U$, $S$, $T$ being of the order 2, 5, 2, respectively. The points $z_0$, $z_2$, $z_1$ are fixed points of $S$, $T$, $TS$, respectively. $S$ maps $A_1$ upon $A_2$ and $T$ maps the part of $A_3$ joining $z_1$ and $z_2$ onto that part joining $\bar{z}_1$ and $z_2$. Therefore the two halves of $A_3$ count as separate arcs, $z_2$ counts as a vertex, and the full set of vertices of $F$

is $z_0, z_1, z_2, \bar{z}_1$. If parts of the boundary are removed in accordance with sec. 14.1.4, then the maps of $F$ under the 60 transformations (4) will cover the entire $z$-plane simply. The network of triangles on $S_0$ or in the $z$-plane is shown in Forsyth (1900, Fig. 104, p. 660 and Fig. 107, p. 667) where six of the triangles alternately white and black (shaded) form a fundamental region.

In the present case all automorphic functions are rational functions of $z$ and it can be shown (see, for example, Fricke, 1926, vol. 2, chapter 3) that they may be expressed in terms of the functions

(7) $u(z) = z^{20} + 1 - 228(z^{15} - z^5) + 494 z^{10}$

(8) $v(z) = z^{30} + 1 + 522(z^{25} - z^5) - 10005(z^{20} + z^{10})$

(9) $w(z) = z(z^{10} + 11 z^5 - 1)$

as follows. Let $k$, $l$, $m$, $n$ be integers, $n \geq 0$ [if $n = 0$ the sum in (10) must be replaced by zero, and the product in (11) by unity]; let $\epsilon_\nu = \pm 1$, and $a_\nu$ and $b_\nu$ any non-zero constants, $\nu = 1, \ldots, n$; and assume that

(10) $20k + 30l + 12m + 60 \sum_{\nu=1}^{n} \epsilon_\nu = 0.$

Then

(11) $\phi(z) = u^k v^l w^m \prod_{\nu=1}^{n} (a_\nu u^3 + b_\nu v^2)^{\epsilon_\nu}$

is an automorphic function of $G_{60}$, and every automorphic function may be represented in this form. Since the three functions $u$, $v$, $w$ are not independent, and satisfy the relation

(12) $u^3 - v^2 + 12^3 w^5 = 0,$

the representation (11) in not unique.

For a description of the location of the zeros and poles of the automorphic function defined by (11), and for the application of the theory of automorphic functions of $G_{60}$ to the solution of the generic quintic equation see Fricke (1926, vol. 2, chapters 2 and 3).

All finite groups of homographic substitutions may be enumerated. For the theory of automorphic functions belonging to these groups see Fricke (1926, vol. 2, chapter 2).

**14.4. Parabolic substitutions**

If all substitutions of a group $G$, with the exception of the identity, are parabolic then it may be shown that all the parabolic substitutions of the group have the same fixed point. Without restricting the generality of our considerations, we shall assume that the common fixed point is at infinity. In this case $D_0$ will be the *finite part* of the plane, i.e., the set of all finite complex numbers $z$ (also called the punctured $z$-plane or the plane punctured at infinity). The discontinuous group itself will be of one of the following two types. Either there is a fixed real or complex number $\omega$ such that

(1) $\sigma_r(z) = z + r\,\omega \qquad r = 0, \pm 1, \pm 2, \ldots;$

or else there are two fixed real or complex numbers $\omega$ and $\omega'$ such that $\omega/\omega'$ is not real, and the substitutions of the group are

(2) $\sigma_{rr'}(z) = z + r\,\omega + r'\omega' \qquad r, r' = 0, \pm 1, \pm 2, \ldots .$

In the case of the group consisting of the substitutions (1)

(3) $$t = \exp\left(\frac{2\pi i z}{\omega}\right)$$

is an automorphic function of $G$. Any meromorphic function (= single-valued function which is analytic save for poles) of $t$ is an automorphic function, and every automorphic function is of this form. Thus, in this case, the automorphic functions of $G$ are meromorphic periodic functions of period $\omega$.

If $G$ consists of the substitutions (2), the automorphic functions of $G$ are meromorphic doubly-periodic (that is, elliptic, see sec. 13.11) functions of $z$ with periods $\omega$, $\omega'$.

At first it might seem as if one could have groups of parabolic substitutions with three or more periods. However, it can be proved (see sec. 13.10) that a meromorphic function of a complex variable which has more than two independent periods is a constant, so that a group with more than two independent translations has no automorphic functions.

*Generalizations. Multiply period functions.* Translation groups with several independent generators will have automorphic functions if instead of a function of a single complex variable we consider meromorphic functions of $p$ complex variables, $p = 2, 3, 4, \ldots$ . Such functions may have $2p$ (or fewer) periods. These are defined in terms of $2p^2$ constants

(4) $\omega_{\mu\alpha} \qquad \mu = 1, 2, \ldots, p; \quad \alpha = 1, 2, \ldots, 2p$

called the *periods*. The $\omega_{\mu\alpha}$ cannot be chosen arbitrarily and are subject to certain conditions. It can be shown that after a suitable linear transformation of the variables and periods

$$(5) \quad \omega_{\mu\nu} = i\pi\,\delta_{\mu\nu}/e_{\mu}, \quad \omega_{\mu,\,p+\nu} = a_{\mu\nu} = a_{\nu\mu} \qquad \mu, \nu = 1, \ldots, p$$

where $\delta_{\mu\nu}$ is the Kronecker symbol, the $e_{\mu}$ are positive integers, and

$$(6) \quad \operatorname{Re} \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} a_{\mu\nu} x_{\mu} x_{\nu} < 0$$

for all real $x_{\alpha}$ satisfying

$$(7) \quad \sum_{\alpha=1}^{p} x_{\alpha}^{2} > 0.$$

A single-valued analytic function $f(u_1, \ldots, u_p)$ of the $p$ complex variables which is regular for all finite values of $u_1, \ldots, u_p$ save for isolated points which are not essential singularities, and which cannot be expressed as a function of less than $p$ linear combinations of the variables, will be called a $2p$*-tuply periodic function* of $u_1, \ldots, u_p$ if for any integer $n_{\mu\alpha}$, $\mu = 1, \ldots, p$, $\alpha = 1, \ldots, 2p$, and

$$(8) \quad \eta_{\mu} = \sum_{\alpha=1}^{2p} n_{\mu\alpha}\omega_{\mu\alpha} \qquad \mu = 1, \ldots, p$$

we have

$$(9) \quad f(u_1 + \eta_1, \ldots, u_p + \eta_p) = f(u_1, \ldots, u_p)$$

for all finite $(u_1, \ldots, u_p)$ at which $f$ is regular, provided that the $\omega_{\mu\alpha}$ are such that at least for one $\mu$

$$\sum_{\alpha=1}^{2p} \lambda_{\alpha}\omega_{\mu\alpha} \neq 0$$

for all real $\lambda_1, \ldots, \lambda_{2p}$ except $\lambda_1 = \cdots = \lambda_{2p} = 0$.

It may be proved that for any given set of periods $\omega_{\mu\alpha}$ satisfying (5), (6) there exist $2p$-tuply periodic functions. There always exist $p$ such functions which are algebraically independent; any $p + 1$ such functions are connected by an algebraic relation (see sec. 13.11 for the case $p = 1$). Every $2p$-tuply periodic function may be expressed as a rational function of suitably chosen theta functions defined as a $p$-tuple infinite series

(10) $\theta(u_1, \dots, u_p)$

$$= \sum_{m_1, \dots, m_p = -\infty}^{\infty} \exp\left( \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} a_{\mu\nu} m_\mu m_\nu + 2 \sum_{\mu=1}^{p} m_\mu u_\mu \right)$$

in which the $a_{\mu\nu}$ are real or complex numbers such that the Re $a_{\mu\nu}$ form a negative definite symmetric matrix, i.e., $a_{\mu\nu} = a_{\nu\mu}$ and

(11) $$\operatorname{Re}\left( \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} a_{\mu\nu} x_\mu x_\nu \right) < 0$$

for all real $x_1, \dots, x_p$ satisfying

$$\sum_{\mu=1}^{p} x_\mu^2 > 0.$$

For the theory of multiply periodic functions and its connection with algebraic functions of a single variable and with the theory of Abelian functions see Baker (1907), Krazer and Wirtinger (1901 - 1921).

### 14.5. Infinite cyclic group with two fixed points

Let $\sigma$ be a hyperbolic or loxodromic substitution. If $\zeta_1$ and $\zeta_2$ are the fixed points of $\sigma$, this substitution may be represented in the form

$$\frac{z' - \zeta_1}{z' - \zeta_2} = \rho e^{in\phi} \frac{z - \zeta_1}{z - \zeta_2}$$

or

$$z' - \zeta_1 = \rho e^{in\phi} (z - \zeta_1)$$

according as $\zeta_2 \neq \infty$ or $\zeta_2 = \infty$, it being assumed that $\zeta_1 \neq \infty$. Here $\rho > 0$, $\rho \neq 1$. If $\phi$ is an integer multiple of $2\pi$, the substitution is hyperbolic; otherwise it is loxodromic.

Consider the group $G$ generated by $\sigma$. The elements of $G$ are $\sigma^n$, $n = 0, \pm 1, \pm 2, \dots$ . The substitution $\sigma^n$ may be represented as

(1) $$\frac{z' - \zeta_1}{z' - \zeta_2} = \rho^n e^{in\phi} \frac{z - \zeta_1}{z - \zeta_2} \qquad \zeta_1, \zeta_2 \neq \infty$$

or

(2) $$z' - \zeta_1 = \rho^n e^{in\phi} (z - \zeta_1) \qquad \zeta_1 \neq \infty, \quad \zeta_2 = \infty,$$

where $\rho$, $\phi$ are the quantities introduced above, and $n$ is any (positive or negative) integer.

The group $G$ is discontinuous in the region $D_0$ which consists of all complex numbers different from $\zeta_1$ and $\zeta_2$ (the complex plane punctured at $\zeta_1$ and $\zeta_2$). In order to obtain a fundamental region $F$, let $C_0$ be any circle which separates $\zeta_1$ and $\zeta_2$ (so that any continuous curve joining $\zeta_1$ and $\zeta_2$ intersects $C_0$), and let $C_0$ be mapped onto the circle $C_n$ by $\sigma^n$. The sequence of circles $C_n$, $n = 0, \pm 1, \pm 2, \ldots$ is invariant under $G$. No two circles of this sequence have a point in common. Any region bounded by two adjacent circles, $C_n$ and $C_{n+1}$, (with one of the circles forming part of the region and the other not) may be taken as the fundamental region $F$.

The automorphic functions of $G$ are the elliptic functions of the complex variable

$$(3) \quad u = \log \frac{z - \zeta_1}{z - \zeta_2} \qquad \zeta_1, \zeta_2 \neq \infty$$

$$u = \log(z - \zeta_1) \qquad \zeta_1 \neq \infty, \quad \zeta_2 = \infty$$

with periods

$$(4) \quad \omega_1 = \log\rho + i\phi, \quad \omega_2 = 2\pi i.$$

The occurrence of doubly periodic functions as automorphic functions of the group $G$ may be explained by the following circumstances. The group $G$ seems essentially the same as that identified by 14.4(1); there is no algebraic difference between the two groups, they are isomorphic. There is a considerable difference in the regions involved, though. The region $D_0$ (plane punctured at $\infty$) and the fundamental region $F$ (infinite strip) of 14.4(1) are *simply connected*; the region $D_0$ (plane punctured at two points) and the fundamental region $F$ (region between two circles without a common point) of this section are *doubly connected*. In a doubly connected region (such as $F$) a function may be analytic everywhere and yet many-valued, thus violating condition (ii) of sec. 14.2. We need one periodicity to make our function single-valued in $F$, and a second, to transplant it, as it were, to the maps of $F$ according to 14.2(2).

For an application to a boundary value problem in electrostatics see Burnside (1891, 1892) where the case of $2n$ (rather than two) bounding circles is investigated.

## 14.6. Elliptic modular functions

### 14.6.1. The modular group

Let $M$ be the group of all homographic substitutions

$$(1)\quad z' = \frac{az+b}{cz+d} \qquad ad - bc = 1$$

with *integer* $a$, $b$, $c$, $d$. $M$ is called the *modular group* (see also sec. 13.24): it is of infinite order, and all of its substitutions map the upper half-plane $\operatorname{Im} z > 0$ onto itself. Let $D_0$ be the upper half-plane. Then $M$ is discontinuous in $D_0$. The set of points

$$(2)\quad \operatorname{Im} z > 0 \quad \text{and either} \quad |z| \geq 1, \quad -\tfrac{1}{2} \leq \operatorname{Re} z \leq 0$$
$$\text{or} \quad |z| > 1, \quad 0 < \operatorname{Re} z < \tfrac{1}{2}$$

may be taken as a fundamental region $F$. The vertices of $F$ are at the points

$$(3)\quad z_1 = -\tfrac{1}{2} + \tfrac{1}{2} i\sqrt{3}, \quad z_2 = i, \quad z_3 = \tfrac{1}{2} + \tfrac{1}{2} i\sqrt{3}, \quad z_4 = \infty.$$

The substitution $\sigma$ defined by $a = b = d = 1$, $c = 0$, or

$$(4)\quad z' = \sigma(z) = z + 1$$

maps the segment (ray) joining $z_1$ to $z_4$ onto the segment joining $z_3$ to $z_4$, and has $z_4$ as its parabolic fixed point. The substitution $\tau$ defined by $a = d = 0$, $b = -1$, $c = 1$, or

$$(5)\quad z' = \tau(z) = -\frac{1}{z}$$

maps the arc of the upper half of the unit circle between $z_1$ and $z_2$ onto the arc joining $z_3$ and $z_2$, and has $z_3$ as a fixed point (the other fixed point being in the lower half-plane).

The group $M$ is generated by $\sigma$, $\tau$. Since $\tau^2 = I$, any substitution of $M$ may be written in the form

$$\sigma^{n_1} \tau \sigma^{n_2} \tau \cdots \sigma^{n_{l-1}} \tau \sigma^{n_l}$$

where

$$l = 1, 2, 3, \ldots, \qquad n_1, n_l = 0, 1, 2, \ldots, \qquad n_2, \ldots, n_{l-1} = 1, 2, 3, \ldots .$$

The maps of $F$ under these substitutions cover the upper half-plane simply.

### 14.6.2. The modular function $J(z)$

The *absolute invariant*, $J(z)$, of the modular group $M$ arises in the theory of elliptic functions (where the variable is usually denoted by $\tau$, see sec. 13.24). It is important both in that theory and in its applications; and a function nearly related to it is the key to Picard's original proof of Picard's theorem. The principal properties of $J(z)$ are as follows:

(i) The function $J(z)$ is single-valued and analytic in $D_0$ (the upper half-plane) and

$$(6) \quad J(z') = J\left(\frac{az+b}{cz+d}\right) = J(z) \text{ in } D_0$$

for all substitutions 14.6(1) of the modular group $M$.

(ii) The function $w = J(z)$ maps $F$ [given by 14.6(2)] simply onto the (entire) $w$-plane in such a manner that the boundary of $F$ is mapped onto the real $w$-axis from $-\infty$ to 1, and

$$(7) \quad J(-\tfrac{1}{2} + \tfrac{1}{2} i\sqrt{3}) = 0, \quad J(i) = 1, \quad J(\infty) = \infty.$$

(iii) By (i) and (ii), $J(z)$ is an automorphic function of $M$, and every automorphic function of $M$ is a rational function of $J(z)$.

We may add that every point on the real $z$-axis is a singularity of $J(z)$, and the real axis is a *natural boundary* of $J(z)$.

*Expression of $J(z)$ in terms of Eisenstein series.* Let $\omega$, $\omega'$ be two real or complex numbers, $\operatorname{Im}(\omega'/\omega) > 0$. We regard $\omega$ and $\omega'$ as half-periods and form Weierstrass' invariants

$$(8) \quad g_2(\omega, \omega') = 60 \,\Sigma'(m\omega + n\omega')^{-4}$$

$$g_3(\omega, \omega') = 140 \,\Sigma'(m\omega + n\omega')^{-6}$$

[see 13.12(13)] where $\Sigma'$ indicates summation over all pairs of integers $(m, n)$ with the exception of $m = n = 0$. We also set

$$(9) \quad \Delta(\omega, \omega') = g_2^3 - 27 g_3^2$$

[see 13.13(7)]. Clearly, $g_2^3/\Delta$ is a homogeneous function of degree zero in $\omega$ and $\omega'$ and hence depends only on

$$(10) \quad z = \omega'/\omega$$

which will be regarded as a complex variable ranging over the upper half-plane. We have

(11) $J(z) = g_2^3/\Lambda$

[see 13.24(4)].

In

(12) $E_2(z) = \omega^4 g_2(\omega, \omega') = 60 \sum{}'(m + nz)^{-4}$

$E_3(z) = \omega^6 g_3(\omega, \omega') = 140 \sum{}'(m + nz)^{-6}$

we collect all those terms in which $m$ and $n$ have a fixed greatest common divisor $d$ so that $n = sd$, $m = -td$, $s \geq 0$ and $s$ and $t$ are coprime. If $s = 0$, $t = 1$. Using the results

(13) $$\sum_{d=1}^{\infty} \frac{1}{d^4} = \frac{\pi^4}{90}, \qquad \sum_{d=1}^{\infty} \frac{1}{d^6} = \frac{\pi^6}{945}$$

which follow from 1.13(16), we finally obtain

(14) $$E_2(z) = \frac{4}{3}\pi^4 \left[1 + \sum_{(s,\, t)=1,\ s>0} (sz - t)^{-4}\right]$$

$$E_3(z) = \frac{8}{27}\pi^6 \left[1 + \sum_{(s,\, t)=1,\ s>0} (sz - t)^{-6}\right].$$

In the last two sums $s$ runs through all positive integers, and for each $s$, $t$ runs through all (positive, negative, and zero) integers coprime to $s$.

The series (14) are examples of *Eisenstein series*. The characteristic property of such series is the restriction placed upon the indices of summation by number-theoretical conditions.

The expressions $g_2$ and $g_3$ in (8) are called *homogeneous modular forms* (that is modular forms expressed in terms of the homogeneous variables $\omega$, $\omega'$) of dimension $-4$ and $-6$, respectively, and $E_2$ and $E_3$ in (14) are called *inhomogeneous modular forms* (that is modular forms expressed in terms of the inhomogeneous variable $z$). For a definition of modular forms see Klein and Fricke (1890, 1892) and sec. 14.8.3.

If $a$, $b$, $c$, $d$ are integers and $ad - bc = 1$, then $s' = as - ct$, $t' = dt - bs$ run through a complete set of pairs of coprime integers if $s$, $t$ run through this set, it being understood that only one of the pairs $s'$, $t'$ and $-s'$, $-t'$ appears in the set. From this it follows that for any substitution 14.6(1) of $M$

$$(15)\quad E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^4\, E_2(z)$$

$$E_3\left(\frac{az+b}{cz+d}\right) = (cz+d)^6\, E_3(z),$$

and

$$(16)\quad J(z) = \frac{[E_2(z)]^3}{[E_2(z)]^3 - 27\,[E_3(z)]^2}$$

satisfies (6).

From (14) it is seen that $E_2(z)$ and $E_3(z)$ are single-valued analytic functions of $z$ in $D_0$ (the upper half-plane) and that the real axis is a locus of singularities of these functions. A more careful discussion shows that $J(z)$ has the same properties [see (i) above].

*Expression of $J(z)$ in terms of theta functions.* We put

$$(17)\quad q = e^{i\pi z} \qquad |q| < 1.$$

Since the substitution $z' = z + 1$ is in $M$, we see that $J(z)$ is a periodic analytic function, with period 1, of $z$, as $z$ ranges over the upper half-plane. Hence $J(z)$ will be an even analytic function of $q$ in the unit circle punctured at $q = 0$, and may be expanded in a series of even powers of $q$.

The expansion in question may be derived from the formula

$$(18)\quad J(z) = \frac{\pi^8}{54}\,\frac{(\theta_2^8 + \theta_3^8 + \theta_4^8)^3}{\theta_1'^8} = \frac{4\pi^8}{27}\,\frac{(\theta_3^8 - \theta_2^4\,\theta_4^4)^3}{\theta_1'^8}$$

which follows from 13.24(5), 13.19(22) and (23) and in which

$$(19)\quad \theta_1' = \theta_1'(0) = 2\pi q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})^3$$

$$\theta_2 = \theta_2(0) = 2q^{1/4} \prod_{n=1}^{\infty} [(1 - q^{2n})(1 + q^{2n})^2]$$

$$\theta_3 = \theta_3(0) = \prod_{n=1}^{\infty} [(1 - q^{2n})(1 + q^{2n-1})^2]$$

$$\theta_4 = \theta_4(0) = \prod_{n=1}^{\infty} [(1 - q^{2n})(1 - q^{2n-1})^2]$$

[see 13.19(16)] are the theta functions of zero argument. From (18) and (19) we have an expansion of the form

$$(20)\quad 1728\,J(z) = q^{-2} + \sum_{n=0}^{\infty} a_n q^{2n} \qquad q = e^{i\pi z}$$

convergent when $0 < |q| < 1$. Clearly the coefficients $a_n$ are integers: for their numerical values for $0 \leq n \leq 24$ see Zuckerman (1939). Another expression which can be obtained from (18) is

$$(21)\quad J(z) = \frac{\left[1 + 240 \sum_{n=1}^{\infty} n^3 q^{2n}/(1-q^{2n})\right]^3}{12^3 q^2 \prod_{n=1}^{\infty} (1-q^{2n})^{24}}.$$

*Connection with hypergeometric series:* From property (ii) of $J(z)$ it follows by means of sec. 2.7.2 that the inverse function of $J(z)$ may be expressed in terms of hypergeometric functions. See also 13.24(2) and (5), and 13.8(5) and (6).

We put

$$(22)\quad F(J) = {}_2F_1\left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J\right)$$

$$F^*(J) = {}_2F_1\left(\frac{5}{12}, \frac{5}{12}; \frac{4}{3}; J\right),$$

where ${}_2F_1$ is Gauss' hypergeometric series defined in 2.1(2), and introduce

$$(23)\quad \gamma = \frac{F(1)}{F^*(1)} = \left[\frac{\Gamma(\frac{11}{12})}{\Gamma(\frac{7}{12})}\right]^2 \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})}$$

$$(24)\quad \lambda = (2-\sqrt{3})\,\gamma.$$

Then, with $J = J(z)$, $F = F(J)$, $F^* = F^*(J)$ it may be proved that

$$(25)\quad z = e^{2\pi i/3}\,\frac{F - \lambda e^{i\pi/3} J^{1/3} F^*}{F - \lambda e^{-\pi i/3} J^{1/3} F^*}$$

This equation gives the value of $z$ for any $J$ inside the unit circle. Outside the unit circle we have

$$(26)\quad 2\pi i z = -\log J - 3\log 12 + \frac{G(\frac{1}{12}, \frac{5}{12}; 1; J^{-1})}{{}_2F_1(\frac{1}{12}, \frac{5}{12}; 1; J^{-1})}$$

$$|J| > 1, \qquad |\arg(1 - J)| < \pi$$

where ${}_2F_1$ is again Gauss' series and

$$(27)\quad G(a, b; 1; u) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [\psi(a+n) + \psi(b+n) - 2\psi(n+1)$$

$$+ \psi(a) + \psi(b) - 2\psi(1)],$$

$\psi$ being the logarithmic derivative of the gamma function (see Fricke, 1930).

For applications of the modular invariant to the theory of numbers see sec. 14.6.5. For an application in the theory of functions of a complex variable (to the proof of Picard's theorem) see, for instance, Hurwitz and Courant (1925).

### 14.6.3. Subgroups of the modular group

We shall now consider certain subgroups of the modular group; these will be defined by congruence properties of the integers $a$, $b$, $c$, $d$ involved in the homographic substitution

$$(28)\quad z' = \frac{az + b}{cz + d} \qquad ad - bc = 1.$$

Let $m$ be a positive integer, and let $M_m$ be the set of all those substitutions (28) of $M$ for which

(29) either $a+1,\ b,\ c,\ d+1$ or $a-1,\ b,\ c,\ d-1$

are integers divisible by $m$. It is easy to see that $M_m$ itself is a group: it is called the *principal congruence subgroup of level* (in German, *Stufe*) $m$ of the modular group $M$. Each $M_m$ is discontinuous in the half-plane $\operatorname{Im} z > 0$, and a fundamental region of $M_m$ may be constructed by forming the union of $\gamma_m$ suitably chosen "copies" of the fundamental region $F$ of $M$ [defined by (2)]. By a "copy" of $F$ we mean here a region upon which $F$ is mapped by a modular substitution. If

$$(30)\quad m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1, \dots, p_k$ are distinct primes, and $a_1, \dots, a_k$ are positive integers, then $\gamma_2 = 6$, and

(31) $\gamma_m = \frac{1}{2} m^3 (1 - p_1^{-2})(1 - p_2^{-2}) \cdots (1 - p_k^{-2})$ $\quad m > 2.$

We shall further consider the cases $m = 2, 5$. For other $M_m$ see Fricke (1926), Klein and Fricke (1890, 1892).

For $m = 2$, we have (28) with $a$, $d$ odd integers, $b$, $c$ even integers. $M_2$ is the $\lambda$-group (sections 13.22, 13.24). The fundamental region $F_2$ of $M_2$ may be defined by

(32) $\operatorname{Im} z > 0, \quad |z - \frac{1}{2}| > \frac{1}{2}, \quad |z + \frac{1}{2}| \geq \frac{1}{2}, \quad -1 \leq \operatorname{Re} z < 1.$

The points

(33) $z_1 = -1, \quad z_2 = 0, \quad z_3 = 1, \quad z_4 = \infty$

are vertices of $F_2$. The boundary of $F_2$ consists of the parts in the upper half plane of the straight lines $\operatorname{Re} z = \pm 1$, and of the circles $|2z \pm 1| = 1$. We denote the components of the boundary by $A_1, \dots, A_4$ as follows:

(34) $A_1$: $\operatorname{Im} z \geq 0, \quad \operatorname{Re} z = -1$
$A_2$: $\operatorname{Im} z \geq 0, \quad |z + \frac{1}{2}| = \frac{1}{2}$
$A_3$: $\operatorname{Im} z \geq 0, \quad |z - \frac{1}{2}| = \frac{1}{2}$
$A_4$: $\operatorname{Im} z \geq 0, \quad \operatorname{Re} z = 1.$

By (32), $A_1$ and $A_2$ belong to $F_2$, and $A_3$ and $A_4$ do not. The $\lambda$-group is generated (in the sense explained in sec. 14.1.4) by the substitutions

(35) $z' = \sigma(z) = z + 2, \quad z' = \tau(z) = \dfrac{z}{2z + 1}.$

We have already seen in sec. 13.24 that $k^2 = \lambda(z)$ is an *automorphic function* of $M_2$. This function is single-valued and analytic in the upper half-plane, is invariant under substitutions of $M_2$, and maps $F_2$ onto the entire $w$-plane; furthermore, every automorphic function of $M_2$ is a rational function of $k^2$. Since $M_2$ is a subgroup of $M$, and $J(z)$ is an automorphic function of $M$, it follows that $J(z)$ is also an automorphic function of $M_2$ and hence a rational function of $k^2$. The explicit expression is

(36) $J(z) = \dfrac{4}{27} \dfrac{(1 - k^2 + k^4)^3}{k^4 (1 - k^2)^2}.$

The function $\lambda(z)$ can most easily be defined in terms of theta functions [see 13.20(14)]. We have

$$(37)\quad \lambda(z)=k^2=\frac{\theta_2^4(0,q)}{\theta_3^4(0,q)}=16q\prod_{m=1}^{\infty}\left(\frac{1+q^{2m}}{1+q^{2m-1}}\right)^8$$

$$=16\left(\frac{\sum_{m=0}^{\infty} q^{(m+\frac{1}{2})^2}}{1+2\sum_{m=1}^{\infty} q^{m^2}}\right)^4$$

where

$$(38)\quad q=e^{i\pi z},\qquad |q|<1.$$

Series expansions of $\lambda(z)$ which are of the type of Eisenstein series may be derived from the theory of Weierstrass' $\wp$-function.

The function

$$(39)\quad w=\lambda(z)$$

maps the region

$$(40)\quad \operatorname{Im} z\geq 0,\qquad 0\leq \operatorname{Re} z\leq 1,\qquad |z-\tfrac{1}{2}|>\tfrac{1}{2}$$

of the $z$-plane onto the upper half of the $w$-plane in such a manner that the points $z=0,\ 1,\ \infty$ correspond, respectively, to $w=1,\ \infty,\ 0$. As in the case of $J(z)$, this means that the inverse function of (39) may be expressed in terms of the hypergeometric function. By 13.19(3) and 13.8(5) we have

$$(41)\quad z=i\ \frac{{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\ 1;\ 1-\lambda)}{{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\ 1;\ \lambda)},$$

where ${}_2F_1$ is Gauss' series.

An independent approach to the theory of $\lambda(z)$ was given by Nehari (1947) who considered the functional equation

$$(42)\quad f(q)=4[f(q^2)]^{\frac{1}{2}}\{1+[f(q^2)]^{\frac{1}{2}}\}^{-2}$$

and showed that the conditions

$$(43)\quad f(0)=0,\qquad f'(0)>0,\qquad f(q)\ \text{analytic for}\ |q|<1$$

determine a unique solution, $f_0(q)$, of (42). We have $f_0(q) = \lambda(z) = k^2$, where $q$ and $z$ are connected by (38), and (42) is essentially Landen's transformation (see sec. 13.23).

We now turn to $M_5$. The fundamental region, $F_5$, of $M_5$ consists of $\gamma_5 = 60$ "copies", in the upper half-plane, of $F$. The 60 modular substitutions which map $F$ on its 60 copies are representatives of the 60 *cosets* of $M_5$ in $M$. [For the notion of cosets of a subgroup see van der Waerden (1949).]

There exists an automorphic function, $\Lambda(z)$, which stands in the same relation to $M_5$ and $F_5$ as $J(z)$ to $M$ and $F$, or $\lambda(z)$ to $M_2$ and $F_2$. The explicit expression defining $\Lambda(z)$ is

$$(44)\quad \Lambda(z) = q^{2/5} \frac{\sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+3m}}{\sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+m}}.$$

Every automorphic function of $M_5$ is a rational function of $\Lambda(z)$; the absolute invariant $J(z)$ is an automorphic function of every subgroup of $M$, and hence of $M_5$, and must thus be expressible as a rational function of of $\Lambda(z)$. The actual expressions are

$$(45)\quad \frac{J}{J-1} = \frac{[u(\Lambda)]^3}{[v(\Lambda)]^2}, \qquad J = -\frac{[u(\Lambda)]^3}{1728[w(\Lambda)]^5},$$

where $u$, $v$, $w$ are the polynomials defined in 14.3(7), (8), (9). The formulas (45) play an important role in F. Klein's celebrated solution of the quintic equation.

For all positive integers $l$, the function $[\lambda(z)]^{1/(2l)}$ is an automorphic function of some subgroup of $M$. If, and only if, $l = 1$, 2, or 4, there is a principal congruence subgroup ($M_4$, $M_8$, and $M_{16}$, respectively) of which $[\lambda(z)]^{1/(2l)}$ is an automorphic function.

### 14.6.4. Modular equations

If $f(z)$ is either $J(z)$ or the corresponding automorphic function of a principal congruence subgroup (for instance $\lambda(z)$ in the case of $M_2$, and $\Lambda(z)$ in the case of $M_5$), then, for any integer $l > 1$, the functions $f(z)$ and $f(lz)$ are connected by an algebraic equation. Such equations are called *modular equations*.

In the case of the absolute invariant we have the following situation. For any integer $l > 1$, the function $J(lz)$ satisfies an algebraic equation of degree $l+1$. The coefficients in this equation are rational functions of $J(z)$, and the coefficients appearing in these rational functions are rational numbers. The roots of this equation are

$$J(lz),\ J\left(\frac{z}{l}\right),\ J\left(\frac{z+1}{l}\right),\ \ldots,\ J\left(\frac{z+l-1}{l}\right).$$

We shall give explicitly the modular equation satisfied by $J(2z)$. Using the abbreviations

$$(46)\quad j = 12^3\, J(z), \qquad j^* = 12^3\, J(2z),$$

the equation is

$$(47)\quad j^{*3} + j^3 - (j\,j^*)^2 + 2^4 \cdot 3 \cdot 31\, j\,j^*(j + j^*) - 2^4 \cdot 3^4 \cdot 5^3 (j^2 + j^{*2})$$
$$+\, 3^4 \cdot 5^3 \cdot 4027\, j\,j^* + 2^8 \cdot 3^7 \cdot 5^6 (j + j^*) - 2^{12} \cdot 3^9 \cdot 5^9 = 0.$$

### 14.6.5. Applications to number theory

Elliptic modular functions and related functions (Eisenstein series, theta functions) play an important role in the theory of numbers. For some applications see sections 17.2, 17.3, 17.4 and Hardy (1940). The absolute invariant, $J(z)$, has the property that $J(\alpha)$ is an integral algebraic number whenever $\alpha$ has a positive imaginary part, and is a root of a quadratic equation with integer coefficients. The algebraic equations with integer coefficients satisfied by certain $J(\alpha)$ are the so-called class equations for imaginary quadratic number-fields [see Fricke (1928), Fueter (1924, 1927)]: see also Schneider (1936), Hecke (1939).

A new and far-reaching development was originated by Hecke (1935, 1937, 1939, 1940a, 1940b): see also Petersson (1939) and, for certain numerical results, Zassenhaus (1941).

For some results which are relevant for the subject of this section, although they appear as special cases of a much more general theory, see Siegel (1935).

### 14.7. General theory of automorphic functions

In this section we shall briefly describe a classification of discontinuous groups of homographic substitutions, and mention some of the general theorems on automorphic functions of a single variable. All results to be mentioned are based on the definitions of the first few sections of this chapter: it has already been explained that these definitions are not the most general ones known in the literature.

### 14.7.1. Classifications of the groups

Automorphic functions are often classified according to the groups to which they belong. A classification of all discontinuous groups (see sec. 14.1.3) of homographic substitutions was given by Poincaré. It was further developed by Fricke who devoted about one third of the first volume of Fricke and Klein (1897) to a detailed classification the results of which are fully stated on p. 164, 165 of Fricke and Klein (1897, vol. I).

As in the introductory sections, let $G$ be a group of homographic substitutions $\sigma_r (r = 0, 1, 2, 3, \dots)$ where

$$(1) \quad \sigma_r(z) = \frac{a_r z + b_r}{c_r z + d_r} \qquad a_r d_r - b_r c_r = 1.$$

If there exists a circle $C_0$ which is mapped onto itself by each $\sigma_r$, the group $G$ is called a *Fuchsian* group. The circle $C_0$ is called a *principal circle* (in German, *Hauptkreis*) of $G$, and $G$ is also called a group with a principal circle. If $G$ has a principal circle, a homographic transformation of the $z$-plane may be used to map $C_0$ on a standard circle. Two such standardizations are used. (i) $C_0$ is the unit circle. Necessary and sufficient conditions for the unit circle to be mapped onto itself by all $\sigma_r$ are

$$(2) \quad d_r = \bar{a}_r, \quad c_r = \bar{b}_r, \quad |a_r| \neq |b_r| \qquad r = 0, 1, 2, \dots$$

where bars denote conjugate complex quantities (see for instance, Copson 1935, sec. 8.31). (ii) $C_0$ is the real axis. Necessary and sufficient conditions for the real axis to be mapped onto itself by all $\sigma_r$ are

$$(3) \quad a_r, \; b_r, \; c_r, \; d_r \text{ real}, \quad a_r d_r - b_r c_r \neq 0 \qquad r = 0, 1, 2, \dots .$$

The modular group and its subgroups are examples of discontinuous groups for which the real axis is a principal circle.

In general, $G$ will have limit points (see sec. 14.1.4): let $l$ be their number. It can be proved that the only possible values of $l$ are 0, 1, 2, and $\infty$. If $l = 0$, clearly $G$ is a finite group (examples of such groups are given in sec. 14.3). If $l = 1$, it can be shown that $G$ is a group of parabolic substitutions, and all substitutions of the group have the same fixed point: such groups are discussed in sec. 14.4, If $l = 2$, we have the case investigated in sec. 14.5, and the slightly more general case where $G$ is similar to the group generated by the two substitutions

$$(4) \quad \sigma(z) = az, \quad \tau(z) = \epsilon/z$$

where $\epsilon$ is a root of unity, i.e., $\epsilon^m = 1$ for some positive integer $m$, see Fricke and Klein (1897). If $l = \infty$, the limit points of $G$ form an infinite point set, and any limiting point (i.e., point of accumulation) of this set is also a limit point (in the sense of the definition of sec. 14.1.4) of $G$.

If a principal circle, $C_0$, exists, and every point of $C_0$ is a limit point of $G$, then $C_0$ is called a *limit circle* of $G$, and $G$ itself is called a *Fuchsian group of the first kind*. If, on the other hand, the limit points are nowhere dense on $C_0$, then $G$ is called a *Fuchsian group of the second kind*. In all other cases involving an infinity of limit points, $G$ is called a *Kleinian group*. If $l = \infty$ and no principal circle exists, it can be proved that $G$ must contain loxodromic substitutions. The modular group and its subgroups discussed in sec. 14.6.3 are examples of groups for which the real axis is the limit circle.

### 14.7.2. General theorems on automorphic functions

Let $G$ be an infinite discontinuous group (see sec. 14.1.3) of homographic substitutions, let $F$ be a fundamental region (see sec. 14.1.4) of $G$, and let $\phi(z)$, $\phi_1(z)$, ... be automorphic functions (in the sense of sec. 14.2) of $G$. The following general theorems hold for automorphic functions, and correspond to the general theorems of sec. 13.11 on elliptic functions [which are automorphic functions of the group 14.4(2) generated by two translations].

Every automorphic functions has poles in $F$. The number of zeros and poles in $F$ is the same. An automorphic function assumes, in $F$, every value the same number of times.

Any two automorphic function of the same group are *algebraically dependent*, that is, for any two automorphic functions $\phi_1(z)$ and $\phi_2(z)$ of $G$ there exists a polynomial, $P(u, v)$, in two variables, with constant coefficients, so that $P(\phi_1(z), \phi_2(z)) = 0$ identically for all values of $z$ for which $\phi_1(z)$ and $\phi_2(z)$ are defined.

For any given group $G$ it is possible to find two automorphic functions, $\phi_1(z)$ and $\phi_2(z)$, with the property that any automorphic function of $G$ is a rational function of $\phi_1(z)$ and $\phi_2(z)$ with constant coefficients. The expression of elliptic functions in terms of $\wp(z)$ and $\wp'(z)$ in sec. 13.14 is an instance of this theorem.

If there exists an automorphic function, $\phi_0(z)$, of $G$ which has a single simple pole in $F$, and is otherwise analytic there, then every automorphic function of $G$ is a rational function of $\phi_0(z)$. For examples of such functions see $J(z)$ in sec. 14.6.2, and $\lambda(z)$ and $\Lambda(z)$ in sec. 14.6.3. It can be proved that a necessary and sufficient condition for such a $\phi_0(z)$

to exist is the condition that the "genus" of $F$ be zero: for the definition of the genus of the fundamental region see sec. 14.8.2; also Fricke and Klein (1897) or Ford (1929).

If an automorphic function, $\phi_0(z)$, of the nature described in the preceding paragraph, exists, and if $z = \eta(w)$ is the inverse function to $w = \phi_0(z)$ (under the circumstances described above such an inverse function exists since $\phi_0(z)$ assumes every value exactly once), then $\eta(w)$ may be represented as the quotient, $y_1/y_2$, of two particular solutions of the linear differential equation

$$(5)\quad \frac{d^2 y}{dw^2} = u(w)\, y$$

in which $u$ is a rational function of $w$. [In more general cases $u$ will be an algebraic function, see Ford (1929, sec. 44).] For a special case where (5) is equivalent to the hypergeometric equation see sec. 14.10. In the case of $J(z)$, $\lambda(z)$, $\Lambda(z)$, the differential equations corresponding to (5) are special hypergeometric equations.

Every limit point (in the sense of sec. 14.1.4) is an essential singularity for every automorphic function of $G$. In particular, in the case of a Fuchsian group of the first kind, the limit circle is the *natural boundary* for all automorphic functions of $G$; analytic continuation beyond the limit circle is impossible.

## 14.8. Existence and construction of automorphic functions

### 14.8.1. General remarks

The theory of automorphic functions has two fundamental problems. The first of these is the enumeration of all possible fundamental regions (or, of all fundamental regions satisfying certain conditions), and the construction of the group belonging to each of these fundamental regions; and the second problem is the construction of all automorphic functions belonging to a given group.

The problem of finding all groups which possess a fundamental region has been solved completely in the case of groups with a limit circle: see Fricke and Klein (1897). The solution requires a thorough knowledge of non-Euclidean geometry. Even for the more difficult problem of finding a unique standard form for the fundamental region of a given group, partial answers are known in the case of groups with a limit circle which are generated by a finite number of substitutions.

With regard to the second problem of finding all automorphic functions belonging to a given group with a given fundamental region, the general theorems of sec. 14.7.2 show that the basic problems are: to find two automorphic functions in terms of which all others can be expressed rationally, and to discover algebraic relations between automorphic functions belonging to the same group. Two powerful methods for achieving this will be indicated in sections 14.8.2, 14.8.3.

In general, it is very difficult to obtain explicit formulas: the theories of modular and elliptic functions are rather exceptional. In particular, with most groups the coefficients of the substitutions of the groups involved cannot be characterized in a simple and explicit fashion.

### 14.8.2. Riemann surfaces

Given $G$ and $F$, the generators of the group $G$ set up a correspondence between pairs of boundary points of the fundamental region $F$ [see sec. 14.1.4(iii)]. If equivalent boundary points are identified, a *Riemann surface* $S$ is obtained: this Riemann surface may have boundary points, corresponding to fixed points of substitutions of $G$ on the boundary of $F$. The genus of this Riemann surface is also the *genus of the fundamental region* $F$. (See Ford, 1929, p. 238.)

Single-valued analytic functions on $S$ correspond to automorphic functions of $G$ so that the construction of automorphic functions of a given group with a given fundamental region is equivalent to the construction of single-valued analytic functions on a (not necessarily open) Riemann surface. For an outline of this method see Hurwitz and Courant (1925). The problems of uniformization (see sec. 14.9) have played an important role in the development of this approach to the theory of automorphic functions.

In particular cases the construction may be performed explicitly. The simplest examples are the Riemann-Schwarz triangle functions. For these, and for theorems on differential equations satisfied by inverse functions of automorphic functions see sec. 14.10.

### 14.8.3. Automorphic forms. Poincaré's theta series

Poincaré, and after him Ritter (1892, 1894) and Fricke (Fricke and Klein, 1912) developed the theory of automorphic functions by a method resembling Weierstrass' construction of elliptic functions.

Let $G$ be a discontinuous group of homographic substitutions as in sec. 14.2, and let $F$ be a fundamental region of $G$.

Let $s$ be a constant, and for each $r = 0, 1, 2, \ldots$, let $v(\sigma_r)$ be a real or complex number of absolute value unity [so that $v(\sigma_r)$ is a function on $G$ to the unit circle of the complex plane]. With the notations and definitions of sec. 14.2, a function $\psi(z)$ will be called an *automorphic form* of the class $\{G, -s, v\}$ if it satisfies the following conditions.

(i) $\psi(z)$ is analytic and single-valued in $F$ with the possible exception of a finite number of points.

(ii) If $\psi(z)$ is analytic at $z_0$ in $F$, then it may be continued analytically, within $D_0$, to $z_r = \sigma_r(z_0)$, all possible analytic continuations (within $D_0$) lead to the same value $\psi(z_r)$, and

$$(1) \quad \psi(z_r) = v(\sigma_r)(c_r z_0 + d_r)^s \psi(z_0).$$

(iii) In the neighborhood of a singularity, $\psi(z)$ may be represented in the form 14.2(3).

(iv) $\psi(z)$ is not a constant.

The function $v(\sigma_r)$ is called a *multiplier system*, and it follows from (1) that $v$ is a multiplicative function on $G$, i.e.,

$$(2) \quad v(\sigma_r \sigma_{r'}) = v(\sigma_r)\, v(\sigma_{r'}).$$

The condition $|v(\sigma_r)| = 1$ is a customary assumption. The automorphic form satisfying (1) is said to be of dimension $-s$. An automorphic function is an automorphic form of dimension zero having $v(\sigma_r) = 1$ as its multiplier system. Automorphic forms belonging to a subgroup of the modular group are also called *modular forms*.

The construction of automorphic functions may be reduced to that of automorphic forms. If $\psi_1(z)$ and $\psi_2(z)$ are automorphic forms of class $\{G, -s_1, v_1\}$ and $\{G, -s_2, v_2\}$ respectively and if

$$[v_1(\sigma_r)]^{s_2}\, [v_2(\sigma_r)]^{s_1} = 1 \qquad r = 0, 1, 2, \ldots$$

then

$$\phi(z) = [\psi_1(z)]^{s_2}\, [\psi_2(z)]^{s_1}$$

is either a constant or an automorphic function of $G$.

It can be shown that every automorphic form may be represented by a *Poincaré theta series*. We shall construct such a series under the assumption that $z = \infty$ is not a limit point of $G$. The series is then of the form

$$(3) \quad \theta(z; G) = \sum_{r=0}^{\infty} [v(\sigma_r)]^{-1} (c_r z + d_r)^{-2l} H(z_r),$$

where $z_r$, $v(\sigma_r)$, $c_r$, $d_r$ have the meaning ascribed to these symbols in sec. 14.2 and in the present section, $l$ is an integer $\geq 2$, and $H(z)$ is a *rational* function of $z$ which is analytic at all limit points of $G$. The infinite series converges uniformly and absolutely in every closed subset of $F$ in which $H(z)$ is analytic, and it may be shown by means of (2) and the relation

$$\sigma_r(z_{r'}) = \sigma_r[\sigma_{r'}(z)] = \sigma_{r''}(z) = z_{r''}$$

that Poincaré's theta series (3) represents an automorphic form of the class $\{G, -2l, v\}$.

In connection with the construction of automorphic forms in terms of theta series a difficulty arises in certain cases, especially if $G$ is a group with a limit circle, this difficulty being due to the fact that the function represented by the theta series may vanish identically. In the case of automorphic functions with poles this difficulty may be overcome by constructing theta series with a single pole in $F$; such series do certainly not vanish identically in $F$. On the other hand, it may be necessary to construct automorphic forms which are analytic in $F$ and vanish in parabolic cusps of $F$. In this case $H(z)$ is analytic in $F$ and it may very well happen that the series (3) vanishes identically. This circumstance caused the greatest difficulty which Poincaré had to overcome in his theory of the series (3).

A new foundation of the theory of automorphic forms and of Poincaré's theta series was laid by Petersson (1940) whose method is based on a *metrization* of automorphic forms. Let $G$ be a Fuchsian group of the first kind containing parabolic substitutions. Taking the real axis as the limit circle, the coefficients $a_r$, $b_r$, $c_r$, $d_r$ of the substitutions may be taken to be real. In this situation, Petersson puts $z = x + iy$ and defines a scalar product of two forms as

$$(4) \quad (\psi_1, \psi_2) = \int\int_F \psi_1(z)\, \overline{\psi_2(z)}\, y^{s-2}\, dx\, dy \qquad s > 2,$$

the bar, as usual, indicating complex conjugation. Using the invariance of hyperbolic measure under the group $G$, Petersson computes (4) explicitly if $\psi_1$ is an automorphic form which is analytic in $F$ and vanishes at all parabolic cusps of $F$ ("Spitzenform"), and $\psi_2$ in a Poincaré theta series. The resulting formula is used for a characterization of theta series and for the proof of the fundamental theorems in the theory of these series. If $G$ is a congruence subgroup of the modular group, the theory holds for $s = 2$, $v(\sigma_r) = 1$. For extensions, generalizations, and applications of this method see Petersson (1941, 1944, 1949).

For the case of a Fuchsian group of the first kind containing hyperbolic substitutions only, Dalzell (1932, 1944, 1949a, 1949b) developed a new method for Poincaré's theta series and related functions.

In many cases the theory of Poincaré's theta series has been supplemented by the theory of functions analogous to Weierstrass' sigma and zeta function (while the theta series are analogous to the $\wp$-function). See Ford (1929), the references given in sec. 14.10.2, and also Ritter (1892), Stahl (1888), Dalzell (1932).

In the case of a group without limit circle, Poincaré's theta series may converge absolutely for $l = 1$ and the multiplier system $v(\sigma_r) = 1$ (see sec. 14.10.2).

## 14.9. Uniformization

Let $G$ be a Fuchsian group of the first kind such that the closure of the fundamental region $F$ is contained in the interior of the limit circle (if the limit circle is the real axis, we define the upper half-plane to be the interior). We assume that all substitutions of $G$ (with the exception of the identity, $\sigma_0$) are hyperbolic. We know from sec. 14.7.2 that any two automorphic functions, $\phi_1(z)$ and $\phi_2(z)$, of $G$ are algebraically dependent, i.e., satisfy a relation

(1) $P(\phi_1(z), \phi_2(z)) = 0$

identically in $F$, $P(u, v)$ being a polynomial. This means that the variables $u$, $v$ which are connected by means of the relation

(2) $P(u, v) = 0$

and hence are algebraic functions of each other, may be expressed as single-valued functions,

(3) $u = \phi_1(z), \quad v = \phi_2(z)$

of an auxiliary variable $z$ which is then called a *uniformizing variable* for the algebraic relation (2). Alternatively, (2) may be regarded as defining an *algebraic curve* and (3) as a parametric representation, in terms of single-valued functions, of that curve. It is an important fact that every algebraic relation may be uniformized in this manner, and that automorphic functions are the most general functions that need to be used (see also sec. 13.2). This result may be described in greater detail as follows.

Let $P(u, v)$ be an irreducible polynomial in two variables $u$ and $v$ (i.e., a polynomial which cannot be decomposed into a product of polynomials), and let the variables $u$ and $v$ be connected by the algebraic relation (2). Then there exist two functions, $\phi_1(z)$ and $\phi_2(z)$, of a complex variable $z$, and a region $F^*$ in the $z$-plane, with the following properties: For any pair, $u$, $v$, of complex numbers satisfying (2), there exists a $z$ in $F^*$ such that $u = \phi_1(z)$, $v = \phi_2(z)$, and apart from a finite number of pairs $(u, v)$, this $z$ in $F^*$ is uniquely determined. Moreover, the functions $\phi_1(z)$ and $\phi_2(z)$ may be chosen so that *either* $\phi_1(z)$ and $\phi_2(z)$ are rational functions and $F^*$ is the entire $z$-plane, *or* $\phi_1(z)$ and $\phi_2(z)$ are elliptic functions with a common pair of periods and $F^*$ is a period parallelogram of these functions (only one of the vertices and two of the sides of this parallelogram being parts of $F^*$), *or else* $\phi_1(z)$ and $\phi_2(z)$ are automorphic functions of a Fuchsian group of the first kind all of whose substitutions (with the exception of $\sigma_0$) are hyperbolic, and $F^*$ is a fundamental region of this group.

For the theory and history of uniformization see, for instance, Hurwitz and Courant (1925, Part III, Chap. 9).

## 14.10. Special automorphic functions

Particular automorphic functions were also described in sections 14.3 to 14.6.3.

### 14.10.1. The Riemann-Schwarz triangle functions

In certain cases the differential equation 14.7(5) may be reduced to the hypergeometric equation 2.1(1). The resulting automorphic functions have a limit circle. They are called the Riemann-Schwarz *triangle functions*; see also sec. 2.7.2, Klein and Fricke (1890-1892), Ford (1929 sec. 114).

In order to construct a fundamental region for the group of such a triangle function, and to obtain the group itself, let $C_1$, $C_2$, $C_3$ be three circles, and let $C_0$ be a circle which is orthogonal to $C_1$, $C_2$, $C_3$. We may take $C_0$ as the real axis when the centers of $C_1$, $C_2$, $C_3$ will lie on the real axis (one or several of $C_i$, $i = 1, 2, 3$ may be straight lines perpendicular to the real axis, the center of such a straight line being the point at infinity of the real axis). Let $\Delta$ be a triangle bounded by arcs, $A_1$, $A_2$, $A_3$, of the circles $C_1$, $C_2$, $C_3$: we assume that $\Delta$ is in the upper half-plane. Let $n_1$, $n_2$, $n_3$ be three positive integers, and let the interior angles of $\Delta$ be $\alpha_1$, $\alpha_2$, $\alpha_3$ where

$$(1) \quad \alpha_i = \frac{\pi}{2n_i} \qquad i = 1, 2, 3,$$

and it is assumed that

(2) $\alpha_1 + \alpha_2 + \alpha_3 < \pi.$

Zero angles (or infinite integers $n$) are admitted. We number the angles and vertices in such a manner that $\alpha_1$ is the angle made by $A_2$ and $A_3$, etc., $V_1$ is the vertex at which $A_2$ and $A_3$ meet, etc. Let $\Delta'$ be the triangle obtained by an inversion of $\Delta$ on the circle $C_3$. The points $V_1$ and $V_2$ are also vertices of $\Delta'$; let the remaining vertex of $\Delta'$ be $V_4$. We then take the closed region $\Delta + \Delta'$ as the region $F^*$ of sec. 14.1.4. Clearly our $F^*$ satisfies the condition (i) of sec. 14.1.4, and we shall construct a group $G$ so that conditions (ii)-(iv) are also satisfied.

There is a unique homographic substitution, $\sigma_1$, with real coefficients which maps $V_1$ onto itself and $V_3$ onto $V_4$; likewise a similar substitution, $\sigma_2$, which maps $V_2$ onto itself and $V_3$ onto $V_4$. Clearly, $\sigma_1$ maps $A_2$ onto $A'_2$ and $\sigma_2$ maps $A_1$ onto $A'_1$. The arcs $A_1$, $A'_1$, $A_2$, $A'_2$ bound $F^*$, and condition (iii) of sec. 14.1.4 is satisfied. The group $G$ generated by $\sigma_1$ and $\sigma_2$ clearly satisfies conditions (ii) and (iv). Let $F$ be the region obtained from $F^*$ by removing $V_4$ and the interiors of $A'_1$, $A'_2$. Then $G$ is a Fuchsian group of the first kind whose limit circle is the real axis, and $F$ is a fundamental region of $G$.

The group $G$ possesses an automorphic function $\phi_0(z)$ whose inverse function is a *Schwarz function* (see sec. 2.7.2) and may be expressed as a quotient of two hypergeometric functions. The function $\phi_0(z)$ assumes every value exactly once in $F$, and every automorphic function of $G$ is a rational function of $\phi_0(z)$. Simple examples of such functions are the absolute invariant of sec. 14.6.2 or the corresponding automorphic functions (in sec. 14.6.3) of subgroups of the modular group. If

(3) $\alpha_1 = \dfrac{\pi}{2}, \quad \alpha_2 = 0, \quad \alpha_3 = \dfrac{\pi}{3},$

$G$ is the modular group $M$, and we may take $\phi_0(z) = J(z)$; if

(4) $\alpha_1 = \alpha_2 = \alpha_3 = 0,$

$G$ is the lambda-group $M_2$, and we may take $\phi_0(z) = k^2(z) = \lambda(z)$.

E.T. Whittaker (1899, 1902) has studied another class of automorphic functions which has the property that every member of the class is a rational function of a single automorphic function. See also Ford (1929, sec. 96).

### 14.10.2. Burnside's automorphic functions

Let $C_\mu$, $C'_\mu$, $\mu = 1, \dots, m$ be $2m$ circles, and assume that no two of these circles have a point in common, and no circle separates any two others. These assumptions imply that there is at most one straight line among these circles, and, if there is a straight line, all other circles lie on one side of it: the half-plane bounded by this straight line and containing no circle will be regarded as its interior.

Let $\tau_1, \dots, \tau_m$ be $m$ hyperbolic or loxodromic substitutions such that $\tau_\mu$ maps the interior of $C_\mu$ onto the exterior of $C'_\mu$, and let $G$ be the group generated by $\tau_1, \dots, \tau_m$. The part of the plane exterior to all circles may be taken as the fundamental region $F$. The group $G$ has no principal circle. If $m > 1$, $G$ has an infinity of limit points; if these are removed the remaining part of the $z$-plane is a connected set.

Automorphic functions of $G$ may be constructed in terms of Poincaré theta series, and in this case series of dimension $-2$ converge absolutely. The theory of automorphic functions of $G$ has been developed by Burnside (1891, 1892) who applied his results to a boundary value problem of Laplace's equation. See also Riemann (1876) and, for similar groups and their automorphic functions, Schottky (1887).

### 14.11. Hilbert's modular groups

The theory of modular and automorphic functions has been extended in several ways to functions of more than one variable. The first results are due to Picard (1882). In this section we shall briefly indicate an approach originated by Hilbert, and in the following section describe researches carried out by Siegel. For the general theory of automorphic functions of several variables see also Hurwitz (1905), Fubini (1908, Chap. 3), Sugawara (1940a, b), Hua (1946).

Let $R$ be the field of rationals, let $K_1$ be a finite real algebraic extension of $R$, $K_2, \dots, K_n$ the fields conjugate to $K_1$, and assume that all $K_\rho$, $\rho = 1, \dots, n$ are real. For any $\alpha^{(1)}$ in $K_1$ let $\alpha^{(2)}, \dots, \alpha^{(n)}$ be the conjugates, $\alpha^{(\rho)}$ in $K_\rho$, similar notations being used for $\beta$, $\gamma$, $\delta$. Let $z_\rho$, $\rho = 1, \dots, n$, be $n$ complex variables and let $S$ be the region $\operatorname{Im} z_\rho > 0$, $\rho = 1, \dots, n$, in the space of $n$ complex variables (this space having $2n$ real dimensions). Let $\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, \delta^{(1)}$ be any algebraic integers in $K_1$ such that

(1) $\alpha^{(1)}\delta^{(1)} - \gamma^{(1)}\beta^{(1)} = 1.$

More generally, unity in (1) may be replaced by any totally positive unit of $K_1$. We then define a modular transformation $\sigma$ by the equations

(2) $$z'_\rho = \frac{\alpha^{(\rho)} z_\rho + \beta^{(\rho)}}{\gamma^{(\rho)} z_\rho + \delta^{(\rho)}} \qquad \rho = 1, \ldots, n$$

and see that $\sigma$ maps $S$ onto itself. The set of all such $\sigma$ forms a group $G$ which is called *Hilbert's modular group of* $K_1$.

Blumenthal (1903, 1904) proved that $G$ possesses a fundamental region in $S$ and also that there exist automorphic functions of the $n$ complex variables $z_1, \ldots, z_n$ belonging to $G$. If regularity conditions, analogous to the conditions of sec. 14.2, are imposed, it turns out that any $n + 1$ automorphic functions are connected by an algebraic relation, and that $n + 1$ automorphic functions may be selected in such a manner that any automorphic function of $G$ is a rational function of the $n + 1$ particular functions.

Maass (1941) investigated Hilbert's modular group when $K_1 = R(\sqrt{5})$, the field obtained by adjoining $\sqrt{5}$ to $R$, and consequently $n = 2$. He applied the theory of modular forms of the resulting group to problems in number theory (quadratic forms). For other investigations of Hilbert's modular group and of its automorphic functions, for the extension to this situation of Petersson's theory of Poincaré's theta series see Maass (1940 a, b, 1942, 1948). Maass (1940 a, b) also investigated generalizations of Hilbert's modular group.

For an extension of Blumenthal's results in the direction of Hecke's theory of modular forms of one variable see de Bruijn (1943).

## 14.12. Siegel's functions

A theory of modular functions of $\frac{1}{2}n(n + 1)$ complex variables with $n = 1, 2, \ldots$, was developed by Siegel (1935, 1936, 1937, 1939) who took the arithmetical theory of quadratic forms as the point of departure for the theory of what he called *modular functions of the* $n$th *degree*. Many general theorems of this theory reduce, when $n = 1$, to known results on modular functions or modular forms of a single variable; others lead to new results even when $n = 1$. One outstanding feature of Siegel's theory is the utilization of *symplectic geometry* (geometry of positive definite matrices in the space of symmetric matrices) in $n(n + 1)$ real dimensions, in place of the non-Euclidean (hyperbolic) geometry of the Poincaré half-plane of two real dimensions (Siegel 1943). This leads to a theory of automorphic functions (Siegel 1942, 1943). Another outstanding feature of this theory is the frequent use of arithmetical methods for the proof of results which in the case of a single variable are usually proved by analytical methods. Many of the groups of automorphic functions of a single variable have important arithmetical properties, yet there is an essentially geometrical approach to them: in Siegel's theory arithmetical methods are of central importance in the definition of discontinuous groups.

In this section we shall give some of the basic definitions and results in the simplest case corresponding to the theory of the modular group $M$ and its absolute invariant $J(z)$ in the case of a single variable. A review of the ramifications of Siegel's theory, and of its numerous important results and applications, is far beyond the scope of this section.

*The modular group of degree n.* Matrices whose elements are integers will be called *integral matrices*. Unless a statement is made to the contrary, capital letters in this section will denote square matrices of $n$ rows and columns. The element in the $l$th row and $k$th column of the matrix $A$ will be denoted by $a_{lk}$, and we shall write

(1) $A = [a_{lk}]$ $\qquad l, k = 1, \ldots, n.$

We shall write $N$ for the zero matrix, and $I$ for the unit matrix, of $n$ rows and $n$ columns,

(2) $N = [n_{lk}], \quad I = [i_{lk}], \quad n_{lk} = 0, \quad i_{lk} = \delta_{lk}$ $\qquad l, k = 1, \ldots, n.$

The transposed of $A$ will be denoted by $A'$ so that $a'_{lk} = a_{kl}$; the inverse of $A$ is $A^{-1}$ so that $AA^{-1} = A^{-1}A = I$.

Let $A$, $B$, $C$, $D$ be four $n \times n$ integral matrices, and let

(3) $$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be the $2n \times 2n$ matrix partitioned into $A$, $B$, $C$, $D$ as indicated in (3). We define a $2n \times 2n$ matrix $J$ as

(4) $$J = \begin{bmatrix} N & I \\ -I & N \end{bmatrix}.$$

We shall assume that the integral matrices $A, \ldots, D$ have been so chosen that

(5) $M'JM = J.$

The necessary and sufficient conditions for this are

(6) $AB' = BA', \quad CD' = DC'$

(7) $AD' - BC' = I.$

If $C$ and $D$ satisfy the second condition (6), i.e., if $CD'$ is a symmetric matrix, then $C$ and $D$ are said to form a *symmetric pair*. Let $C_1$, $D_1$ and $C_2$ and $C_2$ be two symmetric pairs of matrices; these are called *associate* if there is a matrix $U$ such that both $U$ and $U^{-1}$ are integral and

(8) $C_1 = UC_2, \quad D_1 = UD_2.$

All symmetric pairs of matrices which are associate to a given pair are said to form a *class*. Let $C$, $D$ be a fixed symmetric pair of integral matrices, and let $U$ range over the set of all non-singular integral matrices. The matrices $C$ and $D$ are called *coprime* if a necessary condition for $U^{-1}C$ and $U^{-1}D$ to be integral is that $U^{-1}$ itself be integral (this condition is always sufficient).

All $2n \times 2n$ integral matrices $M$ satisfying (5) form a group. The two elements

(9) $$\pm \begin{bmatrix} I & N \\ N & I \end{bmatrix}$$

of this group form a normal (or invariant) subgroup of order two. The quotient group of the group of all $M$ relative to the subgroup (9), i.e., the group of all $M$ satisfying (5) if $M_1$ and $M_2 = -M_1$ are identified, is called the *modular group of degree n* and will be denoted by $\mathfrak{M}$. The elements of $\mathfrak{M}$ will be called substitutions, and each of these is determined by four integral matrices $A$, $B$, $C$, $D$ satisfying (6), (7). The matrices $A$, $B$, $C$, $D$ and $-A$, $-B$, $-C$, $-D$ determine the same substitution.

Let $Z$ be a (complex) symmetric matrix. We put

(10) $z_{lk} = z_{kl} = x_{lk} + iy_{lk} \qquad l, k = 1, \ldots, n$

and correspondingly

(11) $Z = X + iY$

where the $x_{lk}$ and $y_{lk}$ are real numbers, and $X$ and $Y$ are real matrices. We shall regard the $z_{lk}$ as complex variables and shall restrict them by the condition that $Y$ be *positive* (i.e., the quadratic form whose coefficients are the elements of $Y$ be positive definite). The matrices $Z$ may be envisaged as points of a space in which the $z_{lk}$, or $x_{lk}$ and $y_{lk}$, are coordinates: this space has $\frac{1}{2}n(n+1)$ complex dimensions, or $n(n+1)$ real dimensions. That part of this space in which $Y$ is a positive matrix forms a subspace which will be called $\mathfrak{H}$, and our variable matrix $Z$ will range over $\mathfrak{H}$ ("the positive cone").

For any integral matrices $A$, $B$, $C$, $D$ satisfying (6) and (7), i.e., for any element of $\mathfrak{M}$, we define the substitution

(12) $\sigma(Z) = (AZ + B)(CZ + D)^{-1}.$

It can be proved that each substitution (12) defines a 1-1 mapping of $\mathfrak{H}$ onto itself, and that the group of these mappings is homomorphic to $\mathfrak{M}$. It can also be shown that $\mathfrak{M}$, considered as a group of mappings of $\mathfrak{H}$ onto itself, possesses a fundamental region $\mathfrak{Z}$ which is bounded by a finite number of analytic hypersurfaces. For a set of generators of $\mathfrak{M}$ see Hua and Reiner (1949).

*Modular forms and modular functions*. Let $L$ be the set of all classes of coprime symmetric pairs of matrices. From each class we choose a representative pair $C$, $D$, and form the generalized Eisenstein series

$$(13)\quad \psi_r(Z) = \sum_L \, [\det(CZ + D)]^{-2r}.$$

It can be shown that for sufficiently large positive integers $r$, the series in (13) converges absolutely for every $Z$ in $\mathfrak{H}$ and defines, in $\mathfrak{H}$, an analytic function of the $\frac{1}{2}n(n+1)$ complex variables $x_{lk}$. The function $\psi_r(Z)$ thus defined is called a *modular form* belonging to $\mathfrak{M}$.

If $r$, $s$ are sufficiently large integers, the modular forms $\psi_r$ and $\psi_s$ exist, and

$$(14)\quad \psi_r^s \, \psi_s^{-r}$$

is a *modular function* of $\mathfrak{M}$ with fundamental region $\mathfrak{Z}$. It can be shown that there exist $\frac{1}{2}n(n+1)$ algebraically independent modular functions of the form (14), and that any $\frac{1}{2}n(n+1)+1$ such functions are connected by an algebraic relation with rational coefficients.

The modular forms (13) may also be expanded in theta series.

Petersson's (1940) theory of Poincaré's theta series has been generalized by Maass (1951). In this generalization, the hyperbolic metric of Poincaré's half-plane is replaced by Siegel's sympleptic metric of the positive cone $\mathfrak{H}$.

*Two identities*. We conclude this brief introduction to Siegel's theory by giving two remarkable identities, both for the case $n = 2$.

The first of these identities is due to Siegel (1937) and expresses a modular form in terms of a theta double series. Let $L$ be the set of all classes of coprime symmetric pairs of $2 \times 2$ matrices, and select a representative pair $C$, $D$ from each class. Let $L_2$ be the subset of all representative pairs for which

$$CD' \equiv N \pmod 2,$$

i.e., the elements of $CD'$ are even integers (if this condition is satisfied for one representative of the class it will be satisfied also for any other representative). Put

$$Z = \begin{bmatrix} u & v \\ v & w \end{bmatrix} = X + iY$$

where $u$, $v$, $w$ are complex variables, $X$, $Y$ are real matrices and $Y$ is positive, and let $a$, $b$ run through all integers. Siegel's identity states

(15) $$\sum_{L_2} [\det(CZ + D)]^{-4} = \{\sum_{a,b} \exp[i\pi(ua^2 + 2vab + wb^2)]\}^8.$$

The second of these identities is due to Witt (1941) and is an identity between two modular forms of degree 2. With the notation used in (13), the identity may be written as

(16) $$\psi_4(Z) = [\psi_2(Z)]^2.$$

Witt's identity is analogous to the well-known formula

(17) $$\sum_{a,b} (az + b)^{-8} = [\sum_{a,b} (az + b)^{-4}]^2$$

in the theory of Eisenstein series of a single complex variable $z$, where $a$, $b$ run through all pairs of coprime integers $a$, $b$ such that $a \geq 0$ and that $b = 1$ when $a = 0$.

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CHAPTER XV

# LAMÉ FUNCTIONS

## 15.1. Introduction

Lamé functions arise in solutions of Laplace's equations in certain systems of curvilinear coordinates. Separation of variables in the three-dimensional Laplace equation is discussed fully in Bôcher's (1891) book, and in recent papers by Levinson, Bogert, and Redheffer (1949), and Moon and Spencer (1952 a, b, 1953). For the separation of variables in more general differential equations see Eisenhart (1934) where there are also references to earlier writers.

Strutt's monograph (1932) gives a summary of the theory of Lamé functions as of 1932, many applications, and an extensive bibliography. For further information about these functions see also Whittaker and Watson (1927, Chapter XXIII) and Hobson (1931, Chapter XI).

### 15.1.1. Coordinates of confocal quadrics

Let $a > b > c > 0$ be fixed numbers, and let $\theta$ be a variable parameter. The equation

$$(1) \qquad \frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1$$

represents a confocal family of quadrics, $x$, $y$, $z$ being rectangular Cartesian coordinates. The quadric represented by (1) is

an ellipsoid if $-c^2 < \theta$

a hyperboloid of one sheet if $-b^2 < \theta < -c^2$

a hyperboloid of two sheets if $-a^2 < \theta < -b^2$

an imaginary quadric if $\theta < -a^2$

For $\theta = -a^2, -b^2, -c^2$ we obtain degenerate quadrics.

Since (1) is a cubic equation in $\theta$, three quadrics of the confocal family pass through each point $(x, y, z)$ for which $xyz \neq 0$ (this excludes the planes of the degenerate quadrics). A discussion of the sign of the left-hand side of (1) as $\theta$ varies shows that exactly one of the three roots lies in each of the intervals $(-c^2, \infty)$, $(-b^2, -c^2)$, $(-a^2, -b^2)$, showing that through every point (not in one of the coordinate planes) there passes one ellipsoid, one one-sheeted hyperboloid, and one two-sheeted hyperboloid of the confocal family.

Let $\lambda$, $\mu$, $\nu$ be the three roots of (1) for given non-zero $x$, $y$, $z$, and let

(2) $\lambda > -c^2 > \mu > -b^2 > \nu > -a^2$.

We may introduce $\lambda$, $\mu$, $\nu$ as curvilinear coordinates in Laplace's equation

(3) $$\Delta W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$$

which transformed to our curvilinear coordinates becomes

(4) $$\frac{4f(\lambda)}{(\lambda-\mu)(\lambda-\nu)} \frac{\partial}{\partial\lambda}\left[f(\lambda)\frac{\partial W}{\partial\lambda}\right] + \frac{4f(\mu)}{(\mu-\lambda)(\mu-\nu)} \frac{\partial}{\partial\mu}\left[f(\mu)\frac{\partial W}{\partial\mu}\right]$$
$$+ \frac{4f(\nu)}{(\nu-\lambda)(\nu-\mu)} \frac{\partial}{\partial\nu}\left[f(\nu)\frac{\partial W}{\partial\nu}\right] = 0$$

where

(5) $$f(\theta) = [(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{1/2}.$$

Now, $\lambda$, $\mu$, $\nu$ depend only on $x^2$, $y^2$, $z^2$ and hence are the same for the eight points $(\pm x, \pm y, \pm z)$. In order to have a one-to-one correspondence between Cartesian and curvilinear coordinates, we introduce *uniformizing variables*, expressing $\lambda$, $\mu$, $\nu$, and hence $x$, $y$, $z$, in terms of Jacobian elliptic functions of three new variables, $\alpha$, $\beta$, $\gamma$. We put

(6) $$k^2 = \frac{a^2-b^2}{a^2-c^2}, \quad k'^2 = \frac{b^2-c^2}{a^2-c^2} \qquad 0 < k,\ k' < 1$$

In what follows, $k$ will be the modulus of the elliptic functions. We then set

(7) $$\lambda = -(a \operatorname{cn} \alpha)^2 - (b \operatorname{sn} \alpha)^2$$
$$\mu = -(a \operatorname{cn} \beta)^2 - (b \operatorname{sn} \beta)^2$$
$$\nu = -(a \operatorname{cn} \gamma)^2 - (b \operatorname{sn} \gamma)^2$$

In terms of our new curvilinear coordinates we have

$$(8)\quad x = k^2(a^2-c^2)^{\frac{1}{2}} \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma$$

$$y = -\frac{k^2}{k'}(a^2-c^2)^{\frac{1}{2}} \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma$$

$$z = \frac{i}{k'}(a^2-c^2)^{\frac{1}{2}} \operatorname{dn}\alpha \operatorname{dn}\beta \operatorname{dn}\gamma,$$

and Laplace's equation (3) becomes

$$(9)\quad \frac{a^2-c^2}{(a^2-b^2)^2}\,\frac{1}{[(\operatorname{sn}\alpha)^2-(\operatorname{sn}\beta)^2][(\operatorname{sn}\beta)^2-(\operatorname{sn}\gamma)^2][(\operatorname{sn}\gamma)^2-(\operatorname{sn}\alpha)^2]}$$

$$\times\left\{[(\operatorname{sn}\gamma)^2-(\operatorname{sn}\beta)^2]\frac{\partial^2 W}{\partial\alpha^2}+[(\operatorname{sn}\alpha)^2-(\operatorname{sn}\gamma)^2]\frac{\partial^2 W}{\partial\beta^2}\right.$$

$$\left.+[(\operatorname{sn}\beta)^2-(\operatorname{sn}\alpha)^2]\frac{\partial^2 W}{\partial\gamma^2}\right\}=0$$

If we let $\alpha$ vary between $i\mathbf{K}'$ and $\mathbf{K}+i\mathbf{K}'$, $\beta$ between $\mathbf{K}$ and $\mathbf{K}+2i\mathbf{K}'$, and $\gamma$ between 0 and $4\mathbf{K}$, it can be verified by means of the formulas and diagrams given in sec. 13.18 that the inequalities (2) are satisfied, and, moreover, that (8) represents a one-to-one correspondence between the Cartesian coordinates $x$, $y$, $z$, and our curvilinear coordinates $\alpha$, $\beta$, $\gamma$ which will be called *ellipsoidal coordinates*, or *coordinates of confocal quadrics*.

The end-points of the intervals in which $\alpha$, $\beta$, $\gamma$ vary are of special importance. They represent $\infty$ and the degenerate quadrics of our system, and may be enumerated as follows:

$\alpha = i\mathbf{K}'$ infinity;

$\alpha = \mathbf{K}+i\mathbf{K}'$ degenerate ellipsoid covering (twice) the area of the focal ellipse;

$\beta = \mathbf{K}$ and $\beta = \mathbf{K}+2i\mathbf{K}'$ two halves of the degenerate hyperboloid covering (twice) the area "between" the two branches of the focal hyperbola;

$\beta = \mathbf{K}+i\mathbf{K}'$ degenerate hyperboloid covering (twice) the area in the $x$, $y$-plane "outside" the focal ellipse;

$\gamma = 0$, $\mathbf{K}$, $2\mathbf{K}$, $3\mathbf{K}$, $4\mathbf{K}$ parts of the degenerate two-sheeted hyperboloids of the system which cover (twice) the area "outside" the focal hyperbola, $\gamma = 0$ and $\gamma = 4\mathbf{K}$ representing the same surface.

The degenerate surfaces act as branch-cuts, and the postulate of continuity of a function across these branch-cuts has the character of boundary conditions.

We may mention that $\alpha$ corresponds to $r$ in spherical polar coordinates, $\beta$ to $\theta$, and $\gamma$ to $\phi$.

Instead of the "Jacobian" uniformizing variables, many authors use "Weierstrassian" variables (see, for instance, Whittaker and Watson, 1927, sec. 23.31).

Laplace's equation (9) has *normal solutions*

(10) $W = A(\alpha)\, B(\beta)\, C(\gamma).$

Substituting in (9) we obtain

(11) $$[(\text{sn } \gamma)^2 - (\text{sn } \beta)^2]\,\frac{A''}{A} + [(\text{sn } \alpha)^2 - (\text{sn } \gamma)^2]\,\frac{B''}{B}$$

$$+ [(\text{sn } \beta)^2 - (\text{sn } \alpha)^2]\,\frac{C''}{C} = 0$$

and since this is an identity in $\alpha$, $\beta$, $\gamma$, there must be constants $h$ and $l$ such that

(12) $$\frac{A''}{A} = l(\text{sn } \alpha)^2 - h, \quad \frac{B''}{B} = l(\text{sn } \beta)^2 - h, \quad \frac{C''}{C} = l(\text{sn } \gamma)^2 - h$$

We write $l = k^2 n(n+1)$ and see that $A$, $B$, $C$ satisfy *Lamé's equation*

(13) $$\frac{d^2 \Lambda(z)}{dz^2} + \{h - n(n+1)\,[k \text{ sn}(z, k)]^2\}\,\Lambda(z) = 0$$

with appropriate variables.

Suppose that (10) represents a solution of Laplace's equation which is continuous and has a continuous gradient on an ellipsoid $\alpha$ = const. Since $\gamma = 0$ and $\gamma = 4\mathbf{K}$ represent the same curve on that ellipsoid, it follows that

(14) $$C(0) = C(4\mathbf{K}), \qquad \frac{\partial C}{\partial \gamma}(0) = \frac{\partial C}{\partial \gamma}(4\mathbf{K}).$$

Since the coefficients in Lamé's equation are periodic mod $4\mathbf{K}$, it follows that $C(\gamma)$ must also be *periodic* mod $4\mathbf{K}$. If $C(\gamma)$ is any mod $4\mathbf{K}$ periodic solution of Lamé's equation, so are $C(2\mathbf{K} - \gamma)$ and $C(\gamma) \pm C(2\mathbf{K} - \gamma)$, and we may restrict oursleves to periodic solutions which are even or odd functions of $\gamma - \mathbf{K}$: we shall express this by saying that our solutions are even or odd with respect to $\mathbf{K}$.

The curves $\beta = \mathbf{K}$ and $\beta = \mathbf{K} + 2i\mathbf{K}'$ are branch-cuts on the ellipsoid; the points $(\mathbf{K}, \gamma)$, $(\mathbf{K}, 2\mathbf{K} - \gamma)$ are identical. The continuity conditions are

$$(15)\quad B(\mathbf{K})\, C(\gamma) = B(\mathbf{K})\, C(2\mathbf{K} - \gamma)$$

$$\frac{\partial B}{\partial \beta}(\mathbf{K})\, C(\gamma) = -\frac{\partial B}{\partial \beta}(\mathbf{K})\, C(2\mathbf{K} - \gamma).$$

If $C(\gamma)$ is even with respect to $\mathbf{K}$, we have $\partial B(\mathbf{K})/\partial\beta = 0$ so that $B(\beta)$ is also even with respect to $\mathbf{K}$; and if $C(\gamma)$ is odd with respect to $\mathbf{K}$, we have $B(\mathbf{K}) = 0$ so that $B(\beta)$ is also odd with respect to $\mathbf{K}$. We have the same situation at $\beta = \mathbf{K} + 2i\mathbf{K}'$, so that if $C(\gamma)$ is even (odd) with respect to $\mathbf{K}$ then $B(\beta)$ must be even (odd) both with respect to $\mathbf{K}$ and with respect to $\mathbf{K} + 2i\mathbf{K}'$. In either case $B(\beta)$ is a *periodic function* mod $4i\mathbf{K}'$. Moreover, $B(\theta)$ and $C(\theta)$ have the same parity at $\theta = \mathbf{K}$, and satisfy the same differential equation: it follows that they are constant multiples of each other. We are thus lead to inquire into the existence of *doubly periodic solutions* of (13). It will be seen later (sec. 15.5.2) that such solutions exist only if $n$ is an integer, and $h$ has one of a sequence of characteristic values. It may be mentioned that an analysis of the solutions in spherical polar, or Cartesian coordinates leads also to the conclusion that $n$ must be an integer.

The choice of $A(a)$ depends on the type of ellipsoidal harmonics under consideration. For *internal harmonics* we wish (10) to be regular inside an ellipsoid $a =$ const. Now, $a = \mathbf{K} + i\mathbf{K}'$ is a branch-cut (the focal ellipse), the points $(\mathbf{K} + i\mathbf{K}', \beta, \gamma)$ and $(\mathbf{K} + i\mathbf{K}', 2\mathbf{K} + 2i\mathbf{K}' - \beta, \gamma)$ are identical, and as above, we conclude that $A(\theta)$ and $B(\theta)$ must have the same parity at $\mathbf{K} + i\mathbf{K}'$ and hence are constant multiples of each other. For *external harmonics* we wish (10) to be regular outside of an ellipsoid $a =$ const., in particular at infinity, $a = i\mathbf{K}'$, and $A(a)$ must be that solution of the Lamé equation which vanishes at $i\mathbf{K}'$. Lastly, for an ellipsoidal harmonic regular between two ellipsoids of the confocal family, we take a linear combination of these two solutions.

### 15.1.2. Coordinates of confocal cones

We introduce coordinates $r$, $\beta$, $\gamma$ which are connected with Cartesian coordinates $x$, $y$, $z$ and spherical polars $r$, $\theta$, $\phi$ by means of the relations

$$(16)\quad x = r\sin\theta\cos\phi = kr \operatorname{sn}\beta \operatorname{sn}\gamma$$

$$y = r\sin\theta\sin\phi = i\frac{k}{k'}\, r \operatorname{cn}\beta \operatorname{cn}\gamma$$

$$z = r\cos\theta \qquad = \frac{1}{k'}\, r \operatorname{dn}\beta \operatorname{dn}\gamma.$$

As in sec. 15.1.1, $\beta$ varies between $\mathbf{K}$ and $\mathbf{K} + 2i\mathbf{K}'$, $\gamma$ varies between 0 and $4\mathbf{K}$, and $r > 0$. The coordinate surfaces are the concentric spheres $r =$ const. and the confocal cones

(17) $$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 0$$

where $\theta$ is $\mu$ or $\nu$ as given in (7), and $k$ is determined by (6). These coordinates are known as *sphero-conal* coordinates: see Hobson 1892 and 1931 Chapter XI.

Laplace's equation in these coordinates is

(18) $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) - \frac{1}{k^2 r^2[(\operatorname{sn}\beta)^2 - (\operatorname{sn}\gamma)^2]}\left(\frac{\partial^2 W}{\partial \beta^2} - \frac{\partial^2 W}{\partial \gamma^2}\right) = 0,$$

normal solutions are of the form

(19) $$W = R(r)\, B(\beta)\, C(\gamma)$$

and lead to the differential equation

(20) $$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = n(n+1),$$

the equations for $B$ and $C$ being the same as in (12), with $l = k^2 n(n+1)$.

If (19) is to represent a function continuous and possessing a continuous gradient on a sphere $r =$ const., the same considerations as in sec. 15.1.1 lead to the conclusion that $B(\theta) = C(\theta)$ must be a doubly periodic solution of Lamé's equation, and hence that $n$ is an integer, and $h$ one of the characteristic values. Alternatively, (16) sets up a relationship between spherical polar, and sphero-conal coordinates, this leads to a relationship between spherical and ellipsoidal surface harmonics, and to the conclusion that $n$ is an integer, and $h$ has exactly $2n + 1$ characteristic values.

The situation is entirely different if (19) is to represent a function regular inside a cone $\beta =$ const., where we take $\beta$ between $\mathbf{K}$ and $\mathbf{K} + i\mathbf{K}'$. If (19) is to be regular inside the entire half-cone $\beta =$ const., we must have $n = -\frac{1}{2} + ip$ where $p$ is real and arbitrary, if (19) is regular inside the cone between the spheres $r = r_1$ and $r = r_2$ and vanishes on these spheres, we must have $n = -\frac{1}{2} + ip$ where $p$ is a root of the transcendental equation $\sin[p \log(r_1/r_2)] = 0$. In either case $n$ is complex and $\operatorname{Re} n = -\frac{1}{2}$. Since $\gamma = 0$ and $\gamma = 4\mathbf{K}$ are the same surface, $C(\gamma)$ must be periodic mod $4\mathbf{K}$, and $h$ must have one of an infinite sequence of characteristic values. Continuity across $\beta = \mathbf{K}$ makes $B(\theta)$ and $C(\theta)$ have the same parity at **K** and hence constant multiples of each other, but the functions involved here are no longer doubly periodic.

### 15.1.3. Coordinates of confocal cyclides of revolution

In cylindrical coordinates $\rho$, $\phi$, $z$, Laplace's equation becomes

$$(21)\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\phi^2}+\frac{\partial^2 W}{\partial z^2}=0.$$

We now introduce new coordinates, $u$, $v$ in the meridian plane by putting $z = z(u, v)$, $\rho = \rho(u, v)$. Wangerin (1875) has determined the most general systems of *orthogonal curvilinear coordinates* $u$, $v$ in which Laplace's equation is *separable*, i.e. possesses *normal solutions* of the form

$$(22)\quad W = w(u, v)\, U(u)\, V(v)\, \Phi(\phi)$$

where $w(u, v)$ is a fixed function, and $U$, $V$, $\Phi$ are solutions of ordinary differential equations. Wangerin's discussion was repeated by Snow (1952), and also by R. Lagrange (1939, 1944). We shall give a brief indication of this discussion, and then a more detailed account of the systems of curvilinear coordinates obtained by Wangerin, and of the boundary value problems which they suggest.

First one proves the following result. If $u$, $v$ are orthogonal coordinates, and Laplace's equation has solutions of the form (22), then $w = \rho^{-1/2}$, and the coordinates $u$, $v$ may be taken in such a manner that the mapping of the $z$, $\rho$-plane in the $u$, $v$-plane is *conformal*. We accordingly put

$$(23)\quad z + i\rho = f(u + iv)$$

where $f$ is an analytic function; and we also put

$$(24)\quad W = \rho^{-1/2}\,\Psi(u, v)\, e^{\pm im\phi} = \rho^{-1/2}\, U(u)\, V(v)\, e^{\pm im\phi}$$

in Laplace's equation (21). The partial differential equation satisfied by $\Psi$ is

$$(25)\quad \frac{\partial^2\Psi}{\partial u^2}+\frac{\partial^2\Psi}{\partial v^2}-(m^2-\tfrac{1}{4})\,F(u, v)\,\Psi=0$$

where

$$(26)\quad F(u, v)=\frac{|f'(u+iv)|^2}{[\operatorname{Im} f(u+iv)]^2}=\frac{|f'|^2}{\rho^2}$$

and we will have solutions of the form $U(u)\,V(v)$ if $F$ is of the form $F(u, v) = F_1(u) + F_2(v)$, the ordinary differential equations satisfied by $U$ and $V$ being

$$\frac{d^2 U}{du^2} + [h - (m^2 - \tfrac{1}{4})\, F_1(u)]\, U = 0$$

$$\frac{d^2 V}{dv^2} - [h + (m^2 - \tfrac{1}{4})\, F_2(v)]\, V = 0.$$

It is then proved: $F(u, v) = F_1(u) + F_2(v)$ if and only if $f$ satisfies the ordinary differential equation

$$f'^2 = a_0 + a_1 f + a_2 f^2 + a_3 f^3 + a_4 f^4 = P_4(f)$$

in which $a_0, \ldots, a_4$ are real constants. Thus, $f$ is either an elementary function or an *elliptic function*. Moreover, the form of the differential equation for $f$ does not change if $f$ is replaced by $(Af + B)/(Cf + D)$ where $A, B, C, D$ are real constants and $AD - BC \neq 0$, and we may use such a transformation to reduce the equation to normal form.

We shall assume that $P_4$ has four distinct zeros when the reduction to normal form can be effected by the process described in sec. 13.5. Three cases arise according as all zeros are real, all zeros are complex, or two real and two complex zeros occur. The standard forms of $f$ in the three cases are

$$\operatorname{sn}(u + iv, k), \quad i \operatorname{sn}(u + iv, k), \quad \operatorname{cn}(u + iv, k).$$

We shall now discuss each of these three cases separately. A bar will denote complex conjugation, and the following abbreviations will be used:

(27) $s = \operatorname{sn}(u + iv, k)$, $s_1 = \operatorname{sn}(u, k)$, $s_2 = \operatorname{sn}(iv, k)$, $s_2' = \operatorname{sn}(v, k')$

$c = \operatorname{cn}(u + iv, k)$, $c_1 = \operatorname{cn}(u, k)$, $c_2 = \operatorname{cn}(iv, k)$, $c_2' = \operatorname{cn}(v, k')$

$d = \operatorname{dn}(u + iv, k)$, $d_1 = \operatorname{dn}(u, k)$, $d_2 = \operatorname{dn}(iv, k)$, $d_2' = \operatorname{dn}(v, k')$

$$F(u, v) = \frac{|f'(u + iv)|^2}{[\operatorname{Im} f(u + iv)]^2} \qquad n = \pm m - \tfrac{1}{2}$$

The aim of the following discussion is to show that in each case $F(u, v)$ appears in the form

$$[a \operatorname{sn}(bu + c, a/b)]^2 + [a_1 \operatorname{sn}(b_1 v + c_1, a_1/b_1)]^2$$

so that (25) becomes

$$\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} - n(n+1)\{[a \operatorname{sn}(bu + c, a/b)]^2$$

$$+ [a_1 \operatorname{sn}(b_1 v + c_1, a_1/b_1)]^2\} \Psi = 0.$$

For normal solutions, $\Psi = U(u)\, V(v)$, and $U$ and $V$ satisfy equations which can easily be reduced to Lamé's equation, the variables in the latter being $bu + c$ and $b_1 v + c_1$ respectively. We then investigate the boundary conditions which must be imposed upon $U$ and $V$.

The coordinates $u$, $v$ used in this discussion will be those which arise most naturally from the general theory. They are not necessarily the most suitable ones to use in a given problem, and it will be seen in sec. 15.8 how the transformation theory of elliptic functions can be used to change over to new, and more suitable coordinates.

Case I. *Four real foci on the axis*

We put

$$(28) \quad z + i\rho = \frac{As + B}{Cs + D}, \qquad A, B, C, D \text{ real}, \quad AD - BC \neq 0$$

and find by a straightforward computation using 13.17(16), 13.23(13), and Table 7 in sec. 13.18

$$(29) \quad F(u, v) = -\frac{4cd\bar{c}\bar{d}}{(s - \bar{s})^2} = \frac{k'^4 s_1^2}{c_1^2 d_1^2} + \frac{d_2'^2}{s_2'^2 c_2'^2}$$

$$= -\left\{(1-k) \operatorname{sn}\left[i(1+k)u, \frac{1-k}{1+k}\right]\right\}^2$$

$$+ \left\{(1-k) \operatorname{sn}\left[(1+k)(v - i\mathbf{K}), \frac{1-k}{1+k}\right]\right\}^2$$

For the further discussion we take $A = D = 0$, $B = C = 0$ in (28). The mapping $z + i\rho = \operatorname{sn}(u + iv)$ was described in sec. 13.25. We see from the diagram given there that the half-plane $\rho > 0$ is mapped on a rectangle in the $(u, v)$-plane with corners at $(\pm\mathbf{K}, 0)$ and $(\pm\mathbf{K}, i\mathbf{K}')$. Thus $-\mathbf{K} \leq u \leq \mathbf{K}$ and $0 \leq v \leq \mathbf{K}'$. The curves $u$ = const. and $v$ = const. in the $z$, $\rho$-plane are confocal bicircular quartics whose foci are at $z = \pm 1, \pm k^{-1}$, $\rho = 0$. Note that $F$ becomes infinite for $u = \pm\mathbf{K}$ or for $v = 0, \mathbf{K}'$ so that the end-points of the intervals of $u$ and $v$ correspond to singular points of the ordinary differential equations for $U$ and $V$.

For a potential which is regular inside (or outside) a surface $u =$ const., $\rho^{-1/2}\, V(v)$ must remain finite at both end-points $v = 0$, $\mathbf{K}'$, and we shall see later that this determines certain characteristic values of $h$ as well as the solution $V(v)$ to be used. For a potential regular outside [inside] a surface $u = c < 0$, or inside [outside] a surface $u = c > 0$, $U(\mathbf{K})[U(-\mathbf{K})]$ must be finite, and this determines the choice of $U$. Similar statements hold for potentials regular inside or outside a surface $v =$ const.

Case II. *No real foci on the axis*

Here

$$(30)\quad z + i\rho = \frac{Ais + B}{Cis + D}, \qquad A, B, C, D \text{ real}, \quad AD - BC \neq 0$$

$$(31)\quad F(u, v) = \frac{4c\bar{c}d\bar{d}}{(s+\bar{s})^2} = \frac{c_1^2 d_1^2}{s_1^2} - \frac{k'^4 s_2^2}{c_2^2 d_2^2}$$

$$= -\left\{(1-k)\,\mathrm{sn}\left[i(1+k)(u-\mathbf{K}), \frac{1-k}{1+k}\right]\right\}^2 + \left\{(1-k)\,\mathrm{sn}\left[(1+k)v, \frac{1-k}{1+k}\right]\right\}^2$$

For the further discussion we again take $A = D = 1$, $B = C = 0$. The mapping of the quarter-plane $z < 0$, $\rho > 0$ on the rectangle with corners at $(0, 0)$, $(\mathbf{K}, 0)$, $(\mathbf{K}, \mathbf{K}')$, $(0, \mathbf{K}')$ in the $u$, $v$-plane is described by the diagram in sec. 13.25. To complete the mapping we reflect in the $(z, \rho)$-plane on $z = 0$, and in the $u$, $v$-plane either on $v = 0$ or on $u = \mathbf{K}$. The curves $u =$ const., $v =$ const. in the $z$, $\rho$-plane are confocal bicircular quartics with real foci at $z = 0$, $\rho = 1, k^{-1}$.

For a potential regular inside or outside a surface $u =$ const., we map the half-plane $\rho > 0$ on the rectangle with vertices $(0, \pm\mathbf{K}')$, $(\mathbf{K}, \pm\mathbf{K}')$ in the $u$, $v$-plane. $v = \mathbf{K}'$ and $v = -\mathbf{K}'$ are both maps of $z = 0$, $\rho > k^{-1}$. By the same argument as in sec. 15.1.1 it is seen that $V(v)$ must be a periodic solution of the appropriate differential equation, the period being $2\mathbf{K}'$. This condition determines characteristic values of $h$, and the corresponding characteristic functions $V(v)$. For a potential regular inside $u =$ const. the continuity condition across $u = \mathbf{K}$ (i.e. $z = 0$, $1 < \rho < k^{-1}$) demand that $U$ at $\mathbf{K}$ and $V$ at 0 have the same parity; for a potential regular outside $u =$ const., $U$ must remain finite at $u = 0$. This case has been discussed in detail by Poole (1929, 1930) who used a slightly different mapping.

For a potential regular inside or outside a surface $v =$ const., we map the half-plane $\rho > 0$ on the rectangle with vertices $(0, 0)$, $(2\mathbf{K}, 0)$, $(2\mathbf{K}, \mathbf{K}')$, $(0, \mathbf{K}')$. Here $\rho^{-1/2}\, U(u)$ must remain finite at both $u = 0$ and $u = 2\mathbf{K}$, and this condition determines characteristic values of $h$, and characteristic functions $U(u)$. $V(v)$ is then determined by its parity at 0 (for a potential regular inside $v =$ const.) or at $\mathbf{K}'$ (for a potential regular outside $v =$ const.).

Case III. *Two real foci on the axis*

Here

$$(32) \quad z + i\rho = \frac{Ac + B}{Cc + D}, \qquad A, B, C, D \text{ real}, \quad AD - BC \neq 0$$

$$(33) \quad F(u, v) = -\frac{4sd\,\bar{s}\,\bar{d}}{(c - \bar{c})^2} = \frac{c_1^2}{s_1^2 d_1^2} - \frac{c_2^2}{s_2^2 d_2^2}$$

$$= \left\{(k - ik')\,\mathrm{sn}\left[(k + ik')(u + \mathbf{K}), \frac{k - ik'}{k + ik'}\right]\right\}^2$$

$$- \left\{(k - ik')\,\mathrm{sn}\left[i(k + ik')(v - i\mathbf{K}), \frac{k - ik'}{k + ik'}\right]\right\}^2$$

In this case the modulus of the elliptic functions appearing in Lamé's equation is not a real fraction but a complex number of modulus 1, and the transformation theory of elliptic functions (see Table 11 in sec. 13.22) must be used to reduce all functions to a real modulus between 0 and 1. We take $A = D = 1$, $B = C = 0$, The curves $u =$ const., $v =$ const. in the $z$, $\rho$-plane are confocal bicircular quartics whose foci are at the points $z = \pm 1$, $\rho = 0$ and $z = 0$, $\rho = k'/k$. Further details of the mapping will be seen from the diagram in sec. 13.25.

For a potential regular inside or outside a surface $u =$ const., we map the half-plane $\rho > 0$ on the rectangle with vertices $(0, -2\mathbf{K}')$, $(\mathbf{K}, -2\mathbf{K}')$, $(\mathbf{K}, 0)$, $(0, 0)$ in the $u$, $v$-plane. The condition that $\rho^{-1/2}\, V(v)$ remain finite both at $v = 0$ and at $v = -2\mathbf{K}'$ determines characteristic values of $h$ and characteristic functions $V(v)$. For a potential regular inside $u =$ const. we have a branch-cut at $z = 0$, $\rho < k'/k$, or $u = \mathbf{K}$, and continuity across this branch-cut determines the parity of $U$ at $\mathbf{K}$ to be the same as the parity of $V$ at $-\mathbf{K}'$. For a potential regular outside of $u =$ const., $\rho^{-1/2}\, U$ is determined by the condition that it remain finite at $u = 0$.

For a potential regular inside or outside a surface $v =$ const. we map the half-plane $\rho > 0$ on the rectangle with vertices $(0, -\mathbf{K}')$, $(2\mathbf{K}, -\mathbf{K}')$,

$(2\mathbf{K}, 0)$, $(0, 0)$ in the $u$, $v$-plane. $\rho^{-1/2}\, U(u)$ must remain finite both at $u = 0$ and at $u = 2\mathbf{K}$, and this condition determines characteristic values of $h$ and characteristic functions $U(u)$. For a potential regular inside $v = \text{const}$, $\rho^{-1/2}\, V(v)$ must be finite at $v = 0$, and for a potential regular outside $v = \text{const}$. the parity of $V$ at $-\mathbf{K}'$ must be the same as the parity of $U$ at $\mathbf{K}$.

## 15.2. Lamé's equation

In the preceding sections it was shown that the solution of a number of boundary value problems depends on the differential equation

$$(1) \quad \frac{d^2 \Lambda}{dz^2} + \{h - n(n+1)[k \operatorname{sn}(z, k)]^2\}\, \Lambda = 0$$

which we shall call the *Jacobian form of Lamé's equation*, or briefly *Lamé's equation*. This form of the equation was used by Hermite, E.T. Whittaker, Ince, and other authors and is preferable to other forms (to be given below) from the point of view of numerical computations.

In (1), $k$ is mostly between 0 and 1, but we have encountered one case in sec. 15.1.3 where $k$ is complex and $|k| = 1$. $z$ is a complex variable, but in most boundary value problems $z$ varies along one of the lines $\operatorname{Re} z = N\mathbf{K}$, $\operatorname{Im} z = N\mathbf{K}'$, $N$ integer. $h$ is a parameter, characteristic values of which are determined either by a periodicity condition, or by a "finiteness condition". $n$ is sometimes an integer, sometimes half of an odd integer (as in sec. 15.1.3), and sometimes (as in one problem in 15.1.2) a complex number whose real part is $-\tfrac{1}{2}$.

We have encountered several types of solutions. First there are solutions with a given parity at one of the quarter-period points $M\mathbf{K} + iN\mathbf{K}'$ ($M$, $N$ integers), or solutions which are to remain finite at one of the poles $2M\mathbf{K} + i(2N+1)\mathbf{K}'$ ($M$, $N$ integers) of $\operatorname{sn} z$. Such solutions exist, and are determined up to a constant factor, for any given values of $h, n, k$. Then there are the solutions with a prescribed period (which is also a period of $\operatorname{sn} z$). We shall see later that for given $n$, $k$ there is aninfinite sequence of characteristic values of $h$ for which such solutions exist. In sections 15.1.1 and 15.1.2 we found occasion to use solutions with two prescribed periods. It will be seen later that such solutions exist only when $2n$ is an integer. Lastly, in sec. 15.1.3 we were lead to solutions which are to remain finite at two poles. We shall see later that for given $n$, $k$ there is an infinite sequence of characteristic values of $h$ for which such functions exist.

Beside the Jacobian form (1) of Lamé's equation there are other important forms of this equation. If we put

$$(2)\quad k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \qquad z = i\mathbf{K}' + u(e_1 - e_3)^{1/2},$$

$$(e_1 - e_3)h + n(n+1)e_3 = H$$

and use 13.16(4) in conjunction with Table 7 in sec. 13.18, we obtain the *Weierstrassian form of Lamé's equation*

$$(3)\quad \frac{d^2\Lambda}{du^2} + [H - n(n+1)\wp(u)]\Lambda = 0$$

which was used by Halphén and other French mathematicians, and is extensively used in modern theoretical work.

A *trigonometric form* may be obtained by the substitution

$$(4)\quad \operatorname{sn} z = \cos\zeta, \qquad \zeta = \tfrac{1}{2}\pi - \operatorname{am} z$$

which leads to

$$(5)\quad [1 - (k\cos\zeta)^2]\frac{d^2\Lambda}{d\zeta^2} + k^2\cos\zeta\sin\zeta\,\frac{d\Lambda}{d\zeta}$$

$$+ [h - n(n+1)(k\cos\zeta)^2]\Lambda = 0.$$

This form was used by G.H. Darwin and Ince.

Several *algebraic forms* are also available. With

$$(6)\quad (\operatorname{sn} z)^2 = x$$

we obtain from (1)

$$(7)\quad \frac{d^2\Lambda}{dx^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-k^{-2}}\right)\frac{d\Lambda}{dx} + \frac{hk^{-2} - n(n+1)x}{4x(x-1)(x-k^{-2})}\Lambda = 0,$$

and with

$$(8)\quad \wp(u) = p$$

we obtain from (3)

$$(9)\quad \frac{d^2\Lambda}{dp^2} + \frac{1}{2}\left(\frac{1}{p-e_1} + \frac{1}{p-e_2} + \frac{1}{p-e_3}\right)\frac{d\Lambda}{dp} + \frac{H - n(n+1)p}{4(p-e_1)(p-e_2)(p-e_3)}\Lambda = 0.$$

Other algebraic forms may be obtained by rational transformations of these. Algebraic forms were used by Stieltjes, F. Klein, Bôcher, and others.

The algebraic forms of Lamé's equation are of the Fuchsian type, having four regular singularities. There are three finite regular singularities)[at 0, 1, $k^{-2}$ for (7), at $e_1$, $e_2$, $e_3$ for (9)] with indices 0, ½, and a regular singularity at infinity with indices $-\frac{1}{2}n$, $\frac{1}{2}n + \frac{1}{2}$. For the theory of Fuchsian equations see for instance Ince (1927, p. 370ff.) or Poole (1936, p. 74 ff.)

There are general theories covering the other forms too. For the theory of differential equations with doubly periodic coefficients see Ince (1927, p. 375 ff.) or Poole (1936, p. 170ff.); for equations with simply periodic coefficients see Ince (1927, p. 381 ff.) or Poole (1936, p. 178 ff.).

Unless the contrary is stated, we shall regard $h$, $k$, $n$ as given (real or complex) constants, and the variables as complex variables.

### 15.3. Heun's equation

It can be shown that any Fuchsian equation of the second order with four singularities can be reduced to the form

$$(1)\quad \frac{d^2w}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{dw}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)}\, w = 0$$

where

$$(2)\quad \alpha + \beta - \gamma - \delta - \epsilon + 1 = 0,$$

Here $x = 0, 1, a, \infty$ are the singularities of (1), the indices at these singularities depend on $\alpha, \ldots, \epsilon$, and the constant $q$ is the so-called *accessory parameter* whose presence is due to the fact that a Fuchsian equation of the second order with four (or more) singularities is not completely determined by the position of the singularities and the indices. (See Ince, 1927, p. 370ff.; Poole, 1936, p. 77 ff.) The reduction is effected by a linear fractional transformation of the independent variable and a suitable transformation of the dependent variable according to (4) and (5). Equation (1) is known as Heun's equation (Heun, 1889, Whittaker and Watson, 1927, p. 576 ff.).

Heun's equation may be characterized by a $P$-symbol (see sec. 2.6.1, or Ince, 1927, p. 372),

$$(3)\quad P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{Bmatrix}$$

but it should be noted that the $P$-symbol does not characterize the equation completely, and in any transformation of the equation, the transformation of the accessory parameter must be ascertained by explicit computation.

For the four-column $P$-symbol we have the *linear transformations*

$$(4)\quad \left(\frac{x-a}{x-d}\right)^{\rho}\left(\frac{x-b}{x-d}\right)^{\sigma}\left(\frac{x-c}{x-d}\right)^{\tau} P\begin{Bmatrix} a & b & c & d & \\ \alpha' & \beta' & \gamma' & \delta' & x \\ \alpha'' & \beta'' & \gamma'' & \delta'' & \end{Bmatrix}$$

$$= P\begin{Bmatrix} a & b & c & d & \\ \alpha'+\rho & \beta'+\sigma & \gamma'+\tau & \delta'-\rho-\sigma-\tau & x \\ \alpha''+\rho & \beta''+\sigma & \gamma''+\tau & \delta''-\rho-\sigma-\tau & \end{Bmatrix}$$

$$(5)\quad P\begin{Bmatrix} a & b & c & d & \\ \alpha' & \beta' & \gamma' & \delta' & x \\ \alpha'' & \beta'' & \gamma'' & \delta'' & \end{Bmatrix} = P\begin{Bmatrix} M(a) & M(b) & M(c) & M(d) & \\ \alpha' & \beta' & \gamma' & \delta' & M(x) \\ \alpha'' & \beta'' & \gamma'' & \delta'' & \end{Bmatrix}$$

where

$$(6)\quad \alpha'+\alpha''+\beta'+\beta''+\gamma'+\gamma''+\delta'+\delta''=2$$

$$M(x)=\frac{Ax+B}{Cx+D},\quad AD-BC\neq 0.$$

If two of the exponent-differences are equal to $\frac{1}{2}$, we have a *quadratic transformation*. For instance, if

$$(7)\quad \delta=\epsilon=\tfrac{1}{2},\quad \gamma=\alpha+\beta$$

in (1), (3) we have

$$(8)\quad P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & \tfrac{1}{2} & \tfrac{1}{2} & \beta & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & \alpha & 0 & 0 & \dfrac{x-a}{x-1} \\ \tfrac{1}{2} & \beta & 1-\gamma & \tfrac{1}{2} & \end{Bmatrix}$$

$$= P\begin{Bmatrix} -a^{\frac{1}{2}} & -1 & 1 & a^{\frac{1}{2}} & \\ 0 & \alpha & \alpha & 0 & \left(\dfrac{x-a}{x-1}\right)^{\frac{1}{2}} \\ 1-\gamma & \beta & \beta & 1-\gamma & \end{Bmatrix}$$

$$= P\begin{Bmatrix} 0 & 1 & A^2 & \infty & \\ \alpha & 0 & 0 & \alpha & X \\ \beta & 1-\gamma & 1-\gamma & \beta & \end{Bmatrix}$$

where in the last $P$-symbol

$$(9)\quad A = \frac{1+a^{\frac{1}{2}}}{1-a^{\frac{1}{2}}}, \quad X = A\,\frac{(x-1)^{\frac{1}{2}} + (x-a)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}} - (x-a)^{\frac{1}{2}}}.$$

If three of the exponent-differences are equal to ½ (as in the case of the algebraic forms of Lamé's equation) then there are three distinct quadratic transformations, and each can be followed by a second quadratic transformation thus leading to *biquadratic* transformations.

We now turn to the analytical representations of solutions of (1).

Let $a_1, \ldots, a_4$ be the singularities of the four-column $P$-symbol, $\alpha_i'$ and $\alpha_i''$ the exponents at $a_i$, and $\Sigma(\alpha_i' + \alpha_i'') = 2$. In analogy with Kummer's 24 series for the three-column $P$-symbol (sec. 2.9), we have 192 series of the form

$$(10)\quad \left(\frac{x-a_j}{x-a_l}\right)^{\sigma}\left(\frac{x-a_k}{x-a_l}\right)^{\tau}\sum_{m=0}^{\infty} A_m\left(\frac{a_j-a_l}{a_j-a_i}\;\frac{x-a_i}{x-a_l}\right)^{\rho+m}$$

where $i, j, k, l$ is a permutation of 1, 2, 3, 4; $\rho$ is $\alpha_i'$ or $\alpha_i''$; $\sigma$ is $\alpha_j'$ or $\alpha_j''$; and $\tau$ is $\alpha_k'$ or $\alpha_k''$. Actually only 96 of the 192 series are distinct. These series were studied by Heun (1889), Snow (1952, Chapter VII), and others. When quadratic transformations exist [as in the case of equation (8)], they lead to further power series expansions.

Alternatively, solutions of (1) can be expanded in series of hypergeometric functions. Such expansions have been studied by Svartholm (1939) and Erdélyi (1942, 1944). A typical expansion may be indicated as

$$(11)\quad P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{matrix}\right\}$$

$$= \sum_{m=0}^{\infty} A_m P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \lambda+m & x \\ 1-\gamma & 1-\delta & \mu-m & \end{matrix}\right\}$$

where

$$(12)\quad \lambda+\mu=\gamma+\delta-1=\alpha+\beta-\epsilon.$$

It turns out (Erdélyi, 1944) that there are essentially two possibilities for choosing $\lambda$ and $\mu$. Series of type I (Erdélyi, 1942) have $\lambda=\alpha, \mu=\beta-\epsilon$, converge outside of an ellipse with foci at 0, 1 and passing through $a$, and represent that branch of (3) which belongs to the exponent $\alpha$ at $\infty$. There are three distinct expansions of this type for each branch of (3). Series of type II (Svartholm 1939) have $\mu=0$ , $\gamma-1$, $\delta-1$, or $\gamma+\delta-2$. They are series of Jacobi polynomials, do not converge in general; but they do converge in the exceptional case of *Heun functions* (see below) when the accessory parameter has one of its characteristic values.

In all the above-mentioned expansions the coefficients $X_r$ satisfy three-term recurrence relations

$$(13)\quad \beta_0 X_0 + \gamma_1 X_1 = 0$$

$$\alpha_r X_{r-1} + \beta_r X_r + \gamma_{r+1} X_{r+1} = 0 \qquad r=1, 2, \ldots$$

where $\alpha_r$, $\beta_r$, $\gamma_r$ are known expressions in $r$ and the parameters, $\gamma_r \neq 0$, and

$$(14)\quad \alpha_r \to \alpha, \quad \beta_r \to \beta, \quad \gamma_r \to \gamma \quad \text{as} \quad r\to\infty.$$

If $t_1$ and $t_2$ are the roots of the quadratic equation

$$(15)\quad \alpha+\beta t+\gamma t^2=0$$

and $|t_1|<|t_2|$, then $\lim X_r/X_{r-1}$ exists and is in general equal to $t_2$: if the parameters of the problem satisfy a certain condition,

$$\lim X_r/X_{r-1}=t_1 \quad \text{as} \quad r\to\infty$$

(Perron 1929, sec. 57). The recurrence relation may be written as

$$\frac{X_r}{X_{r-1}} = \frac{-a_r}{\beta_r + \gamma_{r+1} X_{r+1}/X_r} \qquad r = 1, 2, \ldots$$

and by repeated application we are lead to the infinite continued fraction

$$(16) \quad q_r = -\cfrac{a_r}{\beta_r - \cfrac{a_{r+1}\gamma_{r+1}}{\beta_{r+1} - \cfrac{a_{r+2}\gamma_{r+2}}{\beta_{r+2} - \cfrac{a_{r+3}\gamma_{r+3}}{\beta_{r+3} - \cdots}}}}$$

It can be shown that this continued fraction is divergent when $|t_1| = |t_2|$ and $t_1 \neq t_2$, it is convergent when $|t_1| < |t_2|$ or $t_1 = t_2$, and $q_r \to t_1$ as $r \to \infty$.

If $|t_1| < |t_2|$ and the parameters satisfy the equation $\beta_0 = q_1 \gamma_1$, then $X_r/X_{r-1} \to t_1$ as $r \to \infty$, and the $X_r$ may be computed by means of the $q_r$: if $|t_1| < |t_2|$ and $\beta_0 \neq q_1 \gamma_1$, then $X_r/X_{r-1} \to t_2$; and if $|t_1| = |t_2|$, $t_1 \neq t_2$, $\lim X_r/X_{r-1}$ does not exist.

In the application to Heun's equation (and hence also to Lamé's equation), $\beta_0$ and $q_r$ depend on the accessory parameter ($q$ or $h$, as the case may be). In general, $\beta_0 \neq q_1 \gamma_1$, $X_r/X_{r-1} \to t_2$, the domain of convergence of the power series and of series of type I of hypergeometric functions is restricted, and includes only one of the four singularities of the equation: series of type II of hypergeometric functions do not exist in this case. If the accessory parameter has one of a sequence of *characteristic values*, then $\beta_0 = q_1 \gamma_1$, $X_r/X_{r-1} \to t_1$, the series converge ina more extensive region which includes at least two singularities, the corresponding characteristic solutions behave in a prescribed manner at two singularities, and will be called *Heun* (or *Lamé*) *functions*. In this case the series of type II of hypergeometric functions also converge and represent a Heun (or Lamé) function.

Theorems on the existence and distribution of characteristic values of the accessory parameter follow from the general (singular) Sturm-Liouville theory.

In general, $\beta_0 = q_1 \gamma_1$ will be a transcendental equation for the accessory parameter, but an exceptional case arises when $a_R = 0$ for some positive integer $R$. When $r \leq R$, $q_r$ is a finite continued fraction,

$\beta_0 = q_1 \gamma_1$ is an algebraic equation for the accessory parameter, if $\beta_0 = q_1 \gamma_1$ then $X_R = 0$ and from (13) also $X_{R+1} = X_{R+2} = \cdots = 0$. In this case the series expansions terminate and we have *Heun* (or *Lamé*) *polynomials* or *algebraic Heun* (or *Lamé*) functions. Alternatively, when $a_R = 0$, we may put $X_0 = X_1 = \cdots = X_{R-1} = 0$, determine the accessory parameters from the equation $\beta_R = \gamma_{R+1} q_{R+1}$ (which is a transcendental equation) and obtain transcendental Heun (or Lamé) functions.

## 15.4. Solutions of the general Lamé equation

We shall now apply the results of the preceding section to Lamé's equation, and put

(1) $s = \operatorname{sn} z, \quad c = \operatorname{cn} z, \quad d = \operatorname{dn} z.$

Throughout this section, $n$ and $h$ are arbitrary.

From 15.2 (7),

$$(2) \quad \Lambda = P\begin{Bmatrix} 0 & 1 & k^{-2} & \infty & \\ 0 & 0 & 0 & -\tfrac{1}{2}n & s^2 \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2}n+\tfrac{1}{2} & \end{Bmatrix},$$

and various transformations of this follow from 15.3 (4), (5), (8); in particular from 15.3 (8),

$$(3) \quad \Lambda = P\begin{Bmatrix} 0 & 1 & \left(\dfrac{1+k}{1-k}\right)^2 & \infty & \\ -\tfrac{1}{2}n & 0 & 0 & -\tfrac{1}{2}n & \dfrac{1+k}{1-k}\,\dfrac{d+kc}{d-kc} \\ \tfrac{1}{2}n+\tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2}n+\tfrac{1}{2} & \end{Bmatrix}$$

Further quadratic transformations of (2) lead to

$$(4) \quad \Lambda = P\begin{Bmatrix} -1 & 1 & k^{-1} & -k^{-1} & \\ 0 & 0 & -\tfrac{1}{2}n & -\tfrac{1}{2}n & \dfrac{c}{d} \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2}n+\tfrac{1}{2} & \tfrac{1}{2}n+\tfrac{1}{2} & \end{Bmatrix}$$

$$(5) \quad \Lambda = P\begin{Bmatrix} k' & -k' & ik & -ik & \\ 0 & 0 & -\tfrac{1}{2}n & -\tfrac{1}{2}n & \dfrac{d}{s} \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2}n+\tfrac{1}{2} & \tfrac{1}{2}n+\tfrac{1}{2} & \end{Bmatrix}$$

$$(6)\quad \Lambda = P\begin{Bmatrix} ik' & -ik' & i & -i & \\ 0 & 0 & -\frac{1}{2}n & -\frac{1}{2}n & \dfrac{c}{s} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}n+\frac{1}{2} & \frac{1}{2}n+\frac{1}{2} & \end{Bmatrix}$$

From (2), (4), (5), (6) and the results of sec. 15.3, a large variety of expansions of solutions of Lamé's equation follow. An unpublished list by Erdélyi gives 30 variables which may be used in series like 15.3 (10), with four different factors for each variable. Taking account of the fact that $\rho$ may be either 0 or $\frac{1}{2}$ with the first 18 of these variables, there are altogether 192 distinct series. For some of the simplest power series, and the recurrence relations which their coefficients satisfy see Ince, 1940 a, and the literature quoted there. For expansions in Legendre functions see Erdélyi, 1942 a. Expansions in exponential or trigonometric functions follow from 15.2 (5), and other trigonometric forms of Lamé's equation, by the theory of differential equations with periodic coefficients (Ince, 1927, p. 381 ff., Poole, 1936, p. 182 ff.). Such expansions have been discussed by Ince (1940 b) and Erdélyi (1942 a).

### 15.5. Lamé functions

We shall now assume that $k$, $n$ are given, $0 < k < 1$ and $n(n+1)$ is real so that either $n$ is real or $n = -\frac{1}{2} + ip$ where $p$ is real. We shall study *periodic solutions* of Lamé's equation, and shall show that such solutions exist for certain (characteristic) values of $h$: they will be called *periodic Lamé functions*, or briefly, Lamé functions.

### 15.5.1. Lamé functions of real periods

Since $\operatorname{sn}^2 z$ has the primitive real period $2\mathbf{K}$, the primitive real period of any Lamé function of real period must be of the form $P = 2p\mathbf{K}$ where $p = 1, 2, \ldots$ . Now, $\operatorname{sn}^2 z$ is an even function of $z - \mathbf{K}$ and when $\Lambda(z)$ is a periodic solution of Lamé's equation then so are the functions $\Lambda(2\mathbf{K} - z)$, $\Lambda(z) \pm \Lambda(2\mathbf{K} - z)$, and we may restrict ourselves to the investigation of Lamé functions which are even or odd functions of $z - \mathbf{K}$. A Lamé function of real period will be denoted by $\mathrm{Ec}_n(z, k^2)$ if it is an even function of $z - \mathbf{K}$, and by $\mathrm{Es}_n(z, k^2)$ if it is an odd function of $z - \mathbf{K}$. More specifically, we shall write $\mathrm{Ec}_n^m(z, k^2)$ and $\mathrm{Es}_n^m(z, k^2)$ for functions of period $P = 2p\mathbf{K}$ which have exactly $pm$ zeros in $0 \le z < 2p\mathbf{K}$ (or any half-open real interval of length $P$). The characteristic values of $h$ belonging to $\mathrm{Ec}_n^m$ and $\mathrm{Es}_n^m$ will be denoted by $a_n^m(k^2)$ and $b_n^m(k^2)$ respectively. This notation was introduced by Ince (1940 a) and modified by Erdélyi (1941 a). There being no generally accepted normalization we leave a constant factor undetermined in $\mathrm{Ec}_n^m(z)$ and $\mathrm{Es}_n^m(z)$. For this reason, we suppress constant factors in relations such as (31) below.

*Solutions of periods* $2\mathbf{K}$ *and* $4\mathbf{K}$. In either of these cases $\mathrm{Ec}(-\mathbf{K}+t)=\mathrm{Ec}(3\mathbf{K}+t)$ on account of periodicity, and this is equal to $\mathrm{Ec}(-\mathbf{K}-t)$ on account of parity, so that $\mathrm{Ec}(z)$ is an even function of both $z-\mathbf{K}$ and $z+\mathbf{K}$. We thus have the boundary conditions

$$(1)\quad \Lambda'(-\mathbf{K})=\Lambda'(\mathbf{K})=0 \qquad \text{for} \qquad \Lambda=\mathrm{Ec}(z)$$

Conversely, if a solution $\Lambda(z)$ of 15.2(1) satisfies (1), then it is an even function of both $z-\mathbf{K}$ and $z+\mathbf{K}$, and it must have period $4\mathbf{K}$. Similarly,

$$(2)\quad \Lambda(-\mathbf{K})=\Lambda(\mathbf{K})=0 \qquad \text{for} \qquad \Lambda=\mathrm{Es}(z)$$

On account of the symmetry relations at $\pm\mathbf{K}$, it is sufficient to investigate Lame functions of periods $2\mathbf{K}$ and $4\mathbf{K}$ in the interval, $(-\mathbf{K},\mathbf{K})$. We shall show that this interval may be reduced to $(0,\mathbf{K})$.

If $E(z)$ is either $\mathrm{Ec}(z)$ or $\mathrm{Es}(z)$, then $E(z)$ and $E(-z)$ satisfy the same differential equation and, by (1) and (2), the same boundary conditions, and must be constant multiples of each other. Thus, $E(z)$ is either an even, or an odd function of $z$, and we have the following four cases ($m=0, 1, 2, \ldots$):

$$(3)\quad \Lambda(0)=\Lambda(\mathbf{K})=0, \qquad \Lambda=\mathrm{Es}_n^{2m+2}(z), \qquad \text{period } 2\mathbf{K}$$

$$(4)\quad \Lambda'(0)=\Lambda(\mathbf{K})=0, \qquad \Lambda=\mathrm{Es}_n^{2m+1}(z), \qquad \text{period } 4\mathbf{K}$$

$$(5)\quad \Lambda(0)=\Lambda'(\mathbf{K})=0, \qquad \Lambda=\mathrm{Ec}_n^{2m+1}(z), \qquad \text{period } 4\mathbf{K}$$

$$(6)\quad \Lambda'(0)=\Lambda'(\mathbf{K})=0, \qquad \Lambda=\mathrm{Ec}_n^{2m}(z), \qquad \text{period } 2\mathbf{K}$$

with the appropriate symmetry relations.

Our functions may also be determined as solutions of a boundary value problem on the interval $(0, 2\mathbf{K})$.

$$(7)\quad \Lambda(0)=\Lambda(2\mathbf{K})=0 \qquad \text{for} \qquad \Lambda=\mathrm{Es}_n^{2m}(z) \quad \text{or} \quad \mathrm{Ec}_n^{2m+1}(z)$$

$$(8)\quad \Lambda'(0)=\Lambda'(2\mathbf{K})=0 \qquad \text{for} \qquad \Lambda=\mathrm{Es}_n^{2m+1}(z) \text{ or } \mathrm{Ec}_n^{2m}(z).$$

The existence of exactly one solution of each of the problems (3) to (6) for each $m=0, 1, 2, \ldots$ now follows from the Sturm-Liouville theory (see, for instance, Ince, 1927, sec. 10.61). Since the characteristic numbers of a Sturm-Liouville problem form an increasing sequence, we have from (1), (2), (7), and (8)

(9) $a_n^0 < a_n^1 < a_n^2 < \cdots, \quad a_n^m \to \infty \quad \text{as} \quad m \to \infty$

(10) $b_n^1 < b_n^2 < \cdots, \quad b_n^m \to \infty \quad \text{as} \quad m \to \infty$

(11) $a_n^1 < b_n^2 < a_n^3 < b_n^4 < \cdots$

(12) $a_n^0 < b_n^1 < a_n^2 < b_n^3 < \cdots .$

Thus, the relative position of the characteristic values is fairly well established, except that no statement can be made about the relative position of $a_n^m$ and $b_n^m$. Ince (1940a, b) computed characteristic values for integer values of $2n$, but it should be noted that his notation differs slightly from the one adopted here: $a_n^{2m+1}$ and $b_n^{2m+1}$ should be interchanged in order to convert Ince's notation to ours.

For the construction of Lamé functions, Ince (1940a) first used power series. Later (1940b) he discovered the expansions in trigonometric series which are more rapidly convergent, especially when $k$ is near 1.

The expansions in trigonometric series are based on 15.2(5) and on the similar differential equation satisfied by $\Lambda(z)/\text{dn}\, z$. For each Lamé function of period $2\mathbf{K}$ or $4\mathbf{K}$ we get two expansions which are listed below. The abbreviations

(13) $\zeta = \tfrac{1}{2} - \text{am}\, z, \quad H = 2h - k^2 n(n+1)$

are used throughout, and $m$ is a non-negative integer.

Trigonometric series for Lamé functions of period $2\mathbf{K}$, $4\mathbf{K}$:

(14) $\text{Ec}_n^{2m}(z) = \tfrac{1}{2}A_0 + \sum_{r=1}^{\infty} A_{2r}\cos(2r\zeta) = \text{dn}\, z\,[\tfrac{1}{2}C_0 + \sum_{r=1}^{\infty} C_{2r}\cos(2r\zeta)]$

(15) $\text{Ec}_n^{2m+1}(z) = \sum_{r=0}^{\infty} A_{2r+1}\cos[(2r+1)\zeta] = \text{dn}\, z \sum_{r=0}^{\infty} C_{2r+1}\cos[(2r+1)\zeta]$

(16) $\text{Es}_n^{2m}(z) = \sum_{r=1}^{\infty} B_{2r}\sin(2r\zeta) = \text{dn}\, z \sum_{r=1}^{\infty} D_{2r}\sin(2r\zeta)$

(17) $\text{Es}_n^{2m+1}(z) = \sum_{r=0}^{\infty} B_{2r+1}\sin[(2r+1)\zeta] = \text{dn}\, z \sum_{r=0}^{\infty} D_{2r+1}\sin[(2r+1)\zeta]$

The recurrence formulas for the coefficients in (14) to (17) are ($r = 1, 2, 3, \ldots$):

(18) $-HA_0 + (n-1)(n+2)k^2 A_2 = 0$

$\tfrac{1}{2}(n-2r+2)(n+2r-1)k^2 A_{2r-2} - [H - 4r^2(2-k^2)]A_{2r}$

$+ \tfrac{1}{2}(n-2r-1)(n+2r+2)k^2 A_{2r+2} = 0$

(19) $-HC_0 + n(n+1)k^2 C_2 = 0$

$$\tfrac{1}{2}(n-2r+1)(n+2r)k^2 C_{2r-2} - [H-4r^2(2-k^2)]C_{2r} + \tfrac{1}{2}(n-2r)(n+2r+1)k^2 C_{2r+2} = 0$$

(20) $-[H-2+k^2-\tfrac{1}{2}n(n+1)k^2]A_1 + \tfrac{1}{2}(n-2)(n+3)k^2 A_3 = 0$

$$\tfrac{1}{2}(n-2r+1)(n+2r)k^2 A_{2r-1} - [H-(2r+1)^2(2-k^2)]A_{2r+1} + \tfrac{1}{2}(n-2r-2)(n+2r+3)k^2 A_{2r+3} = 0$$

(21) $-[H-2+k^2-\tfrac{1}{2}n(n+1)k^2]C_1 + \tfrac{1}{2}(n-1)(n+2)k^2 C_3 = 0$

$$\tfrac{1}{2}(n-2r)(n+2r+1)k^2 C_{2r-1} - [H-(2r+1)^2(2-k^2)]C_{2r+1} + \tfrac{1}{2}(n-2r-1)(n+2r+2)k^2 C_{2r+3} = 0$$

(22) $-(H-8+4k^2)B_2 + \tfrac{1}{2}(n-3)(n+4)k^2 B_4 = 0$

$$\tfrac{1}{2}(n-2r)(n+2r+1)k^2 B_{2r} - [H-(2r+2)^2(2-k^2)]B_{2r+2} + \tfrac{1}{2}(n-2r-3)(n+2r+4)k^2 B_{2r+4} = 0$$

(23) $-(H-8+4k^2)D_2 + \tfrac{1}{2}(n-2)(n+3)k^2 D_4 = 0$

$$\tfrac{1}{2}(n-2r-1)(n+2r+2)k^2 D_{2r} - [H-(2r+2)^2(2-k^2)]D_{2r+2} + \tfrac{1}{2}(n-2r-2)(n+2r+3)k^2 D_{2r+4} = 0$$

(24) $-[H-2+k^2+\tfrac{1}{2}n(n+1)k^2]B_1 + \tfrac{1}{2}(n-2)(n+3)k^2 B_3 = 0$

$$\tfrac{1}{2}(n-2r+1)(n+2r)k^2 B_{2r-1} - [H-(2r+1)^2(2-k^2)]B_{2r+1} + \tfrac{1}{2}(n-2r-2)(n+2r+3)k^2 B_{2r+3} = 0$$

(25) $-[H-2+k^2+\tfrac{1}{2}n(n+1)k^2]D_1 + \tfrac{1}{2}(n-1)(n+2)k^2 D_3 = 0$

$$\tfrac{1}{2}(n-2r)(n+2r+1)k^2 D_{2r-1} - [H-(2r+1)^2(2-k^2)]D_{2r+1} + \tfrac{1}{2}(n-2r-1)(n+2r+2)k^2 D_{2r+3} = 0.$$

After division by $4r^2$, each of the eight recurrence relations is of the form 15.3(13) with $X_r = A_{2r}, A_{2r+1}, \ldots, D_{2r+1}$ respectively. In all eight cases $\alpha = \gamma = \frac{1}{2}k^2$, $\beta = k^2 - 2$, and the roots of the quadratic equation 15.3(15) are

$$(26)\quad t_{1,2} = \left(\frac{1 \pm k'}{k}\right)^2$$

For periodic Lamé functions $X_r/X_{r-1}$ tends to the smaller root, and the convergence of (14) to (17) for real $\zeta$ is comparable with that of a geometric progression with ratio $(1 - k')/(1 + k')$.

For the characteristic values of $h$, the continued fraction 15.3(16) gives an equation in each case: these equations were given by Ince (1940b). In general, these equations are transcendental equations, and the method of numerical solution is explained in Ince (1932, p. 359). If, however, $n$ is an integer, some of the continued fractions terminate, and we obtain (for $n = 0, 1, 2, \ldots$) altogether $2n + 1$ Lamé functions which are represented by terminating trigonometric series, and are therefore polynomials in $s$, $c$, $d$: these Lamé functions are known as *Lamé polynomials*. Note that even in this case there exists an infinite sequence of transcendental Lamé functions (Ince, 1940a).

Lamé functions of real period may also be represented by series of Legendre functions (see Erdélyi, 1948 and the literature quoted there). We obtain finite expansions in the case of Lamé polynomials, and infinite series for transcendental Lamé functions. The coefficients in these series are simple multiples of the coefficients in the trigonometric expansions. The expansions in Legendre functions are most useful in the construction of Lamé functions of the second kind (see below).

Ince (1940b) has discussed the *coexistence question*. His results can be summarized as follows. If $n$ is not an integer, there can never be two distinct periodic solutions belonging to the same characteristic value of $h$. If $n$ is an integer and we have a Lamé polynomial, then the second solution is never periodic. On the other hand, if $n$ is an integer and we have a transcendental Lamé function then an even and an odd solution always belong to the same characteristic value of $h$. Thus, (9) to (12) may be supplemented by

$$(27)\quad a_n^m \neq b_n^m \quad \text{for all } m = 0, 1, 2, \ldots, \quad \text{if } n \text{ is not an integer}$$

$$\text{or if } n \text{ is an integer and } m = 0, 1, \ldots, |n + \tfrac{1}{2}| - \tfrac{1}{2};$$

$$a_n^m = b_n^m \quad \text{if } m \text{ and } n \text{ are integers and } m > |n + \tfrac{1}{2}| - \tfrac{1}{2}.$$

Ince (1940a) also investigated the asymptotic behavior of the characteristic values when $n$ is large and found that for large real $n$

$$(28)\quad a_n^{2m} \sim b_n^{2m+1} \sim (4m+1)k\,[n(n+1)]^{\frac{1}{2}}$$

$$a_n^{2m+1} \sim b_n^{2m+2} \sim (4m+3)k\,[n(n+1)]^{\frac{1}{2}}$$

*Solutions of other real periods.* Solutions of primitive period 8**K** may be represented in terms of Fourier series such as

$$\Sigma\, A_r \,{\cos \atop \sin}\, (2r - \tfrac{1}{2})\zeta$$

which lead to recurrence relations for the coefficients, and an equation involving a continued fraction for the determination of the characteristic values of $h$. When $2n$ is an odd integer, the continued fractions terminate and we have an algebraic equation for $h$. The Lamé functions of period 8**K** which correspond to the roots of this algebraic equation are algebraic functions of $s$, $c$, $d$ and are known as *algebraic Lamé functions*. (For algebraic Lamé functions see Lambe 1951, 1952 and the literature quoted there.) For both algebraic and transcendental Lamé functions $a_n^{m+\frac{1}{2}} = b_n^{m+\frac{1}{2}}$ for $m = 0, 1, 2, \ldots$ and all $n$.

Solutions of primitive period $2p$**K** may be represented in terms of Fourier series such as

$$\Sigma\, A_r \,{\cos \atop \sin} \left(2r - \frac{q}{p}\right)\zeta$$

which lead to the appropriate recurrence relations etc. Except when $p = 1$, 2, or 4, the equation determining $h$ is always a transcendental equation, and the Fourier series never terminate.

*Functions of the second kind.* Let $h$ have one of its characteristic values, $a_n^m$ or $b_n^m$. Then one solution of Lamé's equation is a (periodic) Lamé function, $E(z)$, say. Except when $2n$ is an integer and $m > |n + \frac{1}{2}| - \frac{1}{2}$, Lamé's equation has only one periodic solution, and it is necessary to construct a *Lamé function of the second kind*. For many purposes a suitable function of the second kind will be that solution of Lamé's equation which belongs to the exponent $\frac{1}{2}n + \frac{1}{2}$ at $\infty$ in 15.4 (2). We take Re $n \geq -\frac{1}{2}$.

Various constructions of Lamé functions of the second kind are available. Equation 15.4 (2) suggests an expansion in descending powers of $s$, and the theory of Heun's equation provides several alternative power series expansions. Also, if $E(z)$ is the (periodic) Lamé function of the first kind,

$$E(z) \int_{i\mathbf{K}'}^{z} [E(u)]^{-2}\, du$$

will represent the Lamé function of the second kind. This representation is often used in the older literature (see, for instance, Whittaker and Watson 1927, sec. 23.71).

If the Lamé function of the first kind has been represented by a series of Legendre functions of the first kind in which the variable is proportional to $s$, $c$, or $d$, then the corresponding Lamé function of the second kind may be obtained simply by replacing each Legendre function of the first kind by the corresponding Legendre function of the second kind. This solution is of especial importance when $2n$ and $2m$ are integers and $0 \le m \le |n + \tfrac{1}{2}| - \tfrac{1}{2}$. In this case the Lamé function of the first kind is a Lamé polynomial (if $2n$ is even) or an algebraic Lamé function (if $2n$ is odd), in either case it is represented by a terminating series of Legendre functions of the first kind, and the corresponding Lamé function of the second kind will be represented by a finite combination of Legendre functions of the second kind. This representation is most useful for constructing *external ellipsoidal harmonics* (see sec. 15.1.1).

### 15.5.2. Lamé functions of imaginary periods. Transformation formulas

Since $\operatorname{sn}^2 z$ has the primitive imaginary period $2i\mathbf{K}'$, the primitive period of any Lamé function of imaginary period must be of the form $2ip\mathbf{K}'$ where $p = 1, 2, \dots$ . The existence and properties of such functions could be established in a manner analogous to that of the preceding section by setting up certain Sturm-Liouville problems, e.g., for the interval $(\mathbf{K}, \mathbf{K} + i\mathbf{K}')$. Instead of this, we shall deduce all the requisite information from the results of the preceding section by means of the *imaginary transformation* of Lamé's equation.

We put

$$(29) \quad z' = i(z - \mathbf{K} - i\mathbf{K}'), \qquad h' = n(n+1) - h$$

in 15.2(1) and use Table 7 in sec. 13.18 and Table 11 in sec. 13.22 to obtain

$$[k \operatorname{sn}(z, k)]^2 = \left[\frac{\operatorname{dn}(iz', k)}{\operatorname{cn}(iz', k)}\right]^2 = [\operatorname{dn}(z', k')]^2 = 1 - [k' \operatorname{sn}(z', k')]^2$$

and hence

$$(30) \quad \frac{d^2 \Lambda}{dz'^2} + \{h' - n(n+1)[k' \operatorname{sn}(z', k')]^2\} \Lambda = 0.$$

Clearly, every solution of (30) satisfies 15.2(1) and *vice versa*. Moreover, solutions of (30) which as functions of $z'$ have a real period will have an imaginary period when considered as functions of $z$. From the results of sec. 15.5.1 we obtain the following information.

It is sufficient to consider Lamé functions of imaginary period $2ip\mathbf{K}'$, $p = 1, 2, \ldots$, which are even or odd functions of $z - \mathbf{K} = -i(z' - \mathbf{K}')$. Even functions will be denoted by $\mathrm{Ec}_n'^m(z, k^2)$, and odd functions by $\mathrm{Es}_n'^m(z, k^2)$ if they have exactly $pm$ zeros when $z = \mathbf{K} + it$ and $t$ ranges over $0 \le t < 2p\mathbf{K}'$ (or any half-open interval of length $2p\mathbf{K}'$). The characteristic values of $h' = n(n+1) - h$ belonging to $\mathrm{Ec}_n'^m$ and $\mathrm{Es}_n'^m$ will be denoted by $a_n'^m(k^2)$ and $b_n'^m(k^2)$ respectively.

If $0 < k < 1$ and $n(n+1)$ is real, we have for each $m = 0, 1, 2, \ldots$ exactly one $\mathrm{Ec}_n'^m$ and for each $m = 1, 2, \ldots$ one $\mathrm{Es}_n'^m$. These functions have the period $2i\mathbf{K}'$ if $m$ is even, and $4i\mathbf{K}'$ if $m$ is odd. They, and the characteristic values of $h'$ belonging to them, can be expressed as

(31) $$\mathrm{Ec}_n'^m(z, k^2) = \mathrm{Ec}_n^m(z', k'^2), \qquad \mathrm{Es}_n'^m(z, k^2) = \mathrm{Es}_n^m(z', k'^2)$$

(32) $$a_n'^m(k^2) = a_n^m(k'^2), \qquad b_n'^m(k^2) = b_n^m(k'^2).$$

Two distinct solutions of periods $2i\mathbf{K}'$ or $4i\mathbf{K}'$ belong to the same characteristic value of $h'$ (or $h$) if and only if $n$ is an integer and the functions in question are transcendental Lamé functions of imaginary periods (i.e., $m > |n + \tfrac{1}{2}| - \tfrac{1}{2}$).

Information about the relative position and asymptotic behavior of the characteristic values may be obtained from (9)-(12), (27), (28) by means of (32).

*Lamé polynomials*, being polynomials in $s$, $c$, and $d$, have both a real and an imaginary period. An analysis of the zeros leads to the following identities

(33) $$\mathrm{Ec}_n^m(z, k^2) = \mathrm{Ec}_n'^{n-m}(z, k^2) = \mathrm{Ec}_n^{n-m}(z', k'^2)$$

$$\mathrm{Es}_n^m(z, k^2) = \mathrm{Es}_n'^{n-m+1}(z, k^2) = \mathrm{Es}_n^{n-m+1}(z', k'^2)$$

(34) $$a_n^m(k^2) + a_n'^{n-m}(k^2) = a_n^m(k^2) + a_n^{n-m}(k'^2) = n(n+1)$$

$$b_n^m(k^2) + b_n'^{n-m+1}(k^2) = b_n^m(k^2) + b_n^{n-m+1}(k'^2) = n(n+1)$$

which are valid provided $n$ is an integer, $m = 0, 1, \dots, |n + \frac{1}{2}| - \frac{1}{2}$, and the Lamé functions have been normalized suitably (Erdélyi, 1941 a). In particular, for $k^2 = k'^2 = \frac{1}{2}$,

$$(35)\quad a_n^m(\tfrac{1}{2}) + a_n^{n-m}(\tfrac{1}{2}) = b_n^m(\tfrac{1}{2}) + b_n^{n-m+1}(\tfrac{1}{2}) = n(n+1)$$

$$(36)\quad a_{2n}^n(\tfrac{1}{2}) = n(2n+1), \qquad b_{2n+1}^{n+1}(\tfrac{1}{2}) = (n+1)(2n+1) \qquad n = 0, 1, 2, \dots$$

Similar relations hold for *algebraic Lamé functions* (Erdélyi, 1941b).

$$(37)\quad \mathrm{Ec}_n^{m+\frac{1}{2}}(z, k^2) = \mathrm{Ec}'^{\,n-m}_n(z, k^2) = \mathrm{Ec}_n^{n-m}(z', k'^2)$$

$$\mathrm{Es}_n^{m+\frac{1}{2}}(z, k^2) = \mathrm{Es}'^{\,n-m}_n(z, k^2) = \mathrm{Es}_n^{n-m}(z', k'^2)$$

$$(38)\quad a_n^{m+\frac{1}{2}}(k^2) + a'^{\,n-m}_n(k^2) = a_n^{m+\frac{1}{2}}(k^2) + a_n^{n-m}(k'^2) = n(n+1)$$

$$(39)\quad a_n^{m+\frac{1}{2}}(\tfrac{1}{2}) + a_n^{n-m}(\tfrac{1}{2}) = n(n+1)$$

provided $n - \frac{1}{2}$ is an integer, $m = 0, \dots, |n - \frac{1}{2}|$ and the Lamé functions have been normalized suitably. Note that corresponding even and odd algebraic Lamé functions belong to the same characteristic values, and hence $a = b$ for these functions. From (39) we also have

$$(40)\quad a_{2m+\frac{1}{2}}^{m+\frac{1}{2}}(\tfrac{1}{2}) = \tfrac{1}{2}(2m + \tfrac{1}{2})(2m + 3/2) \qquad m = 0, 1, 2, \dots$$

We can now discuss the *coexistence question* for solutions of periods $2\mathbf{K}$, $4\mathbf{K}$, $2i\mathbf{K}'$, $4i\mathbf{K}'$ (see Erdélyi, 1941a). We already know that two solutions of real periods coexist (belong to the same characteristic value) if and only if $n$ is an integer, and the functions in question are transcendental Lamé functions of the same real period. Likewise, two solutions of imaginary period coexist if and only if $n$ is an integer and the functions in question are transcendental Lamé functions of the same imaginary period. Moreover, in the case of Lamé polynomials, a Lamé function of real period, and a Lamé function of imaginary period coincide. Lamé polynomials are doubly-periodic Lamé functions, and it can be shown that they are the only doubly-periodic solutions of periods $4\mathbf{K}$, $4i\mathbf{K}'$ of Lamé's equation. An analysis of the information about the relative position of the characteristic values also shows that two distinct Lamé functions one of which has a real period $2\mathbf{K}$ or $4\mathbf{K}$, and another an imaginary period $2i\mathbf{K}'$ or $4i\mathbf{K}'$ can never belong to the same value of $h$.

Summing up, if $E_n(z)$ is a Lamé function of period $2\mathbf{K}$, $4\mathbf{K}$, $2i\mathbf{K}'$, or $4i\mathbf{K}'$, and $n$ is not an integer, then $E_n(z)$ has only a real, or only an imaginary period, and it is the only periodic solution of Lamé's equation. On the other hand, if $n$ is an integer, then $E_n(z)$ is either a Lamé polynomial and doubly-periodic (in which case the corresponding Lamé function of the second kind is not periodic), or else $E_n(z)$ is a transcendental simply-periodic Lamé function and coexists with another Lamé function of the same period.

### 15.5.3. Integral equations for Lamé functions

Integral equations for Lamé functions have been discovered by Whittaker (1915a,b), and have been investigated by Ince (1922, 1940a,b), Erdélyi (1943) and others. The corresponding integral equations for Heun functions have been investigated by Lambe and Ward (1934) and Erdélyi (1942b).

Let $N(\beta, \gamma)$ satisfy the partial differential equation

$$(41)\quad \frac{\partial^2 N}{\partial \beta^2} - n(n+1)[k \operatorname{sn}(\beta, k)]^2 N = \frac{\partial^2 N}{\partial \gamma^2} - n(n+1)[k \operatorname{sn}(\gamma, k)]^2 N$$

and let $\Lambda(\gamma)$ be a solution of Lamé's equation

$$(42)\quad \frac{d^2 \Lambda}{d\gamma^2} + \{h - n(n+1)[k \operatorname{sn}(\gamma, k)]^2\} \Lambda = 0.$$

We then have, by integration by parts,

$$(43)\quad \left\{\frac{d^2}{d\beta^2} + h - n(n+1)[k \operatorname{sn}(\beta, k)]^2\right\} \int_a^b N(\beta, \gamma)\Lambda(\gamma)\,d\gamma$$

$$= \int_a^b \left(\frac{\partial^2 N}{\partial \gamma^2} + \{h - n(n+1)[k \operatorname{sn}(\gamma, k)]^2\} N\right) \Lambda(\gamma)\,d\gamma$$

$$= \left[\frac{\partial N(\beta, \gamma)}{\partial \gamma}\,\Lambda(\gamma) - N(\beta, \gamma)\,\frac{d\Lambda}{d\gamma}\right]_a^b$$

$$+ \int_a^b N(\beta, \gamma)\left(\frac{d^2 \Lambda}{d\gamma^2} + \{h - n(n+1)[k \operatorname{sn}[(\gamma, k)]\} \Lambda\right) d\gamma,$$

and it follows that $\int_a^b N(\beta, \gamma)\Lambda(\gamma)\,d\gamma$ is a solution of Lamé's equation provided that the "integrated parts", $[\,\cdots\,]_a^b$, vanish.

Now, let $h = a_n^m$ or $b_n^m$, and let $\Lambda(\gamma) = E_n^m(\gamma)$ be a solution of period $2\mathbf{K}$ or $4\mathbf{K}$ corresponding to $h$: assume also that $N(\beta, \gamma)$ is a solution of (41) which is a periodic function of both $\beta$ and $\gamma$ mod $4\mathbf{K}$. Then our argument shows that

$$\int_{-2\mathbf{K}}^{2\mathbf{K}} N(\beta, \gamma)\, E_n^m(\gamma)\, d\gamma$$

is a solution of Lamé's equation, is periodic mod $4\mathbf{K}$, and belongs to the same characteristic value as $E_n^m(\gamma)$. If $n$ is not an integer, or else if $n$ is an integer and $m \leq n$ so that $E_n^m(\gamma)$ is a Lamé polynomial, $E_n^m(\gamma)$ is the only periodic solution of (42), and we obtain an *integral equation* for $E_n^m$,

$$(44) \quad \int_{-2\mathbf{K}}^{2\mathbf{K}} N(\beta, \gamma)\, E_n^m(\gamma)\, d\gamma = \lambda_n^m\, E_n^m(\beta)$$

$$n = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, n$$
$$\text{or } n \text{ not an integer}, \quad m = 0, 1, 2, \ldots,$$

If $n$ is a non-negative integer and $m > n$, the Lamé equation has two distinct periodic solutions, and the integral will be a linear combination of $\mathrm{Ec}_n^m(\beta)$ and $\mathrm{Es}_n^m(\beta)$. In this case we obtain integral equations for two distinct periodic solutions but these need not be $\mathrm{Ec}_n^m$ and $\mathrm{Es}_n^m$. However, integral equations for $\mathrm{Ec}_n^m$ ($\mathrm{Es}_n^m$) may be obtained by taking $N(\beta, \gamma)$ an even [odd] function of $\beta - \mathbf{K}$.

The construction of suitable kernels $N(\beta, \gamma)$ is facilitated by the remark that upon the introduction of new independent variables $\theta$, $\phi$ according to

$$(45) \quad \begin{aligned} \sin\theta \cos\phi &= k \operatorname{sn}\beta \operatorname{sn}\gamma \\ \sin\theta \sin\phi &= i\frac{k}{k'} \operatorname{cn}\beta \operatorname{cn}\gamma \\ \cos\theta &= \frac{1}{k'} \operatorname{dn}\beta \operatorname{dn}\gamma, \end{aligned}$$

it is seen from 15.1 (16) and 15.1 (18) that the partial differential equation (41) becomes the partial differential equation of spherical surface harmonics, so that $N(\beta, \gamma)$ is any solution of the latter equation expressed in sphero-conal coordinates. If $n$ is an integer, and $N(\beta, \gamma)$ is a (regular) spherical surface harmonic, and hence (according to sec. 15.1.2) also a (regular) ellipsoidal surface harmonic, all characteristic functions belonging to non-zero $\lambda_n^m$ of $N$ are Lamé polynomials.

We now list a few simple kernels together with their characteristic functions (determined by consideration of the parity of the kernel as a function of $\beta$ and of $\beta - \mathrm{K}$).

$$(46)\quad N = P_n(\cos\theta) = P_n\left(\frac{1}{k'}\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\right) \qquad (\mathrm{Ec}_n^{2m})$$

$$(47)\quad N = P_n^1(\cos\theta)\cos\phi = k\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma\,P_n'\left(\frac{1}{k'}\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\right) \qquad (\mathrm{Ec}_n^{2m+1})$$

$$(48)\quad N = P_n^1(\cos\theta)\sin\phi = i\,\frac{k}{k'}\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma\,P_n'\left(\frac{1}{k'}\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\right) \qquad (\mathrm{Es}_n^{2m})$$

$$(49)\quad N = P_n^2(\cos\theta)\sin(2\phi) = 2i\,\frac{k^2}{k'}\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma \times P_n''\left(\frac{1}{k'}\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma\right) \qquad (\mathrm{Es}_n^{2m+1})$$

If $n$ is an integer, the characteristic functions of the kernels (46) to (49) are Lamé polynomials: kernels appropriate for transcendental Lamé functions involve $Q_n$. It may also be mentioned that further simple kernels involving Legendre functions of $k\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma$ or $i(k/k')\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma$ are also known.

### 15.5.4. Degenerate cases

If $k = 0$, Lamé's equation becomes

$$(50)\quad \frac{d^2\Lambda}{dz^2} + h\Lambda = 0,$$

we have $\mathbf{K} = \frac{1}{2}\pi$, and the solutions of (50) satisfying (3) to (6) are

$$(51)\quad \mathrm{Ec}_n^m(z, 0) = \cos[m(z - \tfrac{1}{2}\pi)]$$

$$\mathrm{Es}_n^m(z, 0) = \sin[m(z - \tfrac{1}{2}\pi)].$$

They both belong to the characteristic value

$$(52)\quad a_n^m(0) = b_n^m(0) = m^2$$

If $k = 1$, we see from 13.18(4) that Lamé's equation becomes

$$(53)\quad \frac{d^2\Lambda}{dz^2} + [h - n(n+1)(\tanh z)^2]\Lambda = 0$$

and $\mathbf{K} = \infty$, $\mathbf{K}' = \frac{1}{2}\pi$. In this case Ince (1940a) has shown that

(54) $a_n^{2m}(1) = b_n^{2m+1}(1) = (4m+1)n - 4m^2$

$a_n^{2m+1}(1) = b_n^{2m+2}(1) = (4m+3)n - (2m+1)^2$

(55) $\mathrm{Ec}_n^{2m}(z, 1) = \mathrm{Es}_n^{2m+1}(z, 1) = P_n^{n-2m}(\tanh z)$

$\mathrm{Ec}_n^{2m+1}(z, 1) = \mathrm{Es}_n^{2m+2}(z, 1) = P_n^{n-2m-1}(\tanh z).$

Finally, let $n \to \infty$ and simultaneously $k \to 0$ in such a manner that

(56) $n(n+1)k^2 \to -4\theta$

In this case $\mathrm{sn}(z, 0) = \sin z$, and, from 15.2(4), $\zeta = \frac{1}{2} - z$. Equation 15.2(5) becomes

(57) $$\frac{d^2\Lambda}{d\zeta^2} + [h + 4\theta(\cos\zeta)^2]\,\Lambda = 0$$

which is a form of *Mathieu's equation*. Lamé functions of real periods become Mathieu functions: the imaginary period $\mathbf{K}' = \infty$ in this case.

## 15.6. Lamé-Wangerin functions

We have seen in sec. 15.1.3 that some of the potential problems formulated in the systems of coordinates introduced by Wangerin lead to postulating solutions which are finite at two singularities of Lamé's equation. We shall now show that such solutions are possible only for certain characteristic values of $h$: and we shall call the ensuing characteristic solutions *finite Lamé functions* or *Lamé-Wangerin functions* in order to distinguish them from the (periodic) Lamé functions discussed in the preceding sections. Comparatively little is known about Lamé-Wangerin functions, and most of the material to be presented here is taken from a note (1948a) and unpublished work by Erdélyi.

A Lamé-Wangerin function is a solution of Lamé's equation 15.2(1) and has the property that $(\mathrm{sn}\,z)^{\frac{1}{2}}\,\Lambda(z)$ is bounded in a region which contains at least two poles of sn $z$. More specifically, we shall denote by $F_n^m(z, k^2)$ a Lamé-Wangerin function for which $(\mathrm{sn}\,z)^{\frac{1}{2}}\,F_n^m(z, k^2)$ is bounded, and has exactly $m$ zeros on the open interval $(i\mathbf{K}', 2\mathbf{K} + i\mathbf{K}')$; it then follows that $(\mathrm{sn}\,z)^{\frac{1}{2}}\,F_n^m(z, k^2)$ is also bounded in a region which includes this interval, indeed in an infinite strip which contains the line $z = i\mathbf{K}' + 2\mathbf{K}t$, $-\infty < t < \infty$. The characteristic value of $h$ which belongs to $F_n^m$ will be denoted by $c_n^m(k^2)$.

We shall assume that $k$ and $n$ are given and are such that for real $t$

$$n(n+1)[k\,\mathbf{K}\,\mathrm{sn}(i\mathbf{K}'+2\mathbf{K}t,\,k^2)]^2$$

is real, so that Lamé's differential equation 15.2(1), when expressed in terms of $t$ as the independent variable, is a differential equation with real coefficients. Without loss of generality we shall take $\mathrm{Re}\, n \geq -\frac{1}{2}$. Our assumptions are always satisfied when $0 < k < 1$ and $n(n+1)$ is real, but it is seen from 15.1(33) that the case of complex $k$ also occurs. From 13.23(13) and Table 11 in sec. 13.22 it is easy to verify that the functions involved in 15.1(33) satisfy our reality condition.

If $F(z)$ is a Lamé-Wangerin function, so are the functions

$$F(2\mathbf{K}+2i\mathbf{K}'-z) \quad \text{and} \quad F(z) \pm F(2\mathbf{K}+2i\mathbf{K}'-z),$$

and we may restrict ourselves to the investigation of Lamé-Wangerin functions which are even or odd functions of $z-\mathbf{K}-i\mathbf{K}'$. If $F_n^m(z,k^2)$ is such a function, it will be an even or odd function of $z-\mathbf{K}-i\mathbf{K}'$ according as $m$ is even or odd. Thus we arrive at the following boundary conditions

$$(1) \quad (\mathrm{sn}\, z)^{\frac{1}{2}}\,\Lambda(z) \quad \text{bounded at} \quad z = i\mathbf{K}'$$
$$\Lambda'(\mathbf{K}+i\mathbf{K}') = 0 \quad \text{for} \quad \Lambda = F_n^{2m}(z)$$

$$(2) \quad (\mathrm{sn}\, z)^{\frac{1}{2}}\,\Lambda(z) \quad \text{bounded at} \quad z = i\mathbf{K}'$$
$$\Lambda(\mathbf{K}+i\mathbf{K}') = 0 \quad \text{for} \quad \Lambda = F_n^{2m+1}(z)$$

Since $z = i\mathbf{K}'$ is a singular point of Lamé's equation, the existence and properties of Lamé-Wangerin functions must be deduced from the *singular* Sturm-Liouville theory. However, the nature of the singularity at $z = i\mathbf{K}'$ and of the boundary condition there enables us to use the simplest kind of singular Sturm-Liouville theory, retaining practically all the features of the regular theory. It follows from the work of McCrea and Newing (1933) that for each $m = 0, 1, \ldots$ there is exactly one Lamé-Wangerin function, and that the characteristic values of $\mathbf{K}^2 h$ belonging to these functions form an increasing unbounded sequence,

$$(3) \quad \mathbf{K}^2 c_n^0 \leq \mathbf{K}^2 c_n^1 \leq \mathbf{K}^2 c_n^2 \leq \cdots \qquad \mathbf{K}^2 c_n^m \to \infty \ \text{ as } \ m \to \infty$$

If $\mathrm{Re}\, n > -\frac{1}{2}$ or $n = -\frac{1}{2}$, no two Lamé-Wangerin functions belong to the same characteristic value, and we have a strictly increasing sequence,

$$(4) \quad \mathbf{K}^2 c_n^0 < \mathbf{K}^2 c_n^1 < \mathbf{K}^2 c_n^2 < \cdots \qquad \mathrm{Re}\, n > -\tfrac{1}{2} \ \text{ or } \ n = -\tfrac{1}{2}.$$

If $0 < k < 1$, so that $\mathbf{K}$ is real, the $c_n^m$ themselves are real, and we have

(5) $c_n^0 < c_n^1 < c_n^2 < \cdots, \qquad c_n^m \to \infty \qquad\qquad 0 < k < 1, \quad n \geq -\frac{1}{2}$

For the construction of Lamé-Wangerin functions, the $P$-symbols given in sec. 15.4 suggest a variety of expansions of the form 15.3(10). We shall give here the series in descending powers of $s$ which are convergent when $0 < k < 1$.

Any Lamé-Wangerin function belongs at $s = \infty$ to the exponent $\frac{1}{2}n + \frac{1}{2}$ in 15.4(2), and 15.3(10) suggests power series in $s^{-2}$ multiplied by $s^{-n-1-2\rho-2\sigma} c^{2\rho} d^{2\sigma}$ where $\rho$ and $\sigma$ have the values 0 or $\frac{1}{2}$. Clearly, $\sigma = 0$ for $F_n^{2m}$ and $\sigma = \frac{1}{2}$ for $F_n^{2m+1}$. We thus obtain the power series

(6) $$F_n^{2m}(z) = \sum_{r=0}^{\infty} A_r s^{-n-2r-1} = c \sum_{r=0}^{\infty} C_r s^{-n-2r-2}$$

(7) $$F_n^{2m+1}(z) = d \sum_{r=0}^{\infty} A_r s^{-n-2r-2} = cd \sum_{r=0}^{\infty} D_r s^{-n-2r-3}$$

The recurrence relations for the coefficients are ($r = 1, 2, 3, \ldots$):

(8) $$[h - (n+1)^2 (1+k^2)] A_0 + 2(2n+3) k^2 A_1 = 0$$

$$(n+2r-1)(n+2r) A_{r-1} + [h - (n+2r+1)^2 (1+k^2)] A_r + 2(r+1)(2n+2r+3) k^2 A_{r+1} = 0$$

(9) $$[h - (n+2)^2 - (n+1)^2 k^2] B_0 + 2(2n+3) k^2 B_1 = 0$$

$$(n+2r)(n+2r+1) B_{r-1} + [h - (n+2r+2)^2 - (n+2r+1)^2 k^2] B_r + 2(r+1)(2n+2r+3) k^2 B_{r+1} = 0$$

(10) $$[h - (n+1)^2 - (n+2)^2 k^2] C_0 + 2(2n+3) k^2 C_1 = 0$$

$$(n+2r)(n+2r+1) C_{r-1} + [h - (n+2r+1)^2 - (n+2r+2)^2 k^2] C_r + 2(r+1)(2n+2r+3) k^2 C_{r+1} = 0$$

(11) $[h-(n+2)^2(1+k^2)]D_0+2(2n+3)k^2D_1=0$

$$(n+2r+1)(n+2r+2)D_{r-1}+[h-(n+2r+2)^2(1+k^2)]D_r$$
$$+2(r+1)(2n+2r+3)k^2D_{r+1}=0.$$

After division by $4r^2$, each of these recurrence relations is of the form 15.3(13). In all four cases $\alpha=1$, $\beta=-(1+k^2)$, $\gamma=k^2$ and the roots of the quadratic equation 15.3(15) are $t_1=1$, $t_2=k^{-2}$. For general values of $h$, the ratio of two successive coefficients approaches $k^{-2}$, and the series (6), (7) do not converge at $z=\mathbf{K}+i\mathbf{K}'$ where $s^{-2}=k^2$. If $h$ has one of its characteristic values, the ratio of two successive coefficients approaches 1, and the series converge in the region $|s|>1$ which includes the entire line $\operatorname{Im} z=\mathbf{K}'$.

The series (6), (7) are unsuitable when $|k|=1$, i.e. in Case III of sec. 15.1.3. In this case analogous series in descending powers of $c$ may be used with advantage.

Series which are more suitable for numerical computation may be derived from the $P$-symbols 15.4(4), (5), (6). At $z=i\mathbf{K}'$, $c/s=-i$, Lamé-Wangerin functions belong to the exponent $\frac{1}{2}n+\frac{1}{2}$ at $c/s=-i$ in 15.4(6), and 15.3(10) suggests series in powers of $(c+is)/(c-is)=(c+is)^2$ multiplied by

$$\left(\frac{c+is}{c-is}\right)^{\frac{1}{2}n+\frac{1}{2}}\left(\frac{c-ik's}{c-is}\right)^{\rho}\left(\frac{c+ik's}{c-is}\right)^{\sigma},$$

where $\rho$ and $\sigma$ have the values 0 or $\frac{1}{2}$. Clearly, $\rho=\sigma=0$ for even $m$, $\rho=\sigma=\frac{1}{2}$ for odd $m$. Moreover, if we introduce $\zeta$ as in 15.5(13), we have

(12) $\operatorname{sn} z=\cos\zeta,\quad \operatorname{cn} z=\sin\zeta,\quad c\pm is=\pm ie^{\mp i\zeta}$

We thus obtain the alternative expansions

(13) $F_n^{2m}(z)=\sum_{r=0}^{\infty}A_r\exp[-(n+2r+1)\zeta i]$

(14) $F_n^{2m+1}(z)=\operatorname{dn} z\sum_{r=0}^{\infty}B_r\exp[-(n+2r+2)\zeta i]$

whose coefficients satisfy the recurrence relations

(15) $[H-(n+1)^2(2-k^2)]A_0+(2n+3)k^2A_1=0$

$$(2r-1)(n+r)k^2A_{r-1}+[H-(n+1+2r)^2(2-k^2)]A_r$$
$$+(r+1)(2n+2r+3)k^2A_{r+1}=0$$

(16) $[H-(n+2)^2(2-k^2)]B_0+(2n+3)k^2B_1=0$

$$(2r+1)(n+r+1)k^2B_{r-1}+[H-(n+2r+2)^2(2-k^2)]B_r$$

$$+(r+1)(2n+2r+3)k^2B_{r+1}=0$$

where $H=2h-n(n+1)k^2$ and $r=1,2,3,\ldots$ .

After division by $2r^2$, these recurrence relations are of the form 15.3(13) with $\alpha=\gamma=k^2$, $\beta=-2(2-k^2)$, and the roots of the quadratic equation 15.3(15) are

$$t_1=\frac{1-k'}{1+k'},\quad t_2=\frac{1+k'}{1-k'}.$$

If Re $k'>0$, we have $|t_1|<|t_2|$. Let us consider the convergence of the series (13), (14) when $0<k,\ k'<1$. When $z=i\mathbf{K}'+u$, $0\le u\le\mathbf{K}$, it can be seen from Table 7 in sec. 13.18 that

$$c+is=\frac{ik\ \mathrm{sn}\ u}{1+\mathrm{dn}\ u}$$

and consequently $|c+is|^2\le(1-k')/(1+k')$. If $h$ has one of its characteristic values, the ratio of two successive coefficients of (13) or (14) approaches $t_1=(1-k')/(1+k')$ so that the convergence of (13) or (14) on the line Im $z=\mathbf{K}'$ is at least as good as that of a geometric progression with ratio $[(1-k')/(1+k')]^2$. Note that

$$e^{-i\zeta}=-i(c+is)=\frac{k\ \mathrm{sn}\ u}{1+\mathrm{dn}\ u}$$

is real on the line Im $z=\mathbf{K}'$.

Other power series expansions, and series of exponential functions, and also expansions in terms of Legendre functions may be obtained in the manner indicated in sections 15.3 and 15.4.

Integral equations for Lamé-Wangerin functions can be obtained very much in the same manner as for periodic Lamé functions. As in sec. 15.5.3, let $N(\beta,\gamma)$ satisfy the partial differential equation 15.5(41) and consider the integral

$$\int_{i\mathbf{K}'}^{2\mathbf{K}+i\mathbf{K}'}N(\beta,\gamma)\,F_n^m(\gamma)\,d\gamma.$$

The computation indicated in 15.5(43) shows that this integral satisfies Lamé's equation with $h = c^m_n$ provided that the nucleus is so chosen that

$$N(\beta, \gamma) \frac{d F^m_n(\gamma)}{d\gamma} \to 0, \quad \frac{\partial N(\beta, \gamma)}{\partial \gamma} F^m_n(\gamma) \to 0$$

as $\gamma \to i\mathbf{K}'$ or $\gamma \to 2\mathbf{K} + i\mathbf{K}'$. If, moreover, $N(\beta, \gamma)$ belongs to the exponent $\frac{1}{2}n + \frac{1}{2}$ at $\beta = i\mathbf{K}'$ and at $\beta = 2\mathbf{K} + i\mathbf{K}'$, uniformly for all $\gamma$ in the range of integration, then the above integral is a constant multiple of the Lamé-Wangerin function, and we have the integral equation

$$(17) \quad \int_{i\mathbf{K}'}^{2\mathbf{K}+i\mathbf{K}'} N(\beta, \gamma) F^m_n(\gamma)\, d\gamma = \lambda^m_n F^m_n(\beta)$$

The construction of suitable kernels $N(\beta, \gamma)$ is based upon the remark that the change of variables 15.5(45) transforms 15.5(41) into the partial differential equation of spherical surface harmonics. It should be noted (see the diagrams in sec. 13.25) that on the interval $(i\mathbf{K}', 2\mathbf{K} + i\mathbf{K}')$, $s$ is positive, $c$ is negative imaginary, and $d$ is real, so that in 15.5(45)

$$(18) \quad \cos\theta \text{ real}, \qquad \sin\theta\cos\phi > 0, \qquad i\sin\theta\sin\phi > 0.$$

Now, for every sufficiently regular function $f$, and every constant $a$,

$$f(x\cos a + y\sin a - iz)$$

is a solution of Laplace's equation (in Cartesian coordinates $x$, $y$, $z$), and we may take $f(u) = u^{-n-1}$, thus seeing that

$$(19) \quad (\sin\theta\cos\phi\cos a + \sin\theta\sin\phi\sin a - i\cos\theta)^{-n-1}$$
$$= \left(k \operatorname{sn}\beta \operatorname{sn}\gamma \cos a + i\frac{k}{k'} \operatorname{cn}\beta \operatorname{cn}\gamma \sin a - \frac{i}{k'} \operatorname{dn}\beta \operatorname{dn}\gamma\right)^{-n-1}$$

is a surface harmonic, provided $a$ has been so chosen that the expression in the parantheses does not vanish as $\beta$ and $\gamma$ range over $(i\mathbf{K}', 2\mathbf{K} + i\mathbf{K}')$. Moreover, $N$, $\partial N/\partial\gamma \to 0$ as $\beta$ or $\gamma$ approach one of the end-points of the interval so that (19) represents suitable nuclei. We choose, in particular, $a = \pm\frac{1}{2}\pi$, and obtain the integral equations

$$(20) \quad \int_{i\mathbf{K}'}^{2\mathbf{K}+i\mathbf{K}'} (\pm \operatorname{dn}\beta \operatorname{dn}\gamma - k \operatorname{cn}\beta \operatorname{cn}\gamma)^{-n-1} F^m_n(\gamma)\, d\gamma = \lambda^m_n F^m_n(\beta).$$

We can also construct integral equations for even or odd Lamé-Wangerin functions only, over the interval $(i\mathbf{K}', \mathbf{K} + i\mathbf{K}')$. The appropriate kernels are the sum or difference of the two kernels in (20).

### 15.7. Ellipsoidal and sphero-conal harmonics

We shall now briefly discuss the application of our results to the construction of ellipsoidal and sphero-conal harmonics.

Let us introduce *ellipsoidal coordinates* $\alpha, \beta, \gamma$ in place of rectangular coordinates $x$, $y$, $z$. The transformation formulas are 15.1(8), and the ranges of $\alpha$, $\beta$, $\gamma$ are described in the lines following 15.1(9). We shall call $B(\beta)\, C(\gamma)$ an *ellipsoidal surface harmonic* if $B$ and $C$ satisfy Lamé's equation, and $BC$ is continuous and has a continuous gradient, on all ellipsoids $\alpha$ = const. We shall call $A(\alpha)\, B(\beta)\, C(\gamma)$ an *internal ellipsoidal harmonic* if $A$, $B$, $C$ satisfy Lamé's equation and $ABC$ is continuous and has continuous derivatives inside an ellipsoid $\alpha$ = const., and we shall call $A(\alpha)\, B(\beta)\, C(\gamma)$ an *external ellipsoidal harmonic* if similar conditions prevail outside an ellipsoid, and $A(i\mathbf{K}') = 0$.

We have seen in sec. 15.1.1 that for an ellipsoidal surface harmonic we must have $B(\theta) = C(\theta)$, and that this function must be a doubly-periodic Lamé function with periods $4\mathbf{K}$ and $4i\mathbf{K}'$. By sec. 15.5.2, the only Lamé functions which have periods $4\mathbf{K}$ and $4i\mathbf{K}'$ are Lamé polynomials, and for these $n$ must be an integer which we may take non-negative, and $m \le n$. Thus we arrive at the $2n + 1$ *ellipsoidal surface harmonics of degree* $n$,

$$(1)\quad \mathrm{Sc}_n^m(\beta, \gamma) = \mathrm{Ec}_n^m(\beta)\, \mathrm{Ec}_n^m(\gamma) \qquad n = 0, 1, 2, \dots, \qquad m = 0, 1, \dots, n$$

$$\mathrm{Ss}_n^m(\beta, \gamma) = \mathrm{Es}_n^m(\beta)\, \mathrm{Es}_n^m(\gamma) \qquad n = 1, 2, 3, \dots, \qquad m = 1, 2, \dots, n$$

In terms of the variables $\theta$, $\phi$ of 15.5(45) these functions are spherical surface harmonics, and the number, $2n + 1$, of linearly independent surface harmonics could have been established by this connection.

*Ellipsoidal surface harmonics form an orthogonal system*, i.e.

$$(2)\quad \int\!\!\int_{\mathcal{E}} \mathrm{Sc}_n^m(\beta, \gamma)\, \mathrm{Sc}_\nu^\mu(\beta, \gamma)\, [(\mathrm{sn}\,\beta)^2 - (\mathrm{sn}\,\gamma)^2]\, d\beta\, d\gamma$$

$$= \int\!\!\int_{\mathcal{E}} \mathrm{Ss}_n^m(\beta, \gamma)\, \mathrm{Ss}_\nu^\mu(\beta, \gamma)\, [(\mathrm{sn}\,\beta)^2 - (\mathrm{sn}\,\gamma)^2]\, d\beta\, d\gamma = 0$$

except when $n = \nu$ and $m = \mu$, and

$$(3)\quad \int\!\!\int_{\mathcal{E}} \mathrm{Sc}_n^m(\beta, \gamma)\, \mathrm{Ss}_\nu^\mu(\beta, \gamma)\, [(\mathrm{sn}\,\beta)^2 - (\mathrm{sn}\,\gamma)^2]\, d\beta\, d\gamma = 0.$$

Here $\mathcal{E}$ indicates the surface of the ellipsoid where $\beta$ ranges from $\mathbf{K}$ to $\mathbf{K} + 2i\mathbf{K}'$, and $\gamma$ from 0 to $4\mathbf{K}$. Equation (3) follows from the different parity of Sc and Ss at $\gamma = \mathbf{K}$. To prove (2) for $n \neq \nu$, we recall that

$$\left(\frac{\partial^2}{\partial \beta^2} - \frac{\partial^2}{\partial \gamma^2}\right) \mathrm{S}_n = n(n+1)[(k \operatorname{sn} \beta)^2 - (k \operatorname{sn} \gamma)^2] \mathrm{S}_n$$

where $\mathrm{S}_n$ is $\mathrm{Sc}_n^m(\beta, \gamma)$ or $\mathrm{Ss}_n^m(\beta, \gamma)$, and hence

$$\mathrm{S}_\nu \left(\frac{\partial^2}{\partial \beta^2} - \frac{\partial^2}{\partial \gamma^2}\right) \mathrm{S}_n - \mathrm{S}_n \left(\frac{\partial^2}{\partial \beta^2} - \frac{\partial^2}{\partial \gamma^2}\right) \mathrm{S}_\nu$$

$$= [n(n-1) - \nu(\nu - 1)]\, k^2 [(\operatorname{sn} \beta)^2 - (\operatorname{sn} \gamma)^2]\, \mathrm{S}_n \mathrm{S}_\nu.$$

Integrating over $\mathcal{E}$ we have

$$[n(n-1) - \nu(\nu - 1)] \iint_{\mathcal{E}} [(\operatorname{sn} \beta)^2 - (\operatorname{sn} \gamma)^2]\, \mathrm{S}_n \mathrm{S}_\nu \, d\beta \, d\gamma = 0,$$

and hence (2) for $n \neq \nu$, For $n = \nu$ and $m \neq \mu$ we remark that $\mathrm{Ec}_n^m$ and $\mathrm{Ec}_n^\mu$ (and likewise $\mathrm{Es}_n^m$ and $\mathrm{Es}_n^\mu$) are two characteristic functions of the same Sturm-Liouville problem 15.5(1) [or 15.5(2)], and that by 15.5(9) [and 15.5(10)] they belong to different characteristic values. Equation (2) for $n = \nu$, $m \neq \mu$ then follows from the orthogonal property of Sturm-Liouville functions.

The orthogonal property of ellipsoidal surface harmonics enables us to determine the coefficients in the expansion in a series of ellipsoidal surface harmonics of an arbitrary function given on $\mathcal{E}$. The validity of the expansion can be deduced by means of the connection between ellipsoidal and spherical surface harmonics.

For *internal ellipsoidal harmonics*, we have seen in sec. 15.1.1 that $A(\theta) = B(\theta) = C(\theta)$ so that internal ellipsoidal harmonics are of one of the forms

$$\text{(4)} \quad \mathrm{Hc}_n^m(\alpha, \beta, \gamma) = \mathrm{Ec}_n^m(\alpha)\, \mathrm{Ec}_n^m(\beta)\, \mathrm{Ec}_n^m(\gamma)$$

$$n = 0, 1, 2, \ldots, \quad m = 0, 1, 2, \ldots, n$$

$$\mathrm{Hs}^m(\alpha, \beta, \gamma) = \mathrm{Es}_n^m(\alpha)\, \mathrm{Es}_n^m(\beta)\, \mathrm{Es}_n^m(\gamma)$$

$$n = 1, 2, 3, \ldots, \quad m = 1, 2, \ldots, n$$

The Lamé polynomials occurring here may be written in the form $s^\rho c^\sigma d^\tau$ times a polynomial of degree $\frac{1}{2}(n - \rho - \sigma - \tau)$ in $s^2$, and consequently

$\mathrm{Hc}_n^m$ and $\mathrm{Hs}_n^m$ are polynomials of degree $n$ in the Cartesian coordinates $x, y, z$ (harmonic polynomials).

E.T. Whittaker (Whittaker and Watson, 1927, sec. 23.62) found elegant integral representations of internal ellipsoidal harmonics.

$$(5)\quad \int_0^{4\mathbf{K}} P_n(w)\,\mathrm{Ec}_n^m(\tau)\,d\tau = \lambda\,\mathrm{Hc}_n^m(a, \beta, \gamma)$$

$$\int_0^{4\mathbf{K}} P_n(w)\,\mathrm{Es}_n^m(\tau)\,d\tau = \lambda\,\mathrm{Hs}_n^m(a, \beta, \gamma)$$

where

$$(6)\quad w = \frac{k'x\ \mathrm{sn}\ \tau + y\ \mathrm{cn}\ \tau - iz\ \mathrm{dn}\ \tau}{k'(a^2 - c^2)^{1/2}}$$

$$= k^2\ \mathrm{sn}\ a\ \mathrm{sn}\ \beta\ \mathrm{sn}\ \gamma\ \mathrm{sn}\ \tau - \frac{k^2}{k'^2}\ \mathrm{cn}\ a\ \mathrm{cn}\ \beta\ \mathrm{cn}\ \gamma\ \mathrm{cn}\ \tau$$

$$+ \frac{1}{k'^2}\ \mathrm{dn}\ a\ \mathrm{dn}\ \beta\ \mathrm{dn}\ \gamma\ \mathrm{dn}\ \tau$$

isthe sphericaldistanceoftwo points on the unit sphere whose Cartesian coordinates are

$$(7)\quad \left(k\ \mathrm{sn}\ a\ \mathrm{sn}\ \beta,\ i\,\frac{k}{k'}\ \mathrm{cn}\ a\ \mathrm{cn}\ \beta,\ \frac{1}{k'}\ \mathrm{dn}\ a\ \mathrm{dn}\ \beta\right)$$

and

$$(8)\quad \left(k\ \mathrm{sn}\ \gamma\ \mathrm{sn}\ \tau,\ i\,\frac{k}{k'}\ \mathrm{cn}\ \gamma\ \mathrm{cn}\ \tau,\ \frac{1}{k'}\ \mathrm{dn}\ \gamma\ \mathrm{dn}\ \tau\right).$$

To prove (5), we remark that $P_n(w)$ is a solution of Laplace's equation, and so are the integrals on the left-hand sides of (5). Moreover, these integrals are polynomials in sn $a$, sn $\beta$, sn $\gamma$, cn $a$, ... , dn $\gamma$. Lastly, $P_n(w)$, as a function of the point (8), is a spherical surface harmonic of degree $n$, and by 15.5(44) the integrals must be multiplies of $\mathrm{Ec}_n^m(\gamma)$, $\mathrm{Es}_n^m(\gamma)$. Since $a$, $\beta$, $\gamma$ occur symmetrically in $w$, (5) follows.

*External ellipsoidal harmonics* differ from (4) in that $\mathrm{Ec}_n^m(a)$, $\mathrm{Es}_n^m(a)$ are replaced by the corresponding Lamé functions of the second kind (see the end of sec. 15.5.1). Such harmonics may also be represented by the integrals

$$\int_0^{4\mathbf{K}} Q_n(w)\,\mathrm{Ec}_n^m(\tau)\,d\tau, \qquad \int_0^{4\mathbf{K}} Q_n(w)\,\mathrm{Es}_n^m(\tau)\,d\tau$$

where $Q_n$ is the Legendre function of the second kind and $w$ is given by (6).

In sphero-conal coordinates 15.1(16) we have the surface harmonics (1) which, if $\beta$ and $\gamma$ are sphero-conal coordinates, are spherical surface harmonics. Internal and external sphero-conal harmonics are, respectively, of the form

$$r^n \operatorname{Sc}_n^m(\beta, \gamma), \quad r^n \operatorname{Ss}_n^m(\beta, \gamma) \qquad \text{(internal)}$$

$$r^{-n-1} \operatorname{Sc}_n^m(\beta, \gamma), \quad r^{-n-1} \operatorname{Ss}_n^m(\beta, \gamma) \qquad \text{(external)}$$

where $n$ is a non-negative integer and $m < n$.

### 15.8. Harmonics associated with cyclides of revolution

In order to show the application of Lamé-Wangerin functions to the construction of harmonic functions associated with confocal systems of cyclides of revolution, we shall discuss in greater detail Case I of sec. 15.1.3, that is to say, the case of a confocal system of cyclides of revolution with four (real) foci on the axis of rotation. In particular, we shall construct harmonic functions regular inside one of the surfaces $u = \text{const.} > 0$.

The reduction of the differential equation for $f$ to normal form suggests in this case the introduction of curvilinear coordinates $u, v$ by means of the transformation

$$(1) \quad z + i\rho = s = \operatorname{sn}(u + iv, k)$$

and 15.1(29) shows that the separation of variables leads to the ordinary differential equations

$$(2) \quad \frac{d^2 U}{du^2} - (1+k)^2 \left\{ h - (m^2 - \tfrac{1}{4}) \left[ \frac{1-k}{1+k} \operatorname{sn}\left( i(1+k)u, \frac{1-k}{1+k} \right) \right]^2 \right\} U = 0$$

$$\frac{d^2 V}{dv^2} + (1+k)^2 \left\{ h - (m^2 - \tfrac{1}{4}) \left[ \frac{1-k}{1+k} \operatorname{sn}\left( (1+k)(v - i\mathbf{K}), \frac{1-k}{1+k} \right) \right]^2 \right\} V = 0$$

It has been shown in sec. 15.1.3 that the boundary conditions are that $\rho^{-\frac{1}{2}} V$ should remain finite both at $v = 0$ and $v = \mathbf{K}'$ (where the second equation (2) has singularities), and that $\rho^{-\frac{1}{2}} U$ should remain finite at $u = \mathbf{K}$ (where the first equation has a singularity).

Now, equations (2) are of the form of Lamé's equation, and the requisite solutions may be obtained by means of sec. 15.6. This procedure is satisfactory when $k$ is near 1 (when two of the foci of the confocal system are near to each other); for smaller values of $k$ it is of advantage to deal with Lamé's equation with modulus $k$, rather than $(1-k)/(1+k)$ as in (2). This can be achieved by using curvilinear coordinates $u$, $v$ which are different from those introduced by (1).

By a combination of transformation $B$ of Table 11, sec. 13.22 with Landen's transformation 13.23(13) it is seen that

$$\operatorname{sn}(u, k) = -i \operatorname{sc}(iu, k') = -\frac{2i}{1+k} \frac{\operatorname{sn}(\dot{u}, \dot{k})}{\operatorname{cn}(\dot{u}, \dot{k}) + \operatorname{dn}(\dot{u}, \dot{k})}$$

where

$$\dot{u} = i(1+k)u, \quad \dot{k} = \frac{1-k}{1+k},$$

and this suggests the introduction of curvilinear coordinates, $u$, $v$ by means of the equation

$$(3) \quad z + i\rho = \frac{iak's}{c+d} = ia \frac{d-c}{k's} = f(u+iv)$$

where

$$(4) \quad s = \operatorname{sn}(u+iv, k), \quad c = \operatorname{cn}(u+iv, k), \quad d = \operatorname{dn}(u+iv, k),$$

and the foci

$$z = \pm a\left[\frac{1-k}{1+k}\right]^{1/2}, \quad \pm a\left[\frac{1+k}{1-k}\right]^{1/2}$$

of the confocal system determine $a > 0$, and $k$, $0 < k < 1$. From now on $u$ and $v$ will be the curvilinear coordinates introduced by (3), and the abbreviations 15.1(27) will be used.

By means of the formulas of sec. 13.17 we obtain the real form of the transformation (3),

$$(5) \quad z = \frac{iak's_2}{c_1d_2 + c_2d_1}, \quad \rho = \frac{ak's_1}{c_1d_2 + c_2d_1},$$

and also

$$(6) \quad F(u, v) = \frac{|f'(u+iv)|^2}{\rho^2} = \frac{1 - k^2 s_1^2 s_2^2}{s_1^2}$$

$$= \frac{1}{s_1^2} - k^2 s_2^2 = [k \operatorname{sn}(u + i\mathbf{K}', k)]^2 - [k \operatorname{sn}(iv, k)]^2.$$

The ordinary differential equations for $U$ and $V$ become

$$(7)\quad \frac{d^2 U}{du^2} + \{h - (m^2 - \tfrac{1}{4})\,[k\ \mathrm{sn}\,(u + i\mathbf{K}', k)]^2\}\,U = 0$$

$$(8)\quad -\frac{d^2 V}{dv^2} + \{h - (m^2 - \tfrac{1}{4})\,[k\ \mathrm{sn}\,(iv, k)]^2\}\,V = 0.$$

The transformation (3) maps the rectangle with vertices $\pm i\mathbf{K}'$, $2\mathbf{K} \pm i\mathbf{K}'$ in the $u$, $v$-plane on the half-plane $\rho > 0$. In the diagram on page 87, corresponding points are denoted by the same letter. The line $v = v_0 > 0$ is mapped on a part of a bicircular quartic whose foci, $\mathfrak{A}$ and $\mathfrak{B}$, are the points $z = -a\,[(1-k)/(1+k)]^{1/2}$, $z = -a\,[(1+k)/(1-k)]^{1/2}$, $\rho = 0$, and we shall construct harmonic functions which are regular inside this bicircular quartic. The condition that $\rho^{-1/2}\,UV$ remain bounded on the axis of rotation inside $v = v_0$ entails the conditions that $[\mathrm{sn}\,(u + i\mathbf{K}', k)]^{1/2}\,U(u)$ remain bounded on the interval $(0, 2\mathbf{K})$, and $[\mathrm{sn}\,(iv, k)]^{1/2}\,V(v)$ remain bounded on the interval $(v_0, \mathbf{K}')$.

Now, the differential equation (7) is Lamé's equation with $n = m - \frac{1}{2}$ and $z = u + i\mathbf{K}'$. Solutions for which $(\mathrm{sn}\ z)^{1/2}\,U(u)$ is bounded at $z = i\mathbf{K}'$ and $z = 2\mathbf{K} + i\mathbf{K}'$ exist if and only if $h = c^r_{m-1/2}(k^2)$, and the only such solution is

$$U(u) = F^r_{m-1/2}(u + i\mathbf{K}', k^2) \qquad r = 0, 1, 2, \ldots,$$

In equation (8) we now have $h = c^r_{m-1/2}(k^2)$ so that one solution of this equation is

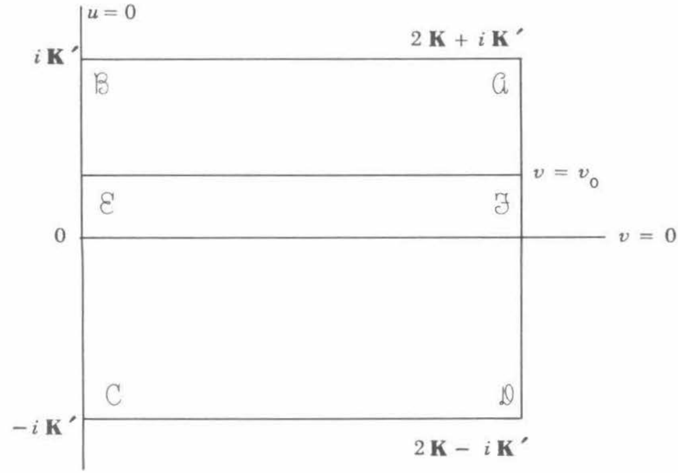
$$V(v) = F^r_{m-1/2}(iv, k^2).$$

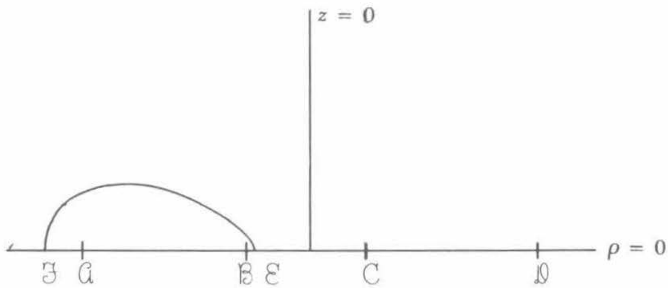
Moreover, this solution has the property that $[\mathrm{sn}\,(iv, k)]^{1/2}\,V(v)$ is bounded at $v = \mathbf{K}'$, and is determined (up to a constant factor) by this property. Then equation 15.1 (24) shows that the only normal solutions of Laplace's equation in the curvilinear coordinates introduced by (3) are of the form

$$(9)\quad W_{m,r} = \left(\frac{c_1 d_2 + c_2 d_1}{s_1}\right)^{1/2} F^r_{m-1/2}(u + i\mathbf{K}', k^2)$$

$$\times F^r_{m-1/2}(iv, k^2)\,e^{\pm im\phi}$$

$$m = 0, 1, 2, \ldots, \qquad r = 0, 1, 2, \ldots$$




The mapping (5)

Other potential problems in coordinates of confocal cyclides of revolution may be handled in a similar manner. In view of a certain lack of clarity in the literature of this subject it deserves mention that none of the boundary value problems mentioned in sec. 15.1.3, in fact no known boundary value problem in the coordinates introduced by Wangerin, leads to algebraic Lamé functions (although such functions exist for certain values of $h$ since $n+\frac{1}{2}$ is an integer). With the exception of the harmonics of a flat ring which were discussed by Poole (1929, 1930) and shown by him to depend on periodic Lamé functions, all the other boundary value problems of sec. 15.1.3 involve finite Lamé functions (i.e. Lamé-Wangerin functions).

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# CHAPTER XVI

# MATHIEU FUNCTIONS, SPHEROIDAL AND ELLIPSOIDAL WAVE FUNCTIONS

## 16.1. Introduction

The functions to be discussed in this chapter arise when the wave equation $\Delta W + \kappa^2 W = 0$ is solved by separation of variables in certain systems of curvilinear coordinates. For the general problem of separation of variables in the wave equation and in related partial differential equations see the literature quoted in sec. 15.1.

For Mathieu functions there is a standard work, by McLachlan (1947), which contains many applications and a bibliography. A book on the theory and applications of Mathieu functions and spheroïdal wave functions, by Meixner and Schäfke, is in preparation. A monograph by Strutt (1932) summarizes the theory of all the functions discussed in this chapter, indicates their applications, and gives an extensive list of references. A supplement to this list was also published by Strutt (1935). For Mathieu functions see also Whittaker and Watson (1927, Chapter XIX).

In the present chapter we shall give a brief description of the principal properties of the functions concerned, and references to the newer literature. For a more detailed presentation of these functions, and for the older literature, see the works mentioned above. In the sections on Mathieu functions we shall follow McLachlan's book, and in the sections on spheroidal wave functions, Meixner's papers. Very little is known about ellipsoidal wave functions, and what there is, is summarized in Strutt's monograph.

### 16.1.1. Coordinates of the elliptic cylinder

We introduce curvilinear coordinates $u$, $v$ in place of the Cartesian coordinates $x$, $y$ by means of the equations

(1) $x = c \cosh u \cos v, \quad y = c \sinh u \sin v$

where $c$ is a positive constant. In the $x,y$-plane, the curves $u$ = const. form a confocal family of ellipses, and the curves $v$ = const., a confocal family of hyperbolas, the foci of the confocal system being the points $(\pm c, 0)$. Each curve $v$ = const. is one quarter of a hyperbola, and we obtain the whole $x,y$-plane if we take the ranges of $u$ and $v$ as $0 \leq u < \infty$, $0 \leq v < 2\pi$. $v = 0$ and $v = 2\pi$ are the same curve (that portion of the $x$-axis from $x = c$ to $x = +\infty$). The curve $u = 0$ is a degenerate ellipse (the segment $-c \leq x \leq c$ covered twice) which acts as a branch-cut, the points $u = 0$, $v = v_1$ and $u = 0$, $v = 2\pi - v_1$ being identical. In the $x,y,z$-space we have correspondingly confocal families of elliptic and hyperbolic cylinders.

In the coordinates introduced by (1),

$$(2) \quad \Delta W + \kappa^2 W = \frac{2c^{-2}}{\cosh(2u) - \cos(2v)} \left( \frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} \right) + \frac{\partial^2 W}{\partial z^2} + \kappa^2 W = 0;$$

and if there are normal solutions of the form

$$(3) \quad W = U(u)\, V(v)\, Z(z),$$

the functions $U$, $V$, $Z$ must satisfy the ordinary differential equations

$$(4) \quad \frac{d^2 U}{du^2} - [h - 2\theta \cosh(2u)]U = 0$$

$$(5) \quad \frac{d^2 V}{dv^2} + [h - 2\theta \cos(2v)]V = 0$$

$$(6) \quad \frac{d^2 Z}{dz^2} + l^2 Z = 0.$$

Here $h$, $\theta$, and $l$ are separation constants, $h$ is arbitrary while

$$(7) \quad \kappa^2 = l^2 + 4c^{-2}\theta.$$

Equation (5) is known as *Mathieu's equation*; (4) may be reduced to Mathieu's equation by an imaginary change of variable, and is known as the *modified Mathieu equation*.

For a wave function $W$ which is continuous, and has a continuous derivative, on an elliptic cylinder $u = u_0$, we must have $V(2\pi) = V(0)$, $V'(2\pi) = V'(0)$. Since $2\pi$ is a period of the coefficients of (5), it follows that $V(v)$ must be a periodic function of $v$ with period $2\pi$. It will be seen later that for a given $\theta$ there is an infinite sequence of characteristic values of $h$ for which such periodic solutions exist: they are called *Mathieu functions*. If $V(v)$ is a mod $2\pi$ periodic solution of (5), so are $V(-v)$ and $V(v) \pm V(-v)$ so that we may restrict ourselves to the consideration of Mathieu functions which are even or odd functions of $v$.

Suppose now that $W$ is continuous, and has a continuous gradient, inside the elliptic cylinder $u = u_0$. Since $u=0$, $v=v_1$ and $u=0$, $v=2\pi-v_1$ represent the same point on the opposite sides of the branch-cut, we must have

$$U(0)\,V(v_1) = U(0)\,V(2\pi - v_1), \qquad U'(0)\,V(v_1) = -U'(0)\,V(2\pi - v_1)$$

for $0 \le v_1 < 2\pi$. If $V(v)$ is an even Mathieu function then $V(2\pi - v_1) = V(-v_1) = V(v_1)$, the first of these conditions is always satisfied, and the second demands that $U'(0) = 0$. From (4) it then follows that $U(u)$ is an even function of $u$, so that $U(u) = V(iu)$ up to a constant factor. Similarly, if $V(v)$ is an odd function of $v$, then $U(u)$ must be an odd function of $u$, and again $U(u) = V(iu)$. The solution of (4) thus determined is called a *modified Mathieu function of the first kind*.

For a wave function $W$ which is continuous, and has a continuous gradient, outside an elliptic cylinder $u = u_0$, usually the behavior at infinity is prescribed (for instance, by Sommerfeld's radiation condition). Now, for large values of $u$,

$$\rho = (x^2 + y^2)^{1/2} = c\,[(\cosh u \cos v)^2 + (\sinh u \sin v)^2]^{1/2}$$

is approximately $\frac{1}{2} c e^u$, and those solutions of (4) which behave asymptotically like $\exp(\frac{1}{2} i \kappa c e^u)$ or $\exp(-\frac{1}{2} i \kappa c e^u)$ are called *modified Mathieu functions of the third kind*.

### 16.1.2. Prolate spheroidal coordinates

We now introduce *prolate spheroidal coordinates* $u, v, \phi$ by the equations

$$(8) \quad x = c \sinh u \sin v \cos\phi, \qquad y = c \sinh u \sin v \sin\phi,$$
$$z = c \cosh u \cos v$$

where $c$ is a positive constant. The surfaces $u$ = const. form a confocal system of prolate spheroids, and the surfaces $v$ = const., a confocal system of two-sheeted hyperboloids, the foci of the confocal system being the points $x = y = 0$, $z = \pm c$. The respective ranges of $u$, $v$, $\phi$ are: $0 \le u < \infty$, $0 \le v \le \pi$, $0 \le \phi < 2\pi$. The surfaces $\phi$ = const. are meridian planes, $\phi = 0$ and $\phi = 2\pi$ being the same. $u = 0$ is a degenerate ellipsoid which reduces to the segment $x = y = 0$, $-c \le z \le c$, and $v = 0$ and $v = \pi$

are the two halves of the degenerate hyperboloid of the system reducing respectively to $x = y = 0$, $z \geq c$ and $x = y = 0$, $z \leq -c$. Thus, the entire axis of revolution is a singular line of the coordinate system.

In the coordinates introduced by (8),

$$(9)\quad \Delta W + \kappa^2 W = \frac{c^{-2}}{(\cosh u)^2 - (\cos v)^2}\left(\frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} + \operatorname{ctnh} u \frac{\partial W}{\partial u} + \operatorname{ctn} v \frac{\partial W}{\partial v}\right) + \frac{1}{(c \sinh u \sin v)^2}\frac{\partial^2 W}{\partial \phi^2} + \kappa^2 W = 0,$$

and if there are normal solutions of the form

$$(10)\quad W = U(u)\, V(v)\, e^{\pm im\phi},$$

the functions $U$, $V$ must satisfy the ordinary differential equations

$$(11)\quad \frac{d^2U}{du^2} + \operatorname{ctnh} u \frac{dU}{du} - [h - (\kappa c \sinh u)^2 + (m \operatorname{csch} u)^2]U = 0$$

$$(12)\quad \frac{d^2V}{dv^2} + \operatorname{ctn} v \frac{dV}{dv} + [h + (\kappa c \sin v)^2 - (m \csc v)^2]V = 0$$

where $h$ is again a separation constant. Equation (12) will be called the trigonometric form of the equation of spheroidal wave functions: (11) may be reduced to (12) by an imaginary change of the variable.

For a wave function $W$ which is continuous inside or outside a spheroid $u = u_0$, $W$ must be a periodic function of $\phi$ with period $2\pi$, and hence $m$ in (10) must be an integer. Also, $W$ must be bounded on ellipsoids $u =$ const., that is to say, $V(v)$ must be a solution of (12) which is bounded for $0 \leq v \leq \pi$. As in the case of Legendre's equation 3.1(2) to which (12) reduces when $\kappa = 0$, such solutions exist only for certain characteristic values of $h$: the bounded solutions of (12) are called *spheroidal wave functions*. If $W$ is to be continuous inside a spheroid $u = u_0$ then it must be bounded on the degenerate spheroid $u = 0$; this determines the choice of $U$ and shows that $U(u)$ is a constant multiple of $V(iv)$, that is to say $U(u)$ is a *modified spheroidal wave function of the first kind*. On the other hand, if $W$ is a wave function regular outside a spheroid $u = u_0$ then usually its behavior at infinity is prescribed to be asymptotically that of $r^{-1} \exp(\pm i\kappa r)$ where

$$r = (x^2 + y^2 + z^2)^{1/2} = c\,[(\sinh u \sin v)^2 + (\cosh u \cos v)^2]^{1/2}$$

is approximately $\frac{1}{2} c e^u$ when $u$ is large. The solutions of (11) determined by their behavior at infinity are *modified spheroidal wave functions of the third kind*. The solutions of (11), with $h$ having one of its characteristic values, should more precisely be called *modified wave functions of the prolate spheroid*.

### 16.1.3. Oblate spheroidal coordinates

*Oblate spheroidal coordinates* $u$, $v$, $\phi$ are introduced by the equations

$$(13)\quad x = c \cosh u \sin v \cos \phi, \quad y = c \cosh u \sin v \sin \phi,$$
$$z = c \sinh u \cos v$$

where $c$ is a positive constant. The surfaces $u = \text{const.}$ form a confocal family of oblate spheroids, the surfaces $v = \text{const.}$, a confocal family of one-sheeted hyperboloids, and the surfaces $\phi = \text{const.}$ are meridian planes. The focal circle of the confocal system is the circle $x^2 + y^2 = c^2$, $z = 0$. The respective ranges of $u$, $v$, $\phi$ are: $0 \le u < \infty$, $0 \le v \le \pi$, $0 \le \phi < 2\pi$, $\phi = 0$ and $\phi = 2\pi$ being the same meridian plane. $u = 0$ is a degenerate ellipsoid which covers the area inside the focal circle twice. $v = 0$ and $v = \pi$ are two halves of a degenerate hyperboloid reducing respectively to the positive and negative $z$-axis, and $v = \frac{1}{2}\pi$ is a degenerate hyperboloid which lies in the plane $z = 0$ and covers the area outside the focal circle twice. Thus, the entire $x,y$-plane is a singular surface of the coordinate system.

In the coordinates introduced by (13),

$$(14)\quad \Delta W + \kappa^2 W = \frac{c^{-2}}{(\cosh u)^2 - (\sin v)^2} \left( \frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} + \tanh u \frac{\partial W}{\partial u} + \operatorname{ctn} v \frac{\partial W}{\partial v} \right) + \frac{1}{(c \cosh u \sin v)^2} \frac{\partial^2 W}{\partial \phi^2} + \kappa^2 W = 0,$$

and if there are normal solutions of the form

$$(15)\quad W = U(u)\, V(v)\, e^{\pm i m \phi},$$

the functions $U$ and $V$ must satisfy the ordinary differential equations

$$(16)\quad \frac{d^2 U}{du^2} + \tanh u \frac{dU}{du} - [h - (\kappa c \cosh u)^2 - (m \operatorname{sech} u)^2] U = 0$$

$$(17)\quad \frac{d^2 V}{dv^2} + \operatorname{ctn} v \frac{dV}{dv} + [h - (\kappa c \sin v)^2 - (m \csc v)^2] V = 0$$

where $h$ is again a separation constant. Equation (17) is the differential equation of spheroidal wave functions with $\kappa^2 c^2$ replaced by $-\kappa^2 c^2$; (16) may be reduced to (17) by the substitution $u = i(v - \frac{1}{2}\pi)$.

As in sec. 16.1.2, $m$ must be an integer, $V$ must be a spheroidal wave function, and $h$ must have one of its characteristic values. The solutions of (16) may be called *modified wave functions of the oblate spheroid*, and it should be noted that the modification appropriate to oblate spheroids, $u = iv - \frac{1}{2}\pi i$, differs from that appropriate to prolate spheroids, $u = iv$. Since $V(\pi - v)$, $V(v) \pm V(\pi - v)$ are also spheroidal wave functions, we may take $V(v)$ to be an even or odd function of $v - \frac{1}{2}\pi$. For a wave function which is regular inside a spheroid $u = u_0$, a consideration similar to that given in sec. 16.1.1 shows that continuity across the degenerate spheroid of the coordinate system (where the points $u = 0$, $v = v_1$ and $u = 0$, $v = \pi - v_1$ coincide) demands that $U(u)$ be an even or odd function of $u$ according as $V(v)$ is an even or odd function of $v - \frac{1}{2}$, that is to say, that $U(u) = V(iv - \frac{1}{2}\pi i)$: we call these solutions of (16) *modified spheroidal wave functions of the first kind*. Wave functions for the exterior of an oblate spheroid are determined in terms of their behavior at $u = \infty$, and the functions $U$ involved are *modified spheroidal wave functions of the third kind*.

### 16.1.4. Ellipsoidal coordinates

We define *ellipsoidal coordinates* $\alpha$, $\beta$, $\gamma$ by 15.1(8) where $a > b > c > 0$, and $k$ is given by 15.1(6). For the description of the coordinate surfaces and for the ranges of $\alpha$, $\beta$, $\gamma$ see sec. 15.1.1. It is seen from 15.1(9) that in ellipsoidal coordinates $\alpha$, $\beta$, $\gamma$ the partial differential equation $\Delta W + \kappa^2 W = 0$ is

$$(18)\quad [(\operatorname{sn}\gamma)^2 - (\operatorname{sn}\beta)^2] \frac{\partial^2 W}{\partial \alpha^2} + [(\operatorname{sn}\alpha)^2 - (\operatorname{sn}\gamma)^2] \frac{\partial^2 W}{\partial \beta^2}$$

$$+ [(\operatorname{sn}\beta)^2 - (\operatorname{sn}\alpha)^2] \frac{\partial^2 W}{\partial \gamma^2} + (a^2 - b^2) k^2 \kappa^2 [(\operatorname{sn}\alpha)^2$$

$$- (\operatorname{sn}\beta)^2] [(\operatorname{sn}\beta)^2 - (\operatorname{sn}\gamma)^2] [(\operatorname{sn}\gamma)^2 - (\operatorname{sn}\alpha)^2] W = 0,$$

and if there are normal solutions of the form

(19) $W = A(\alpha)\, B(\beta)\, C(\gamma)$,

then the functions $A, B, C$ must satisfy the ordinary differential equations

(20) $$\frac{d^2 A}{d\alpha^2} + [h - l\,(\operatorname{sn}\alpha)^2 + (a^2 - b^2)\,k^2\kappa^2(\operatorname{sn}\alpha)^4]\,A = 0$$

(21) $$\frac{d^2 B}{d\beta^2} + [h - l\,(\operatorname{sn}\beta)^2 + (a^2 - b^2)\,k^2\,\kappa^2\,(\operatorname{sn}\beta)^4]\,B = 0$$

(22) $$\frac{d^2 C}{d\gamma^2} + [h - l\,(\operatorname{sn}\gamma)^2 + (a^2 - b^2)\,k^2\,\kappa^2(\operatorname{sn}\gamma)^4]\,C = 0,$$

where $h$ and $l$ are separation constants. These three equations are of the same form, they differ only in the ranges of their independent variables. The common form is known as the *equation of ellipsoidal wave functions* or *Lamé's wave equation*. Solutions of these equations which satisfy appropriate boundary conditions are known as *ellipsoidal wave functions* or *Lamé wave functions*. When $\kappa = 0$, the equations reduce to Lamé's equation (Chapter XV) and Lamé wave functions reduce to Lamé functions.

If $W$ is continuous and has continuous derivatives inside or outside an ellipsoid $u = u_0$, the boundary conditions to be imposed upon $B$ and $C$ are those obtained in sec. 15.1.1. These boundary conditions determine characteristic values of $h$ and $l$, and also the corresponding Lamé wave functions of the first kind. For a wave function regular inside an ellipsoid $u = u_0$, the boundary condition for $A$ is also the same as in sec. 15.1.1 so that $A$ is a Lamé wave function of the first kind. For a wave function regular outside an ellipsoid the asymptotic behavior at infinity, i.e., near $\alpha = i\mathbf{K}'$ is given, and $A$ can be expressed as a Lamé wave function of the third kind.

MATHIEU FUNCTIONS

### 16.2. The general Mathieu equation and its solutions

We shall adopt

(1) $$\frac{d^2u}{dz^2} + [h - 2\,\theta\cos(2z)]\,u = 0$$

as the standard form of Mathieu's equation. This is the form used by Ince (1932 and other papers) and several other authors. There is no generally accepted standard form. Whittaker and Watson (1927, Chapter XIX) put $h=a$, $\theta=-8q$, Stratton et al. (1941) have $h=b-\frac{1}{2}c^2$, $4\theta=c^2$, Jahnke-Emde (1938) put $h=4a$, $\theta=8q$, and in the tables prepared by the National Bureau of Standards (1951) $h=b-\frac{1}{2}s$, $\theta=\frac{1}{4}s$.

Ince (1923) also studied the equation

$$\frac{d^2u}{dz^2}+[h-2\theta\cos(2z)-\nu(\nu-1)(\csc z)^2]\,u=0$$

which he called the *associated Mathieu equation*. Since the substitution $u=(\sin z)^{\frac{1}{2}}\,v$ carries this equation into

$$\frac{d^2v}{dz^2}+\operatorname{ctn} z\,\frac{dv}{dz}+\left[h-\tfrac{1}{4}-2\theta\cos(2z)-\frac{(\nu-\frac{1}{2})^2}{(\sin z)^2}\right]v=0$$

which is the differential equation of spheroidal wave functions, the associated Mathieu equation will not be discussed here.

In this section we shall consider both $h$ and $\theta$ as given (real or complex) constants. Equation (1) is then referred to as the *general Mathieu equation* to distinguish it from the equation of Mathieu functions in which only $\theta$ is prescribed while $h$ has one of its characteristic values. For the sake of brevity, we shall call (1) Mathieu's equation.

With

(2) $x=(\sin z)^2$

we obtain

(3) $$4x(1-x)\frac{d^2u}{dx^2}+2(1-2x)\frac{du}{dx}+(h-2\theta+4\theta x)\,u=0$$

and we shall call this equation the *algebraic Mathieu equation*. This algebraic form and related equations were used in the investigations of Lindemann, Stieltjes, and others. The algebraic Mathieu equation has two regular singular points, at $x=0$ and $x=1$, both with exponents 0 and $\frac{1}{2}$, and one irregular singular point at infinity. Because of this irregular singularity, (3) is comparatively untractable, although it can be used to derive certain series expansions of the solutions, both series in powers of $x$ or $1-x$, and series of hypergeometric functions. The equation is a limiting case of Heun's equation (sec. 15.3).

Mathieu's equation (1) is a differential equation with periodic coefficients. From the general theory of such equations (Ince 1927, p. 381ff., Poole 1936, p. 178ff.) it follows that (1) has a solution of the form

(4) $e^{\mu z}\,P(z)$

where $P(z)$ is a periodic function with period $\pi$, and $\mu$ is a constant, called the *characteristic exponent*, which depends on $h$ and $\theta$(Floquet's theorem). Clearly

(5) $e^{-\mu z} P(-z)$

is also a solution of (1). In general (4) and (5) are linearly independent and form a fundamental system of solutions of (1). The only exception arises when $i\mu$ is an integer: this is the case of periodic Mathieu functions which will be discussed in sections 16.4 to 16.8.

Solutions of the form (4) and (5) are sometimes called *solutions of the first kind*. Other significant solutions of Mathieu's equation are those which vanish when $z \to i\infty$ or $z \to -i\infty$: such solutions are called *solutions of the third kind*.

There are several methods for the determination of $\mu$. We shall outline some of these, and refer the reader to Blanch (1946) and to Chapters IV and V of McLachlan's book for further details and for a description of numerical methods.

Poincaré bases the determination of $\mu$ on the two solutions, $u_1$ and $u_2$, of (1) defined by the initial conditions

(6) $u_1(0) = 1, \quad u_1'(0) = 0; \qquad u_2(0) = 0, \quad u_2'(0) = 1.$

These two solutions are linearly independent, their Wronskian is unity, and $u_1$ $[u_2]$ is an even [odd] function of $z$. If $P(0) \neq 0$ we have

$$u_1(z) = \frac{e^{\mu z} P(z) + e^{-\mu z} P(-z)}{2P(0)},$$

and if $P'(0) + \mu P(0) \neq 0$ we have

$$u_2(z) = \frac{e^{\mu z} P(z) - e^{-\mu z} P(-z)}{2[P'(0) + \mu P(0)]}$$

At least one of these two expressions is meaningful. We now differentiate $u_2$, and put $z = \pi$ in both $u_1$ and $u_2'$: since $P(\pm\pi) = P(0)$, $P'(\pm\pi) = P'(0)$, we obtain

(7) $\cosh(\mu\pi) = u_1(\pi) = u_2'(\pi)$

It is evident from (7) that $\mu$ is determined up to its sign and an integral multiple of $2i$. (7) can be used for the determination of $\mu$ if $u_1(\pi)$ or $u_2'(\pi)$ can be evaluated with sufficient accuracy. (See also sec. 16.3.)

Hill expands the solution (4) in the form

$$(8) \quad \sum_{-\infty}^{\infty} c_n e^{(\mu+2ni)z}$$

Substitution in (1) leads to the recurrence relations

$$(9) \quad -\theta c_{n-1} + [h + (\mu + 2ni)^2] c_n - \theta c_{n+1} = 0 \qquad n = 0, \pm 1, \pm 2, \dots$$

for the coefficients $c_n$. We write (9) in the form

$$(10) \quad c_n + \gamma_n(\mu)(c_{n-1} + c_{n+1}) = 0 \qquad n = 0, \pm 1, \pm 2, \dots$$

where

$$(11) \quad \gamma_n = \gamma_n(\mu) = \theta/[(2n - \mu i)^2 - h]$$

The (infinite) determinant of the system (10) is

$$(12) \quad \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \gamma_{-2}(\mu) & 0 & 0 & 0 & \cdots \\ \cdots & \gamma_{-1}(\mu) & 1 & \gamma_{-1}(\mu) & 0 & 0 & \cdots \\ \cdots & 0 & \gamma_0(\mu) & 1 & \gamma_0(\mu) & 0 & \cdots \\ \cdots & 0 & 0 & \gamma_1(\mu) & 1 & \gamma_1(\mu) & \cdots \\ \cdots & 0 & 0 & 0 & \gamma_2(\mu) & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \Delta(\mu)$$

and $\mu$ is determined by the equation $\Delta(\mu) = 0$. The infinite determinant (12) is clearly absolutely convergent, and it represents a meromorphic function of $\mu$. This function has simple poles at $\mu = \pm i(h^{1/2} + 2n)$, $n = 0, \pm 1, \pm 2, \dots$, Since $\gamma_n(\mu + 2ki) = \gamma_{n+k}(\mu)$, $k$ integer, and $\gamma_n(-\mu) = \gamma_{-n}(\mu)$, we see that $\Delta(\mu)$ is an even periodic function of period $2i$. Thus,

$$(13) \quad \Delta(\mu) - \frac{C}{\cosh(\mu\pi) - \cos(\pi h^{1/2})}$$

is an even periodic meromorphic function of $\mu$. If $C$ is determined so that (13) has no pole at $\mu = ih^{1/2}$, then (13) will have no pole whatsoever, and hence will be a constant. Since $\Delta(\mu) \to 1$ as $\mu \to \infty$, the value of this constant is unity. To determine $C$, we put $\mu = 0$ and obtain

(14) $\Delta(\mu) = 1 - \dfrac{[1-\Delta(0)]\,[1-\cos(h^{1/2}\pi)]}{\cosh(\mu\pi) - \cos(h^{1/2}\pi)}$

$$= \frac{\cosh(\mu\pi) - 1 + \Delta(0)\,[1-\cos(h^{1/2}\pi)]}{\cosh(\mu\pi) - \cos(h^{1/2}\pi)}$$

Since $\mu$ is determined by the equation $\Delta(\mu) = 0$, we have

(15) $\cosh(\mu\pi) = 1 + 2\Delta(0)\,[\sin(\tfrac{1}{2}h^{1/2}\pi)]^2.$

For further work on infinite determinants arising in connection with Mathieu's equation and similar differential equations see Magnus (1953).

If $h$ and $\theta$ are both real, it is seen from (7) or (15) that $\cosh(\mu\pi)$ is also real. If $-1 < \cosh(\mu\pi) < 1$, then $\mu$ is imaginary, $\mu i$ is not an integer, and (4) and (5) show that every solution of Mathieu's equation is bounded on the real $z$-axis. *Stable regions* are those regions of the $h,\theta$-plane in which $-1 < \cosh(\mu\pi) < 1$. If $\cosh(\mu\pi) > 1$, then $\mu$ may be taken as real (and non-zero), if $\cosh(\mu\pi) < -1$, then $\mu - i$ may be taken as real (and non-zero): in either case it can be seen from (4) and (5) that Mathieu's equation has no solution bounded on the real axis. Those regions of the $h,\theta$-plane in which $\cosh(\mu\pi) > 1$ or $\cosh(\mu\pi) < -1$ are called *unstable regions*. Stable and unstable regions are separated by curves along which $\cosh(\mu\pi) = \pm 1$, one solution of Mathieu's equation is bounded (and periodic), and the general solution is unbounded: for this exceptional case see sections 16.4 to 16.8. For *stability charts* showing stable and unstable regions of the $h,\theta$-plane see Strutt (1932, p. 24), McLachlan (1947, p. 40, 41), and p. xliv, xlv of the NBS tables. For computation of stability charts see also Blanch (1946), Schäfke (1950).

Most numerical methods for the solution of Mathieu's equation with moderate values of $\lambda$ and $\theta$ are based on the recurrence relation (9) or on some variant of it. From (9),

$$\frac{c_n}{c_{n-1}} = \frac{\theta}{h - (2n - i\mu)^2 - \theta c_{n+1}/c_n}$$

$$= \frac{-\theta(2n - i\mu)^{-2}}{1 - h(2n - i\mu)^{-2} + \theta(2n - i\mu)^{-2}\, c_{n+1}/c_n}$$

and repeated application of this, as in sec. 15.3, leads to a convergent infinite continued fraction, $R_n$, say, so that

(16) $\dfrac{c_n}{c_{n-1}} = R_n(\mu).$

On the other hand, from (9) we also have

$$\frac{c_n}{c_{n+1}} = \frac{-\theta(2n - i\mu)^{-2}}{1 - h(2n - i\mu)^{-2} + \theta(2n - i\mu)^{-2}\, c_{n-1}/c_n}$$

and repeated application of this leads to

$$(17)\quad \frac{c_n}{c_{n+1}} = L_n(\mu) = R_{-n}(-\mu)$$

where $L_n(\mu)$ is again an infinite continued fraction. The equation for the determination of $\mu$ is

$$(18)\quad L_0(\mu)\, R_1(\mu) = 1,$$

and in the course of computing $\mu$ from (18) all ratios (16) and (17) are automatically obtained so that

$$(19)\quad c_n = c_0 R_1(\mu)\, R_2(\mu) \ldots R_n(\mu) \qquad n = 1, 2, 3, \ldots$$

$$(20)\quad c_{-n} = c_0 L_{-1}(\mu)\, L_{-2}(\mu) \ldots L_{-n}(\mu) \qquad n = 1, 2, 3, \ldots$$

From (16) and (17)

$$(21)\quad \lim_{n\to\infty} \frac{n^2 c_n}{c_{n-1}} = \lim_{n\to-\infty} \frac{n^2 c_n}{c_{n+1}} = -\frac{\theta}{4}$$

so that the series (8) converges absolutely and uniformly in any region in which $e^{\pm iz}$ is bounded, for instance in any horizontal strip of the complex $z$-plane.

In a stable region $\mu = i\rho$, $\rho$ is real, so are all the $c_n$ provided $c_0$ is taken as real, and from (8) we have two linearly independent real solutions

$$(22)\quad \sum_{-\infty}^{\infty} c_n \cos[(\rho + 2n)z], \qquad \sum_{-\infty}^{\infty} c_n \sin[(\rho + 2n)z].$$

In an unstable region either $\mu$ or $\mu - i$ is real, in either case (8) is a real solution, and two linearly independent real solutions are given by

$$(23)\quad \sum_{-\infty}^{\infty} c_n e^{(\mu+2ni)z}, \qquad \sum_{-\infty}^{\infty} c_n e^{-(\mu+2ni)z}.$$

While (8) is the best expansion when $z$ is real, other expansions lead to more rapidly convergent series for complex values of $z$, and they are also suitable for representing solutions of the third kind.

Erdélyi (1942) puts

$$(24)\quad \phi_\nu(z) = \left[ e^{i\pi} \frac{\cos(z-\beta)}{\cos(z+\beta)} \right]^{\nu/2} J_\nu \{ 2[\theta \cos(z-\beta)\cos(z+\beta)]^{1/2} \}$$

where $\beta$ is an arbitrary fixed real or complex number. By a straightforward computation using the recurrence relations and differentiation formulas for Bessel functions

$$(25)\quad \frac{d^2\phi_\nu}{dz^2} - 2\theta\phi_\nu \cos(2z) = -\theta\phi_{\nu-2} - \nu^2\phi_\nu - \theta\phi_{\nu+2}$$

and it follows that

$$(26)\quad \sum_{-\infty}^{\infty} c_n \phi_{2n-i\mu}(z)$$

is a formal solution of Mathieu's equation provided the coefficients $c_n$ satisfy (9), i.e. are the same as the $c_n$ of (8). From the asymptotic formulas for Bessel functions we have

$$(27)\quad \lim_{n\to\infty} \frac{\phi_{2n-i\mu-2}}{n^2\phi_{2n-i\mu}} = \lim_{n\to-\infty} \frac{\phi_{2n-i\mu}}{n^2\phi_{2n-i\mu+2}} = \frac{-4}{\theta[\cos(z-\beta)]^2}$$

and (21) and (27) show that (26) is convergent, and represents a solution, when $|\cos(z-\beta)| > 1$. The region of convergence consists of two disjoint parts, one entirely in the half-plane $\operatorname{Im}(z-\beta) > 0$, and the other in the half-plane $\operatorname{Im}(z-\beta) < 0$. From (24), $\phi_\nu = [\cos(z-\beta)]^\nu$ times an entire function of $z$. As $z$ changes to $z + 2\pi$ in the half-plane $\operatorname{Im}(z-\beta) > 0$, $\cos(z-\beta)$ encircles the origin in the negative sense, and it follows that, in its region of convergence in the half-plane $\operatorname{Im}(z-\beta) > 0$, (26) represents the solution of the first kind (5). By a similar consideration, in the region of convergence in the half-plane $\operatorname{Im}(z-\beta) < 0$, (26) represents (4).

Particular forms of (26) are obtained for $\beta = 0$ and $\beta = \pi/2$. They are, respectively,

$$(28)\quad e^{\frac{1}{2}\pi\mu} \sum_{-\infty}^{\infty} (-1)^n c_n J_{2n-i\mu}(2\theta^{1/2}\cos z)$$

$$(29)\quad \sum_{-\infty}^{\infty} c_n J_{2n-i\mu}(2\theta^{1/2}\, i\sin z).$$

As $\beta \to i\infty$, (8) appears as a limiting form of (26).

We now replace $J_\nu$ by $H_\nu^{(j)}$, $j = 1, 2$ in (24) and call the resulting functions $\psi_\nu^{(j)}$. Since Bessel functions of the first and third kinds satisfy the same recurrence relations and obey the same differentiation formulas,

$$(30)\quad \sum_{-\infty}^{\infty} c_n \psi_{2n-i\mu}^{(j)}$$

will be a formal solution of Mathieu's equation, the $c_n$ being the same as in (8). An investigation of the convergence of (30) by the ratio test shows that convergence obtains if $|\cos(z-\beta)| > 1$ and $|\cos(z+\beta)| > 1$. There is always a region of convergence in the half-plane $\operatorname{Im} z > |\operatorname{Im}\beta|$, and another region of convergence in $\operatorname{Im} z < -|\operatorname{Im}\beta|$. In both these regions, (30) represents solutions of the third kind as can be seen by investigating the asymptotic behavior of (30) as $z \to i\infty$ (see Meixner 1949a). If $|\operatorname{Im}\beta|$ is sufficiently large, more precisely if $\sinh|\operatorname{Im}\beta| > 1$, there is a third region of convergence which includes the entire real $z$-axis and is situated in the strip $|\operatorname{Im} z| < |\operatorname{Im}\beta|$. In this region of convergence (30) represents a solution of the first kind, (4) or (5) according as $\operatorname{Im}\beta$ is positive or negative.

Expansions of solutions of Mathieu's equation in series of products of Bessel functions were introduced by Sieger (1908) and Dougall (1916). In this case we put

$$(31)\quad \phi_{\nu,\lambda}(z) = e^{i\nu\pi} J_{\nu+\lambda}(\theta^{1/2} e^{iz})\, J_\nu(\theta^{1/2} e^{-iz})$$

and obtain, by a straightforward computation,

$$(32)\quad \frac{d^2\phi_{\nu,\lambda}}{dz^2} - 2\theta\phi_{\nu,\lambda}\cos(2z) = -\theta\phi_{\nu-1,\lambda} - (2\nu+\lambda)^2\phi_{\nu,\lambda} - \theta\phi_{\nu+1,\lambda}$$

This relation shows that

$$(33)\quad \sum_{-\infty}^{\infty} c_n \phi_{n,-i\mu}$$

is a formal solution of Mathieu's equation, the coefficients being those determined by (9). Since

$$(34)\quad \lim_{n\to\infty} \frac{n^2\phi_{n+1,-i\mu}}{\phi_{n,-i\mu}} = -\frac{\theta}{4}, \qquad \lim_{n\to-\infty} \frac{\phi_{n,-i\mu}}{\phi_{n+1,-i\mu}} = -e^{-2iz}$$

it follows from (21) that (33) is convergent in the entire $z$-plane. Since (33) is of the form $e^{\mu z}$ times an entire function of $z$, it represents the solution of the first kind (4).

There is a considerable number of series of products of Bessel functions, for instance

$$(35) \quad \sum_{-\infty}^{\infty} c_n H^{(j)}_{n-i\mu}(\theta^{1/2} e^{iz}) J_{-n}(\theta^{1/2} e^{-iz}) \qquad j = 1, 2$$

Further series are modifications and combinations of (33) and (35). See also sections 16.5 and 16.6.

## 16.3. Approximations, integral relations, and integral equations for solutions of the general Mathieu equation

*Approximations for small* $|\theta|$. When $\theta=0$, the two (degenerate) solutions of the first kind of Mathieu's equation 16.2 (1) are exp $(\pm ih^{1/2}z)$ so that $\mu = ih^{1/2}$ in this case. For small values of $|\theta|$ the determinant in 16.2 (15) may be evaluated as

$$(1) \quad \Delta(0) = 1 + \frac{\pi\theta^2}{(1-h)\,h^{1/2}} \operatorname{ctn}\left(\frac{\pi h^{1/2}}{2}\right) + O(\theta^4)$$

so that equation 16.2 (15) becomes

$$(2) \quad \cosh(\mu\pi) = \cos(h^{1/2}\pi) + \frac{\pi\theta^2}{(1-h)\,h^{1/2}} \sin(h^{1/2}\pi) + O(\theta^4)$$

and can be used for the computation of $\mu$. Alternatively, $u_1$ as determined by 16.2 (1) and 16.2 (6) may be expanded in powers of $\theta$,

$$u_1(z) = \sum_{n=0}^{\infty} \theta^n f_n(z)$$

where

$$f_0(z) = \cos(h^{1/2} z),$$

$$f_n(z) = 2h^{-1/2} \int_0^z \cos(2t) \sin[h^{1/2}(z-t)]\, f_{n-1}(t)\, dt$$

$$n = 1, 2, \ldots$$

and then 16.2(7) may be used for the computation of the characteristic exponent $\mu$. Once $\mu$ is known, the coefficients of the expansion 16.2(8) may be computed from the continued fraction, or else $P(z)$ of 16.2(4) may be expanded in powers of $\theta$, and the terms of this expansion may be determined recurrently from 16.2(1).

For another method of approximation for small $|\theta|$ see Whittaker and Watson (1927 sec. 19.7) or Strutt (1932, p. 26).

*Asymptotic forms for large* $|h|$, $|\theta|$. We shall assume that $h$ and $\theta$ are both real.

If $h > 2|\theta|$, we use Liouville's transformation

(3) $$\zeta = \int_0^z [h - 2\theta\cos(2t)]^{1/2}\,dt, \qquad \eta = [h - 2\theta\cos(2z)]^{1/4}\,u$$

to turn Mathieu's equation 16.2(1) into

(4) $$\frac{d^2\eta}{d\zeta^2} + [1 + r(\zeta)]\,\eta = 0$$

where

(5) $$r(\zeta) = \frac{4\theta^2 - 2h\,\theta\sin(2z) + \theta^2[\sin(2z)]^2}{[h - 2\theta\cos(2z)]^3}$$

If $h$ is large, then $r(\zeta)$ is small in comparison with unity,that solution of (4) which corresponds to $u_1$ is approximately a constant multiple of $\cos\zeta$, and 16.2(7) becomes

(6) $$\cosh(\mu\pi) = \cos\{\int_0^\pi [h - 2\theta\cos(2t)]^{1/2}\,dt\} + O(h^{-1/2})$$
$$h \to \infty, \quad 2|\theta| \le h - \epsilon, \quad \epsilon > 0.$$

If $h < -2|\theta|$, we use a slightly different transformation

$$\zeta = \int_0^z [-h + 2\theta\cos(2t)]^{1/2}\,dt, \qquad \eta = [-h + 2\theta\cos(2z)]^{1/4}\,u$$

and obtain again (6). Actually, (6) is valid for arbitrary complex values of $h$, provided $2|\theta| \le |h| - \epsilon$, $\epsilon > 0$.

When $h$ and $\theta$ are real, and $-2\theta < h < 2\theta$, then $r(\zeta)$ as given by (5) is no longer bounded, and for those values of $z$ near to $\tfrac{1}{2}\cos^{-1}[h/(2\theta)]$ certainly not negligible. Also, the integral occurring in (6) is neither real nor imaginary. Strutt (1932, p. 28) states that in this case

(7) $$\cosh(\mu\pi) = \cos\{\mathrm{Re}\int_0^\pi [h - 2\theta\cos(2t)]^{1/2}\,dt\}$$
$$\times \cosh\{\mathrm{Im}\int_0^\pi [h - 2\theta\cos(2t)]^{1/2}\,dt\} + O(h^{-1/2}) \qquad h \to \infty$$

A detailed investigation of the solutions of Mathieu's equation 16.2(1) for large real $h$, $\theta$, and complex $z$, was given by Langer (1934).

*Asymptotic forms for large* $|\sin z|$. The point $x = \infty$ is an irregular singularity of the algebraic form 16.2(3) of Mathieu's equation. There are formal series of the form

$$\exp(\pm 2\theta^{1/2} x^{1/2}) \sum a_n x^{-1/4 - 1/2 n}$$

satisfying 16.2(3); these are called *subnormal solutions* (Ince 1927, sec. 17.53). Although these series diverge, it follows from the general theory of linear differential equations that they represent certain solutions of 16.2(3) asymptotically as $x \to \infty$.

Reversing the transformation 16.2(2) we see that there are formal series

(8) $$\exp(\pm 2\theta^{1/2} \sin z) \sum a_n (\sin z)^{-1/2 - n}$$

which satisfy Mathieu's equation 16.2(1), and that there are certain solutions of Mathieu's equation (solutions of the third kind) which are asymptotically represented by one or the other of the series (8) as $\operatorname{Im} z \to \pm\infty$. Any solution of Mathieu's equation is represented by a linear combination of the two series (8) but the constants involved in that linear combination may be different for different vertical strips of the $z$-plane. See also Dougall (1916) and Whittaker and Watson (1927, sec. 19.8).

For asymptotic expansions of the solutions of the first kind in descending powers of $e^{iz}$ rather than $\sin z$ see Erdélyi (1936, 1938). The asymptotic behavior of the solutions of Mathieu's equation as $\operatorname{Im} z \to \pm\infty$ may also be determined by means of the series of Bessel functions representing the various solutions. The requisite general theorems were proved by Meixner (1949a).

*Integral relations and integral equations.* Let $N(z, \zeta)$ be a nucleus satisfying the partial differential equation

(9) $$\frac{\partial^2 N}{\partial z^2} - 2\theta \cos(2z) N = \frac{\partial^2 N}{\partial \zeta^2} - 2\theta \cos(2\zeta) N,$$

and let

(10) $$g(z) = \int_a^b N(z, \zeta) f(\zeta)\, d\zeta.$$

Then

$$(11)\quad \frac{d^2 g}{dz^2} + [h - 2\theta\cos(2z)]\,g = \int_a^b \left\{\frac{\partial^2 N}{\partial \zeta^2} + [h - 2\theta\cos(2\zeta)]N\right\} f\, d\zeta$$

$$= \left[\frac{\partial N}{\partial \zeta} f - N\frac{df}{d\zeta}\right]_a^b + \int_a^b N\left\{\frac{d^2 f}{d\zeta^2} + [h - 2\theta\cos(2\zeta)]f\right\} d\zeta$$

by repeated integrations by parts. If the nucleus $N$ and the limits of integration, $a$ and $b$, are chosen so that

$$(12)\quad \left[\frac{\partial N}{\partial \zeta} f - N\frac{df}{d\zeta}\right]_{\zeta=a}^{\zeta=b} = 0,$$

then (11) shows that $g(z)$ will be a solution of Mathieu's equation provided that $f(z)$ is a solution of that equation.

The case $\cosh(\mu\pi) = \pm 1$ is that of periodic Mathieu functions and will be discussed later (see sections 16.4, 16.8). In this section we assume that $\cosh(\mu\pi) \neq \pm 1$ so that the two solutions of the first kind

$$(13)\quad u_0(z) = e^{\mu z} P(z), \qquad u_0(-z) = e^{-\mu z} P(-z)$$

are linearly independent. We know from (8) that

$$(14)\quad u_0(z) = c_1 (\sin z)^{-1/2} \exp(2\theta^{1/2} \sin z)\,[1 + O(|\sin z|^{-1})]$$
$$+ c_2 (\sin z)^{-1/2} \exp(-2\theta^{1/2} \sin z)\,[1 + O(|\sin z|^{-1})]$$

as $z \to \pm i\infty$ where the constants $c_1$ and $c_2$ may change as we move from one vertical strip to another one.

In (10) we shall choose $f(\zeta) = u_0(\zeta)$ and

$$(15)\quad N(z, \zeta) = \exp[2\theta^{1/2}(\sin z \sin\zeta \sin\beta + i\cos z\cos\zeta\cos\beta)]$$

where $\beta$ is a fixed real or complex number: (15) satisfies (9). The asymptotic behavior of the expression in the square brackets in (12) can now be investigated by means of (14) and (15) when $\mathrm{Im}\,\zeta \to \pm\infty$. Set

$$(16)\quad \arg\{\theta^{1/2}[\cos(z-\beta)+1]\} = a_1, \qquad \arg\{\theta^{1/2}[\cos(z-\beta)-1]\} = a_2$$
$$\arg\{\theta^{1/2}[\cos(z+\beta)+1]\} = a_3, \qquad \arg\{\theta^{1/2}[\cos(z+\beta)-1]\} = a_4.$$

It turns out that

$$\frac{\partial N}{\partial \zeta} u_0 - N \frac{du_0}{d\zeta} \to 0 \quad \text{as} \quad \operatorname{Im} \zeta \to \infty,$$

provided that $\rho = \operatorname{Re} \zeta$ satisfies

(17) $\sin(\rho - \alpha_1) < 0, \quad \sin(\rho - \alpha_2) < 0,$

and that

$$\frac{\partial N}{\partial \zeta} u_0 - N \frac{du_0}{d\zeta} \to 0 \quad \text{as} \quad \operatorname{Im} \zeta \to -\infty$$

provided that $\rho' = \operatorname{Re} \zeta$ satisfies

(18) $\sin(\rho' + \alpha_3) > 0, \quad \sin(\rho' + \alpha_4) > 0.$

The two inequalities (17) are consistent if $\operatorname{Im}(z - \beta) \neq 0$, and the two inequalities (18) are consistent if $\operatorname{Im}(z + \beta) \neq 0$. If $\rho$ is any solution of (17) then also $\rho + 2n\pi$ is a solution where $n$ is integer, and similarly for $\rho'$. This investigation shows that the paths of integration which may be used in (10) are very similar to those occurring in Sommerfeld's integral representations of Bessel functions (see sec. 7.3.5).

Let $\rho$ satisfy (17) and consider

$$g(z) = \int_{\rho + i\infty}^{\rho + 2\pi + i\infty} N(z, \zeta)\, u_0(\zeta)\, d\zeta,$$

the path of integration being like $C_3$ of sec. 7.3.5. Then (12) is satisfied, and $g(z)$ is a solution of Mathieu's equation and hence of the form

(19) $g(z) = C_1\, u_0(z) + C_2\, u_0(-z)$

As $z$ changes into $z + 2\pi$ in the half-plane $\operatorname{Im}(z - \beta) < 0$, $\alpha_1$ and $\alpha_2$, and hence also $\rho$, are increased by $2\pi$.

(20) $g(z + 2\pi) = C_1\, e^{2\mu\pi} u_0(z) + C_2\, e^{-2\mu\pi} u_0(-z) = \int_{\rho+2\pi + i\infty}^{\rho + 4\pi + i\infty}$

In the last integral replace $\zeta$ by $\zeta + 2\pi$, obtaining

(21) $g(z + 2\pi) = \int_{\rho + i\infty}^{\rho+2\pi + i\infty} N(z, \zeta)\, u_0(\zeta + 2\pi)\, d\zeta = e^{2\mu\pi} g(z)$

From the comparison of (20) and (21) it follows that $C_2 = 0$. Hence the *singular integral equation*

(22) $\int_{\rho + i\infty}^{\rho+2\pi+i\infty} N(z, \zeta)\, u_0(\zeta)\, d\zeta = \lambda u_0(z) \qquad \operatorname{Im}(z - \beta) < 0$

is satisfied by the solution of the first kind. The close relation to Sommerfeld's integral representation 7.3(23) of Bessel functions of the first kind is seen if $\beta = 0$ is taken in (22) when that equation becomes

$$(23)\quad u_0(z) = \text{const.} \int_{\rho+i\infty}^{\rho+2\pi+i\infty} \exp(2i\,\theta^{1/2} \cos z \cos \zeta)\, u_0(\zeta)\, d\zeta$$

$$\operatorname{Im} z < 0$$

These integral equations can also be used to elucidate the connection between the various expansions of solutions of the first kind given in sec. 16.2. If 16.2(8) is substituted for $u_0$ under the integral sign in (23), and then 7.3(23) is used, 16.2(28) is obtained, and 16.2(26) is similarly obtained from (22). Thus the interesting fact that all expansions of sec. 16.2 have the same coefficients is a direct consequence of the integral equations satisfied by $u_0(z)$.

Instead of a path of the type $C_3$ of sec. 7.3.5 we may use paths of the type $C_1$ or $C_2$. Let $\rho$, $\rho'$ satisfy (17) and (18), and consider

$$(24)\quad g(z) = \int_{\rho'-i\infty}^{\rho+i\infty} N(z, \zeta)\, u_0(\zeta)\, d\zeta \qquad \operatorname{Im}(z \pm \beta) \neq 0$$

First let us assume that $z$ is confined to the strip $|\operatorname{Im} z| < |\operatorname{Im} \beta|$. As $z$ increases by $2\pi$ in this strip, either both $\rho$ and $\rho'$ increase by $2\pi$ or both $\rho$ and $\rho'$ decrease by $2\pi$ according as $\operatorname{Im} \beta$ is positive or negative. Thus

$$(25)\quad \int_{\rho'-i\infty}^{\rho+i\infty} N(z, \zeta)\, u_0(\zeta)\, d\zeta = \lambda u_0(\pm z) \qquad |\operatorname{Im} z| < |\operatorname{Im} \beta|$$

is another singular integral equation satisfied by $u_0(z)$, and leads to expansions of the form 16.2(30) for solutions of the first kind in the strip $|\operatorname{Im} z| < |\operatorname{Im} \beta|$ (see also sec. 16.2). On the other hand, if $\operatorname{Im} z > |\operatorname{Im} \beta|$, or $\operatorname{Im} z < -|\operatorname{Im} \beta|$, then either $\rho$ increases by $2\pi$ and $\rho'$ decreases by $2\pi$, or *vice versa*, as $z$ increases by $2\pi$. In this case the path of integration in (24) changes its shape as well as its position, and the integral no longer represents a solution of the first kind. From the behavior of $N$ as $\operatorname{Im} z \to \infty$ it follows that

$$(26)\quad u_3(z) = \int_{\rho'-i\infty}^{\rho+i\infty} N(z, \zeta)\, u_0(\zeta)\, d\zeta \qquad \operatorname{Im} z > |\operatorname{Im} \beta|$$

vanishes exponentially as $\operatorname{Im} z \to \infty$, and hence is a solution of the third kind. Integral relations of this kind between solutions of the first kind and those of the third kind lead to expansions like 16.2(30) for solutions of the third kind.

There are also singular integral equations for solutions of the third kind, and integral relations which express a solution of the first kind as an integral involving $u_3$.

## 16.4. Periodic Mathieu functions

If $i\mu$ is an integer, then the solution of the first kind, 16.2(4), is a *periodic function*: $\pi$ is a period of this function if $i\mu$ is an even integer, and $\pi$ is a half-period (i.e., the solution changes its sign when $z$ is increased by $\pi$) if $i\mu$ is an odd integer so that the period in the latter case is $2\pi$. Unless otherwise specified, *periodic* will always mean period $\pi$ or half-period $\pi$. Periodic solutions are required in many applications of Mathieu's equation, and sections 16.4-16.8 will be devoted to periodic Mathieu functions, and to the corresponding solutions of the second and third kinds.

Those curves in the real $h$, $\theta$-plane along which $i\mu$ is an integer are called *characteristic curves*; they divide the $h$, $\theta$-plane into stable and unstable regions (see sec. 16.2). Given $\theta$, those values of $h$ for which periodic solutions exist are called *characteristic values*, and the periodic solutions are called *Mathieu functions* or Mathieu functions of the first kind. No generally accepted definition or notation of Mathieu functions exists. We shall adopt Ince's notation (1932) which is also used by McLachlan (1947) and by many other authors. It should be noted however (i) that many older authors use a normalization which is different from that proposed by Goldstein, adopted by Ince and McLachlan and followed here; and (ii) that Stratton et al. (1941) and the NBS tables (1951) use a different notation and a different normalization. On p. xxxviii of the NBS tables there is a detailed comparison of three notations.

Throughout our discussion we take $\theta$ to be real so that the characteristic values of $h$, and the characteristic functions, are real. The case of complex parameters has been discussed by Strutt (1935, 1948).

If $u(z)$ is a Mathieu function, then so are the functions

$$u(-z), \qquad u(z) \pm z(-z),$$

and we may restrict ourselves to Mathieu functions which are even or odd functions of $z$. An even Mathieu function with $n$ zeros in the interval $0 \leq z < \pi$, or in any half-open interval of length $\pi$ on the real axis, will be denoted by $\mathrm{ce}_n(z, \theta)$, an odd Mathieu function by $\mathrm{se}_n(z, \theta)$. The corresponding characteristic values of $h$ will be denoted by $a_n(\theta)$ and $b_n(\theta)$ respectively. Often we shall write $\mathrm{ce}_n(z)$, $\mathrm{se}_n(z)$, $a_n$, $b_n$, omitting $\theta$.

Mathieu functions are the characteristic functions of the Sturm-Liouville problems involving the differential equation

$$(1) \qquad \frac{d^2 u}{dz^2} + [h - 2\theta \cos(2z)]\, u = 0$$

and the boundary conditions

(2) $u(0) = u(\pi) = 0$ for $\mathrm{se}_n(z, \theta)$

(3) $\dfrac{du}{dz}(0) = \dfrac{du}{dz}(\pi) = 0$ for $\mathrm{ce}_n(z, \theta)$

From the general Sturm-Liouville theory (see for instance Ince 1927, Chapter X) it follows that for each $n = 1, 2, \ldots$ there is a characteristic function $\mathrm{se}_n(z, \theta)$ determined up to a constant factor, and that for each $n = 0, 1, 2, \ldots$ there is a $\mathrm{ce}_n(z, \theta)$ determined up to a constant factor. We complete the definition of Mathieu functions by choosing the arbitrary constant factor so that

(4) $\mathrm{ce}_n(0, \theta) > 0, \quad \int_0^{2\pi} [\mathrm{ce}_n(z, \theta)]^2 \, dz = \pi$

$$\frac{d\,\mathrm{se}_n}{dz}(0, \theta) > 0, \quad \int_0^{2\pi} [\mathrm{se}_n(z, \theta)]^2 \, dz = \pi$$

If $\mathrm{e}(z)$ is either $\mathrm{ce}_n(z)$ or $\mathrm{se}_n(z)$, then $\mathrm{e}(z)$ and $\mathrm{e}(\pi - z)$ satisfy the same differential equation and the same boundary conditions, and must be constant multiples of each other so that $\mathrm{e}(z)$ is either an even or an odd function of $\frac{1}{2}\pi - z$, and we have the following four cases:

(5) $u(0) = u\left(\dfrac{\pi}{2}\right) = 0, \qquad \mathrm{e} = \mathrm{se}_{2m+2}(z), \qquad$ period $\pi$

(6) $u(0) = \dfrac{du}{dz}\left(\dfrac{\pi}{2}\right) = 0, \qquad \mathrm{e} = \mathrm{se}_{2m+1}(z), \qquad$ period $2\pi$

(7) $\dfrac{du}{dz}(0) = u\left(\dfrac{\pi}{2}\right) = 0, \qquad \mathrm{e} = \mathrm{ce}_{2m+1}(z), \qquad$ period $2\pi$

(8) $\dfrac{du}{dz}(0) = \dfrac{du}{dz}\left(\dfrac{\pi}{2}\right) = 0, \qquad \mathrm{e} = \mathrm{ce}_{2m}(z), \qquad$ period $\pi$

For each $m = 0, 1, 2, \ldots$ there is exactly one characteristic function of each of these four boundary value problems, and $m$ is the number of zeros in the interval $0 < z < \frac{1}{2}\pi$.

From (5) to (8) we also have

$$(9)\quad u\left(-\frac{\pi}{2}\right)=u\left(\frac{\pi}{2}\right)=0 \qquad \text{for} \qquad ce_{2m+1}(z) \quad \text{and} \quad se_{2m+2}(z)$$

$$(10)\quad \frac{du}{dz}\left(-\frac{\pi}{2}\right)=\frac{du}{dz}\left(\frac{\pi}{2}\right)=0 \qquad \text{for} \qquad ce_{2m}(z) \quad \text{and} \quad se_{2m+1}(z)$$

and finally,

$$(11)\quad u(-\pi)=u(\pi), \quad \frac{du}{dz}(-\pi)=\frac{du}{dz}(\pi) \qquad \text{for all Mathieu functions.}$$

If we use the comparison theorems for characteristic values of Sturm-Liouville problems we obtain: $a_n < a_{n+1}$ from (3), $b_n < b_{n+1}$ from (2), $a_{2m+1} < b_{2m+2} < a_{2m+3}$ from (9), and $a_{2m} < b_{2m+1} < a_{2m+2}$ from (10). Thus we know the relative positions of the characteristic values except for the relative positions of $a_n$ and $b_n$. Ince has proved that $a_n \neq b_n$ if $\theta \gtrless 0$, and from the numerical tables $a_n > b_n$ when $\theta < 0$. Thus we have

$$(12)\quad a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < \cdots \qquad \theta > 0$$

$$a_0 < b_1 < a_1 < b_2 < a_2 < b_3 < a_3 < \cdots \qquad \theta < 0$$

$$a_n,\ b_n \to \infty \quad \text{as} \quad n \to \infty .$$

For further investigation, estimates, and asymptotic forms of the characteristic values see Strutt (1943).

The symmetry relations given in Table 1 follow from the above boundary conditions.

TABLE 1. SYMMETRY RELATIONS FOR MATHIEU FUNCTIONS

| $e(z)$ | $e(-z)$ | $e(\pi-z)$ | $e(\pi+z)$ |
|---|---|---|---|
| $ce_{2m}$ | $ce_{2m}$ | $ce_{2m}$ | $ce_{2m}$ |
| $ce_{2m+1}$ | $ce_{2m+1}$ | $-ce_{2m+1}$ | $-ce_{2m+1}$ |
| $se_{2m+1}$ | $-se_{2m+1}$ | $se_{2m+1}$ | $-se_{2m+1}$ |
| $se_{2m+2}$ | $-se_{2m+2}$ | $-se_{2m+2}$ | $se_{2m+2}$ |

Mathieu's equation (1) is invariant under the transformation $\theta = -\theta'$, $z = \frac{1}{2}\pi - z'$. It then follows from (5) to (8) and (4) that

$$(13)\quad a_{2m}(-\theta) = a_{2m}(\theta), \qquad b_{2m+2}(-\theta) = b_{2m+2}(\theta), \qquad a_{2m+1}(-\theta) = b_{2m+1}(\theta),$$

$$(14)\quad \mathrm{ce}_{2m}(z, -\theta) = (-1)^m\, \mathrm{ce}_{2m}(\tfrac{1}{2}\pi - z, \theta)$$

$$\mathrm{se}_{2m+2}(z, -\theta) = (-1)^m\, \mathrm{se}_{2m+2}(\tfrac{1}{2}\pi - z, \theta)$$

$$\mathrm{ce}_{2m+1}(z, -\theta) = (-1)^m\, \mathrm{se}_{2m+1}(\tfrac{1}{2}\pi - z, \theta)$$

Since Mathieu functions are characteristic functions of certain Sturm-Liouville problems, they have the following *orthogonal properties*.

$$(15)\quad \int_0^{\frac{1}{2}\pi} \mathrm{ce}_{2k}(z)\,\mathrm{ce}_{2m}(z)\,dz = \int_0^{\frac{1}{2}\pi} \mathrm{ce}_{2k+1}(z)\,\mathrm{ce}_{2m+1}(z)\,dz$$

$$= \int_0^{\frac{1}{2}\pi} \mathrm{se}_{2k+1}(z)\,\mathrm{se}_{2m+1}(z)\,dz = \int_0^{\frac{1}{2}\pi} \mathrm{se}_{2k+2}(z)\,\mathrm{se}_{2m+2}(z)\,dz = 0$$

$$k, m = 0, 1, 2, \ldots, \qquad k \neq m$$

$$(16)\quad \int_0^{\pi} \mathrm{ce}_n(z)\,\mathrm{ce}_l(z)\,dz = \int_0^{\pi} \mathrm{se}_{n+1}(z)\,\mathrm{se}_{l+1}(z)\,dz = 0$$

$$l, n = 0, 1, 2, \ldots, \qquad l \neq n$$

$$(17)\quad \int_0^{2\pi} \mathrm{ce}_n(z)\,\mathrm{se}_{l+1}(z)\,dz = 0 \qquad\qquad l, n = 0, 1, 2, \ldots$$

If $i\mu$ is a rational fraction then 16.2(4) and 16.2(5) are periodic solutions of Mathieu's equation, the period being a multiple of $\pi$. Such solutions are sometimes called *Mathieu functions of fractional order* (see McLachlan 1947, Chapter IV). Orthogonal properties of such solutions have been obtained by Schäfke (1953).

*Integral equations* for Mathieu functions may be obtained from the work of the preceding section. If $f$ is any periodic Mathieu function, $b = a + 2\pi$, $N$ is a solution of 16.3(9) which is periodic in $\zeta$, then 16.3(12) is satisfied, and 16.3(10) is a solution of Mathieu's equation. If $N$ is also a periodic function of $z$ then 16.3(10) is a periodic solution of (1) and hence the multiple of a Mathieu function. As a nucleus we may use 16.3(15) with an arbitrary $\beta$, or special values of $\beta$, combinations of nuclei 16.3(15), partial derivatives of these nuclei with respect to $\beta$, and the like. The interval can be reduced by utilizing the symmetry properties of Mathieu functions. In Table 2 we list the intervals and nuclei for the principal integral equations of the form

(18) $\int_a^b N(z, \zeta)\, e(\zeta)\, d\zeta = \lambda e(z)$

for Mathieu functions. Other nuclei may be obtained by giving special values to $\beta$ (when $\beta = 0$ or $\beta = \pi/2$, it may be necessary first to divide by $\sin\beta$ or $\cos\beta$), or by integrating with respect to $\beta$. Nuclei involving Bessel functions (Erdélyi 1942a, McLachlan 1947, Chapter X) may be obtained in this manner.

TABLE 2. INTEGRAL EQUATIONS FOR MATHIEU FUNCTIONS

| $a$ | $b$ | $N(z,\zeta)$ | $e(z)$ |
|---|---|---|---|
| 0 | $\pi$ | $\exp(2i\,\theta^{1/2}\cos z\,\cos\zeta\,\cos\beta)\,\cosh(2\theta^{1/2}\sin z\,\sin\zeta\,\sin\beta)$ | $ce_n(z)$ |
| 0 | $\pi$ | $\exp(2i\,\theta^{1/2}\cos z\,\cos\zeta\,\cos\beta)\,\sinh(2\theta^{1/2}\sin z\,\sin\zeta\,\sin\beta)$ | $se_{n+1}(z)$ |
| 0 | $\frac{1}{2}\pi$ | $\cos(2\theta^{1/2}\cos z\,\cos\zeta\,\cos\beta)\,\cosh(2\theta^{1/2}\sin z\,\sin\zeta\,\sin\beta)$ | $ce_{2m}(z)$ |
| 0 | $\frac{1}{2}\pi$ | $\sin(2\theta^{1/2}\cos z\,\cos\zeta\,\cos\beta)\,\cosh(2\theta^{1/2}\sin z\,\sin\zeta\,\sin\beta)$ | $ce_{2m+1}(z)$ |
| 0 | $\frac{1}{2}\pi$ | $\cos(2\theta^{1/2}\cos z\,\cos\zeta\,\cos\beta)\,\sinh(2\theta^{1/2}\sin z\,\sin\zeta\,\sin\beta)$ | $se_{2m+1}(z)$ |
| 0 | $\frac{1}{2}\pi$ | $\sin(2\theta^{1/2}\cos z\,\cos\zeta\,\cos\beta)\,\sinh(2\theta^{1/2}\sin z\,\sin\zeta\,\sin\beta)$ | $se_{2m+2}(z)$ |

### 16.5. Expansions of Mathieu functions and functions of the second kind

From the periodicity of Mathieu functions, and their symmetry properties listed in Table 1, it follows that these functions may be expanded in *Fourier series* as follows

(1) $ce_{2m}(z, \theta) = \sum_{r=0}^{\infty} A_{2r} \cos(2rz)$

(2) $ce_{2m+1}(z, \theta) = \sum_{r=0}^{\infty} A_{2r+1} \cos[(2r+1)z]$

(3) $se_{2m+1}(z, \theta) = \sum_{r=0}^{\infty} B_{2r+1} \sin[(2r+1)z]$

(4) $se_{2m+2}(z, \theta) = \sum_{r=0}^{\infty} B_{2r+2} \sin[(2r+2)z].$

These are the forms to which 16.2(8) reduces when $i\mu$ is an integer. If necessary, the order of the Mathieu function, and the value of $\theta$, may be indicated by writing $A_{2r}^{2m}(\theta)$ for $A_{2r}$, etc.

Substitution of the expansions (1) to (4) in Mathieu's equation 16.4(1) leads to the following recurrence relations for the determination of $A_{2r}, \ldots, B_{2r+2}$.

(5) $$hA_0 - \theta A_2 = 0$$

$$(h-4)A_2 - \theta(2A_0 + A_4) = 0$$

$$(h-4r^2)A_{2r} - \theta(A_{2r-2} + A_{2r+2}) = 0 \qquad h = a_{2m}(\theta), \quad r = 2, 3, \ldots$$

(6) $$(h - \theta - 1)A_1 - \theta A_3 = 0$$

$$[h - (2r+1)^2]\,A_{2r+1} - \theta(A_{2r-1} + A_{2r+3}) = 0$$

$$h = a_{2m+1}(\theta), \quad r = 1, 2, \ldots$$

(7) $$(h + \theta - 1)B_1 - \theta B_3 = 0$$

$$[h - (2r+1)^2]\,B_{2r+1} - \theta(B_{2r-1} + B_{2r+3}) = 0$$

$$h = b_{2m+1}(\theta), \quad r = 1, 2, \ldots$$

(8) $$(h-4)B_2 - \theta B_4 = 0$$

$$[h - (2r+2)^2]\,B_{2r+2} - \theta(B_{2r} + B_{2r+4}) = 0$$

$$h = b_{2m+2}(\theta), \quad r = 1, 2, \ldots$$

As in the case of 15.3(13), each of the recurrence relations leads to an expression of the ratio of two successive coefficients as an infinite continued fraction involving $h$, and substitution into the first equation of each of the systems (5) to (8) leads to a transcendental equation for $h$ which may be used for determining the characteristic values. In the case of (5), for instance, the transcendental equation for $h$ is

$$h = \cfrac{-\theta^2/2}{1 - \cfrac{h}{4} - \cfrac{\theta^2/64}{1 - \cfrac{h}{16} - \cfrac{\theta^2/576}{1 - \cfrac{h}{36} - \cdots}}} \qquad h = a_{2m}(\theta)$$

Once $h$ has been determined, the ratios of successive coefficients are known. For the determination of the coefficients themselves, (5) to (8) must be supplemented by the relations

$$(9)\quad \sum_{r=0}^{\infty} A_{2r} > 0, \quad 2[A_0]^2 + \sum_{r=0}^{\infty} [A_{2r}]^2 = 1$$

$$(10)\quad \sum_{r=0}^{\infty} A_{2r+1} > 0, \quad \sum_{r=0}^{\infty} [A_{2r+1}]^2 = 1$$

$$(11)\quad \sum_{r=0}^{\infty} (2r+1) B_{2r+1} > 0, \quad \sum_{r=0}^{\infty} [B_{2r+1}]^2 = 1$$

$$(12)\quad \sum_{r=0}^{\infty} (2r+2) B_{2r+2} > 0, \quad \sum_{r=0}^{\infty} [B_{2r+2}]^2 = 1$$

which follow from 16.4(4). For more detailed descriptions of the numerical computations see Ince (1932), Blanch (1946), and McLachlan (1947). For a list of numerical tables see Bickley (1945) and also the list of references in the NBS tables (1951).

From the infinite continued fractions

$$(13)\quad \lim_{r\to\infty} \frac{r^2 A_{2r+2}}{A_{2r}} = \lim_{r\to\infty} \frac{r^2 A_{2r+1}}{A_{2r-1}}$$

$$= \lim_{r\to\infty} \frac{r^2 B_{2r+1}}{B_{2r-1}} = \lim_{r\to\infty} \frac{r^2 B_{2r+2}}{B_{2r}} = -\frac{\theta}{4}$$

so that the series (1) to (4) converge in the entire complex $z$-plane.

Expansions of Mathieu functions in *series of Bessel functions* may be obtained from 16.2(26), (28), (29) by putting $i\mu = 0, 1$ and taking account of the symmetry of Mathieu functions, or else from the integral equations listed in Table 2 by substituting the Fourier expansions (1) to (4) under the integral sign. The following expansions follow from the integral equations when the limiting forms $\beta = 0$, $\beta = \frac{1}{2}\pi$ of the nuclei are used.

$$(14)\quad \mathrm{ce}_{2n}(z, \theta) = \frac{\mathrm{ce}_{2n}(\frac{1}{2}\pi, \theta)}{A_0^{2n}(\theta)} \sum_{r=0}^{\infty} (-1)^r A_{2r} J_{2r}(2\theta^{\frac{1}{2}} \cos z)$$

$$= \frac{\mathrm{ce}_{2n}(0, \theta)}{A_0^{2n}(\theta)} \sum_{r=0}^{\infty} (-1)^r A_{2r} I_{2r}(2\theta^{\frac{1}{2}} \sin z)$$

(15) $$\mathrm{ce}_{2m+1}(z,\theta) = -\frac{\mathrm{ce}'_{2m+1}(\tfrac{1}{2}\pi,\theta)}{\theta^{1/2} A_1^{2m+1}(\theta)} \sum_{r=0}^{\infty} (-1)^r A_{2r+1} J_{2r+1}(2\theta^{1/2}\cos z)$$

$$= \frac{\mathrm{ce}_{2m+1}(0,\theta)}{\theta^{1/2} A_1^{2m+1}(\theta)} \operatorname{ctn} z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1} I_{2r+1}(2\theta^{1/2}\sin z)$$

(16) $$\mathrm{se}_{2m+1}(z,\theta) = \frac{\mathrm{se}_{2m+1}(\tfrac{1}{2}\pi,\theta)}{\theta^{1/2} B_1^{2m+1}(\theta)} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1} J_{2r+1}(2\theta^{1/2}\cos z)$$

$$= \frac{\mathrm{se}'_{2m+1}(0,\theta)}{\theta^{1/2} B_1^{2m+1}(\theta)} \sum_{r=0}^{\infty} (-1)^r B_{2r+1} I_{2r+1}(2\theta^{1/2}\sin z)$$

(17) $$\mathrm{se}_{2m+2}(z,\theta) = -\frac{\mathrm{se}'_{2m+2}(\tfrac{1}{2}\pi,\theta)}{\theta B_2^{2m+2}(\theta)} \tan z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2} J_{2r+2}(2\theta^{1/2}\cos z)$$

$$= \frac{\mathrm{se}'_{2m+2}(0,\theta)}{\theta B_2^{2m+2}(\theta)} \operatorname{ctn} z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2} I_{2r+2}(2\theta^{1/2}\sin z)$$

In these formulas $e' = de/dz$. The constant factor $\lambda$ in 16.4 (18) has been determined in each case by setting $z = 0$ or $z = \tfrac{1}{2}\pi$ in the expansion or in the derivative of the expansion. The infinite series of Bessel functions converge for all values of $z$.

There is a considerable number of expansions of Mathieu functions in *series of products of Bessel functions* of the kind 16.2 (33), (35). The most important among these are

(18) $$\mathrm{ce}_{2m}(z,\theta) = \frac{p_{2m}}{A_0^{2m}} \sum_{r=0}^{\infty} (-1)^r A_{2r} J_r(\theta^{1/2} e^{iz}) J_r(\theta^{1/2} e^{-iz})$$

(19) $$\mathrm{ce}_{2m+1}(z,\theta) = \frac{p_{2m+1}}{A_1^{2m+1}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}$$
$$\times [J_r(\theta^{1/2} e^{iz}) J_{r+1}(\theta^{1/2} e^{-iz}) + J_{r+1}(\theta^{1/2} e^{iz}) J_r(\theta^{1/2} e^{-iz})]$$

(20) $$\mathrm{se}_{2m+1}(z,\theta) = -\frac{s_{2m+1}}{iB_1^{2m+1}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}$$
$$\times [J_r(\theta^{1/2} e^{iz}) J_{r+1}(\theta^{1/2} e^{-iz}) - J_{r+1}(\theta^{1/2} e^{iz}) J_r(\theta^{1/2} e^{-iz})]$$

(21) $\operatorname{se}_{2m+2}(z,\theta)=\dfrac{s_{2m+2}}{iB_2^{2m+2}}\sum_{r=0}^{\infty}(-1)^r B_{2r+2}$

$$\times\left[J_r(\theta^{1/2}e^{iz})\,J_{r+2}(\theta^{1/2}e^{-iz})-J_{r+2}(\theta^{1/2}e^{iz})\,J_r(\theta^{1/2}e^{-iz})\right]$$

The *multipliers* $p_n$ and $s_n$ have been determined by McLachlan (1947 p. 368ff.) who compared the asymptotic form of the two sides of the expansions (18) to (21). By means of the results of sec. 16.7 one obtains

(22) $A_0^{2m}\,p_{2m}=\operatorname{ce}_{2m}(0)\operatorname{ce}_{2m}(\tfrac{1}{2}\pi)$

$$\theta^{1/2}A_1^{2m+1}\,p_{2m+1}=-\operatorname{ce}_{2m+1}(0)\operatorname{ce}'_{2m+1}(\tfrac{1}{2}\pi)$$

$$\theta^{1/2}B_1^{2m+1}\,s_{2m+1}=\operatorname{se}'_{2m+1}(0)\operatorname{se}_{2m+1}(\tfrac{1}{2}\pi)$$

$$\theta\,B_2^{2m+2}\,s_{2m+2}=\operatorname{se}'_{2m+2}(0)\operatorname{se}'_{2m+2}(\tfrac{1}{2}\pi)$$

The series in (18) to (24) converge for all values of $z$. These and other expansions in series of products of Bessel functions may be obtained from integral equations whose nuclei involve Bessel functions (McLachlan 1947, p. 193ff.).

Ince has proved (see for instance McLachlan 1947, Chapter VII) that the general solution of Mathieu's equation with $\theta \neq 0$ is never periodic. Thus, if $e(z)$ is any Mathieu function of the first kind, any second solu-ion of Mathieu's equation will be non-periodic. Since Mathieu functions of the second kind are of minor importance, we shall not give many details, and refer the reader to McLachlan's book, or to the analogous work in connection with the modified Mathieu equation in sec. 16.6.

There are several ways of constructing *Mathieu functions of the second kind*. A degenerate form of Floquet's theorem states that in the case that $i\mu$ is an integer, and $e(z)$ is the corresponding Mathieu function of the first kind, a second solution may be determined in the form $z\,e(z)+f(z)$ where $f(z)$ is periodic and is represented by a sine series if $e(z)$ is a cosine series and *vice versa*. Another method is based on integral relations such as 16.3(26). The simplest, and perhaps most efficient method is based on the remark that the series of Bessel functions given in this section remain formal solutions of Mathieu's equation if the Bessel functions of the first kind are replaced by Bessel functions of the second or the third kinds. The series resulting in this manner from (14) to (17) converge only if $|\cos z|>1$ or $|\sin z|>1$ respectively, and are not suitable for the computation of Mathieu functions of the second or third kinds for real values of $z$. On the other hand, the series of products of Bessel functions, in which one of the Bessel functions is of the first kind and the other of the second or third kinds, such as 16.2(35), converge for all values of $z$. Moreover, these series are well suited for numerical computations.

## 16.6. Modified Mathieu functions

The differential equation

$$(1)\quad \frac{d^2u}{dz^2} - [h - 2\theta \cosh(2z)]\, u = 0$$

is known as the *modified Mathieu equation*: it differs from 16.2(1) only in that $z$ has been replaced by $iz$, and accordingly, the results of sections 16.2 and 16.3 apply with slight changes. Frequently (1) appears in conjunction with Mathieu's equation when $h$ has one of the characteristic values $a_n$ or $b_n$. We shall restrict ourselves to this case.

*Modified Mathieu functions of the first kind* may be defined as

$$(2)\quad \mathrm{Ce}_n(z, \theta) = \mathrm{ce}_n(iz, \theta) \qquad h = a_n(\theta)$$

$$\mathrm{Se}_n(z, \theta) = -i\ \mathrm{se}_n(iz, \theta) \qquad h = b_n(\theta)$$

Expansions of modified Mathieu functions in Fourier series, in series of Bessel functions, and in series of products of Bessel functions now follow from the preceding section and are recorded in McLachlan (1947, sections 2.30, 2.31, Chapters VIII and XIII).

Modified Mathieu functions of the second kind are obtained on replacing Bessel functions of the first kind by Bessel functions of the second kind in 16.5(14) to (17), and, similarly, Bessel functions of the third kind appear in the definitions of modified Mathieu functions of the third kind. The notation adopted by McLachlan indicates by Fe the functions corresponding to Ce, by Ge the functions corresponding to Se and adds a y for functions of the second kind, a k for functions of the third kind.

*Modified Mathieu functions of the second kind*

$$(3)\quad \mathrm{Fey}_{2m}(z, \theta) = \frac{\mathrm{ce}_{2m}(\frac{1}{2}\pi, \theta)}{A_0^{2m}(\theta)} \sum_{r=0}^{\infty} (-1)^r A_{2r} Y_{2r}(2\theta^{\frac{1}{2}} \cosh z)$$

$$= \frac{\mathrm{ce}_{2m}(0, \theta)}{A_0^{2m}(\theta)} \sum_{r=0}^{\infty} A_{2r} Y_{2r}(2\theta^{\frac{1}{2}} \sinh z)$$

$$= \frac{p_{2m}}{A_0^{2m}} \sum_{r=0}^{\infty} (-1)^r A_{2r} J_{2r}(\theta^{\frac{1}{2}} e^{-z})\, Y_{2r}(\theta^{\frac{1}{2}} e^{z}) \qquad h = a_{2m}(\theta)$$

(4) $$\mathrm{Fey}_{2m+1}(z,\theta) = -\frac{\mathrm{ce}'_{2m+1}(\tfrac{1}{2}\pi,\theta)}{\theta^{1/2}A_1^{2m+1}(\theta)}\sum_{r=0}^{\infty}(-1)^r A_{2r+1}Y_{2r+1}(2\theta^{1/2}\cosh z)$$

$$= \frac{\mathrm{ce}_{2m+1}(0,\theta)}{\theta^{1/2}A_1^{2m+1}(\theta)}\operatorname{ctnh} z\sum_{r=0}^{\infty}(2r+1)A_{2r+1}Y_{2r+1}(2\theta^{1/2}\sinh z)$$

$$= \frac{p_{2m+1}}{A_1^{2m+1}}\sum_{r=0}^{\infty}(-1)^r A_{2r+1}[J_r(\theta^{1/2}e^{-z})Y_{r+1}(\theta^{1/2}e^{z})$$

$$+ J_{r+1}(\theta^{1/2}e^{-z})Y_r(\theta^{1/2}e^{z})] \qquad h = a_{2m+1}(\theta)$$

(5) $$\mathrm{Gey}_{2m+1}(z,\theta) = \frac{\mathrm{se}_{2m+1}(\tfrac{1}{2}\pi,\theta)}{\theta^{1/2}B_1^{2m+1}}\tanh z$$

$$\times\sum_{r=0}^{\infty}(-1)^r(2r+1)B_{2r+1}Y_{2r+1}(2\theta^{1/2}\cosh z)$$

$$= \frac{\mathrm{se}'_{2m+1}(0,\theta)}{\theta^{1/2}B_1^{2m+1}}\sum_{r=0}^{\infty}B_{2r+1}Y_{2r+1}(2\theta^{1/2}\sinh z)$$

$$= \frac{s_{2m+1}}{B_1^{2m+1}}\sum_{r=0}^{\infty}(-1)^r B_{2r+1}[J_r(\theta^{1/2}e^{-z})Y_{r+1}(\theta^{1/2}e^{z})$$

$$- J_{r+1}(\theta^{1/2}e^{-z})Y_r(\theta^{1/2}e^{z})] \qquad h = b_{2m+1}(\theta)$$

(6) $$\mathrm{Gey}_{2m+2}(z,\theta) = -\frac{\mathrm{se}'_{2m+2}(\tfrac{1}{2}\pi,\theta)}{\theta B_2^{2m+2}}\tanh z$$

$$\times\sum_{r=0}^{\infty}(-1)^r(2r+2)B_{2r+2}Y_{2r+2}(2\theta^{1/2}\cosh z)$$

$$= \frac{\mathrm{se}'_{2m+2}(0,\theta)}{\theta B_2^{2m+2}}\coth z\sum_{r=0}^{\infty}(2r+2)B_{2r+2}Y_{2r+2}(2\theta^{1/2}\sinh z)$$

$$= -\frac{s_{2m+2}}{B_2^{2m+2}}\sum_{r=0}^{\infty}(-1)^r B_{2r+2}[J_r(\theta^{1/2}e^{-z})Y_{r+2}(\theta^{1/2}e^{z})$$

$$- J_{r+2}(\theta^{1/2}e^{-z})Y_r(\theta^{1/2}e^{z})] \qquad h = b_{2m+2}(\theta)$$

In each of these four groups of expansions the first series converges when $|\cosh z| > 1$, the second when $|\sinh z| > 1$, and the third for all values of $z$: in the first two series we also assume $\operatorname{Re} z > 0$.

There are several *modified Mathieu functions of the third kind*. The functions obtained when $Y_\nu$ is replaced by $H_\nu^{(j)}$, $j = 1, 2$, in the series for $\operatorname{Fey}_n$ and $\operatorname{Gey}_n$ are denoted respectively by $\operatorname{Me}_n^{(j)}$ and $\operatorname{Ne}_n^{(j)}$, $j = 1, 2$, and the functions obtained when $Y_{2r}(w)$ is replaced by $(-1)^r \pi^{-1} K_{2r}(-iw)$ and $Y_{2r+1}(w)$ is replaced by $(-1)^r \pi^{-1} K_{2r+1}(-iw)$ in the first two series representing $\operatorname{Fey}_n$ and $\operatorname{Gey}_n$ are denoted by $\operatorname{Fek}_n$ and $\operatorname{Gek}_n$ respectively. Since we have

$$J_\nu(w) + i\,Y_\nu(w) = H_\nu^{(1)}(w) = \frac{2}{\pi}\, i^{-\nu-1}\, K_\nu(-iw)$$

from 7.2(5) and 7.2(17), the various modified Mathieu functions are obtained by the relations

$$(7)\quad \operatorname{Ce}_{2m}(z, \theta) + i\, \operatorname{Fey}_{2m}(z, \theta) = \operatorname{Me}_{2m}^{(1)}(z, \theta) = -2i\, \operatorname{Fek}_{2m}(z, \theta)$$

$$\operatorname{Ce}_{2m+1}(z, \theta) + i\, \operatorname{Fey}_{2m+1}(z, \theta) = \operatorname{Me}_{2m+1}^{(1)}(z, \theta) = -2\, \operatorname{Fek}_{2m+1}(z, \theta)$$

$$\operatorname{Se}_{2m+1}(z, \theta) + i\, \operatorname{Gey}_{2m+1}(z, \theta) = \operatorname{Ne}_{2m+1}^{(1)}(z, \theta) = -2\, \operatorname{Gek}_{2m+2}(z, \theta)$$

$$\operatorname{Se}_{2m+2}(z, \theta) + i\, \operatorname{Gey}_{2m+2}(z, \theta) = \operatorname{Ne}_{2m+2}^{(1)}(z, \theta) = -2i\, \operatorname{Gek}_{2m+2}(z, \theta)$$

For expansions of the various modified Mathieu functions of the third kind see McLachlan (1947, sections 8.14, 8.30, 13.30, 13.40).

The *asymptotic behavior* of modified Mathieu functions as $z \to \infty$ may be read off their expansions in series of Bessel functions, or in series of products of Bessel functions (see sec. 16.7).

There are numerous *integral relations* between Mathieu functions and modified Mathieu functions, and also between modified Mathieu functions themselves. If $N(z, \zeta)$ is a nucleus for the interval $(a, b)$ as in 16.4(18), then

$$\int_b^a N(iz, \zeta)\, \mathrm{e}(\zeta)\, d\zeta$$

is a multiple of a modified Mathieu function of the first kind. The integral relations thus arising from the limiting cases $\beta = 0$, $\beta = \frac{1}{2}\pi$ of the nuclei of Table 2 of sec. 16.4 are listed in McLachlan (1947, sec. 10.20).

Let $\theta > 0$, $z > 0$. Then the integrals

$$\int_0^\infty \exp(2i\,\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Ce}_n(\zeta,\theta)\,d\zeta$$

$$\int_0^\infty \sinh z\sinh\zeta\exp(2i\,\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Se}_{n+1}(\zeta,\theta)\,d\zeta$$

are convergent. If the Fourier series are substituted for the modified Mathieu functions of the first kind, and the resulting integrals are evaluated by means of 7.12(21), the following integral relations result

(8) $$\pi A_0^{2m}\,\mathrm{Fek}_{2m}(z,\theta) = \mathrm{ce}_{2m}(\tfrac{1}{2}\pi,\theta)\int_0^\infty \exp(2i\,\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Ce}_{2m}(\zeta,\theta)\,d\zeta$$

$\theta > 0, \quad z > 0$

(9) $$\pi A_1^{2m+1}\,\mathrm{Fek}_{2m+1}(z,\theta) = -\theta^{-1/2}\,\mathrm{ce}'_{2m+1}(\tfrac{1}{2}\pi,\theta)\int_0^\infty \exp(2i\,\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Ce}_{2m+1}(\zeta,\theta)\,d\zeta$$

$\theta > 0, \quad z > 0$

(10) $$\pi B_1^{2m+1}\,\mathrm{Gek}_{2m+1}(z,\theta) = -2i\,\mathrm{se}_{2m+1}(\tfrac{1}{2}\pi,\theta) \times\int_0^\infty \sinh z\sinh\zeta\exp(2i\,\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Se}_{2m+1}(\zeta,\theta)\,d\zeta$$

$\theta > 0, \quad z > 0$

(11) $$\pi B_2^{2m+2}\,\mathrm{Gek}_{2m+2}(z,\theta) = -2i\,\theta^{-1/2}\,\mathrm{se}'_{2m+2}(\tfrac{1}{2}\pi,\theta) \times\int_0^\infty \sinh z\sinh\zeta\exp(2i\,\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Se}_{2m+2}(\zeta,\theta)\,d\zeta$$

$\theta > 0, \quad z > 0$

We now separate in (8) to (11) the real and imaginary parts by means of equations (7) and obtain a further group of integral relations

(12) $$\pi A_0^{2m}\,\mathrm{Fey}_{2m}(z,\theta) = -2\,\mathrm{ce}_{2m}(\tfrac{1}{2}\pi,\theta)\int_0^\infty \cos(2\theta^{1/2}\cosh z\cosh\zeta)\,\mathrm{Ce}_{2m}(\zeta,\theta)\,d\zeta$$

$\theta > 0, \quad z > 0$

(13) $$\pi A_1^{2m+1} \operatorname{Fey}_{2m+1}(z, \theta) = 2\theta^{-\frac{1}{2}} \operatorname{ce}'_{2m+1}(\tfrac{1}{2}\pi, \theta) \int_0^\infty \sin(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Ce}_{2m+1}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

(14) $$\pi B_1^{2m+1} \operatorname{Gey}_{2m+1}(z, \theta) = 4 \operatorname{se}_{2m+1}(\tfrac{1}{2}\pi, \theta) \times \int_0^\infty \sinh z \sinh\zeta \cos(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Se}_{2m+1}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

(15) $$\pi B_2^{2m+2} \operatorname{Gey}_{2m+2}(z, \theta) = -4\theta^{-\frac{1}{2}} \operatorname{se}'_{2m+2}(\tfrac{1}{2}\pi, \theta) \times \int_0^\infty \sinh z \sinh\zeta \sin(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Se}_{2m+2}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

and also the integral equations

(16) $$\pi A_0^{2m} \operatorname{Ce}_{2m}(z, \theta) = 2 \operatorname{ce}_{2m}(z, \theta) \int_0^\infty \sin(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Ce}_{2m}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

(17) $$\pi A_1^{2m+1} \operatorname{Ce}_{2m+1}(z, \theta) = 2\theta^{-\frac{1}{2}} \operatorname{ce}'_{2m+1}(\tfrac{1}{2}\pi, \theta) \int_0^\infty \cos(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Ce}_{2m+1}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

(18) $$\pi B_1^{2m+1} \operatorname{Se}_{2m+1}(z, \theta) = -4 \operatorname{se}_{2m+1}(\tfrac{1}{2}\pi, \theta) \times \int_0^\infty \sinh z \sinh\zeta \sin(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Se}_{2m+1}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

(19) $$\pi B_2^{2m+2} \operatorname{Se}_{2m+2}(z, \theta) = -4\theta^{-\frac{1}{2}} \operatorname{se}'_{2m+2}(\tfrac{1}{2}\pi, \theta) \times \int_0^\infty \sinh z \sinh\zeta \cos(2\theta^{\frac{1}{2}} \cosh z \cosh\zeta) \operatorname{Se}_{2m+2}(\zeta, \theta)\, d\zeta \qquad \theta > 0, \quad z > 0$$

For integral relations for negative $\theta$ see McLachlan (1947, Chapter X). For integral relations whose nuclei involve Bessel functions see McLachlan (1947, Chapter X) and Meixner (1951 a). Meixner (ibid.) also gives some integrals involving products of Mathieu functions.

**16.7. Approximations and asymptotic forms**

*Approximations for small* $|\theta|$. Mathieu's equation reduces to a differential equation with constant coefficients when $\theta = 0$, and we have

$$(1) \quad a_n(0) = b_n(0) = n^2$$

$$ce_0(z, 0) = 2^{-\frac{1}{2}}, \quad ce_n(z, 0) = \cos(nz), \quad se_n(z, 0) = \sin(nz) \qquad n = 1, 2, \ldots$$

Starting from (1), the characteristic values and characteristic functions may be expanded in powers of $\theta$. Strutt (1932, p. 36) proved that

$$(2) \quad a_n(\theta) = n^2 + O(|\theta|^n), \quad b_n(\theta) = n^2 + O(|\theta|^n) \qquad \theta \to 0$$

so that the characteristic curves belonging to $ce_n$ and $se_n$ are at contact of order $n - 1$ at the point $h = n^2$, $\theta = 0$; this is the only common point of these two characteristic curves. Strutt (1932, p. 31 ff.) also gives the expansion of $a_n(\theta)$ as far as $\theta^6$, of $ce_n(z, \theta)/A_n^n$ as far as $\theta^4$ and of some coefficients $A_r^n/A_n^n$ as far as $\theta^4$ or $\theta^5$. Numerical bounds for the $O$-term in (2) were given by Weinstein (1935).

*Asymptotic forms for large* $|z|$. The asymptotic behavior of Mathieu functions as $\operatorname{Im} z \to \infty$, or of modified Mathieu functions as $\operatorname{Re} z \to \infty$, may be ascertained from the Bessel function expansions by means of a general theorem proved by Meixner (1949a) which states that under certain conditions the asymptotic expansions of series like 16.6(3) may be obtained by substituting the asymptotic expansions of the Bessel functions on the right-hand side.

To obtain the leading terms of the asymptotic expansions of modified Mathieu functions as $\operatorname{Re} z \to \infty$, we remark that

$$J_\nu(2\theta^{\frac{1}{2}} \cosh z) \sim \theta^{-\frac{1}{4}} (\pi \cosh z)^{-\frac{1}{2}} \cos(2\theta^{\frac{1}{2}} \cosh z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$$
$$\sim (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \theta^{-\frac{1}{4}} e^{-\frac{1}{2}z} \cos(\theta^{\frac{1}{2}} e^z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$$
$$\operatorname{Re} z \to \infty, \quad -\pi < \operatorname{Im} z < \pi$$

by 7.13(3). Substituting this in 16.5(14),

$$Ce_{2m}(z, \theta) = ce_{2m}(iz, \theta)$$
$$\sim \frac{ce_{2m}(0)\, ce_{2m}(\frac{1}{2}\pi)}{(\frac{1}{2}\pi)^{\frac{1}{2}} \theta^{\frac{1}{4}} A_0^{2m}} e^{-\frac{1}{2}z} \cos(\theta^{\frac{1}{2}} e^z - \tfrac{1}{4}\pi)$$

On the other hand, using 16.5(18), and noting that for large $\operatorname{Re} z$ the first term of the series dominates the others,

$$Ce_{2m}(z, \theta) = ce_{2m}(iz, \theta) \sim p_{2m} (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \theta^{-\frac{1}{4}} e^{-\frac{1}{2}z} \cos(\theta^{\frac{1}{2}} e^z - \tfrac{1}{4}\pi).$$

By comparison of the last two equations we obtain the first relation in 16.5(22), and the others may be confirmed similarly. In order to obtain the asymptotic forms of modified Mathieu functions of the second kind from 16.6(3) to (6), we use 7.13(4) instead of 7.13(3), and this amounts to replacing

$$\cos(\theta^{1/2} e^{z} - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) \quad \text{by} \quad \sin(\theta^{1/2} e^{z} - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$$

In this manner the following results are obtained:

$$(3)\quad \mathrm{Ce}_{2m}(z, \theta) \sim p_{2m}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \cos(\theta^{1/2} e^{z} - \tfrac{1}{4}\pi)$$

$$\mathrm{Ce}_{2m+1}(z, \theta) \sim p_{2m+1}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \cos(\theta^{1/2} e^{z} - \tfrac{3}{4}\pi)$$

$$\mathrm{Se}_{2m+1}(z, \theta) \sim s_{2m+1}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \cos(\theta^{1/2} e^{z} - \tfrac{3}{4}\pi)$$

$$\mathrm{Se}_{2m+2}(z, \theta) \sim s_{2m+2}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \cos(\theta^{1/2} e^{z} - \tfrac{1}{4}\pi)$$

$$\mathrm{Re}\, z \to \infty, \qquad -\pi < \tfrac{1}{2}\arg\theta + \mathrm{Im}\, z < \pi$$

$$(4)\quad \mathrm{Fey}_{2m}(z, \theta) \sim p_{2m}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \sin(\theta^{1/2} e^{z} - \tfrac{1}{4}\pi)$$

$$\mathrm{Fey}_{2m+1}(z, \theta) \sim p_{2m+1}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \sin(\theta^{1/2} e^{z} - \tfrac{3}{4}\pi)$$

$$\mathrm{Gey}_{2m+1}(z, \theta) \sim s_{2m+1}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \sin(\theta^{1/2} e^{z} - \tfrac{3}{4}\pi)$$

$$\mathrm{Gey}_{2m+2}(z, \theta) \sim s_{2m+2}(\tfrac{1}{2}\pi)^{-1/2}\, \theta^{-1/4}\, e^{-1/2 z} \sin(\theta^{1/2} e^{z} - \tfrac{1}{4}\pi)$$

$$\mathrm{Re}\, z \to \infty, \qquad -\pi < \tfrac{1}{2}\arg\theta + \mathrm{Im}\, z < \pi$$

Asymptotic series in descending powers of $e^{z}$ or $\cosh z$ may be obtained from the modified Mathieu equation 16.6(1): see McLachlan (1947, Chapter XI).

*Asymptotic forms for large* $|\theta|$. The asymptotic behavior of Mathieu functions and of the characteristic values of $h$ for large real values of $\theta$ has been investigated by Jeffreys, Goldstein, and Ince. The results of these investigations, and references to the literature, are given in Strutt (1932, p. 37 ff.) and McLachlan (1947, sections 11.40 to 11.44).

The principal results are:

$$(5)\quad a_{n}(\theta) \sim b_{n+1}(\theta) \sim -2\theta + 2(2n+1)\theta^{1/2} - \tfrac{1}{4}(2n^{2} + 2n + 1) \qquad \theta \to \infty$$

(6) $$\mathrm{ce}_n(z,\theta) \sim C_n (\cos z)^{-n-1} \{[\cos(\tfrac{1}{2}z + \tfrac{1}{4}\pi)]^{2n+1} \exp(2\theta^{1/2} \sin z) + [\sin(\tfrac{1}{2}z + \tfrac{1}{4}\pi)]^{2n+1} \exp(-2\theta^{1/2} \sin z)\}$$

$$\mathrm{se}_{n+1}(z,\theta) \sim S_{n+1} (\cos z)^{-n-1} \{[\cos(\tfrac{1}{2}z + \tfrac{1}{4}\pi)]^{2n+1} \exp(2\theta^{1/2} \sin z) - [\sin(\tfrac{1}{2}z + \tfrac{1}{4}\pi)]^{2n+1} \exp(-2\theta^{1/2} \sin z)\}$$

$$-\tfrac{1}{2}\pi < z < \tfrac{1}{2}\pi, \qquad \theta \to \infty$$

(7) $$\mathrm{Ce}_n(z,\theta) \sim C_n\, 2^{1/2-n} (\cosh z)^{-1/2} \times \cos[2\theta^{1/2} \sinh z - (2n+1)\tan^{-1}(\tanh \tfrac{1}{2}z)]$$

$$\mathrm{Se}_{n+1}(z,\theta) \sim S_{n+1}\, 2^{1/2-n} (\cosh z)^{-1/2} \times \sin[2\theta^{1/2} \sinh z - (2n+1)\tan^{-1}(\tanh \tfrac{1}{2}z)]$$

$$z > 0, \qquad \theta \to \infty$$

For large $z$, (7) and (3) may be compared to give

(8) $$C_n = (-1)^m\, 2^{n-1/2}\, \theta^{-1/4}\, \pi^{-1/2}\, p_n$$

$$S_n = (-1)^m\, 2^{n-3/2}\, \theta^{-1/4}\, \pi^{-1/2}\, s_n$$

where $m = \left[\frac{n}{2}\right]$, i.e., $n = 2m$ or $n = 2m+1$ according as $n$ is even or odd.

Langer (1934) investigated the asymptotic behavior of Mathieu functions when $\theta$ is large and real while $z$ may be complex.

Equations (6) describe the behavior of Mathieu functions when

$$-1 < \cos z < 1,$$

and (7), when $\cos z > 1$. Both formulas fail near $\cos z = 1$. In order to have formulas valid in a range including this point, Meixner (1948) and Sips (1949) expand Mathieu functions in a series of parabolic cylinder functions. These expansions are of the form

(9) $$\mathrm{ce}_n(z,\theta) = \sum_{r=0}^{\infty} \alpha_r D_r(2\theta^{1/2} \cos z)$$

$$\mathrm{se}_{n+1}(z,\theta) = \sin z \sum_{r=0}^{\infty} \beta_r D_r(2\theta^{1/2} \cos z)$$

where $r$ ranges over even or odd integers according as $n$ is even or odd, and the $\alpha_r$, $\beta_r$ satisfy five-term recurrence relations. When $\theta$ is large, the dominant terms in the expansions (9) are those corresponding to $r = n$, and we have

(10) $\mathrm{ce}_n(z,\theta) \sim \alpha_n D_n(2\theta^{1/2}\cos z)$

$\mathrm{se}_{n+1}(z,\theta) \sim \beta_n \sin z\, D_n(2\theta^{1/2}\cos z) \qquad \theta \to \infty$

The $\alpha_n$ and $\beta_n$ may be determined by putting $z = \frac{1}{2}\pi$ (after differentiation if necessary) and using the values of $D_\nu(0)$, $D_\nu'(0)$ obtained from 8.2(4).

## 16.8. Series, integrals, expansion problems

Most of the known *infinite series* involving Mathieu functions may be interpreted as superpositions of solutions of the wave equation. As in sec. 16.1.1, let $x, y$ be Cartesian coordinates, and $u$, $v$ elliptic coordinates, while $\rho$, $\phi$ are polar coordinates so that

(1) $x + iy = c\cosh(u + iv) = \rho e^{i\phi}$.

Typical solutions of the two-dimensional wave equation

(2) $$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \kappa^2 W = 0$$

in elliptic coordinates are $U(u)\,V(v)$ where $V$ is a Mathieu function, $U$ an associated Mathieu function, and

(3) $\theta = (\frac{1}{2}\kappa c)^2$

in Mathieu's equation. Typical solutions in polar coordinates are

$$Z_\nu(\kappa\rho)\, e^{i\nu\phi}$$

where $Z_\nu$ is a Bessel function of order $\nu$. The remark that elliptic cylindrical waves may be generated by the superposition of (circular) cylindrical waves, and *vice versa*, leads to a number of important infinite series; and elliptic cylindrical waves may similarly be related to plane waves.

Consider

(4) $$W = \sum_{r=0}^{\infty} (-1)^r A_{2r}^{2m}(\theta)\, H_{2r}^{(j)}(\kappa\rho)\cos(2r\phi) \qquad j = 1, 2$$

as a function of $u$ and $v$, recalling that

(5) $$\kappa\rho = 2[\theta\cosh(u+iv)\cosh(u-iv)]^{1/2}$$

$$e^{2i\phi} = \frac{\cosh(u+iv)}{\cosh(u-iv)}$$

from (1). Thus, (4) is an expansion of the form 16.2(30), and for real $u$, $v$ (or, more generally, for $|\mathrm{Im}\, v| < |\mathrm{Re}\, u|$), and fixed $u$, it represents a multiple of $\mathrm{ce}_{2m}(v,\ \theta)$. Since

$$W = U(u)\ \mathrm{ce}_{2m}(v,\ \theta),$$

we have from 16.1.1 that $U(u)$ is an associated Mathieu function. The asymptotic behavior of (4) as $u \to \infty$, and hence $\rho \to \infty$, shows that $U(u)$ must be an associated Mathieu function of the third kind, in fact

$$W = \text{const. } \mathrm{Me}_{2m}^{(j)}(u,\ \theta)\ \mathrm{ce}_{2m}(v,\ \theta) \qquad j = 1,\ 2$$

We determine the constant factor by making $u \to \infty$, $\rho \to \infty$, using 7.13(1), (2) on the left-hand side, and 16.7(3), (4) on the right-hand side. This computation, and analogous work with $\mathrm{ce}_{2m+1}$, $\mathrm{se}_{2m+1}$, $\mathrm{se}_{2m+2}$ leads to the following expansions in which $\theta$ is omitted from the symbols of Mathieu functions and also from the coefficients.

$$(6) \quad \mathrm{Me}_{2m}^{(j)}(u)\ \mathrm{ce}_{2m}(v) = p_{2m} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{2m}\ H_{2r}^{(j)}(\kappa\rho) \cos(2r\phi)$$

$$\mathrm{Me}_{2m+1}^{(j)}(u)\ \mathrm{ce}_{2m+1}(v) = p_{2m+1} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{2m+1}\ H_{2r+1}^{(j)}(\kappa\rho) \cos[(2r+1)\phi]$$

$$\mathrm{Ne}_{2m+1}^{(j)}(u)\ \mathrm{se}_{2m+1}(v) = s_{2m+1} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2m+1}\ H_{2r+1}^{(j)}(\kappa\rho) \sin[(2r+1)\phi]$$

$$\mathrm{Ne}_{2m+2}^{(j)}(u)\ \mathrm{se}_{2m+2}(v) = -s_{2m+2} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{2m+2}\ H_{2r+2}^{(j)}(\kappa\rho) \sin[(2r+2)\phi]$$

$$j = 1,\ 2$$

Here $p$ and $s$ have the same meaning as in 16.5(22).

For $v = 0$ and $v = \frac{1}{2}\pi$, (6) reduces to 16.5(14)-(17), and as $u \to \infty$, (6) reduces to 16.5(1)-(4) so that the most important series expansions of Mathieu functions are particular or limiting cases of (6).

Meixner (1949a, see also Schäfke 1953) generalized (6) in two respects. He admitted polar coordinates whose pole does not coincide with the center of the confocal family of ellipses and hyperbolas, and he expanded a product $U(u)\ V(v)$ where $V(v)$ is a solution of the first kind of the general Mathieu equation, i.e., with arbitrarily given $h$ and $\theta$, and $U(u)$ is a solution of the third kind of the corresponding modified equation. His expansions are of the form

$$U(u)\ V(v) = \sum_{r=-\infty}^{\infty} d_r H_{r-i\mu}^{(j)}(\kappa\rho)\ e^{(r-i\mu)i\phi}$$

where

$$\kappa\rho = 2\{\theta[\cosh(u+iv)-a][\cosh(u-iv)-a]\}^{1/2}$$

$$e^{2i\phi} = \frac{\cosh(u+iv)-a}{\cosh(u-iv)-a}$$

and $\mu$ is the characteristic exponent of the general Mathieu equation. The coefficients $d_r$ are given in Meixner's paper as

$$d_r = \sum_{n=-\infty}^{\infty} (-1)^n c_n J_{2n-r}(2a\theta)$$

where the $c_n$ are the coefficients occurring in 16.2(8), and $V(z)$ is that solution of the general Mathieu equation which is represented by 16.2(8).

The representation of elliptic cylindrical waves as a superposition of plane waves leads to integrals rather than series. Consider

$$(7) \quad W = \int_0^{2\pi} \exp[i\kappa(x\cos a + y\sin a)] \operatorname{ce}_n(a, \theta)\, da$$

as a function of $u$, $v$. We see from Table 2 in sec. 16.4 that for fixed $u$, $W$ is a multiple of $\operatorname{ce}_n(v)$, while for fixed $v$, it is a multiple of $\operatorname{Ce}_n(u)$. Thus,

$$W = \text{const.} \operatorname{Ce}_n(u) \operatorname{ce}_n(v),$$

and the constant may be determined by putting $x = y = 0$, i.e., $u = 0$, $v = \frac{1}{2}\pi$, in $W$ or $\partial W/\partial v$ according as $n$ is even or odd. We follow a similar process with $\operatorname{se}_{n+1}$, use the symmetry relations of Table 1, sec. 16.4, and obtain

$$(8) \quad \operatorname{Ce}_{2m}(u)\operatorname{ce}_{2m}(v) = 2\pi^{-1} p_{2m} \int_0^{\frac{1}{2}\pi} \cos(\kappa x\cos a)\cos(\kappa y\sin a)\operatorname{ce}_{2m}(a)\, da$$

$$\operatorname{Ce}_{2m+1}(u)\operatorname{ce}_{2m+1}(v) = 2\pi^{-1} p_{2m+1} \int_0^{\frac{1}{2}\pi} \sin(\kappa x\cos a)\cos(\kappa y\sin a)\operatorname{ce}_{2m+1}(a)\, da$$

$$\operatorname{Se}_{2m+1}(u)\operatorname{se}_{2m+1}(v) = 2\pi^{-1} s_{2m+1} \int_0^{\frac{1}{2}\pi} \cos(\kappa x\cos a)\sin(\kappa y\sin a)\operatorname{se}_{2m+1}(a)\, da$$

$$\operatorname{Se}_{2m+2}(u)\operatorname{se}_{2m+2}(v) = -2\pi^{-1} s_{2m+2} \int_0^{\frac{1}{2}\pi} \sin(\kappa x\cos a)\sin(\kappa y\sin a)\operatorname{se}_{2m+2}(a)\, da$$

where $x$ and $y$ are given by 16.1(1), $\theta$ by (3), and $p$ and $s$ by 16.5(22).

Similar integrals involving Bessel functions instead of trigonometric functions were given by Sips (1953, 1954).

Inversion of the relations obtained above yields sums of infinite series of products of Mathieu functions. Equations (8) may be regarded as determining the Fourier coefficients in the expansion of

$$\begin{matrix}\cos\\ \sin\end{matrix}(\kappa x \cos\alpha)\ \begin{matrix}\cos\\ \sin\end{matrix}(\kappa y \sin\alpha)$$

in a series of Mathieu functions; and lead to the following expansions.

(9) $$\cos(\kappa x \cos\alpha)\cos(\kappa y \sin\alpha) = 2\sum_{m=0}^{\infty} p_{2m}^{-1}\,\mathrm{ce}_{2m}(\alpha)\,\mathrm{Ce}_{2m}(u)\,\mathrm{ce}_{2m}(v)$$

$$\sin(\kappa x \cos\alpha)\cos(\kappa y \sin\alpha)$$

$$= 2\sum_{m=0}^{\infty} p_{2m+1}^{-1}\,\mathrm{ce}_{2m+1}(\alpha)\,\mathrm{Ce}_{2m+1}(u)\,\mathrm{ce}_{2m+1}(v)$$

$$\cos(\kappa x \cos\alpha)\sin(\kappa y \sin\alpha)$$

$$= 2\sum_{m=0}^{\infty} s_{2m+1}^{-1}\,\mathrm{se}_{2m+1}(\alpha)\,\mathrm{Se}_{2m+1}(u)\,\mathrm{se}_{2m+1}(v)$$

$$\sin(\kappa x \cos\alpha)\sin(\kappa y \sin\alpha)$$

$$= -2\sum_{m=0}^{\infty} s_{2m+2}^{-1}\,\mathrm{se}_{2m+2}(\alpha)\,\mathrm{Se}_{2m+2}(u)\,\mathrm{se}_{2m+2}(v)$$

Here again $x, y, \kappa, c$ and $u, v, \theta$ are connected as in 16.1(1) and 16.8(3), $\theta$ has been omitted from the symbol of Mathieu functions, and $p$, $s$ are given by 16.5(22). From (9) a large number of expansions may be derived by differentiating with respect to $\alpha$, $u$, or $v$, and choosing special values for some of the parameters. Some of these expansions are listed in McLachlan's book (1947, sections 10.60, 10.61).

The inversion of (6) leads to the expansion of

$$H_\nu^{(j)}(\kappa\rho)\begin{matrix}\cos\\ \sin\end{matrix}(\nu\phi) \qquad j = 1, 2$$

in a series of products of Mathieu functions and associated Mathieu functions (see, for instance, Sips 1953, 1954), and the result may be interpreted as the generation of circular cylindrical waves by the superposition of elliptic cylindrical waves. Sips has also expansions involving products of four Mathieu functions: these are needed in case the axis of the circular cylinder is different from the axis of the elliptic cylinder. The generalization to expansions involving products of solutions of the general Mathieu equation was given by Schäfke (1953).

Lastly, the generation of elliptic cylindrical waves by the superposition of other elliptic cylindrical waves leads to the so-called *addition theorem* of Mathieu functions (Schäfke 1953).

A somewhat different type of infinite series of Mathieu functions and of products of Mathieu functions was investigated by Ince (1939). Using special cases of (9) and of derivatives of (9), Ince expanded

$$\frac{\mathrm{se}_{2m+1}(z)}{\sin z}$$

in a series $\Sigma\, a_r\, \mathrm{ce}_{2r}(z)$ and gave numerous other expansions involving Mathieu functions and their derivatives in combination with trigonometric functions. When $\theta = 0$, Ince's expansions reduce to the addition theorems and differentiation formulas of trigonometric functions, and other trigonometric identities.

For *integral relations* with trigonometric nuclei see sections 16.4, 16.6, and (8); also McLachlan (1947, Chapters X, XIV). Integrals involving Bessel functions are given in McLachlan (1947, Chapter X), Sips (1949 a), Meixner (1951 a), Schäfke (1953). The latter author has evaluated an integral of a product of three Mathieu functions. Both Meixner and Schäfke extended their results to solutions of the general Mathieu equation.

The orthogonal properties of Mathieu functions are recorded in 16.4 (15), (16), (17). It follows from the general theory of Sturm-Liouville problems that each of the four systems $\{\mathrm{ce}_{2m}\}$, $\{\mathrm{ce}_{2m+1}\}$, $\{\mathrm{se}_{2m+1}\}$, $\{\mathrm{se}_{2m+2}\}$ is complete in the interval $(0, \tfrac{1}{2}\pi)$, each of the two systems $\{\mathrm{ce}_n\}$, $\{\mathrm{se}_{n+1}\}$ is complete in $(0, \pi)$, and the system $\{\mathrm{ce}_n, \mathrm{se}_{n+1}\}$ is complete in $(0, 2\pi)$: here $m, n = 0, 1, 2, \ldots$ . *An arbitrary function which can be expanded in a Fourier series can also be expanded in a series of Mathieu functions.* The coefficients in the latter expansion may be computed by means of the orthogonal properties of Mathieu functions. Important examples of such expansions are (9) and the expansions of (circular) cylindrical waves in series of Mathieu functions.

Characteristic value problems for (non-periodic) solutions of the general Mathieu equations have been considered by Strutt (1943) who gave bounds for the characteristic values, asymptotic forms, expansion formulas, and expansion theorems. In Strutt's work, $\cos(2z)$ in 16.2(1) is replaced by any real periodic function (period $p$) which can be expanded in a convergent Fourier series: the resulting differential equation is

*Hill's equation*, and the boundary value problem consisting of Hill's equation and the boundary conditions

$$u(z_0 + p) = \sigma u(z_0), \qquad u'(z_0 + p) = \sigma u'(z_0),$$

$\sigma$ given, is called *Hill's problem* by Strutt. (In the case of periodic Mathieu functions, $\sigma = \pm 1$.)

Expansions in series of products of Mathieu functions and associated Mathieu functions arise in connection with the (two-dimensional) wave equation (2). Suppose we consider (2) inside of an ellipse $u = u_0$, and impose the boundary condition $W(u_0, v) = 0$ (appropriate to the problem of vibrations of an elliptic membrane). Solutions of (2) are of the form

$$\psi c_n(u, v) = \mathrm{Ce}_n(u, \theta)\,\mathrm{ce}_n(v, \theta)$$

$$\psi s_{n+1}(u, v) = \mathrm{Se}_{n+1}(u, \theta)\,\mathrm{se}_{n+1}(v, \theta) \qquad n = 0, 1, \ldots$$

Those values of $\kappa$ for which $\mathrm{Ce}_n(u_0, \theta) = 0$ or $\mathrm{Se}_{n+1}(u_0, \theta) = 0$ are the *characteristic values* of (2) for the region $u \leq u_0$. These correspond to certain characteristic values of $\theta$, and the resulting characteristic functions may be denoted by $\psi c_n^m$, $\psi s_{n+1}^m$, $n = 0, 1, 2, \ldots, m = 1, 2, \ldots$. The element of area is $[\cosh(2u) - \cos(2v)]\,du\,dv$, and we have the following orthogonal property

$$\int_0^{u_0}\int_0^{2\pi} \psi c_n^m\, \psi c_k^l [\cosh(2u) - \cos(2v)]\,du\,dv$$

$$= \int_0^{u_0}\int_0^{2\pi} \psi s_{n+1}^m\, \psi s_{k+1}^l [\cosh(2u) - \cos(2v)]\,du\,dv = 0$$

$$k, n = 0, 1, \ldots; \quad m, l = 1, 2, \ldots; \quad k \neq n \text{ or } m \neq l$$

$$\int_0^{u_0}\int_0^{2\pi} \psi c_n^m\, \psi s_{k+1}^l [\cosh(2u) - \cos(2v)]\,du\,dv = 0$$

$$k, n = 0, 1, \ldots; \quad l, m = 1, 2, \ldots$$

For the computation of the integrals involving $[\psi c_n^m]^2$ and $[\psi s_n^m]^2$ see McLachlan (1947, sec. 9.40). The expansion of an arbitrary function in a series of $\psi c$ and $\psi s$ in the region $u \leq u_0$ now follows. There are corresponding expansions for other boundary conditions.

SPHEROIDAL WAVE FUNCTIONS

## 16.9. The differential equation of spheroidal wave functions and its solution

We shall adopt

$$(1)\quad (1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + [\lambda + 4\theta(1-z^2) - \mu^2(1-z^2)^{-1}]\,y = 0$$

as the standard form of the *differential equation of spheroidal wave functions*. There is no generally accepted standard form. Meixner, in his recent work (1950, 1951), uses (1) with $4\theta = \gamma^2$, Bouwkamp, Strutt (1932), and Meixner in his earlier work (1944, 1947, 1948) have, respectively, $k^2z^2$, $-k^2c^2z^2$, and $-\gamma^2z^2$ in place of $4\theta(1-z^2)$ so that their $\lambda$ corresponds to $\lambda + 4\theta$ in (1). Stratton et al. (1941) use the differential equation satisfied by $(1-z^2)^{\frac{1}{2}\mu}y$. We shall, in this section, regard $\theta$, $\lambda$, $\mu$ as given, real or complex parameters, and $z$ as a complex variable. $\mu$ will be called the *order* of the spheroidal wave functions.

With

$$(2)\quad z = \cos v$$

we obtain

$$(3)\quad \frac{d^2y}{dv^2} + \operatorname{ctn} v\,\frac{dy}{dv} + [\lambda + 4\theta(\sin v)^2 - \mu^2(\csc v)^2]\,y = 0,$$

the *trigonometric form* of the differential equation of spheroidal wave functions [see also 16.1(11), (12), (16), (17)].

We shall now discuss several special and limiting cases of (1) since these suggest the choice of relevant solutions.

If $\theta = 0$, i.e., $\kappa = 0$ in the wave equation 16.1(9) and (14), then equation (1) reduces to Legendre's equation 3.2(1) with $\lambda = \nu(\nu+1)$. For the solutions in the cut $z$-plane see sec. 3.2, and for the appropriate solutions on the cut see sec. 3.4.

If $\mu = \frac{1}{2}$, a simple computation shows that in terms of the variable $v$, $(\sin v)^{-\frac{1}{2}}\,y(v)$ satisfies Mathieu's equation with $\theta$ having the same meaning as in 16.2(1), and $h = \lambda + \frac{1}{4} + 2\theta$.

With

$$(4)\quad \zeta = 2\theta^{\frac{1}{2}} z$$

as the independent variable, (1) changes into

$$(5)\quad (\zeta^2 - 4\theta)\frac{d^2y}{d\zeta^2} + 2\zeta\frac{dy}{d\zeta} + \left(\zeta^2 - \lambda - 4\theta - \frac{4\theta\mu^2}{\zeta^2 - 4\theta}\right) y = 0,$$

and if $\theta = 0$ in (5), the solutions can be expressed in terms of Bessel functions. In particular, if $\theta = 0$ in (5), this equation has the following four solutions

$$(6) \quad \psi_\nu^{(1)}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} J_{\nu+1/2}(\zeta), \qquad \psi_\nu^{(2)}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} Y_{\nu+1/2}(\zeta)$$

$$\psi_\nu^{(3)}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} H_{\nu+1/2}^{(1)}(\zeta), \qquad \psi_\nu^{(4)}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} H_{\nu+1/2}^{(2)}(\zeta)$$

where $\lambda = \nu(\nu + 1) = (\nu + \tfrac{1}{2})^2 - \tfrac{1}{4}$; see also 7.2(44) for the notation.

These special and limiting cases are important not only because they exhibit the relation of spheroidal wave functions to other special functions but also because they indicate the behavior of solutions of (1) near the singularities, and suggest the choice of special solutions of (1) as well as expansions of these solutions in series of Legendre or Bessel functions. For the relation of (1) to the differential equations of confluent hypergeometric functions and parabolic cylinder functions see Meixner (1948, 1951), Sips (1949).

The differential equation (1) has three singular points, $z = 1, -1$, and $\infty$. $z = \pm 1$ are regular singular points, the exponents at each of them being $\pm \tfrac{1}{2}\mu$. $z = \infty$ is an irregular singular point, and (5) suggests that there are two solutions of (1) which behave at $\infty$ like $z^\nu$ times a single-valued function, and $z^{-\nu-1}$ times a single-valued function. The exponent $\nu$ appearing here is called the *characteristic exponent* of (1): it is a function of $\theta$, $\lambda$, $\mu$ and, like the characteristic exponent of Mathieu's equation, it is determined by a relation of the form $\cos(2\pi\nu) = f(\lambda, \mu^2, \theta)$. Often it is more convenient to represent $\lambda$ as a function of $\theta$, $\mu$, and $\nu$, and the notation used by Meixner is $\lambda_\nu^\mu(\theta)$. Clearly,

$$(7) \quad \lambda_\nu^\mu(0) = \nu(\nu + 1), \qquad \lambda_\nu^\mu(\theta) = \lambda_\nu^{-\mu}(\theta) = \lambda_{-\nu-1}^{\pm\mu}(\theta).$$

For a discussion of the functional relationship between $\lambda$, $\mu$, $\nu$, $\theta$ see Schmid (1948, 1949), Schäfke (1950), Meixner (1951).

We shall assume that $\lambda = \lambda_\nu^\mu(\theta)$ in (1), and express the solutions in terms of $\theta$, $\mu$, $\nu$.

A first group of solutions will be obtained as expansions in series of Bessel functions. (5) suggests expansions of the form

$$(8) \quad S_\nu^{\mu\,(j)}(z, \theta) = (1 - z^{-2})^{-1/2\mu} s_\nu^\mu(\theta) \sum_{r=-\infty}^{\infty} a_{\nu,r}^\mu(\theta) \psi_{\nu+2r}^{(j)}(2\theta^{1/2} z)$$

$$j = 1, 2, 3, 4$$

where $\psi^{(j)}$ are the functions defined in (6). As a rule, we shall write $a_r$ for $a^{\mu}_{\nu,r}(\theta)$ and simplify other notations similarly. Substitution of (8) in (1) leads to a recurrence relation for the coefficients $a_r$ which is given by Meixner (1951) as

(9) $$\frac{(\nu+2r-\mu)(\nu+2r-\mu-1)}{(\nu+2r-3/2)(\nu+2r-1/2)}\theta a_{r-1}+\frac{(\nu+2r+\mu+2)(\nu+2r+\mu+1)}{(\nu+2r+3/2)(\nu+2r+5/2)}\theta a_{r+1}$$
$$+\left[\lambda^{\mu}_{\nu}(\theta)-(\nu+2r)(\nu+2r+1)+\frac{(\nu+2r)(\nu+2r+1)+\mu^2-1}{(\nu+2r-1/2)(\nu+2r+3/2)}2\theta\right]a_r=0$$
$$r=0,\ \pm 1,\ \pm 2,\ \ldots$$

From now on we assume that $\nu+\frac{1}{2}$ is not an integer. (It appears that the case thus excluded has not been fully investigated.)

The recurrence relation (9) is similar to 16.2(9). After division by a suitable factor it leads to an infinite determinant whose vanishing is the condition which determines the functional relationship between $\theta, \lambda, \mu, \nu$. Alternatively, infinite continued fractions $R_n$ and $L_n$ may be derived as in 16.2(16),(17). We shall assume that $a^{\mu}_{\nu,0}(\theta)$ has been so chosen that

(10) $$a^{\mu}_{\nu,0}(\theta)=a^{\mu}_{-\nu-1,0}(\theta)=a^{-\mu}_{\nu,0}(\theta).$$

It then follows that

(11) $$a^{\mu}_{\nu,r}(\theta)=a^{\mu}_{-\nu-1,-r}(\theta)=\frac{(\nu-\mu+1)_{2r}}{(\nu+\mu+1)_{2r}}a^{-\mu}_{\nu,r}(\theta).$$

From the continued fractions we have, as in 16.2(21),

(12) $$\lim_{r\to\infty}\frac{r^2 a_r}{a_{r-1}}=\lim_{r\to-\infty}\frac{r^2 a_r}{a_{r+1}}=\frac{\theta}{4}$$

unless the sequence of coefficients $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ terminates to the right or the left, when the first or the second limit in (12) becomes meaningless. This cannot happen unless $\nu+\mu$ or $\nu-\mu$ is an integer. From the asymptotic formulas for Bessel functions we have

(13) $$\lim_{r\to\infty}\frac{\psi^{(1)}_{\nu+2r-2}}{r^2\psi^{(1)}_{\nu+2r}}=\lim_{r\to-\infty}\frac{\psi^{(1)}_{\nu+2r}}{r^2\psi^{(1)}_{\nu+2r+2}}=\frac{4}{\theta z^2}$$

(14) $$\lim_{r\to\infty}\frac{\psi^{(j)}_{\nu+2r}}{r^2\psi^{(j)}_{\nu+2r-2}}=\lim_{r\to-\infty}\frac{\psi^{(j)}_{\nu+2r}}{r^2\psi^{(j)}_{\nu+2r+2}}=\frac{4}{\theta z^2}\qquad j=2,\ 3,\ 4$$

and it follows from (12)-(14) that (8) converges when $|z| > 1$. In this region, $(1 - z^{-2})^{-\frac{1}{2}\mu}$ may be made single-valued by defining it by the binomial expansion, and we may take $-\pi < \arg z \leq \pi$ in (8). In the exceptional cases, when one or the other of the limits (12) ceases to exist, the series of coefficients terminates in one direction, and the question of convergence in that direction does not arise.

H.L. Schmid (1948, 1949) investigated thoroughly a class of recurrence relations which includes (9). His results establish the existence and uniqueness (up to a common factor) of the $a_r$, and also the expansion of $\lambda_\nu^\mu$ and $a_{\nu,r}^\mu$ in convergent series of powers of $\theta$.

The asymptotic behavior of $S^{(j)}$ as $z \to \infty$ may be determined by means of results by Meixner (1949). If $j = 1, 2$, and $\theta > 0$, we let $z \to \infty$ in the upper or lower half-plane, if $j = 3, 4$, $z \to \infty$ in any manner. Then

$$\frac{\psi_{\nu+2r}^{(j)}}{\psi_\nu^{(j)}} \to (-1)^r \qquad \text{as} \qquad z \to \infty$$

by 7.13(1)-(4). If we set

$$(15)\quad s_\nu^\mu(\theta) = [\sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,r}^\mu(\theta)]^{-1}$$

then

$$(16)\quad \lim_{z\to\infty} [S_\nu^{\mu(j)}(z, \theta)/\psi_\nu^{(j)}(2\theta^{\frac{1}{2}} z)] = 1$$

where in the cases $j = 1, 2$, it is assumed that $\operatorname{Im}(\theta^{\frac{1}{2}} z) \neq 0$. This relation may also be written as

$$(17)\quad S_\nu^{\mu(j)}(z, \theta) \sim \psi_\nu^{(j)}(2\theta^{\frac{1}{2}} z) \qquad j = 1, 2, 3, 4, \quad z \to \infty, \quad |\arg(\theta^{\frac{1}{2}} z)| < \pi$$

and in this form the case of positive $\theta^{\frac{1}{2}} z$ need not be excluded. When $j = 3, 4$, the range of $\arg(\theta^{\frac{1}{2}} z)$ can be extended as in 7.13(1), (2), to $(-\pi, 2\pi)$, $(-2\pi, \pi)$ respectively. We shall assume throughout that $s_\nu^\mu$ is determined by (15).

From (6) it follows that $\psi_{\nu+2r}^{\mu(1)}$, and hence $S_\nu^{\mu(1)}$, is of the form $z^\nu$ times a function which is single-valued near $\infty$, so that $S_\nu^{\mu(1)}$ is a *solution of the first kind*. $S_\nu^{\mu(2)}$ may be called a *solution of the second kind*. From (16), (6) and 7.13(1), (2) it is seen that $S_\nu^{\mu(3)}$ and $S_\nu^{\mu(4)}$ vanish exponentially as $z \to \infty$ in the half-planes $\operatorname{Im}(\theta^{\frac{1}{2}} z) > 0$ and $\operatorname{Im}(\theta^{\frac{1}{2}} z) < 0$ respectively: thus $S^{(3,4)}$ are *solutions of the third kind*. Beside $S_\nu^{\mu(j)}$ we have the further solutions $S_\nu^{-\mu(j)}$ and $S_{-\nu-1}^{\pm\mu(j)}$, $j = 1, 2, 3, 4$. Between these 16 solutions there are numerous relations which are consequences either of (16) or of (11) and identities between Bessel functions. We list a few of these relations, omitting $z$ and $\theta$ which are the same throughout.

(18) $S_{\nu}^{\mu(j)} = S_{\nu}^{-\mu(j)}$

(19) $S_{\nu}^{\mu(3)} = S_{\nu}^{\mu(1)} + i\,S_{\nu}^{\mu(2)} = e^{-i\pi(\nu+\frac{1}{2})}\,S_{-\nu-1}^{\mu(3)}$

$S_{\nu}^{\mu(4)} = S_{\nu}^{\mu(1)} - i\,S_{\nu}^{\mu(2)} = e^{i\pi(\nu+\frac{1}{2})}\,S_{-\nu-1}^{\mu(4)}.$

(20) $S_{\nu}^{\mu(2)} = -(\cos\nu\pi)^{-1}\,[S_{\nu}^{\mu(1)}\sin(\nu\pi) + S_{-\nu-1}^{\mu(1)}]$

$S_{\nu}^{\mu(3)} = [i\,\cos(\nu\pi)]^{-1}\,[S_{-\nu-1}^{\mu(1)} - S_{\nu}^{\mu(1)}\,e^{-i\pi(\nu+\frac{1}{2})}]$

$S_{\nu}^{\mu(4)} = [i\,\cos(\nu\pi)]^{-1}\,[S_{\nu}^{\mu(1)}\,e^{i\pi(\nu+\frac{1}{2})} - S_{-\nu-1}^{\mu(1)}]$

(18) follows from (17) since the asymptotic representation in a sector of angular width $> \pi$ determines a solution of (1) uniquely. (19) and (20) follow from (6), (8), (11), (15) in combination with 7.2(4), (5), (6), (9). Meixner (1951) gives these and other relations, in particular formulas for the analytic continuation to values of $\arg(\theta^{\frac{1}{2}}\,z)$ outside $(-\pi,\ \pi)$, and formulas for the Wronskians of the solutions $S_{\nu}^{\mu(j)}$. It turns out, like in the case of Bessel functions, that any two of our four solutions are linearly independent since $\nu + \frac{1}{2}$ has been assumed not to be an integer.

The solutions discussed so far are represented by series convergent for $|z| > 1$, and are especially useful when $z$ is large. We now turn to solutions useful near $\pm 1$, and also on the segment $(-1,\ 1)$, and to expansions convergent inside the unit circle. Meixner denotes these solutions as follows:

(21) $Ps_{\nu}^{\mu}(z,\ \theta) = \sum_{r=-\infty}^{\infty} (-1)^r\,a_{\nu,r}^{\mu}(\theta)\,P_{\nu+2r}^{\mu}(z)$

$Qs_{\nu}^{\mu}(z,\ \theta) = \sum_{r=-\infty}^{\infty} (-1)^r\,a_{\nu,r}^{\mu}(\theta)\,Q_{\nu+2r}^{\mu}(z)$

(22) $\mathrm{Ps}_{\nu}^{\mu}(x,\ \theta) = \sum_{r=-\infty}^{\infty} (-1)^r\,a_{\nu,r}^{\mu}(\theta)\,\mathrm{P}_{\nu+2r}^{\mu}(x)$

$\mathrm{Qs}_{\nu}^{\mu}(x,\ \theta) = \sum_{r=-\infty}^{\infty} (-1)^r\,a_{\nu,r}^{\mu}(\theta)\,\mathrm{Q}_{\nu+2r}^{\mu}(x)$

Here, $P$, $Q$ are the Legendre functions as defined in sec. 3.2 for the cut plane, and $\mathrm{P}$, $\mathrm{Q}$ are the Legendre functions on the cut defined in sec. 3.4. Accordingly, in (21) $z$ is in the complex plane cut along the real axis from $-\infty$ to 1, and we take $|\arg(z \pm 1)| < \pi$ in (21); and in (22) $x$ is on the cut, $-1 < x < 1$, although these solutions could be continued analytically into the complex plane cut along the real axis from $-\infty$ to $-1$ and from 1 to $\infty$.

Substitution of (21) and (22) in (1) leads to the recurrence relation (9) so that the $a_r$ are the same coefficients as before. We assume from now on that

$$(23)\quad a^{\mu}_{\nu,0}(0) = 1$$

and also that (10) and (11) hold, so that

$$(24)\quad Ps^{\mu}_{\nu}(z, 0) = P^{\mu}_{\nu}(z), \qquad Qs^{\mu}_{\nu}(z, 0) = Q^{\mu}_{\nu}(z)$$

$$\mathrm{Ps}^{\mu}_{\nu}(x, 0) = \mathrm{P}^{\mu}_{\nu}(x), \qquad \mathrm{Qs}^{\mu}_{\nu}(x, 0) = \mathrm{Q}^{\mu}_{\nu}(x)$$

From (12) and sec. 3.9.1 it follows that (21) and (22) converge everywhere with the possible exception of $\pm 1$ and $\infty$. From 3.2(3), 3.6(2) it follows that $Ps$ is $(z-1)^{-\frac{1}{2}\mu}$ times a function single-valued near $z = 1$ if $\mu \neq 0, 1, 2, \ldots$, and $Ps$ is $(z-1)^{\frac{1}{2}m}$ times a function single-valued near $z = 1$ if $\mu = m = 0, 1, 2, \ldots$. From 3.2(5) it follows that $Qs$ is $z^{-\nu-1}$ times a function single-valued near $z = \infty$ provided that $\mu + \nu$ is not a negative integer. Thus, $Qs$ is a *solution of the first kind.*

Between the sixteen solutions $Ps^{\pm\mu}_{\nu}$, $Qs^{\pm\mu}_{\nu}$, $Ps^{\pm\mu}_{-\nu-1}$, $Qs^{\pm\mu}_{-\nu-1}$, $\mathrm{Ps}^{\pm\mu}_{\nu}$, $\mathrm{Qs}^{\pm\mu}_{\nu}$, $\mathrm{Ps}^{\pm\mu}_{-\nu-1}$, $\mathrm{Qs}^{\pm\mu}_{-\nu-1}$ there are numerous relations. These follow from, and resemble, the analogous relations for Legendre functions given in sections 3.3.1 and 3.4. Examples of such relations are

$$(25)\quad Ps^{\mu}_{\nu} = Ps^{\mu}_{-\nu-1}, \qquad \mathrm{Ps}^{\mu}_{\nu} = \mathrm{Ps}^{\mu}_{-\nu-1}$$

$$(26)\quad e^{i\mu\pi}\,\Gamma(\nu+\mu+1)\,Qs^{-\mu}_{\nu} = e^{-i\mu\pi}\,\Gamma(\nu-\mu+1)\,Qs^{\mu}_{\nu}$$

$$(27)\quad \mathrm{Ps}^{\mu}_{\nu}(-x) = \cos[(\mu+\nu)\pi]\,\mathrm{Ps}^{\mu}_{\nu}(x) - (2/\pi)\sin[(\mu+\nu)\pi]\,\mathrm{Qs}^{\mu}_{\nu}(x)$$

which follow, respectively, from 3.3(1), 3.4(7), 3.3(2), 3.4(14) in conjunction with (11). For a more detailed list of such relations, and for a list of Wronskians, see Meixner (1951).

Finally, we shall indicate the relations between the solutions represented by series of Bessel functions, and those represented by series of Legendre functions. $S_\nu^{\mu(1)}$ and $Qs_{-\nu-1}^{\mu}$ are both solutions of the first kind, they both belong to the exponent $\nu$ at $\infty$, and hence must be constant multiples of one another. Meixner (1951) writes

$$(28)\quad S_\nu^{\mu(1)}(z,\theta)=\pi^{-1}\sin[(\nu-\mu)\pi]\,e^{-(\nu+\mu+1)\pi i}K_\nu^\mu(\theta)\,Qs_{-\nu-1}^{\mu}(z,\theta)$$

and establishes a number of identities satisfied by $K_\nu^\mu(\theta)$: these follow from the identities valid for $S_\nu^{\mu(1)}$ and $Qs_\nu^\mu$. An explicit expression for $K_\nu^\mu$ is based on the remark that it follows from (8), (6), and 7.2(2) that

$$z^{-\nu}(1-z^{-2})^{\frac{1}{2}\mu}S_\nu^{\mu(1)}(z,\theta)$$
$$=\tfrac{1}{2}\pi^{\frac{1}{2}}\,s_\nu^\mu(\theta)\sum_{r=-\infty}^{\infty}\sum_{s=0}^{\infty}(-1)^s a_{\nu,r}^\mu(\theta)\frac{\theta^{\frac{1}{2}\nu+r+s}\,z^{2r+2s}}{s!\,\Gamma(\nu+2r+s+3/2)}$$

while it follows from (21) and 3.2(41) that

$$z^{-\nu}(1-z^{-2})^{\frac{1}{2}\mu}e^{-i\mu\pi}Qs_{-\nu-1}^{\mu}(z,\theta)$$
$$=\pi^{\frac{1}{2}}\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}(-1)^r a_{-\nu-1,r}^{\mu}(\theta)\,2^{\nu-2r-2t}z^{-2r-2t}\frac{\Gamma(\mu-\nu+2r+2t)}{t!\,\Gamma(\frac{1}{2}+2r+t-\nu)}$$

Multiplying both sides of (28) by $z^{-\nu}(1-z^{-2})^{\frac{1}{2}\mu}$, expanding in a Laurent series, and then comparing coefficients of $z^{2k}$ we thus obtain after some simplification

$$(29)\quad K_\nu^\mu(\theta)=\tfrac{1}{2}(\tfrac{1}{4}\theta)^{\frac{1}{2}\nu+k}\,\Gamma(1+\nu-\mu+2k)\,e^{(\nu+k)\pi i}s_\nu^\mu(\theta)$$
$$\times\frac{\displaystyle\sum_{r=-\infty}^{k}\frac{(-1)^r a_{\nu,r}^\mu(\theta)}{(k-r)!\,\Gamma(\nu+k+r+3/2)}}{\displaystyle\sum_{r=k}^{\infty}\frac{(-1)^r a_{\nu,r}^\mu(\theta)}{(r-k)!\,\Gamma(\frac{1}{2}-\nu-k-r)}}$$

Since all $S^{(j)}$ may be expressed in terms of $S^{(1)}$ by (20), and $Ps$ may be expressed in terms of $Qs$ by 3.3 (8), clearly (28) suffices to express any one of the Bessel function series in terms of Legendre function series and *vice versa*. All these relations simplify considerably when $\mu$ and $\nu$ are integers, see sec. 16.11.

### 16.10. Further expansions, approximations, integral relations

*Power series expansions.* Expansions in powers of $z$ or $z^2 - 1$ have been given by Fisher (1937) and others: they do not seem to be very useful either for analytical work or for numerical computations.

Expansions in *series of products of Bessel functions* do not seem to be known except in the case of spheroidal wave functions, see also sec. 16.11.

Meixner (1950) has given *expansions of products of solutions* of 16.9(1) as series of products of Bessel functions and Legendre functions. His expansions are based on the following remark. In a notation which differs slightly from that adopted in sec. 16.1.2, we introduce on the one hand spheroidal coordinates $\xi$, $\eta$, $\phi$, and on the other hand spherical polar coordinates $r$, $\chi$, $\phi$ whose pole is on the axis of revolution. The connection with the Cartesian coordinates is

$$(1)\quad x = c\,[(\xi^2 - 1)(1 - \eta^2)]^{1/2} \cos\phi = r \sin\chi \cos\phi$$

$$y = c\,[(\xi^2 - 1)(1 - \eta^2)]^{1/2} \sin\phi = r \sin\chi \sin\phi$$

$$z = c\,\xi\eta = r\cos\chi + c\,a$$

and we put $4\,\theta = \kappa^2 c^2$. It follows from sec. 16.1.2 that

$$S_\nu^{\mu(j)}(\xi,\,\theta)\,Ps_\nu^\mu(\eta,\,\theta)\,e^{\pm i\mu\phi}, \qquad S_\nu^{\mu(j)}(\xi,\,\theta)\,Qs_\nu^\mu(\eta,\,\theta)\,e^{\pm i\mu\phi}$$

are solutions of $\Delta W + \kappa^2 W = 0$; and so are

$$\psi_\lambda^{(j)}(\kappa r)\,P_\lambda^\mu(\cos\chi)\,e^{\pm i\mu\phi}, \qquad \psi_\lambda^{(j)}(\kappa r)\,Q_\lambda^\mu(\cos\chi)\,e^{\pm i\mu\phi}$$

An investigation of the behavior of these solutions as $\xi \to \infty$ and hence $r \to \infty$, and again as $\eta \to \pm 1$ and hence $\chi \to 0$, $\pi$ suggests expansions of the form

$$(2)\quad S_\nu^{\mu(j)}(\xi,\,\theta)\,Ps_\nu^\mu(\eta,\,\theta) = \sum_{t=-\infty}^{\infty} b_{\nu,\,t}^{\mu}(\theta, a)\,\psi_{\nu+t}^{(j)}(\kappa r)\,P_{\nu+t}^{\mu}(\cos\chi)$$

$$S_\nu^{\mu(j)}(\xi,\,\theta)\,Qs_\nu^\mu(\eta,\,\theta) = \sum_{t=-\infty}^{\infty} b_{\nu,\,t}^{\mu}(\theta,\,a)\,\psi_{\nu+t}^{(j)}(\kappa r)\,Q_{\nu+t}^{\mu}(\cos\chi)$$

where

$$(3)\quad \kappa r = 2\,\theta^{1/2}\,(\xi^2 + \eta^2 + a^2 - 2a\,\xi\eta - 1)^{1/2}$$

$$\cos\chi = (\xi^2 + \eta^2 + a^2 - 2a\,\xi\eta - 1)^{-1/2}\,(\xi\eta - a)$$

Meixner shows that the $b_t$ satisfy a five-term recurrence relation (which reduces to a three-term recurrence relation when $a = \pm 1$ or when $a = 0$), proves the existence of a solution of this recurrence relation, and the convergence, in appropriate regions, of (2), and gives an explicit representation of the $b_t$ in terms of the $a_r$ of 16.9(9) and certain other coefficients $\delta^{\mu,s}_{\nu,t}$ which satisfy a comparatively simple recurrence relation. He discusses the cases of integer values of $\mu$, $\nu$, $\mu \pm \nu$, and shows that all important expansions of solutions of 16.9(1) may be obtained by specializing the parameters in (2). For instance, if $a = 0$ and $\eta \to 1$ in the first expansion (2), we obtain 16.9(8); again if $a = 0$ and $\xi \to \infty$, we obtain 16.9(21).

We obtain new expansions for solutions of 16.9(1) if we take $a = \pm 1$ and $\xi \to \infty$ or $\eta \to 1$ in (2). These expansions, together with their regions of convergence are:

$$(4) \quad Ps^{\mu}_{\nu}(z, \theta) = \exp(\pm 2\theta^{1/2} zi) \sum_{t=-\infty}^{\infty} i^{\pm t} b^{\mu}_{\nu,t}(\theta, 1)\, P^{\mu}_{\nu+t}(z)$$

$$Qs^{\mu}_{\nu}(z, \theta) = \exp(\pm 2\theta^{1/2} zi) \sum_{t=-\infty}^{\infty} i^{\pm t} b^{\mu}_{\nu,t}(\theta, 1)\, Q^{\mu}_{\nu+t}(z) \qquad z \neq 1, -1, \infty$$

$$(5) \quad S^{\mu(j)}_{\nu}(z, \theta) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}\mu} s^{\mu}_{\nu}(\theta) \sum_{t=-\infty}^{\infty} b^{\mu}_{\nu,t}(\theta, 1)\, \psi^{(j)}_{\nu+t}[2\theta^{1/2}(z-1)]$$

$$|z-1| > 2, \qquad j = 1, 2, 3, 4$$

$$S^{\mu(j)}_{\nu}(z, \theta) = \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} s^{\mu}_{\nu}(\theta) \sum_{t=-\infty}^{\infty} b^{\mu}_{\nu,t}(\theta, -1)\, \psi^{(j)}_{\nu+t}[2\theta^{1/2}(z+1)]$$

$$|z+1| > 2, \qquad h = 1, 2, 3, 4$$

The coefficients in all these expansions satisfy three-term recurrence relations, and in some regions these expansions are more useful than those of the preceding section. For $\mathrm{Ps}^{\mu}_{\nu}(x, \theta)$, $\mathrm{Qs}^{\mu}_{\nu}(x, \theta)$ replace $P^{\mu}_{\nu+t}(z)$, $Q^{\mu}_{\nu+t}(z)$ in (4) by $\mathrm{P}^{\mu}_{\nu+t}(x)$, $\mathrm{Q}^{\mu}_{\nu+t}(x)$.

Meixner also obtained more general expansions by making $\xi \to \infty$ or $\eta \to 1$ in (2) without specializing $a$. The ensuing expansions contain an arbitrary parameter: for special values of this arbitrary parameter they reduce to 16.9(8) and 16.9(22) or (4) and (5).

Expansions of $S^{\mu(j)}_{\nu}(z, \theta)$ in series of Bessel functions of argument $2\theta^{1/2}(z^2-1)^{1/2}$ may be obtained by putting $a = \eta = 0$ in (2). Such expansions were given by Fisher (1937), Meixner (1944) and others.

*Approximations for small* $|\theta|$. From 16.9(7), (9), (24), (28), (29), taking $k=0$ in 16.9(29), we have

$$(6)\quad \lambda_\nu^\mu(0)=\nu(\nu+1),\quad Ps_\nu^\mu(z,0)=P_\nu^\mu(z),\quad Qs_\nu^\mu(z,0)=Q_\nu^\mu(z),$$

$$\mathrm{Ps}_\nu^\mu(x,0)=\mathrm{P}_\nu^\mu(x),\quad \mathrm{Qs}_\nu^\mu(x,0)=\mathrm{Q}_\nu^\mu(x),$$

$$a_{\nu,0}^\mu(0)=s_\nu^\mu(0)=1,\quad a_{\nu,r}^\mu(0)=0 \qquad r=\pm 1,\pm 2,\ldots$$

$$\lim_{\theta\to 0}\theta^{-\frac{1}{2}\nu}K_\nu^\mu(\theta)=\frac{e^{\nu\pi i}\Gamma(1+\nu-\mu)\Gamma(1/2-\nu)}{2^{\nu+1}\Gamma(\nu+3/2)}$$

$$\lim_{\theta\to 0}\theta^{-\frac{1}{2}\nu}S_\nu^{\mu(1)}(z,\theta)=-\frac{2^{-\nu-1}\Gamma(1/2-\nu)}{\Gamma(\mu-\nu)\Gamma(\nu+3/2)}\,e^{-\mu\pi i}Q_{-\nu-1}^\mu(z)$$

From the last of these relations and 16.9(20) it is easy to evaluate

$$\lim_{\theta\to 0}\theta^{-\frac{1}{2}\nu}S_\nu^{\mu(j)}(z,\theta) \qquad j=2,3,4$$

For expansions of $\lambda_\nu^\mu(\theta)$ and $a_{\nu,r}(\theta)$ in powers of $\theta$ see Meixner (1944, sec. 6.3).

*Asymptotic forms for large* $|z|$. From 16.9(17), 16.9(6) and 7.13(1), (2)

$$(7)\quad S_\nu^{\mu(3)}(z,\theta)=\tfrac{1}{2}\theta^{-\frac{1}{2}}z^{-1}e^{i(2\theta^{\frac{1}{2}}z-\frac{1}{2}\nu\pi-\frac{1}{2}\pi)}[1+O(|z|^{-1})]$$
$$z\to\infty,\quad -\pi<\arg(\theta^{\frac{1}{2}}z)<2\pi$$

$$(8)\quad S_\nu^{\mu(4)}(z,\theta)=\tfrac{1}{2}\theta^{-\frac{1}{2}}z^{-1}e^{-i(2\theta^{\frac{1}{2}}z-\frac{1}{2}\nu\pi-\frac{1}{2}\pi)}[1+O(|z|^{-1})]$$
$$z\to\infty,\quad -2\pi<\arg(\theta^{\frac{1}{2}}z)<\pi$$

and the asymptotic forms of $S_\nu^{\mu(1)}$, $S_\nu^{\mu(2)}$ follow by means of 16.9(19). Meixner (1951) has obtained asymptotic expansions in descending powers of $z-a$, with $a$ arbitrary, and has given the four term recurrence relations satisfied by the coefficients of his expansions.

The asymptotic form of the $Q$ s follows by 16.9(28), and the $P$ s can be represented as combinations of the $Q$ s by 3.3(3).

*Behavior near* $z=1$. If $\mu$ is not a positive integer, we have from 16.9(21) and 3.2(14)

(9) $$Ps_\nu^\mu(z,\theta)=\frac{2^{\frac{1}{2}\mu}}{\Gamma(1-\mu)}(z-1)^{-\frac{1}{2}\mu}\sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^\mu(\theta)\,[1+O(|z-1|)]$$

$$=\frac{(\frac{1}{2}z-\frac{1}{2})^{-\frac{1}{2}\mu}}{\Gamma(1-\mu)\,s_\nu^\mu(\theta)}\,[1+O(|z-1|)] \qquad z\to 1$$

and similarly

(10) $$\mathrm{Ps}_\nu^\mu(x,\theta)=\frac{(\frac{1}{2}-\frac{1}{2}x)^{-\frac{1}{2}\mu}}{\Gamma(1-\mu)\,s_\nu^\mu(\theta)}\,[1+O(1-x)] \qquad x\to 1-$$

The behavior of the $Qs$ can be deduced from (9) and the behavior of $S_\nu^{\mu(j)}$ follows by means of 16.9(28) and 16.9(20).

*Integral relations.* In order to obtain integral relations between solutions of 16.9(1) we remark that this equation arises when the wave equation $\Delta W+\kappa^2 W=0$ is separated in the coordinates $\xi$, $\eta$, $\phi$ introduced by (1). Let $N(\xi,\eta)\;e^{i\mu\phi}$ be a solution of $\Delta W+\kappa^2 W=0$, and let $f(z)$ be a solution of 16.9(1). By a computation similar to that carried out in sec. 16.3 it is seen that

(11) $$g(\xi)=\int_a^b N(\xi,\eta)\,f(\eta)\,d\eta$$

is a solution of 16.9(1), with $\xi=z$, provided that

(12) $$\left[(1-\eta^2)\left(\frac{\partial N}{\partial\eta}\,f-N\,\frac{df}{d\eta}\right)\right]_a^b=0$$

We choose $f(\eta)=Ps_\nu^{-\mu}(z,\theta)$ and

(13) $$N(\xi,\eta)=(\xi^2-1)^{\frac{1}{2}\mu}(\eta^2-1)^{\frac{1}{2}\mu}\exp(2\theta^{\frac{1}{2}}\xi\eta i)$$

From (9) and the asymptotic behavior of $Ps$ it follows that (12) is satisfied if we take $a=0$, $b=i\infty$, and $\mathrm{Re}(\theta^{\frac{1}{2}}\xi)>|\mathrm{Re}\;\theta^{\frac{1}{2}}|$. Under these circumstances

$$g(\xi)=(\xi^2-1)^{\frac{1}{2}\mu}\int_1^{i\infty}(\eta^2-1)^{\frac{1}{2}\mu}\,Ps_\nu^{-\mu}(\eta,\theta)\exp(2\theta^{\frac{1}{2}}\xi\eta i)\,d\eta$$

is a solution of 16.9(1) with $\xi=z$. Moreover, from (9) and the theory of Laplace integrals it follows that as $\xi\to\infty$ in $\mathrm{Re}(\theta^{\frac{1}{2}}\xi)>|\mathrm{Re}\;\theta^{\frac{1}{2}}|$, $g(\xi)$ behaves asymptotically as

$$2^{\frac{1}{2}\mu}\,\xi^{\mu}\int_1^{i\infty}\frac{(\frac{1}{2}\eta-\frac{1}{2})^{\frac{1}{2}\mu}}{\Gamma(1+\mu)\,s_\nu^{-\mu}(\theta)}\,(\eta-1)^{\frac{1}{2}\mu}\exp(2\theta^{\frac{1}{2}}\,\xi\eta i)\,d\eta$$

$$=\frac{\exp(2\theta^{\frac{1}{2}}\,\xi i+\frac{1}{2}\mu\pi i+\frac{1}{2}\pi i)}{(2\theta^{\frac{1}{2}})^{\mu+1}\;\xi\;s_\nu^{-\mu}(\theta)}$$

so that from (7)

$$g(\xi)=\frac{-e^{\frac{1}{2}(\mu+\nu)\pi i}}{(2\theta^{\frac{1}{2}})^{\mu}\,s_\nu^{-\mu}(\theta)}\;S_\nu^{\mu(3)}(\xi,\,\theta).$$

Thus we obtain the first of the two integral relations

(14) $$S_\nu^{\mu(3)}(\xi,\,\theta)=-e^{-\frac{1}{2}(\mu+\nu)\pi i}\,2^{\mu}\,\theta^{\frac{1}{2}\mu}\,s_\nu^{-\mu}(\theta)\,(\xi^2-1)^{\frac{1}{2}\mu}\times\int_1^{i\infty}(\eta^2-1)^{\frac{1}{2}\mu}\,Ps_\nu^{-\mu}(\eta,\,\theta)\exp(2\theta^{\frac{1}{2}}\,\xi\eta i)\,d\eta$$

$$\mathrm{Re}(\theta^{\frac{1}{2}}\,\xi)>|\mathrm{Re}\;\theta^{\frac{1}{2}}|$$

(15) $$S_\nu^{\mu(4)}(\xi,\,\theta)=-e^{\frac{1}{2}(\mu+\nu)\pi i}\,2^{\mu}\,\theta^{\frac{1}{2}\mu}\,s_\nu^{-\mu}(\theta)\,(\xi^2-1)^{\frac{1}{2}\mu}\times\int_1^{-i\infty}(\eta^2-1)^{\frac{1}{2}\mu}\,Ps_\nu^{-\mu}(\eta,\,\theta)\exp(-2\,\theta^{\frac{1}{2}}\,\xi\eta i)\,d\eta$$

$$\mathrm{Re}(\theta^{\frac{1}{2}}\,\xi)>|\mathrm{Re}\;\theta^{\frac{1}{2}}|$$

The proof of (15) is similar.

## 16.11. Spheroidal wave functions

In the applications to solutions of the wave equation in prolate or oblate spheroidal coordinates (see sections 16.1.2 and 16.1.3), $\mu = m$ is an integer in 16.9(1). Moreover, only those values of $\nu$ and $\lambda$ are of interest for which 16.9(1) possesses a solution which is bounded on the interval $(-1,\,1)$. Without restriction, we may take $m = 0,\,1,\,2,\,\ldots$ . We see from the table in sec. 3.9.2 that the only solution of 16.9(1) which remains bounded at $z = 1$ is $\mathrm{Ps}_\nu^m(x,\,\theta)$ (or a constant multiple thereof). From 16.9(22) and 3.9(13) and (15) we see that this solution is unbounded at $z = -1$ unless $\nu$ is also an integer. Accordingly, from now on we shall restrict ourselves to the differential equation

(1) $$(1-z^2)\frac{d^2y}{dz^2}-2z\,\frac{dy}{dz}+[\lambda_n^m(\theta)+4\theta(1-z^2)-m^2(1-z^2)^{-1}]\,y=0$$

where $m$ and $n$ are integers and $\theta$ is real. On account of 16.9(7), we may take $m,\,n = 0,\,1,\,2,\,\ldots$ and $n \geq m$.

Most of the older, and many of the more recent, papers deal exclusively with the case of integer $\mu$ and $\nu$, and the solutions of (1) are mostly referred to as *spheroidal wave functions*, although some authors use this name for the solutions of the more general equation 16.9(1). The $\lambda_n^m(\theta)$, $m$, $n$ = 0, 1, 2, ... are called the *characteristic values* of $\lambda$, and the bounded solutions $\mathrm{Ps}_n^m(x, \theta)$ which are the corresponding *characteristic functions* are called *spheroidal wave functions of the first kind*. There is a fairly extensive literature on spheroidal wave functions. For a bibliography and a summary of the results up to 1932 see Strutt (1932), for references to more recent literature see Bouwkamp (1947) and Meixner (1951): the latter paper also gives an excellent summary of the results. Some of the more recent papers are listed under Abramowitz, Bouwkamp, Eberlein, Hanson, Leitner and Spence, Meixner, Sips, Spence, Stratton *et al.* at the end of this chapter. For numerical tables see Stratton *et al.* (1941), Bouwkamp (1941, 1947), Meixner (1944), Leitner and Spence (1950). It should be noted that there is no uniform notation, and care is needed in using the results of the aforementioned papers.

The numerical computation of $\lambda_n^m(\theta)$ for moderate values of $\theta$ may be based on the infinite continued fractions mentioned in sec. 16.9: this method has the advantage of producing the ratios $a_r/a_0$ in the course of the computation. For a description of the computational routine see Bouwkamp (1941, 1947) and Blanch (1946). For small values of $\theta$ the characteristic values and the coefficients may be represented by series in ascending powers of $\theta$. Bouwkamp (1950) and Leitner and Spence (1950) give the expansion of $\lambda_n^m(\theta)$ in powers of $\theta$ up to and including $\theta^4$. The numerical values of the coefficients in this expansion have been tabulated by Bouwkamp (1941, 1947, 1950), while Meixner (1944) tabulated the coefficients in the expansion of $\lambda_n^m(\theta)$ up to and including $\theta^5$, and for $a_{n,r}^m(\theta)/a_{n,0}^m(\theta)$ up to and including $\theta^3$.

We assume throughout that $m$ and $n$ are integers, and $0 \leq m \leq n$. In the recurrence relation 16.9(9) satisfied by the coefficients of the expansions 16.9(22), the factor of $a_{r+1}$ vanishes when

$$2r = -m - n - 1 \qquad \text{or} \qquad 2r = -m - n - 2$$

according as $m + n$ is an odd or even integer. From the infinite continued fractions representing the coefficients it follows that

$$(2) \quad a_{n,r}^m(\theta) = 0 \qquad\qquad 2r \leq -m - n - 1$$

From 3.6(3) and (6) it follows that

(3) $\mathrm{P}^m_{n+2r}(x) = 0 \qquad -m-n-1 < 2r < m-n$

so that the first expansion (22) reduces to

(4) $\mathrm{Ps}^m_n(x, \theta) = \sum_{2r \geq m-n} (-1)^r a^m_{n,r}(\theta)\, \mathrm{P}^m_{n+2r}(x)$

or

(5) $\mathrm{Ps}^m_{m+2k}(x, \theta) = \sum_{r=0}^{\infty} (-1)^{k+r} a^m_{m+2k,\, r-k}(\theta)\, \mathrm{P}^m_{m+2r}(x)$

$\mathrm{Ps}^m_{m+2k+1}(x, \theta) = \sum_{r=0}^{\infty} (-1)^{k+r} a^m_{m+2k+1,\, r-k}(\theta)\, \mathrm{P}^m_{m+2r+1}(x)$

$k, m = 0, 1, 2, \ldots$

The coefficients satisfy 16.9(9) with

$\mu = m, \quad \nu = n, \quad$ and $\quad a_r = 0 \quad$ for $\quad 2r \leq -m-n-1.$

We *normalize* (4) so that

(6) $\int_{-1}^{1} [\mathrm{Ps}^m_n(x, \theta)]^2\, dx = \frac{1}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!}$

By 3.12(19) and (21) this is equivalent to normalizing the coefficients so that

(7) $\sum_{2r \geq m-n} \frac{1}{n+2r+\frac{1}{2}} \frac{(n+2r+m)!}{(n+2r-m)!} [a^m_{n,r}(\theta)]^2 = \frac{1}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!}$

and we complete the normalization by

(8) $a^m_{n,0}(\theta) > 0.$

On account of 16.10(6) this normalization is consistent with 16.9(23). The series

(9) $Ps^m_n(z, \theta) = \sum_{2r \geq m-n} (-1)^r a^m_{n,r}(\theta)\, P^m_{n+2r}(z)$

converges for all finite $z$, and the functions (4) and (9) differ only by a factor of $(\pm i)^m$.

From 3.3(7), (10) and 16.9(11) we have

(10) $$Ps_n^{-m}(z, \theta) = \frac{(n-m)!}{(n+m)!} Ps_n^m(z, \theta)$$

$$Ps_n^m(-z, \theta) = (-1)^n Ps_n^m(z, \theta)$$

and numerous other relations for $Ps$ and $\mathrm{Ps}$ follow from known formulas for Legendre functions. From 3.4(20) and 3.4(23) we have

(11) $$\mathrm{Ps}_{m+2k}^m(0, \theta) = \frac{(2m+2k)!}{(2k)!} \mathrm{Ps}_{m+2k}^{-m}(0, \theta)$$

$$= 2^m \pi^{1/2} \sum_{r=-k}^{\infty} \frac{(-1)^r a_{m+2k,r}^m(\theta)}{(k+r)!\,\Gamma(\tfrac{1}{2}-k-m-r)}$$

$$\mathrm{Ps}_{m+2k+1}^{\pm m}(0, \theta) = 0 \qquad k, m = 0, 1, 2, \ldots$$

(12) $$\frac{d\,\mathrm{Ps}_{m+2k}^{\pm m}}{dx}(0, \theta) = 0$$

$$\frac{d\,\mathrm{Ps}_{m+2k+1}^m}{dx}(0, \theta) = \frac{(2m+2k+1)!}{(2k+1)!} \frac{d\,\mathrm{Ps}_{m+2k+1}^{-m}}{dx}(0, \theta)$$

$$= -2^{m+1} \pi^{1/2} \sum_{r=-k}^{\infty} \frac{(-1)^r a_{m+2k+1,r}^m(\theta)}{(k+r)!\,\Gamma(-\tfrac{1}{2}-k-m-r)}$$

$$k, m = 0, 1, 2, \ldots$$

For the solutions 16.9(8) we have in this case

(13) $$S_n^{m(j)}(z, \theta) = S_n^{-m(j)}(z, \theta)$$

$$= (1-z^{-2})^{\frac{1}{2}m} s_n^{-m}(\theta) \sum_{2r \geq m-n} a_{n,r}^{-m}(\theta)\, \psi_{n+2r}^{(j)}(2\theta^{1/2} z)$$

$$j = 1, 2, 3, 4$$

$Qs_{-\nu-1}^{\mu}$ becomes infinite when $\nu - \mu$ is zero or a positive integer but $\sin[(\nu-\mu)\pi]\, Qs_{-\nu-1}^{\mu}$ approaches a finite limit. By 3.3(3)

$$\sin[(\nu-\mu)\pi]\, Qs_{-\nu-1}^{\mu}(z, \theta) \to (-1)^{m+n+1} \pi\, Ps_n^m(z, \theta)$$

as $\mu \to m$, $\nu \to n$, and 16.9(28) gives the connection

(14) $S_n^{m(1)}(z, \theta) = K_n^m(\theta)\, Ps_n^m(z, \theta)$

between the two solutions (9) and (13), thus showing that spheroidal wave functions of the first kind may be represented by series of Bessel functions of the first kind; and these series turn out to be convergent for every finite non-zero $z$. The expression for $K_n^m$ simplifies considerably, Using (9) and (13) and proceeding as in the derivation of 16.9(29), with $k = (m - n)/2$ or $(m - n + 1)/2$ according as $m - n$ is even or odd, we obtain

(15) $\Gamma(m + 3/2)\, K_n^m(\theta)\, \mathrm{Ps}_n^m(0, \theta)$

$$= \tfrac{1}{2}(-1)^n \pi^{1/2} \theta^{1/2 m} s_n^{-m}(\theta)\, a_{n,(m-n)/2}^m(\theta) \qquad n - m \text{ even}$$

(16) $\Gamma(m + 5/2)\, K_n^m(\theta) \dfrac{d\, \mathrm{Ps}_n^m}{dx}(0, \theta)$

$$= \tfrac{1}{2}(-1)^{n+1} \pi^{1/2} \theta^{(m+1)/2} s_n^{-m}(\theta)\, a_{n,(m-n+1)/2}^m(\theta) \qquad n - m \text{ odd}$$

From (14), (15), (16) follow explicit expressions for the values of $S^{(1)}$ and $dS^{(1)}/dz$ at $z = 0$.

Other expansions for spheroidal wave functions of the first kind are

(17) $Ps_n^m(z, \theta) = \exp(\pm 2\theta^{1/2} zi) \sum_{t=m}^{\infty} i^{\pm t} B_{n,t}^m(\theta)\, P_t^m(z)$

which follows from 16.10(4), some expansions which can be derived from 16.10(2), (5), and expansions in series of products of Bessel functions which were given by Meixner (1949).

Spheroidal wave functions of the first kind are orthogonal functions on the interval $(-1, 1)$. For statements about the zeros see Meixner (1944).

Both $S_n^{m(2)}(z, \theta)$ and $Qs_n^m(z, \theta)$ are *spheroidal wave functions of the second kind*. If $|z| > 1$, both of these functions satisfy the functional equation $f(-z) = (-1)^{n+1} f(z)$, and hence they are numerical multiples of each other. Meixner (1951) gives the relation between them in the form

(18) $2\theta^{1/2} K_n^{-m}(\theta)\, S_n^{m(2)}(z, \theta) = (-1)^{m+1} s_n^m(\theta)\, s_n^{-m}(\theta)\, Qs_n^m(z, \theta).$

Other expansions follow from 16.10(2), (4), (5) expansions in series of products of Bessel functions were given by Meixner (1949). *Spheroidal wave functions of the third kind* are $S_n^{m(3,4)}$: they can be expressed in series of Legendre functions by means of 16.9(19), 16.11(14), (18).

We are now in the position to construct appropriate normal solutions of the wave equation in spheroidal coordinates. First, let us take *prolate spheroidal coordinates* $u$, $v$, $\phi$. It has been explained in sec. 16.1.2 that for a wave function which is regular inside a spheroid $u = u_0$, $U$ is a spheroidal wave function of the first kind, and $V$ is a modified spheroidal wave function of the first kind. Thus, *interior prolate spheroidal wave functions* are seen to be of the form

(19) $S_n^{m(1)}(\cosh u, \tfrac{1}{4}\kappa^2 c^2)\, \mathrm{Ps}_n^m(\cos v, \tfrac{1}{4}\kappa^2 c^2)\, e^{\pm im\phi}$

$$m = 0, 1, 2, \ldots, n; \quad n = 0, 1, 2, \ldots$$

while *external prolate spheroidal wave functions* are of the form

(20) $S_n^{m(j)}(\cosh u, \tfrac{1}{4}\kappa^2 c^2)\, \mathrm{Ps}_n^m(\cos v, \tfrac{1}{4}\kappa^2 c^2)\, e^{\pm im\phi}$

$$j = 3, 4;\ m = 0, 1, \ldots, n; \quad n = 0, 1, \ldots$$

where $j = 3$ or $4$ according as the asymptotic behavior at infinity is prescribed as $r^{-1} e^{i\kappa r}$ or $r^{-1} e^{-i\kappa r}$.

For *oblate spheroidal wave functions* we obtain from sec. 16.1.3 similarly

(21) $S_n^{m(j)}(-i \sinh u, \tfrac{1}{4}\kappa^2 c^2)\, \mathrm{Ps}_n^m(\cos v, -\tfrac{1}{4}\kappa^2 c^2)\, e^{\pm im\phi}$

$$j = 1, 3, 4; \quad m = 0, 1, \ldots, n; \quad n = 0, 1, \ldots$$

where $j = 1$ for wave functions for the interior, and $j = 3, 4$ for the exterior, of an ellipsoid $u = u_0$. In (21), $4\theta = -\kappa^2 c^2$, and it is understood that $2\theta^{1/2} = i\kappa c$ is taken in the asymptotic formulas of sec. 16.10.

The expansion of an arbitrary function given on a (prolate or oblate) spheroid $u = u_0$ in a series of the form

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} (A_n^m \cos m\phi + B_n^m \sin m\phi)\, \mathrm{Ps}_n^m(\cos v, \theta)$$

is valid under the same conditions as for spherical surface harmonics, and the coefficients may be computed by using the orthogonal properties of trigonometric functions and of spheroidal wave functions.

**16.12. Approximations and asymptotic forms for spheroidal wave functions**

*Behavior near* $\pm 1$. The behavior of spheroidal wave functions near $\pm 1$ may be investigated by substituting the approximations given in the table in sec. 3.9.2 in the expansions of spheroidal wave functions in series of Legendre functions, and then using 16.11(2), 16.9(11), and 16.9(15) to simplify the formulas. The results are as follows.

(1) $$Ps_n^m(z,\theta) = [K_n^m(\theta)]^{-1} S_n^{m(1)}(z,\theta)$$

$$= \frac{(n+m)!}{(n-m)!}\,\frac{(z-1)^{\frac{1}{2}m}}{2^{\frac{1}{2}m}\, m!\, s_n^{-m}(\theta)} + O(|z-1|^{1+\frac{1}{2}m})$$

$$\mathrm{Ps}_n^m(x,\theta) = \frac{(n+m)!}{(n-m)!}\,\frac{(-1)^m (1-x)^{\frac{1}{2}m}}{2^{\frac{1}{2}m}\, m!\, s_n^{-m}(\theta)} + O(|1-x|^{1+\frac{1}{2}m})$$

$m = 0, 1, \ldots, n; \quad n = 0, 1, \ldots$

(2) $$Qs_n^0(z,\theta) = -2\theta^{\frac{1}{2}}[s_n^o(\theta)]^{-2} K_n^o(\theta)\, S_n^{o(2)}(z,\theta)$$

$$= -\frac{1}{2}[s_n^o(\theta)]^{-1}\log\left(\frac{z-1}{2}\right) - \sum_{2r\ge -n} (-1)^r a_{n,r}^o(\theta)\, h_{n+2r} + O(|z-1|)$$

$$Qs_n^m(z,\theta) = (-1)^{m+1}\, 2\theta^{\frac{1}{2}}[s_n^m(\theta)\, s_n^{-m}(\theta)]^{-1} K_n^{-m}(\theta)\, S_n^{m(2)}(z,\theta)$$

$$= \frac{(-1)^m (m-1)!\, 2^{\frac{1}{2}m-1}}{s_n^m(\theta)\,(z-1)^{\frac{1}{2}m}} + O(|z-1|^{1-\frac{1}{2}m})$$

$m = 1, 2, \ldots, n; \quad n = 1, 2, 3, \ldots$

where

(3) $$h_0 = 0, \qquad h_k = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} \qquad k = 1, 2, \ldots$$

For Qs, replace $z-1$ by $1-x$ in (2).

The behavior of these functions near $-1$ follows from

(4) $$Ps_n^m(-z,\theta) = (-1)^n Ps_n^m(z,\theta), \qquad Qs_n^m(-z,\theta) = (-1)^{n+1} Qs_n^m(z,\theta)$$

$$\mathrm{Ps}_n^m(-x,\theta) = (-1)^{n-m}\,\mathrm{Ps}_n^m(x,\theta), \qquad \mathrm{Qs}_n^m(-x,\theta) = (-1)^{n-m+1}\,\mathrm{Qs}_n^m(x,\theta)$$

*Behavior near infinity*. For $S_n^{m(j)}$ see 16.10(7), (8) and 16.9(19). The $Ps$ and $Qs$ may be expressed in terms of the $S_n^{m(j)}$ by means of 16.11(14), (18).

*Approximations for small* $|\theta|$. For $\lambda$, $Ps$, $Qs$, $a_r$, $s_n^m$ see 16.10(6). From 16.10(6) we also have

(5) $$\lim_{\theta\to 0} \theta^{-\frac{1}{2}n} K_n^m(\theta) = \frac{(n-m)!}{2^n(\frac{1}{2})_n(\frac{3}{2})_n}$$

$$\lim_{\theta\to 0} \theta^{-\frac{1}{2}n} K_n^{-m}(\theta) = \frac{(n+m)!}{2^n(\frac{1}{2})_n(\frac{3}{2})_n} \qquad m=0, 1, \ldots, n; \quad n=0, 1, \ldots$$

and then by 16.11(14), (18)

(6) $$\lim_{\theta\to 0} \theta^{-\frac{1}{2}n} S_n^{m(1)}(z, \theta) = \frac{(n-m)!}{2^n(\frac{1}{2})_n(\frac{3}{2})_n} P_n^m(z)$$

$$\lim_{\theta\to 0} \theta^{\frac{1}{2}n+\frac{1}{2}} S_n^{m(2)}(z, \theta) = \frac{(-1)^{m+1}2^{n-1}}{(n+m)!}\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n Q_n^m(z)$$

$$m=0, 1, \ldots, n; \quad n=0, 1, \ldots$$

*Asymptotic forms for large* $|\theta|$. First $\theta$ will be taken as positive. The substitution

(7) $$y=(1-z^2)^{\frac{1}{2}m}\, Y, \quad 2\theta^{\frac{1}{4}} z = Z$$

carries 16.11(1) into

(8) $$\left(1-\frac{Z^2}{2\theta^{\frac{1}{2}}}\right)\frac{d^2Y}{dZ^2} - \frac{m+1}{2\theta^{\frac{1}{2}}}\frac{dY}{dZ} + \left(\Lambda - \frac{Z^2}{4}\right) Y = 0$$

where

(9) $$\Lambda = \theta^{\frac{1}{2}} + \tfrac{1}{4}\theta^{-\frac{1}{2}}(\lambda_n^m - m - m^2)$$

For large $\theta$, (8) is approximately the differential equation 8.2(1) of parabolic cylinder functions, and the interval $-1<z<1$ corresponds, in the limit as $\theta\to\infty$, to $-\infty<Z<\infty$. Now, $\mathrm{Ps}_n^m$ is a bounded solution of 16.11(1), and also $(1-z^2)^{-\frac{1}{2}m}\,\mathrm{Ps}_n^m(z, \theta)$ is bounded on $-1<z<1$. On the other hand, it is seen from sec. 8.4 that Weber's differential equation has a solution which is bounded on $-\infty < Z < \infty$ if and only if $\Lambda - \frac{1}{2}$ is a non-negative integer. Moreover, this integer is equal to the number of zeros of the bounded solution; since $\mathrm{Ps}_n^m$ has exactly $n - m$ zeros, we conclude that $\Lambda$ is approximately $n - m + \frac{1}{2}$, and $\mathrm{Ps}_n^m$ is approximately a numerical multiple of $(1 - z^2)^{\frac{1}{2}m} D_{n-m}(2\theta^{\frac{1}{4}} z)$. Thus we obtain

$$(10)\quad \lambda_n^m(\theta) = -4\theta + 2\theta^{\frac{1}{2}}(2n - 2m + 1) + O(1) \qquad \theta \to \infty$$

$$\mathrm{Ps}_n^m(x, \theta) \sim c_n^m (1 - x^2)^{\frac{1}{2}m} D_{n-m}(2\theta^{\frac{1}{4}} x) \qquad \theta \to \infty$$

where

$$(11)\quad c_n^m = \mathrm{Ps}_n^m(0, \theta)/D_{n-m}(0) \qquad n - m \text{ even}$$

$$c_n^m = \tfrac{1}{2}\theta^{-\frac{1}{4}} \frac{d\,\mathrm{Ps}_n^m}{dx}(0, \theta) \Big/ \frac{dD_{n-m}}{dZ}(0) \qquad n - m \text{ odd}$$

Explicit expressions for $c_n^m$ follow from 8.2(4) and 16.11(11), (12).

In order to obtain increased accuracy, one may replace (10) by formal infinite series,

$$(12)\quad \lambda_n^m(\theta) = -4\theta + 2\theta^{\frac{1}{2}}(2n - 2m + 1) + \sum_{r=0}^{\infty} \theta^{-\frac{1}{2}r} \lambda_{n,r}^m$$

$$\mathrm{Ps}_n^m(x, \theta) = (1 - x^2)^{\frac{1}{2}m} \sum_{r=-\infty}^{\infty} c_{n,r}^m D_{n-m+2r}(2\theta^{\frac{1}{4}} x)$$

substitute (12) in 16.11(1) and then equate coefficients of like powers of $\theta$. Approximations along these lines were obtained by Meixner (1944, 1947, 1948, 1951), Eberlein (1948), Sips (1949). In particular, Meixner (1951) gives the expansion of $\lambda_n^m$ up to and including the term $\theta^{-5/2}$, and he also states some of the $c_{n,r}^m$. The usefulness of these formulas has been tested numerically.

If $x$ is bounded away from zero, the parabolic cylinder function in (10) may be replaced by its asymptotic representation 8.4(1). In the neighborhood of $x = 0$, the behavior of $\mathrm{Ps}_n^m(x, \theta)$ is more complex since all zeros cluster around this point.

When $\theta$ is negative, the points around which the zeros cluster are $x = \pm 1$, and accordingly, these are the points near which the behavior of $\mathrm{Ps}_n^m(x, \theta)$ is rather complex. To investigate the behavior near $x = 1$, the substitution

$$(13)\quad y = (1 - z^2)^{\frac{1}{2}m} Y, \qquad 4(-\theta)^{\frac{1}{2}}(1 - z) = Z$$

may be used to transform 16.11(1) into

$$(14)\quad Z\left[1-\frac{Z}{8(-\theta)^{1/2}}\right]\frac{d^2Y}{dZ^2}+(m+1)\left[1-\frac{Z}{4(-\theta)^{1/2}}\right]\frac{dY}{dZ}$$
$$+\left\{\Lambda-\frac{Z}{4}\left[1-\frac{Z}{8(-\theta)^{1/2}}\right]\right\}Y=0$$

where

$$(15)\quad 8\,\Lambda=(-\theta)^{-1/2}\,(\lambda_n^m-m-m^2).$$

For large values of $-\theta$, (14) is approximately a differential equation of the form 6.2(1) with

$$a_0=1,\quad a_1=0,\quad a_2=-1/4,\qquad b_0=0,\quad b_1=m+1,\quad b_2=\Lambda.$$

The general solution of this approximate equation is given by 6.2(6) as

$$e^{-1/2 Z}\,\mathfrak{Z}\left(\frac{m+1}{2}-\Lambda,\ m+1,\ Z\right)$$

where $\mathfrak{Z}(a, c, x)$ is the general solution of 6.1(2). Since $Y$ is bounded on $0<z<1$, it must be bounded, as $\theta\to-\infty$, on $0<Z<\infty$. Now, the only solution of the confluent hypergeometric equation with $c=m+1$ which is bounded at $Z=0$ is $\Phi(a, c, Z)$, and it is seen from 6.13(2) that this function increases exponentially as $Z\to\infty$ unless $a$ is zero or a negative integer. Thus, $\tfrac{1}{2}(m+1)-\Lambda=-M$, where $M=0, 1, 2, \dots$, and the solution is approximately a numerical multiple of

$$e^{-1/2 Z}\,\Phi(-M,\ m+1,\ Z)$$

or, by 6.9(36), a numerical multiple of

$$\exp[2(-\theta)^{1/2}\,z]\,L_M^m\,[4(-\theta)^{1/2}\,(1-z)].$$

Now, $M$ is the number of zeros of this solution in $0<z<1$. Since $\mathrm{Ps}_n^m(z,\theta)$ has $(n-m)/2$ or $(n-m-1)/2$ zeros in this interval according as $n-m$ is an even or odd integer, we have $n=m+2M$ or $m+2M+1$ according as $n-m$ is even or odd. Moreover, $\mathrm{Ps}_n^m(z,\theta)$ is an even or odd function of $z$ according as $n-m$ is even or odd, and hence we have the following results:

(16) $\lambda^m_{m+2k}(\theta) = 4(-\theta)^{1/2}(m+2k+1) + O(1) \qquad \theta \to -\infty$

$$\mathrm{Ps}^m_{m+2k}(x,\theta) \sim \tfrac{1}{2} c^m_{m+2k}(1-x^2)^{\frac{1}{2}m}\{\exp[2(-\theta)^{1/2}x]\, L^m_k[4(-\theta)^{1/2}(1-x)]$$

$$+ \exp[-2(-\theta)^{1/2}x]\, L^m_k[4(-\theta)^{1/2}(1+x)]\} \qquad \theta \to -\infty$$

$$c^m_{m+2k} = \mathrm{Ps}^m_{m+2k}(0)/L^m_k[4(-\theta)^{1/2}]$$

(17) $\lambda^m_{m+2k+1}(\theta) = 4(-\theta)^{1/2}(m+2k+1) + O(1) \qquad \theta \to -\infty$

$$\mathrm{Ps}^m_{m+2k+1}(x,\theta) \sim \tfrac{1}{2} c^m_{m+2k+1}(1-x^2)^{\frac{1}{2}m}\{\exp[2(-\theta)^{1/2}x]\, L^m_k[4(-\theta)^{1/2}(1-x)]$$

$$- \exp[-2(-\theta)^{1/2}x]\, L^m_k[4(-\theta)^{1/2}(1+x)]\} \qquad \theta \to -\infty$$

The coefficients $c^m_n$ may be obtained by comparing both sides for small values of $x$.

As in the case $\theta \to \infty$, increased accuracy may be obtained by expanding $\lambda^m_n$ in decreasing powers of $(-\theta)^{1/2}$, and $\mathrm{Ps}^m_n$ in a series of Laguerre polynomials (combined with exponential functions as above,) substituting in 16.11(1) and then equating coefficients of like powers of $\theta$. See Svartholm (1938), Meixner (1944, 1947, 1948, 1951), Sips (1949). In particular, Meixner (1951) gives the expansion of $\lambda^m_n$ up to and including the term $(-\theta)^{-5/2}$, and he also gives a few coefficients in the expansion in series of Laguerre polynomials.

If $x$ is bounded away from $\pm 1$, then the Laguerre polynomials in (16) and (17) may be replaced by the leading terms

$$\frac{(-1)^k}{k!}[4(-\theta)^{1/2}(1 \mp x)]^k$$

Near $\pm 1$ the behavior of Ps is more complex and cannot be described by elementary functions.

*Other asymptotic forms*. The asymptotic behavior of $\lambda^m_n(\theta)$ and $a^m_{n,r}(\theta)$ as $n \to \infty$ has been investigated by Meixner (1944) who showed that the continued fractions lead to expansions in descending powers of $2n+1$. He gives the expansion of $\lambda^m_n$ up to and including the term $(2n+1)^{-5}$, and the expansions of $a_r/a_0$ up to and including the terms $(2n+1)^{-2}$.

Abramowitz (1949) investigated the case of a large $m$, and that of large $m$ and $\theta$ by methods similar to those employed above for the investigation for large $|\theta|$. He also tested his formulas numerically.

## 16.13. Series and integrals involving spheroidal wave functions

*Integral relations and integral equations.* The integral relations established towards the end of sec. 16.10 remain valid for spheroidal wave functions. In addition, there are integral relations with $a = -1$, $b = 1$ since $Ps_n^m$ is bounded on $(-1, 1)$, and has a bounded derivative, and hence 16.10(12) is satisfied for $a = -1$, $b = 1$ whenever $N$ and $\partial N/\partial \eta$ are bounded. We take the nucleus 16.10(13) and consider

$$(1) \quad g(\xi) = (1-\xi^2)^{\frac{1}{2}m} \int_{-1}^{1} (1-\eta^2)^{\frac{1}{2}m} \, \mathrm{Ps}_n^m(\eta, \theta) \exp(2\theta^{\frac{1}{2}} \xi \eta i) \, d\eta$$

By the work of sec. 16.10, this is an ellipsoidal wave function, and since $g(\xi)$ is bounded on $-1 < \xi < 1$, it is a numerical multiple of $\mathrm{Ps}_n^m(\xi, \theta)$. In order to determine the numerical factor involved here, we compute

$$(2) \quad g(0) = \int_{-1}^{1} (1-\eta^2)^{\frac{1}{2}m} \, \mathrm{Ps}_n^m(\eta, \theta) \, d\eta$$

$$\frac{dg}{d\xi}(0) = 2\theta^{\frac{1}{2}} i \int_{-1}^{1} \eta (1-\eta^2)^{\frac{1}{2}m} \, \mathrm{Ps}_n^m(\eta, \theta) \, d\eta$$

by substituting 16.11(4). Now

$$(3) \quad \int_{-1}^{1} (1-\eta^2)^{\frac{1}{2}m} \, \mathrm{P}_{n+2r}^m(\eta) \, d\eta$$

clearly vanishes if $n - m$ is an odd integer because then the integrand is an odd function of $\eta$; and by 3.12(25) the integral also vanishes when $n - m$ is even and $n + 2r \neq m$. Lastly, when $n + 2r = m$, we have by 3.12(25)

$$(4) \quad \int_{-1}^{1} (1-\eta^2)^{\frac{1}{2}m} \, \mathrm{P}_m^m(\eta) \, d\eta = \frac{(-2)^m \, m!}{m + \frac{1}{2}}$$

Similarly,

$$\int_{-1}^{1} \eta (1-\eta^2)^{\frac{1}{2}m} \, \mathrm{P}_{n+2r}^m(\eta) \, d\eta$$

vanishes unless $n + 2r = m + 1$ and

$$(5) \quad \int_{-1}^{1} \eta (1-\eta^2)^{\frac{1}{2}m} \, P_{m+1}^m(\eta) \, d\eta = \frac{(-2)^m \, m!}{m + 3/2}$$

so that

$$(6)\quad g(0) = (-1)^{k+m} \frac{2^m\, m!}{m+\frac{1}{2}}\, a^m_{n,-k}(\theta) \qquad n = m + 2k$$

$$g(0) = 0 \qquad n = m + 2k + 1$$

$$\frac{dg}{d\xi}(0) = 0 \qquad n = m + 2k$$

$$\frac{dg}{d\xi}(0) = 2\theta^{\frac{1}{2}}\, i\, (-1)^{k+m} \frac{2^m\, m!}{m+3/2}\, a^m_{n,-k}(\theta) \qquad n = m + 2k + 1$$

Using these results and the parity of Ps, we obtain from (1) the integral equations

$$(7)\quad (m+\tfrac{1}{2})\, \mathrm{Ps}^m_n(0,\theta)\,(1-\xi^2)^{\frac{1}{2}m} \int_0^1 (1-\eta^2)^{\frac{1}{2}m} \cos(2\theta^{\frac{1}{2}}\xi\eta)\, \mathrm{Ps}^m_n(\eta,\theta)\,d\eta$$

$$= (-1)^{k+m}\, 2^{m-1}\, m!\, a^m_{n,-k}(\theta)\, \mathrm{Ps}^m_n(\xi,\theta) \qquad n = m + 2k$$

$$(8)\quad (m+3/2)\frac{d\mathrm{Ps}^m_n}{d\xi}(0,\theta)(1-\xi^2)^{\frac{1}{2}m} \int_0^1 (1-\eta^2)^{\frac{1}{2}m} \sin(2\theta^{\frac{1}{2}}\xi\eta)\, \mathrm{Ps}^m_n(\eta,\theta)\,d\eta$$

$$= (-1)^{k+m}\, 2^m\, m!\, \theta^{\frac{1}{2}}\, a^m_{n,-k}(\theta)\, \mathrm{Ps}^m_n(\xi,\theta) \qquad n = m + 2k + 1$$

Meixner (1951) gives also the integral relations

$$(9)\quad \int_{-1}^1 \exp(2i\,\theta^{\frac{1}{2}}\sigma\,\xi\eta)\, J_m\{2[\theta(1-\sigma^2)(1-\eta^2)(\xi^2-1)]^{\frac{1}{2}}\}\, \mathrm{Ps}^m_n(\eta,\theta)\,d\eta$$

$$= 2i^{n-m}\, S^{m(1)}_n(\xi,\theta)\, \mathrm{Ps}^m_n(\sigma,\theta)$$

$$(10)\quad s^{-m}_n(\theta)\int_{-1}^1 P^{-m}_m(\cos\chi)\, \psi^{(j)}_m(\kappa r)\, Ps^m_n(\eta,\theta)\,d\eta$$

$$= \frac{(-1)^m\, \theta^{-\frac{1}{2}m}\,(n+m)!}{2^{2m-1}\, m!\,(n-m)!}\,(a^2-1)^{-\frac{1}{2}m}\, S^{m(j)}_n(\xi,\theta)\, S^{m(1)}_n(a,\theta)$$

In (10), $\kappa r$ and $\cos\chi$ have the same meaning as in 16.10(3). When $j = 1$, (10) is valid for all $\xi$, when $j = 2, 3, 4$ only for sufficiently large $\xi$. Both relations can be established by remarking that their nuclei, as functions of $\xi$ and $\eta$, satisfy the partial differential equation of $N$ in sec. 16.10 and hence the integrals, as functions of $\xi$, are ellipsoidal wave functions.

In the case of (9), this wave function is bounded at $\xi = \pm 1$ and hence must be a multiple of $S_n^{m(1)}(\xi)$. The factor involved here may be determined by multiplying by $(\xi^2 - 1)^{-\frac{1}{2}m}$ both sides of (9), making $\xi \to 1$ and using (7), (8), and 16.12(1). In the case of (10), the asymptotic behavior as $\xi \to \infty$ determines the right-hand side.

Other integral formulas may be derived from some of the expansions of earlier sections by using the orthogonal properties of Legendre functions. For instance, it follows from 16.11(4), 16.9(11), and the orthogonal property and normalization, 3.12(19) and (21), of Legendre functions that

$$(11)\quad \int_{-1}^{1} \mathrm{Ps}_n^m(x, \theta)\, \mathrm{P}_l^m(x)\, dx = 0 \qquad \text{if} \qquad l - n \text{ is negative or odd}$$

$$\int_{-1}^{1} \mathrm{Ps}_n^m(x, \theta)\, \mathrm{P}_l^m(x)\, dx = \frac{(-1)^r a_{n,r}^m(\theta)}{l + \frac{1}{2}}\, \frac{(l+m)!}{(l-m)!}$$

$$= \frac{(-1)^r a_{n,r}^{-m}(\theta)}{l + \frac{1}{2}}\, \frac{(n+m)!}{(n-m)!} \qquad \text{if} \quad l - n = 2r, \quad r = 0, 1, 2, \ldots$$

Other integral formulas may be derived from expansions such as 16.10(2) and its various special and limiting cases. Some important integrals may also be obtained by giving special values to $a$, $\sigma$, $\xi$ in (9) and (10), see Meixner (1951).

From the series and integrals already obtained, a number of expansions in series of spheroidal wave functions, or products of such functions follows. (11) may be thought of as determining the Fourier coefficients in the expansion of $\mathrm{P}_l^m(x)$ in a series of spheroidal wave functions, and leads to the expansion

$$(12)\quad \mathrm{P}_l^m(x) = \sum_{r=0}^{\infty} (-1)^r\, \frac{l - 2r + \frac{1}{2}}{l + \frac{1}{2}}\, a_{l-2r,\, r}^{-m}(\theta)\, \mathrm{Ps}_{l-2r}^m(x, \theta)$$

which may also be regarded as the inversion of 16.11(4). Similarly (7)-(10) may be interpreted as determining the Fourier coefficients in the expansions of the nuclei of these integral relations in series of spheroidal wave functions, see Meixner (1951). The expansions of plane, spherical, and cylindrical waves in spheroidal waves were given by Meixner (1944, 1951), Leitner and Spence (1950).

## ELLIPSOIDAL WAVE FUNCTIONS

### 16.14. Lamé's wave equation

The differential equation

$$(1)\quad \frac{d^2\Lambda}{dz^2} + \{h - l[\mathrm{sn}(z, k)]^2 + \omega^2 k^2 [\mathrm{sn}(z, k)]^4\}\,\Lambda = 0$$

(see sec. 16.1.4) will be called the *Jacobian form of Lamé's wave equation:* it is sometimes also called the generalized Lamé equation, or the differential equation of ellipsoidal wave functions. If $\omega = 0$, (1) reduces to Lamé's equation 15.1(6). In this section, all elliptic functions will have the same modulus $k$, and 15.1(6) shows that $0 < k < 1$. Sec. 15.1.1 also shows that in ellipsoidal wave functions only those values of $z$ occur for which $\mathrm{Im}\, z = 0$, or $\mathrm{Im}\, z = \mathbf{K}'$, or else $\mathrm{Re}\, z = \mathbf{K}$ but at first (1) will be considered for arbitrary complex values of $z$.

An *algebraic form* of Lamé's wave equation may be obtained by the change of variables

$$(2)\quad (\mathrm{sn}\, z)^2 = x$$

which transforms (1) into

$$(3)\quad \frac{d^2\Lambda}{dx^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-k^{-2}}\right)\frac{d\Lambda}{dx} + \frac{hk^{-2} - lk^{-2}x + \omega^2 x^2}{4x(x-1)(x-k^{-2})}\,\Lambda = 0$$

The Weierstrassian form of (1) may be obtained by the substitution 15.2(2), trigonometric forms by 15.2(4), combined with $\Lambda = f(z)M$ where $f(z)$ is 1, $\mathrm{sn}\, z$, $\mathrm{cn}\, z$, $\mathrm{dn}\, z$, $\mathrm{cn}\, z\ \mathrm{dn}\, z$, $\mathrm{sn}\, z\ \mathrm{dn}\, z$, $\mathrm{sn}\, z\ \mathrm{cn}\, z$, or $\mathrm{sn}\, z\ \mathrm{cn}\, z\ \mathrm{dn}\, z$, and an alternative algebraic form by 15.2(8) and other rational transformations of (3).

Equation (3) has four singular points: $x = 0,\ 1,\ k^{-2}$ are regular singular points, each of them with exponents 0 and ½, and $x = \infty$ is an irregular singular point. For the general theory of equations with irregular singular points see Ince (1927, p. 417 ff.). Around any of the regular singular points there are solutions in terms of power series, very much like in the case of Heun's equation (sec. 15.3); but no such convergent expansion exists around the irregular singular point. Instead, there are formal expansions of the form

$$(4)\quad e^{\pm i\omega\xi} \sum_{0}^{\infty} c_n\, \xi^{-2-n}$$

where $\xi = x^{1/2}$, $(x-1)^{1/2}$, or $(x-k^{-2})^{1/2}$ (*subnormal solutions*, Ince 1927, sec. 17.53). Although these formal series are divergent, they represent asymptotically, as $x \to \infty$ in certain sectors, solutions of (3).

Equation (3) can also be considered, in several ways, as a confluent form of an equation of the Fuchsian class. The point of departure is either an equation with five regular singularities (Ince 1927, sec. 15.4) or else an equation with six elementary singularities (Ince 1927, p. 592).

From the general theory of differential equations with doubly-periodic coefficients (Ince 1937, p. 375ff., Poole 1936, p. 170ff.) it follows that (1) has a solution of the form

$$(5)\quad u_0(z) = e^{\mu z} \frac{\theta_1\left(\dfrac{z-a}{2\mathbf{K}}\right)}{\theta_1\left(\dfrac{z}{2\mathbf{K}}\right)} P(z)$$

where $a$ and $\mu$ are constants which depend on $h$, $k^2$, $l$, $\omega$, and $P(z)$ is a doubly-periodic function with periods $2\mathbf{K}$, $2i\mathbf{K}'$. [In writing down (5), we used the relation 13.20(1) between the sigma function and theta functions.] Clearly,

$$(6)\quad u_0(-z) = e^{-\mu z} \frac{\theta_1\left(\dfrac{z+a}{2\mathbf{K}}\right)}{\theta_1\left(\dfrac{z}{2\mathbf{K}}\right)} P(-z)$$

is also a solution, and it is seen from (5), (6) and Table 8 in sec. 13.19 that $a$ is determined up to its sign, and integer multiples of $2\mathbf{K}$ and $2i\mathbf{K}'$. Once one of the possible values of $a$ has been chosen, $\mu$ is determined.

In general $u_0(z)$ and $u_0(-z)$ are linearly independent, and the general solution of (1) is a linear combination of (5) and (6). The only exception arises when $u_0(z) = \pm u_0(-z)$, or

$$e^{2\mu z}\,\theta_1\left(\frac{z+a}{2\mathbf{K}}\right) = \pm\,\theta_1\left(\frac{z-a}{2\mathbf{K}}\right)$$

Putting $z = a$, we see from Table 9 of sec. 13.19 that $a/\mathbf{K}$ is a zero of $\theta_1(v)$ and hence $a = m\mathbf{K} + n\mathbf{K}'i$ in this case. Putting $z = \mathbf{K}$ we see from Table 8 of sec. 13.19 that $e^{2\mu\mathbf{K}} = \pm 1$ and hence $2\mathbf{K}\mu = n'\pi i$ in this case, and a brief computation shows that $n = n'$. In any event, it follows that in the exceptional case $u_0$ is either an even or an odd function of $z$, $u_0(z + 2\mathbf{K}) = \pm u_0(z)$, $u_0(z + 2\mathbf{K}'i) = \pm u_0(z)$ so that $2\mathbf{K}$ and $2\mathbf{K}'i$ are periods or half-periods of $u_0(z)$. In this exceptional case a solution of the second (or third) kind must be constructed in order to obtain a general solution of (1).

According to sec. 16.1.4, the boundary conditions for $B(\beta)$ and $C(\gamma)$ in the case of *ellipsoidal wave functions* are the same as in the case of ellipsoidal harmonics, and by sec. 15.1.1 this means that the only case of interest from the point of view of ellipsoidal wave functions is the case when (1) possesses a solution which is a *doubly-periodic* function of $z$, with periods $4\mathbf{K}$ and $4i\mathbf{K}'$. This is precisely the exceptional case of the last paragraph. The doubly-periodic solution is $u_0(z)$, and is called a *Lamé wave function of the first kind*. There are two conditions for the existence of such a solution, one is a condition on $a$, the other on $\mu$. Given $\omega$ [$= (a^2 - b^2)^{1/2} \kappa$ in the case of the wave equation], these two conditions determine *characteristic values* of both $h$ and $l$.

From now on we assume that $\omega$ is fixed in (1), and $h$ and $l$ have characteristic values. As $\omega \to 0$, the characteristic values of $l$ approach $l_n = n(n+1)k^2$ where $n = 0, 1, \dots$, to each $l_n$ there belong $2n + 1$ characteristic values of $h$, these being the characteristic values of $h$ belonging to Lamé polynomials (see sec. 15.1.1). This shows that for $\omega = 0$ the characteristic values of $l$ are degenerate (or multiple): this degeneracy disappears when $\omega \neq 0$ (see also Strutt 1932, p. 61).

If $h$ and $l$ have characteristic values, then $u_0(z)$ is a Lamé wave function of the first kind. We have seen above that in this case $u_0(-z)$ and $u_0(z)$ are linearly dependent, i.e., $u_0$ is either an even or an odd function of $z$, and it may be proved as in sections 15.5.1 and 16.4 that $u_0$ is also an even or an odd function of $z - \mathbf{K}$, and likewise of $z - \mathbf{K} - \mathbf{K}'i$. According to their parity at the points $0$, $\mathbf{K}$, $\mathbf{K} + \mathbf{K}'i$, Lamé wave functions of the first kind may be divided in eight classes, and functions within the same class may be characterized by the number of their zeros on the intervals $(0, \mathbf{K})$, $(\mathbf{K}, \mathbf{K} + \mathbf{K}'i)$. There does not appear to be a standard definition of these functions, nor is there a well-developed notation.

As in sections 15.5 and 16.4, the properties of Lamé wave functions at $z = 0, \mathbf{K}, \mathbf{K} + i\mathbf{K}'$ may be used to set up a number of Sturm-Liouville problems for the intervals $(0, \mathbf{K})$ and $(\mathbf{K}, \mathbf{K} + \mathbf{K}'i)$. As in sec. 15.5, each Lamé wave function is a common characteristic function of two Sturm-Liouville problems, one for each of the two intervals $(0, \mathbf{K})$ and $(\mathbf{K}, \mathbf{K} + \mathbf{K}'i)$. For each of these two Sturm-Liouville problems one obtains *characteristic curves*, that is characteristic values of $h$ in their dependence on $l$, and the characteristic values of $h$ and $l$ are determined by the intersections of these curves in the $h,l$-plane. *Orthogonal properties* of Lamé wave functions follow from these Sturm-Liouville problems in conjunction with the symmetry properties at the points 0, **K**, K + **K**$'i$.

No *integral equations* seem to be known for Lamé wave functions but Möglich (1927) has derived integral equations for *ellipsoidal surface wave functions*. From 16.1 (21), (22) it is seen that

$$(7) \quad \Psi(\beta, \gamma) = B(\beta)\, C(\gamma)$$

satisfies the partial differential equations

$$(8) \quad [(\operatorname{sn}\beta)^2 - (\operatorname{sn}\gamma)^2]^{-1}\left(\frac{\partial^2\Psi}{\partial\beta^2} - \frac{\partial^2\Psi}{\partial\gamma^2}\right) + \{\omega^2 k^2[(\operatorname{sn}\beta)^2 + (\operatorname{sn}\gamma)^2] - l\}\,\Psi = 0$$

Solutions of (8) which are regular on the surface of an ellipsoid (sec. 15.1.1) will be called ellipsoidal surface wave functions. Transformed to the coordinates $\phi$, $\theta$ introduced by 15.5 (45) we shall abbreviate (8) as

$$(9) \quad L_{\theta,\phi}\,\Psi = 0.$$

Now consider

$$(10) \quad \exp[i\kappa(x\sin\theta'\cos\phi' + y\sin\theta'\sin\phi' + z\cos\theta')]$$

which represents, for fixed $\theta'$, $\phi'$, a plane wave, and hence is a solution of $\Delta W + \kappa^2 W = 0$. Using 15.1 (8) and 15.5 (45), and putting $\omega = (a^2 - b^2)^{1/2}\kappa$, (10) becomes

$$(11) \quad K(\theta, \phi; \theta', \phi') = \exp\left[i\omega\left(k\operatorname{sn}a\sin\theta\sin\theta'\cos\phi\cos\phi' + i\frac{k}{k'}\operatorname{cn}a\sin\theta\sin\theta'\sin\phi\sin\phi' + i\operatorname{dn}a\cos\theta\cos\theta'\right)\right]$$

Möglich now shows that for any fixed $a$, $K$ satisfies

$$(12) \quad (L_{\theta,\phi} - L_{\theta',\phi'})K = 0$$

and deduces by a process similar to that employed in sections 15.5.3 and 16.3 that for each fixed $a$ the characteristic functions of the integral equation

(13) $\int_0^{\pi}\int_0^{2\pi} K(\theta, \phi; \theta', \phi')\, \Psi(\theta', \phi') \sin\theta' d\theta' d\phi' = \lambda \Psi(\theta, \phi)$

are ellipsoidal surface wave functions expressed in terms of the coordinates $\theta$, $\phi$ of sec. 15.5(45).

Very little is known about the actual construction of Lamé wave functions. Ellipsoidal surface wave functions reduce to ellipsoidal surface harmonics as $\omega \to 0$, and this suggests an expansion of ellipsoidal surface wave functions in a series of products of Lamé functions (i.e., in a series of ellipsoidal surface harmonics). For small values of $\omega$ the expansion would be expected to converge rapidly (Strutt 1932, p. 60ff.).

Möglich (1927) obtained a number of expansions of Lamé wave functions by expanding the nucleus of the integral equation (13) in various ways, and allotting particular values (mostly 0, $\pm \mathbf{K}$, $\pm \mathbf{K} \pm \mathbf{K}'i$) to $a$. The most noteworthy of his results are expansions of ellipsoidal surface wave functions in series of spherical surface harmonics, expansions of Lamé wave functions in series of Legendre functions of variable $k'^{-1}\,\mathrm{dn}\, z$ (other possible variables being $\mathrm{sn}\, z$, $k\ \mathrm{sn}\, z$, $\mathrm{cn}\, z$, $ikk'^{-1}\,\mathrm{cn}\, z$, and $\mathrm{dn}\, z$), and expansions of Lamé wave functions in series of spherical Bessel functions 16.9(6). These latter series have the advantage of exhibiting the asymptotic behavior of Lamé wave functions as $z \to i\mathbf{K}'$.

*Lamé wave functions of the second and third kinds* may be obtained by replacing $\psi_\nu^{(1)}$ in Möglich's expansions in series of Bessel functions by $\psi_\nu^{(j)}$, $j = 2, 3, 4$ (Möglich has the series with $\psi_\nu^{(4)}$ which he calls integrals of the second kind). For ellipsoidal wave functions, $B$ and $C$ in sec. 16.1.4 are Lamé wave functions of the first kind, while $A$ is a Lamé wave function of the first or the third kind according as the ellipsoidal wave function is constructed for the interior or exterior of an ellipsoid.

For further information on ellipsoidal wave functions see Malurkar (1935) and Möglich (1927).

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# CHAPTER XVII

## AN INTRODUCTION TO THE FUNCTIONS OF NUMBER THEORY

*Preliminary Remarks.* The purpose of this Chapter is merely to give a first information about the more common functions of number theory and to indicate where more results may be found. No comprehensive survey has been attempted, and in particular the whole theory of algebraic numbers has been omitted, as have been all topics which require the definition of a group, or a valuation, or other algebraic concepts.

In order to avoid too many references in the text, a list is given here of those standard works of reference which should be consulted for information on the topic of each individual section. For the whole of Chapter 17, L. E. Dickson (1919-1923) is the most important source. For the individual sections consult:

17.1. L. E. Dickson, 1919, vol. I; Hardy and Wright, 1938, 1945.
17.2. MacMahon, 1915, 1916; Hardy and Wright, 1938, 1945.
17.3. L. E. Dickson, 1919-1923.
17.5. Landau, 1927, vol. I.
17.6. Landau, 1927, vol. I; Hardy and Wright, 1938, 1945.
17.7. Landau, 1927, vol. II; Titchmarsh, 1930, 1951; Ingham, 1932.
17.8. Landau, 1927, vol. I, 1909, vol. I.
17.10. Landau, 1927, vol. II.

### 17.1. Elementary functions of number theory generated by Riemann's zeta function

#### 17.1.1. Notations and definitions

The following notations will be used throughout this Chapter:

| | |
|---|---|
| $l, m, n$ | denote positive integers (unless another definition is given). |
| $m \mid n$ | means that $m$ divides $n$. |
| $m \nmid n$ | means that $m$ is not a divisor of $n$. |

| | |
|---|---|
| $(m, n)$ | denotes the highest common divisor of $m$ and $n$. If $(m, n) = 1$, we say that $m$ is prime to $n$, or that $m$ and $n$ are coprime. |
| $\sum_{d\mid n}$, $\prod_{d\mid n}$ | sum or product taken over all (positive) divisors $d$ of $n$. |
| $\sum_{(m,n)=1}$ | sum taken over all $m$ which are prime to $n$. |
| $p, p_1, p_2$; $q, q_1, q_2$ | denote prime numbers, i.e., numbers $> 1$ which have no divisor except unity and the number itself. |
| $\sum_p$, $\prod_p$ | the sum or the product taken over all prime numbers $p = 2, 3, 5, 7, 11, \ldots$ . |

(1) $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$

is the *standard form* of $n$ written as a product of powers of different prime numbers. Except when $n = 1$, we assume that

(2) $\alpha_1 > 0, \alpha_2 > 0, \ldots, \alpha_\nu > 0$.

| | |
|---|---|
| $\nu(n)$ | denotes the number of different primes dividing $n$; $\nu(1) = 0$. |
| $\phi(n)$ | denotes *Euler's function*. It is the number of positive integers $m$ which are prime to, and do not exceed $n$. |
| $\phi_k(n)$ | $= \sum_{(m,n)=1,\ 1 \le m \le n} m^k$. $\phi_0(n) = \phi(n)$. |
| $J_k(n)$ | for $k = 1, 2, 3, \ldots$, denotes *Jordan's function*. It is the number of different sets of $k$ (equal or distinct) positive integers $\le n$ whose highest common divisor is prime to $n$. A common notation for $J_k(n)$ is $\tau^k(n)$ or $k^{\text{th}}$ totient of $n$. |
| $d(n) = \sum_{l\mid n} 1$ | is the number of divisors of $n$. |
| $d_k(n)$ | for $k = 2, 3, 4, \ldots$, denotes the number of ways of expressing $n$ as the product of $k$ different factors. Expressions in which the order of factors is different are regarded as distinct. |

(3) $\sigma_k(n) = \sum_{d\mid n} d^k$

denotes the sum of the $k^{\text{th}}$ powers of the divisors of $n$, (including 1 and $n$).

(4) $d(n) = d_2(n) = \sigma_0(n)$.

We shall write $\sigma(n)$ for $\sigma_1(n)$.

The following definitions refer to the standard form (1) of $n$.

$\lambda(n)$ denotes *Liouville's function*. If $n$ has the standard form (1), $\lambda(1)=1$ and $\lambda(n)=(-1)^{a_1+\cdots+a_\nu}$.

$\mu(n)$ denotes *Möbius' function*, $\mu(1)=1$, $\mu(n)=(-1)^\nu$ if $a_1=a_2=\cdots=a_\nu=1$. Otherwise $\mu(n)=0$.

$\Lambda(n)$ denotes zero unless $n=p^m$ is a power of a prime. In this case, $\Lambda(n)=\log p$.

*Multiplicative functions.* A function $f(n)$ which is defined for all positive integers $n$ and for which

(5) $f(n)f(m)=f(nm)$ if $(n, m)=1$,

is called *multiplicative*. If $f(n)f(m)=f(mn)$ for all $m$, $n$ then $f(n)$ is called *completely multiplicative*. The terms *factorable* and *distributive* are also used.

The functions which have been defined in this section are also called *arithmetical* functions, this name being applied to any function $f(n)$ defined for all positive integers $n$.

### 17.1.2. Explicit expressions and generating functions

If $n$ is written in the standard form (1), then $\phi(1)=1$, $J_k(1)=1$, and for $n>1$

(6) $\phi(n)=n(1-p_1^{-1})(1-p_2^{-1})\cdots(1-p_\nu^{-1})$

(7) $J_k(n)=n^k(1-p_1^{-k})(1-p_2^{-k})\cdots(1-p_\nu^{-k})$

(8) $d(n)=(a_1+1)(a_2+1)\cdots(a_\nu+1)$

(9) $$\sigma_k(n)=\frac{p_1^{k(\alpha_1+1)}-1}{p_1^k-1}\cdots\frac{p_\nu^{k(\alpha_\nu+1)}-1}{p_\nu^k-1}.$$

For a multiplicative function $f(n)$ there is the fundamental identity

$$\sum_{n=1}^{\infty} f(n)=\prod_p\,[1+f(p)+f(p^2)+\cdots]$$

valid if the series on the left is absolutely convergent. In this case the product on the right is also absolutely convergent, and it is known as the Euler product of the series. If $f(n)$ is completely multiplicative, then $1+f(p)+f(p^2)+\cdots$ is a geometric progression, and we have

$$\sum_{n=1}^{\infty} f(n)=\prod_p\,[1-f(p)]^{-1} \qquad f(n) \text{ completely multiplicative.}$$

Applying the fundamental identity to the completely multiplicative function $n^{-s}$ and to some multiplicative functions related to it, we obtain a number of identities involving Riemann's zeta function. The zeta function is discussed in sec. 17.7, and many of the identities below are obtained in this manner.

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1-p^{-s})^{-1} \qquad \text{Re } s > 1$$

(10) $$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n)\, n^{-s} \qquad \text{Re } s > 1$$

(11) $$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \phi(n)\, n^{-s} \qquad \text{Re } s > 2$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda(n)\, n^{-s} \qquad \text{Re } s > 1$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} |\mu(n)|\, n^{-s} \qquad \text{Re } s > 1$$

(12) $$\frac{[\zeta(s)]^2}{\zeta(2s)} = \sum_{n=1}^{\infty} 2^{\nu(n)}\, n^{-s} \qquad \text{Re } s > 1$$

(13) $$[\zeta(s)]^k = \sum_{n=1}^{\infty} d_k(n)\, n^{-s} \qquad \text{Re } s > 1, \quad k = 2, 3, \ldots$$

(14) $$\frac{[\zeta(s)]^4}{\zeta(2s)} = \sum_{n=1}^{\infty} [d(n)]^2\, n^{-s} \qquad \text{Re } s > 1$$

(15) $$\zeta(s)\,\zeta(s-k) = \sum_{n=1}^{\infty} \sigma_k(n)\, n^{-s} \qquad \text{Re } s > \max(1, \text{Re } k + 1)$$

(16) $$\frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \sigma_a(n)\,\sigma_b(n)\, n^{-s}$$

$$\text{Re } s > \max[1,\ \text{Re } a + 1,\ \text{Re } b + 1,\ \text{Re}(a+b) + 1]$$

(17) $$[\zeta(s)]^k \prod_{p} P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right) = \sum_{n=1}^{\infty} [d_k(n)]^2\, n^{-s} \qquad \text{Re } s > 1, \quad k \geq 2$$

where $P$ denotes Legendre's polynomial (defined in sec. 3.6.2),

(18) $$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-s},$$

where the prime indicates differentiation with respect to $s$. Relations (14), (16) were discovered by Ramanujan, and (17) was proved by Titchmarsh; (16) has been generalized by Chowla (1928).

The functions on the left-hand side of (10), (11), (12), (13), (14), (15), (16), (17), (18), may be considered as generating functions of the coefficients of $n^{-s}$ on the right-hand side because of the following lemma:

LEMMA: *If* $\sum_{n=1}^{\infty} c_n n^{-s} = 0$ *for all real* $s \geq s_0$, *and if the series converges absolutely for* $s = s_0$, *then* $c_n = 0$ *for* $n = 1, 2, 3, \ldots$ (*see Hardy and Wright*, 1945, *sec.* 17.1).

**17.1.3. Relations and properties**

The functions $\phi(n)$, $\mu(n)$, $J_k(n)$ are multiplicative and

(19) $\sum_{d|n} \Lambda(d) = \log n.$

The functions $\phi(n)$ and $\mu(n)$ are connected by *Möbius' inversion formula* (also called Dedekind-Liouville formula). Let $f(n)$ be defined for all $n = 1, 2, 3, \ldots$, and let

(20) $g(n) = \sum_{d|n} f(d).$

Then

(21) $f(n) = \sum_{d|n} \mu(d)\, g\left(\frac{n}{d}\right)$

and conversely. In particular:

(22) $n = \sum_{d|n} \phi(d), \qquad \phi(n) = \sum_{d|n} \frac{n}{d}\, \mu(d).$

Möbius' inversion formula is a consequence of

(23) $\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{if } n > 1 \\ 1 & \text{if } n = 1. \end{cases}$

It can also be written in the form:

(24) $f(x) = \sum_{m=1}^{\infty} \mu(m)\, m^{-s} F(mx),$

if

(25) $F(x) = \sum_{m=1}^{\infty} m^{-s} f(mx)$

where $f(x)$ is defined for all $x>0$, $|f(x)|=O(x^{s_0})$ as $x\to\infty$, and $\operatorname{Re} s > s_0+2$.

Another inversion formula (see Hardy and Wright, 1945, Chap. 16) can be stated by saying that each of the following equations is a consequence of the other

$$G(x)=\sum_{n=1}^{[x]} F\left(\frac{x}{n}\right), \quad F(x)=\sum_{n=1}^{[x]} \mu(n)\, G\left(\frac{x}{n}\right)$$

where $x$ is a real positive variable, $[x]$ is the largest integer $\leq x$ and where an empty sum (e.g., the first one if $x<1$) is interpreted as zero. If $F(x)=1$ for all $x$, this gives the formula of E. Meissel

$$\sum_{m=1}^{n} \mu(m)\left[\frac{n}{m}\right]=1.$$

The Möbius inversion formula has been generalized (see Cesáro, 1887; H. F. Baker, 1889; Gegenbauer, 1893; E. T. Bell, 1926) and it has been used for a definition of an arithmetical integration and differentiation, $g(n)$ in (20) being called the "integral" of $f(n)$ (see L. E. Dickson, 1919, vol. I, Chap. 14). Another connection between $\mu$ and $\phi$ was stated by Rademacher and proved by R. Brauer (1926):

$$(26)\quad \phi(m) \sum_{d\mid m,\ (d,n)=1} \frac{d}{\phi(d)}\, \mu\left(\frac{m}{d}\right)=\mu(m) \sum_{d\mid (m,n)} d\mu\left(\frac{m}{d}\right).$$

For the $\phi$ function we have

$$(27)\quad \sum_{d\mid n} (-1)^{n/d} \phi(d)=\begin{cases} 0 & \text{if } n \text{ is even} \\ -n & \text{if } n \text{ is odd,} \end{cases}$$

$$(28)\quad \sum_{l=1}^{n} \phi(l^r)\left\{1^{r-1}+2^{r-1}+\cdots+\left[\frac{n}{l}\right]^{r-1}\right\}=1^r+2^r+\cdots+n^r$$

where $r=1, 2, 3, \ldots$, and where $[x]$ denotes the largest integer $\leq x$.

$$(29)\quad \sum_{d\mid n} (n/d)\, \phi_k(d)=1^k+2^k+\cdots+n^k \qquad k=0, 1, 2, \ldots$$

$$(30)\quad \phi_1(n)=\tfrac{1}{2} n\, \phi(n) \qquad n>1$$

$$(31)\quad \sum_{d\mid n} (n/d)^3 \phi_3(d)=\Big\{\sum_{d\mid n} (n/d)\, \phi(d)\Big\}^2$$

$$(32)\quad \lim_{n\to\infty}\left\{\frac{1}{n^2}\left[\phi(1)+\phi(2)+\cdots+\phi(n)\right]\right\}=\frac{3}{\pi^2}$$

(33) $\liminf_{n\to\infty}\left(\frac{\phi(n)}{n}\log\log n\right)=e^{-\gamma}$

where $\gamma$ is Euler's constant. Davenport (1932) proved that for $n \to \infty$

$$\phi_\alpha(n)=\frac{n^\alpha}{\alpha+1}\{\phi(n)+O(1)\} \qquad \alpha \geq 0$$

and obtained analogous results for $\alpha < 0$.

The function $\mu(n)$ can be expressed in the form

(34) $\mu(n)=\sum_{(m,n)=1} e^{2\pi im/n}$.

This means that $\mu(n)$ is the sum of the primitive $n^{\text{th}}$ roots of unity, or the sum of those numbers, $\rho$, for which $\rho^n=1$ but $\rho^m \neq 1$ if $1 \leq m < n$. These numbers, $\rho$, are the zeros of a polynomial

(35) $k_n(x)=\prod_{d|n}(x^d-1)^{\mu(n/d)}$

of degree $\phi(n)$.

For the following results see Landau (1927, vol. 2, Chap. 7), and Titchmarsh (1951). Let

(36) $M(n)=\mu(1)+\mu(2)+\cdots+\mu(n)$.

Then for $n\to\infty$

(37) $M(n)=O\left[n^{1/2}\exp\left(\frac{A\log n}{\log\log n}\right)\right]$

where $A$ is a real positive constant. A consequence of this result is

(38) $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0.$

Riemann's hypothesis (see sec. 17.7) is true if and only if

(39) $\sum_{n=1}^{\infty}\mu(n)\,n^{-s}$

is convergent for all $s$ for which $\operatorname{Re} s > \frac{1}{2}$.

For $\Lambda(n)$, the analogue of (38) is

(40) $\sum_{n=1}^{\infty}\frac{\Lambda(n)-1}{n}=-2\gamma,$

where $\gamma$ denotes Euler's constant defined in 1.1 (4). See also Kienast (1926).

For the following account of the properties of $\sigma(n)$ and of $d(n)$ see Hardy-Wright (1945, Chap. 18). We have

$$\sigma(n) = O(n \log \log n)$$

$$\sigma(1) + \sigma(2) + \cdots + \sigma(n) = \frac{1}{12} \pi^2 n^2 + O(n \log \log n).$$

There is a positive constant $A$ such that

$$A < \frac{\sigma(n)\,\phi(n)}{n^2} \leq 1,$$

$$\limsup_{n\to\infty} \{\sigma_a(n)\, n^{-a}\} = \zeta(a) \qquad a > 1$$

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log \log n} = e^{\gamma},$$

(see Gronwall, 1913). For the case where $-1 < a < 0$ see Bellmann (1950).

Vaidyanathaswamy (1930, 1931) proved that

$$\sigma_k(m, n) = \sum_{d \mid (m,n)} \sigma_k\left(\frac{m}{d}\right) \sigma_k\left(\frac{n}{d}\right) d^k \mu(d),$$

and G. N. Watson (1935) showed that $\sigma_{2m+1}(n)$ is divisible by any fixed integer $k$ for almost all values of $n$. The term "almost all" is defined at the beginning of sec. 17.2.

If $\epsilon > 0$ is arbitrary and fixed, then

$$d(n) < 2^{(1+\epsilon)\log n/\log\log n}$$

for all sufficiently large $n$, and

$$d(n) > 2^{(1-\epsilon)\log n/\log\log n}$$

for an infinity of values of $n$. For $n \to \infty$

$$d(1) + d(2) + \cdots + d(n) = n \log n + (2\gamma - 1)n + O(n^{1/3})$$

where $\gamma$ is Euler's constant. For $d(d(n))$ and related questions see Ramanujan (1915).

The asymptotic behavior of

$$d_k(1) + d_k(2) + \cdots + d_k(n)$$

for large $n$ has been investigated by Titchmarsh (1938).

If $Q(n)$ denotes the number of integers $m$, $1 \leq m \leq n$ which are not divisible by the square of an integer > 1, then for $n \to \infty$ we have

$$Q(n) = 6n/\pi^2 + O(n^{1/2}).$$

*General theorems on arithmetical functions.* Bellmann and Shapiro (1948) proved that the functions $n$, $\phi(n)$, $\sigma(n)$, $d(n)$, $2^{\nu}(n)$, $\mu(n)$ are algebraically independent.

Schoenberg investigated the asymptotic properties of classes of arithmetical functions. For investigations of additive arithmetical functions see Erdös and Wintner (1939). For other results see E. T. Bell (1930); D. H. Lehmer (1931).

## 17.2. Partitions

### 17.2.1. Notations and definitions

We shall write

$$(1) \quad a \equiv b \pmod{n}$$

if $a - b$ is an integer which is divisible by $n$.

Let $\{a_\nu\}$, $\nu = 1, 2, 3, \dots$, be a set $S$ of positive integers and let $N(x)$ be the number of those $a_\nu$ which do not exceed $x$. Suppose that

$$(2) \quad \lim_{x \to \infty} x^{-1} N(x) = a$$

exists. If $a = 0$, we shall say that *almost no* integer $n$ belongs to $S$. If $a = 1$, we shall say that *almost all* integers $n$ belong to $S$.

The number of decompositions

$$(3) \quad n = m_1 + m_2 + \cdots + m_k \qquad k = 1, 2, 3, \dots$$

of $n$ into a sum of any number of positive integers $m_1, m_2, \dots, m_k$ where

$$(4) \quad m_1 \geq m_2 \geq \cdots \geq m_k$$

is called the *number of partitions* of $n$ and is denoted by $p(n)$. If $k$ is restricted so that

$$(5) \quad k \leq l$$

we write $p_l(n)$ for the *number of partitions* of $n$ *into at most* $l$ *parts*. If $m_1$ is also restricted, $m_1 \leq N$, we write $p_{l,N}(n)$ for the number of partitions of $n$ into at most $l$ parts none of which exceeds $N$. The number of partitions of $n$ into an even number of unequal parts shall be denoted by $E(n)$ and that into an odd number of unequal parts by $U(n)$.

### 17.2.2. Partitions and generating functions

If $P(n)$ is the number of partitions of $n$ of a certain type, and if, for sufficiently small $|x|$, the infinite series

(6) $$\sum_{n=1}^{\infty} P(n)\, x^n = F(x),$$

converges, then the generating function $F(x)$ is said to *enumerate* $P(n)$. This is meant to include the case where $F(0) \neq 0$; then $P(0)$ shall be defined to be equal to $F(0)$. We have

(7) $$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \{1 - x^k\}^{-1} \qquad |x| < 1$$

(8) $$\sum_{n=0}^{\infty} p_m(n)\, x^n = \prod_{k=1}^{m} (1 - x^k)^{-1} \qquad |x| < 1.$$

Relation (8) expresses the fact that $p_m(n)$ is also the number of partitions of $n$ into parts which do not exceed $m$. It can also be shown that the number of partitions of $n$ into precisely $m$ parts equals the number of partitions of $n$ into parts, the largest of which is precisely $m$.

Many theorems on partitions may be stated in the form of an identity for the enumerating function $F(x)$. These identities are usually of the following type: $F(x)$ is expressed as both an infinite product and a series; both the product and each term of the series can be expanded in a series of powers of $x$. Examples:

(9) $$\prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

(10) $$\prod_{k=1}^{\infty} (1 + x^k) = \prod_{k=1}^{\infty} \{1 - x^{2k-1}\}^{-1}.$$

Euler's identities:

(11) $$\prod_{k=1}^{\infty} (1 + x^{2k-1}) = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2k})},$$

(12) $$\prod_{k=1}^{\infty} (1 + x^{2k}) = 1 + \sum_{k=1}^{\infty} \frac{x^{k(k+1)}}{(1-x^2)(1-x^4)\cdots(1-x^{2k})},$$

(13) $$\prod_{k=1}^{\infty} (1 - x^k)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)},$$

(14) $$\prod_{k=1}^{\infty}(1-x^k)^{-1}=1+\sum_{k=1}^{\infty}\frac{x^{k^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^k)^2},$$

(15) $$\prod_{k=1}^{\infty}(1-x^k)=\sum_{m=-\infty}^{\infty}(-1)^m\, x^{\frac{1}{2}m(3m+1)}.$$

Jacobi's identities:

(16) $$\prod_{k=1}^{\infty}\{(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2})\}$$

$$=1+\sum_{n=1}^{\infty}x^{n^2}(z^{2n}+z^{-2n})=\sum_{m=-\infty}^{\infty}x^{m^2}z^{2m} \qquad z\neq 0,$$

(17) $$\prod_{k=1}^{\infty}\{(1-x^{2k-1})^2(1-x^{2k})\}=\sum_{m=-\infty}^{\infty}(-1)^m\, x^{m^2},$$

(18) $$\prod_{k=1}^{\infty}\left(\frac{1-x^{2k}}{1-x^{2k-1}}\right)=\sum_{n=0}^{\infty}x^{\frac{1}{2}n(n+1)},$$

(19) $$\prod_{k=1}^{\infty}(1-x^k)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)\,x^{\frac{1}{2}n(n+1)},$$

(20) $$\prod_{k=0}^{\infty}\{(1-x^{5k+1})(1-x^{5k+4})(1-x^{5k+5})\}=\sum_{m=-\infty}^{\infty}(-1)^m\, x^{\frac{1}{2}m(5m+3)},$$

(21) $$\prod_{k=0}^{\infty}\{(1-x^{5k+2})(1-x^{5k+3})(1-x^{5k+5})\}=\sum_{m=-\infty}^{\infty}(-1)^m\, x^{\frac{1}{2}m(5m+1)}.$$

Rogers-Ramanujan identities:

(22) $$\prod_{k=0}^{\infty}\{(1-x^{5k+1})^{-1}(1-x^{5k+4})^{-1}\}$$

$$=1+\sum_{m=1}^{\infty}\frac{x^{m^2}}{(1-x)(1-x^2)\cdots(1-x^m)},$$

(23) $$\prod_{k=0}^{\infty}\{(1-x^{5k+2})^{-1}(1-x^{5k+3})^{-1}\}$$

$$=1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

The identities (17) to (22) and also (15) follow from Jacobi's formula (16) for $z=e^{i\pi u}$, $x=e^{i\pi\tau}$; the right-hand side of (16) becomes the

Fourier expansion of $\theta_3(u|\tau)$; and the left-hand side is the expansion of $\theta_3$ in an infinite product, where $\theta_3$ is one of the elliptic theta functions in the usual notation (see Chap. 13). A survey on the connection between partition problems and modular forms was given by Rademacher (1940).

The formulas (9) to (23) can be stated in the form of partition theorems. Examples of such theorems are:

Formula (9) shows that every $n$ can be expressed in exactly one way as a sum of different powers of 2.

Formula (10) states the fact that the number of partitions of $n$ into unequal parts is equal to the number of its partitions into odd parts.

Formula (15) shows that

$$E(n) - U(n) = (-1)^k \quad \text{if} \quad n = \tfrac{1}{2}k(3k \pm 1), \qquad k = 1, 2, 3, \dots, m$$

$$E(n) - U(n) = 0 \quad \text{all other } n,$$

where $E$, $U$ are defined in sec. 17.2.1.

The general term in the sum on the right-hand side in (22) enumerates the number of partitions of $n - m^2$ into at most $m$ parts. Since

$$m^2 = 1 + 3 + \cdots + 2m - 1,$$

we find that it enumerates also the number of partitions of $n$ into at most $m$ parts of minimal difference 2. Therefore, we find that (22) is equivalent to the fact that the number of partitions of $n$ into parts of the form $5m + 1$ and $5m + 4$ is equal to the number of partitions of $n$ into parts with minimal difference 2.

For a corresponding theorem about the number of partitions into parts of the type $6m + 1$, $6m + 5$ see Schur (1926); an asymptotic formula for this number was given by Niven (1940).

For the non-existence of certain identities in the theory of partitions see D. H. Lehmer (1946) and Alder (1948).

### 17.2.3. Congruence properties

Ramanujan (1919, 1921) conjectured and Darling (1921), Mordell (1922) proved that

(24) $p(5n + 4) \equiv 0 \pmod 5$,

(25) $p(7n + 5) \equiv 0 \pmod 7$,

(26) $p(11n + 6) \equiv 0 \pmod{11}$.

These statements can be derived from certain identities, the first two of which are

(27) $$\sum_{n=0}^{\infty} p(5n+4)\,x^n = 5 \prod_{k=1}^{\infty} \frac{(1-x^{5k})^5}{(1-x^k)^6},$$

(28) $$\sum_{n=0}^{\infty} p(7n+5)\,x^n = 7 \prod_{k=1}^{\infty} \frac{(1-x^{7k})^3}{(1-x^k)^4} + 49x \prod_{k=1}^{\infty} \frac{(1-x^{7k})^7}{(1-x^k)^8}.$$

There exists a similar identity for the enumerating function of $p(13n+6)$ which was discovered by Rademacher and Zuckerman (1939). But not all the terms on the right-hand side of this identity are divisible by 13.

Watson (1938) proved that

(29) $p(n) \equiv 0 \pmod{7^b}$

if $n = 7^b n'$, where $(n', 7) = 1$, and $b = 2, 3, 4, \ldots$, and if $24n \equiv 1 \pmod{7^{2b-2}}$. For a survey of results of this type see Rademacher (1940).

D. H. Lehmer (1936, 1938) proved

(30) $p(599) \equiv 0 \pmod{5^4}$,

(31) $p(721) \equiv 0 \pmod{11^3}$,

(32) $p(14031) \equiv 0 \pmod{11^4}$,

and hereby showed that certain conjectures of Ramanujan are justified in some special cases. The number $p(14031)$ has 127 digits and was computed by using the asymptotic formulas of Hardy and Ramanujan (see sec. 17.2.4) for $p(n)$.

### 17.2.4. Asymptotic formulas and related topics

Hardy and Ramanujan (1916, 1918) showed that

(33) $$\lim_{n\to\infty} 4n\, 3^{1/2}\, p(n) \exp[-\pi(2n/3)^{1/2}] = 1.$$

They also obtained an asymptotic series for $p(n)$ up to terms of the order of magnitude $O(n^{-1/4})$; since $p(n)$ is an integer, this result makes it possible to compute $p(n)$ from the asymptotic expansion exactly if $n$ is large enough (D. H. Lehmer, 1938). For simplified proofs see also Knopp and Schur (1925). D. H. Lehmer (1937) showed that the Hardy-Ramanujan series is divergent. Rademacher (1937a, 1937b, 1943) obtained a remarkable convergent series for $p(n)$, namely,

$$p(n) = \frac{1}{2^{1/2}\pi} \sum_{k=1}^{\infty} A_k(n)\, k^{1/2} \frac{d}{dn} f_k\left(n - \frac{1}{24}\right),$$

where

$$f_k(n) = n^{-1/2} \sinh\left[\frac{\pi}{k}\left(\frac{2n}{3}\right)^{1/2}\right]$$

$$A_k(n) = \sum_{\substack{(h,k)=1 \\ 1 \le h \le k}} \exp\left\{-2\pi i n \frac{h}{k} + \pi i \sum_{\mu=1}^{k-1} \frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right)\right\}.$$

A summation formula for $p(n)$ was given by Atkinson (1939).

Husimi (1938) studied integral representations for $p_m(n)$.

Tricomi (1928) investigated the asymptotic behavior of $p_{l,N}(n)$, and Brigham (1950), general asymptotic formulas for partition functions.

For the whole of this subsection see also Rademacher (1940).

### 17.3. Representations as a sum of squares

*General remarks.* The problem of the representation of an integer as a sum of squares is a special case of the problem of its representations by a (positive definite) quadratic form. For this latter problem see Siegel (1935, 1936, 1937) and Minkowski (1911). The representation of $n$ as a sum of squares can also be considered as a special case of the problem of the representation as a sum of a fixed number of $k^{\text{th}}$ powers. For an account of the results in this field see Landau (1927, vol. II).

The evaluation (or approximate evaluation) of the sum

$$\sum_{n \le x} r_k(n)$$

is the problem of counting lattice-points in a $k$-dimensional sphere. For the case $k = 2$, or for the general theory of lattice-points in two-dimensional space, consult Landau (1927, vol. II), and sec. 17.10.

### 17.3.1. Definitions and notations

Let $k \ge 2$ be a fixed integer. Then $r_k(n)$ shall denote the number of representations of $n$ as a sum of $k$ squares of integers,

$$(1) \quad n = l_1^2 + l_2^2 + \cdots + l_k^2$$

where $l_1, \ldots, l_k$ need not be different from each other and may be negative or zero. Two representations shall be considered different if they involve the same numbers $l_1, \ldots, l_k$ in a different order. For example we have $r_2(2) = 4$, since $2 = 1^2 + 1^2 = (-1)^2 + 1^2 = 1^2 + (-1)^2 = (-1)^2 + (-1)^2$. We shall need the sums of powers of certain divisors of $n$. Let $d^*$, $d^{**}$, $d'$, $d''$, $d_+$, $d_-$, $d_1$, $d_2$, $d_3$, $d_4$ be any (positive) divisors of $n$ which satisfy the conditions

(2) $d^* \equiv 1 \pmod 4$, $\qquad d^{**} \equiv 3 \pmod 4$,

(3) $n/d' \equiv 1 \pmod 4$, $\qquad n/d'' \equiv 3 \pmod 4$,

(4) $d_+ \equiv 0 \pmod 2$, $\qquad d_- \equiv 1 \pmod 2$,

(5) $d_1 \equiv 0 \pmod 2$, $\qquad n/d_1 \equiv 0 \pmod 2$,

$d_2 \equiv 1 \pmod 2$, $\qquad n/d_2 \equiv 1 \pmod 2$,

(6) $d_3 \equiv 0 \pmod 2$, $\qquad n/d_3 \equiv 1 \pmod 2$,

$d_4 \equiv 1 \pmod 2$, $\qquad n/d_4 \equiv 0 \pmod 2$,

and let

(7) $E_k(n) = \Sigma\, d^{*k} - \Sigma\, d^{**k}$,

(8) $E'_k(n) = \Sigma\, d'^{\,k} - \Sigma\, d''^{\,k}$ ,

(9) $\Delta_k(n) = \Sigma\, d_-^k$ ,

(10) $\zeta_k(n) = \Sigma\, d_-^k - \Sigma\, d_+^k$ ,

(11) $\xi_k(n) = \Sigma\, d_1^k + \Sigma\, d_2^k - \Sigma\, d_3^k - \Sigma\, d_4^k$ .

We shall also need the coefficients of the expansion of certain products of elliptic theta functions in a series of powers. Let $\theta_\nu(u, \tau)$ $[\nu = 1, 2, 3, 4;\ \theta_4(u, \tau) = \theta_0(u, \tau)]$ denote the four elliptic theta functions (see Chap. 13). We shall write $\theta_\nu$ for $\theta_\nu(0, \tau)$ and $q$ for $e^{i\pi\tau}$. Then we have

(12) $$\theta_4 = \prod_{k=1}^{\infty} (1 - q^{2k})(1 - q^{2k-1})^2,$$

(13) $$\theta_2 = 2q^{1/4} \prod_{k=1}^{\infty} (1 - q^{2k})(1 + q^{2k})^2,$$

(14) $$\theta_3 = \prod_{k=1}^{\infty} (1 - q^{2k})(1 - q^{2k-1})^2 .$$

Using these infinite products for $\theta_4$, $\theta_2$, $\theta_3$ we define the functions $\Omega(m)$, $W(m)$, $G(m)$, $\Theta(m)$ by their generating functions

(15) $16 \sum_{m=0}^{\infty} \Omega(m) q^m = \theta_2^4 \theta_3^4 \theta_4^4$,

(16) $16 \sum_{m=0}^{\infty} W(m) q^m = \theta_2^4 \theta_3^6 \theta_4^4$,

(17) $16 \sum_{m=0}^{\infty} \Theta(m) q^m = \theta_2^4 \theta_3^{10} \theta_4^4$,

(18) $16 \sum_{m=0}^{\infty} G(m) q^m = \theta_2^4 \theta_4^4 \theta_3^6 (\theta_4^4 - \theta_2^4)$.

**17.3.2. Formulas for $r_k(n)$**

*Representation as a sum of an even number of squares.* Glaisher (1907) has given a survey of the known formulas for $r_{2l}(n)$ for $2l = 2, 4, \dots, 18$. His table has been supplemented by Ramanujan (1918) who gave formulas for $r_{20}$, $r_{22}$, $r_{24}$. For $2l \geq 12$, these formulas involve functions of the type of $\Omega(n)$, $W(n)$, $\Theta(n)$, $G(n)$ which do not have a number-theoretical significance. (Formulas which involve only expressions of number-theoretical significance have been developed by Boulyguine (see Dickson, 1939, vol. II, p. 317). For $2l = 10$ and for $2l = 18$, the formulas in the table by Glaisher involve also sums which are taken over powers of certain complex divisors of $r$, a complex divisor of $n$ being a number $a + ib$, where $a$, $b$ are integers and such that $(a^2 + b^2) | n$. These two cases being omitted, Glaisher's table reads (with the notations of sec. 17.3.1):

(19) $r_2(n) = 4E_0(n)$,

(20) $r_4(n) = (-1)^{n-1} 8 \xi_1(n)$,

(21) $r_6(n) = 4\{4E_2'(n) - E_2(n)\}$,

(22) $r_8(n) = (-1)^{n-1} 16 \zeta_3(n)$,

(23) $r_{12}(2n) = -8 \xi_5(n)$,

(24) $r_{12}(2n+1) = 8\{\Delta_5(2n+1) + 2\Omega(2n+1)\}$,

(25) $r_{14}(n) = \dfrac{4}{61}\{64 E_6'(n) - E_6(n) + 364 W(n)\}$,

(26) $r_{16}(n) = (-1)^{n-1} \dfrac{32}{17} \{\zeta_7(n) + 16\, \Theta(n)\}$.

For a formula for $r_{24}$ see sec. 17.4. The formula for $r_2(n)$ is equivalent to an identity in the theory of elliptic theta functions, viz.,

$$(27)\quad \theta_3^2 = \{\sum_{m=-\infty}^{\infty} q^{m^2}\}^2 = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$$

$$= 1 + 4\sum_{n,\,m=1}^{\infty} \{q^{(4m-3)n} - q^{(4m-1)n}\}.$$

As a consequence of (19) we have the following criterion. Let $k(p)$ be the highest power of the prime number $p$ which divides $n$. A necessary and sufficient condition for $n$ to have a representation as a sum of two squares is that $k(p)$ be even whenever $p \equiv 3 \pmod 4$.

Formula (20) is equivalent to Jacobi's celebrated identity

$$(28)\quad \theta_3^4 = \{\sum_{m=-\infty}^{\infty} q^{m^2}\}^4 = -4q\frac{d}{dq}\left(\log\frac{\theta_2}{\theta_4}\right)$$

$$= 1 + 8\sum_{n,\,m=1}^{\infty} \{nq^{nm} - 4nq^{4nm}\}.$$

This may also be stated in the following way: The number of representations of $n$ as a sum of four squares is eight times the sum of those divisors of $n$ which are not divisible by four. For an odd $n$, this is also eight times (and for an even $n$, it is 24 times) the sum of the odd divisors of $n$. This implies Lagrange's theorem: Every integer $n > 0$ has a representation as a sum of four squares. It also shows that $r_k(n) > 0$ for all $n$ and $k = 4, 5, 6, \ldots$ .

*Representation as a sum of an odd number of squares.* This problem is more complicated than the problem of representation as a sum of an even number of squares. Now $n$ can be represented as a sum of three squares if and only if, $n$ is not of the form

$$(29)\quad 4^a(8b+7) \qquad a, b = 0, 1, 2, \ldots .$$

For odd values of $n$, Eisenstein (1847) showed that

$$(30)\quad r_3(4m+1) = 24\sum_{l=1}^{m}\left(\frac{l}{4m+1}\right),$$

$$(31)\quad r_3(4m+3) = 8\sum_{l=1}^{2m+1}\left(\frac{l}{4m+3}\right),$$

where $\left(\frac{l}{k}\right)$ is the Legendre-Jacobi symbol defined in sec. 17.5.

If $m$ is odd and not divisible by the square of a prime number, Eisenstein (1847) announced, and Smith (1894), Minkowski (1911) proved that

(32) $r_5(n) = -80\, s, \quad -80\,\sigma, \quad -112\,\sigma, \quad 80\, s$

according as

$$n \equiv \quad 1, \qquad 3, \qquad 5, \qquad 7 \quad (\text{mod } 8).$$

Using the Legendre-Jacobi symbol of sec. 17.5, we have

$$s = \sum_{\mu=1}^{\frac{1}{2}n-\frac{1}{2}} \mu \left(\frac{\mu}{n}\right), \qquad \sigma = \sum_{\mu=1}^{\frac{1}{2}n-\frac{1}{2}} \mu \left(\frac{\mu}{n}\right).$$

Hardy (1920) proved that the number $\bar{r}_5(n)$ of primitive representations of $n$ as a sum of five squares (i.e., of representations for which the highest common divisor of the five squares is unity) is

(33) $$\bar{r}_5(n) = \frac{c}{\pi^2}\, n^{-3/2} \sum_{l=0}^{\infty} \left(\frac{n}{2l+1}\right) (2l+1)^{-2}$$

where

$$c = 80, \qquad 160, \qquad 112$$

according as

$$n \equiv 0, 1, 4, \quad n \equiv 2, 3, 6, 7, \quad n \equiv 5 \quad (\text{mod } 8).$$

For more general results, in particular for $r_7(n)$, see Mordell (1919b), Stanley (1927), Hardy (1918, 1920, 1927).

Hardy and Ramanujan (1918) have found asymptotic expansions for $r_k(n)$ which are exact when $k = 3, 4, 5, 6, 7, 8$.

## 17.4. Ramanujan's function

We define Ramanujan's function, $\tau(n)$, for $n = 1, 2, 3, \ldots$, by

(1) $$\sum_{n=1}^{\infty} \tau(n)\, x^n = x \prod_{k=1}^{\infty} (1 - x^k)^{24}.$$

Ramanujan's function is connected with $r_{24}(n)$ (defined in sec. 17.3.1) by

(2) $$\frac{691}{16} r_{22}(2n) = \sigma_{11}(2n) - 2\sigma'_{11}(n) - 8\,[259\,\tau(2n) + 512\,\tau(n)]$$

(3) $$\frac{691}{16}\, r_{24}(2n+1) = \sigma_{11}(2n+1) + 2072\,\tau(2n+1)$$

where $\sigma_{11}(m)$ is the sum of the elevenths powers of the divisors of $m$ and

$\sigma'_{11}(m)$ the sum of the eleventh powers of its odd divisors; see Ramanujan (1916), Hardy (1927).

Ramanujan conjectured, and Mordell (1919b) proved that $\tau(n)$ is a multiplicative function (in the sense of sec. 17.1.1) and that

$$(4) \qquad \sum_{n=1}^{\infty} \tau(n)\, n^{-s} = \prod_p [1 - \tau(p)\, p^{-s} + p^{11-2s}]^{-1}$$

where Re $s > 13/2$, and the product is taken over all prime numbers $p$. Mordell also showed that for all $p$

$$(5) \qquad \tau(p^m) = \tau(p)\, \tau(p^{m-1}) - p^{11}\, \tau(p^{m-2}) \qquad m = 2, 3, 4, \ldots .$$

It follows from (5) that $\tau(p^n)$ is a polynomial in $\tau(p)$ and $p^{11}$; this polynomial has been determined by Sengupta (1948). For an expansion of

$$\sum_{n \leq x} \tau(n)\, (x - n)^k$$

in a series involving Bessel functions see Wilton (1929) and sec. 17.11.2; for other series involving $\tau(n)$ see van der Blij (1948).

Ramanujan conjectured and Watson (1935) proved that $\tau(n)$ is divisible by 691 for almost all $n$ (in the sense defined at the beginning of sec. 17.2). This is true although, as Ramanujan showed, $\tau(n)$ is not divisible by 691 if

$$1 \leq n \leq 5000 \qquad n \neq 1381.$$

Walfisz (1938) proved that for almost all $n$, $\tau(n)$ is divisible by

$$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 691.$$

For congruence properties of $\tau(n)$ consult also Wilton (1929), Bambah and Chowla (1947). D. H. Lehmer showed that $\tau(n) \neq 0$ if

$$n < 214928640000.$$

Mordell (1917) proved a formula analogous to (4) for the coefficients $f(n)$ of the series

$$(6) \qquad \left\{ \sum_{n=0}^{\infty} (-1)^n\, q^{(2m+1)^2} \right\}^4 = \sum_{n=1}^{\infty} f(n)\, q^{4n},$$

this formula being

$$(7) \qquad \sum_{n=1}^{\infty} f(n)\, n^{-s} = \prod_p \{1 - 2f(p)\, p^{-s} + p^{5-2s}\}^{-1}.$$

This result was also conjectured by Ramanujan. For other results and generalizations see Rankin (1939).

### 17.5. The Legendre - Jacobi symbol

In this section $p$, $p_1$, $p_2$, ... , denote odd prime numbers and $u$, $v$ denote odd positive integers.

We shall say that the integer $k$ is a quadratic residue (mod $n$) if the congruence

$$(1)\quad x^2 \equiv k \pmod{n}$$

has an integer solution $x$. We define the Legendre-Jacobi symbol $\left(\frac{k}{u}\right)$ for all $k = 0, \pm 1, \pm 2, \dots$, and for all $u = 1, 3, 5, 7, \dots$, as follows. If $u = p$ is an odd prime,

$$(2)\quad \left(\frac{k}{p}\right) = 1 \quad \text{if} \quad p \nmid k \text{ and } k \text{ is a quadratic residue (mod } p),$$

$$(3)\quad \left(\frac{k}{p}\right) = -1 \quad \text{if} \quad p \nmid k \text{ and } k \text{ is not a quadratic residue (mod } p),$$

$$(4)\quad \left(\frac{k}{p}\right) = 0 \quad \text{if} \quad p \mid k.$$

If $u = p_1 p_2 \cdots p_r$ is a product of $r$ odd prime numbers (not necessarily different from each other), we define

$$(5)\quad \left(\frac{k}{u}\right) = \left(\frac{k}{p_1}\right)\left(\frac{k}{p_2}\right)\cdots\left(\frac{k}{p_r}\right).$$

If $u$, $v$ are odd positive integers and $(u, v) = 1$, we have

$$(6)\quad \left(\frac{u}{v}\right)\left(\frac{v}{u}\right) = (-1)^{(\frac{1}{2}u-\frac{1}{2})(\frac{1}{2}v-\frac{1}{2})}$$

$$(7)\quad \left(\frac{-1}{u}\right) = (-1)^{\frac{1}{2}u-\frac{1}{2}}$$

$$(8)\quad \left(\frac{2}{u}\right) = (-1)^{(u^2-1)/8}.$$

Equations (6), (7), (8) are called the *quadratic law of reciprocity* and *its first and second supplementary theorems*. In particular (7), (8), state that $-1$ is a quadratic residue (mod $p$) if and only if $p \equiv 1 \pmod 4$, and 2 is a quadratic residue (mod $p$) if and only if $p \equiv 1$ or $p \equiv 7 \pmod 8$. It should be observed that $\left(\frac{k}{u}\right) = 1$ implies that $k$ is a quadratic residue (mod $u$) only if $u$ is an odd prime.

Generalizations of the Legendre symbol can be defined if the theory of algebraic fields is employed. For this see, for instance, Hasse (1930).

*Jacobsthal's sums.* We define the $q^{\text{th}}$ Jacobsthal sum of $s$ by

$$(9)\quad \Phi_q(s) = \sum_{m=1}^{p-1} \left(\frac{m}{p}\right)\left(\frac{m^q+s}{p}\right) \qquad q = 2, 3, \dots, \quad s = 1, 2, 3, \dots .$$

Let the prime number $p$ be of the form $p = 4f + 1$, where $f$ is a positive integer. Then $p = a^2 + b^2$, where $a$, $b$ are integers. Jacobsthal (1907) proved that

$$(10)\quad a = \tfrac{1}{2}\Phi_2(r), \quad b = \tfrac{1}{2}\Phi_2(n), \quad \tfrac{1}{2}\Phi_2(-1) \equiv \tfrac{1}{2}(p-3) \pmod 8$$

where $r$ denotes any quadratic residue and $n$ denotes any quadratic non-residue (mod $p$). Analogous results for $p = 6f + 1 = a^2 + 3b^2$ were obtained by Schrutka (1911) and Chowla (1949). For various other results and generalizations see Whiteman (1949, 1952); E. Lehmer (1949).

## 17.6. Trigonometric sums and related topics

*Gaussian sums.* Let $n$ be a positive integer. We define for every integer $m$

$$(1)\quad S(m, n) = \sum_{r=0}^{n-1} \exp(2\pi i r^2 m/n).$$

If $(n, n') = 1$, then

$$(2)\quad S(m, nn') = S(mn', n)\, S(mn, n').$$

For $m = 1$

$$(3)\quad S(1, n) = \begin{cases} (1+i)\, n^{1/2} & \text{if } n \equiv 0 \\ n^{1/2} & \text{if } n \equiv 1 \\ 0 & \text{if } n \equiv 2 \\ i n^{1/2} & \text{if } n \equiv 3 \end{cases} \pmod 4.$$

If $n = p$ is a prime number and $(m, p) = 1$, then

$$(4)\quad S(m, p) = \sum_{r=1}^{p-1} \left(\frac{r}{p}\right) \exp\left(\frac{2\pi i r m}{p}\right) = \left(\frac{m}{p}\right) S(1, p)$$

$$= \begin{cases} \left(\dfrac{m}{p}\right) p^{1/2} & \text{if } p \equiv 1 \pmod 4, \\[2ex] \left(\dfrac{m}{p}\right) i p^{1/2} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $\left(\frac{m}{p}\right)$ denotes Legendre's symbol defined in sec. 17.5.

*Ramanujan's sums* are defined by

(6) $$c_n(m) = \sum_{(r,\, n)=1} \exp(2\pi irm/n),$$

where the sum is taken over the set of those of the numbers $r = 1, 2, \dots, n-1$ for which $(r, n) = 1$. Using Möbius' function (see sec. 17.1) we have

(7) $$c_n(m) = \sum_{d|n,\ d|m} d\mu\left(\frac{n}{d}\right)$$

where the sum is taken over all positive integers $d$ which divide both $n$ and $m$. If $(n, n') = 1$, then

(8) $$c_{nn'}(m) = c_n(m)\, c_{n'}(m).$$

A sum involving the $c_n(m)$ is

(9) $$\sum_{m=1}^{\infty} m^{-1} c_n(m) = -\Lambda(n).$$

For a proof consult Hölder (1936). For applications see Ramanujan (1918); the $c_n(m)$ are important for the representation of a number as a sum of squares. For series expansions see Carmichael (1932); for the statistics of Ramanujan's sums see Wintner (1942).

*Kloosterman's sums.* Let $n > 0$ be an integer and let $r$ denote any integer $0 < r \leq n$ such that $(r, n) = 1$. Then there exists a uniquely determined $r'$ such that

(10) $$0 < r' \leq n, \quad rr' \equiv 1, \quad (\text{mod } n).$$

Kloosterman's sum is defined for integer $u, v, n$ by

(11) $$S(u, v, n) = \sum_r \exp\left[\frac{2\pi i}{n}(ur + vr')\right].$$

If $(n, m) = 1$, then

(12) $$S(u, v, n)\, S(u, w, m) = S(u, vm^2 + wn^2, nm).$$

For applications consult Kloosterman (1926), Atkinson (1948). For generalizations see A. Weil (1948), also Salié (1931), D. H. Lehmer (1938), Whiteman (1945).

*Generalizations.* Gaussian sums have been generalized in many respects. For generalizations applied to the theory of quadratic forms consult Siegel (1935, 1936, 1937, 1941). Expressions of the type

(13) $\sum_{r=0}^{n-1} \exp\left(\frac{2\pi i m}{n} r^k\right)$ $(m, n) = 1$

for fixed values of $k > 2$ have been used by Hardy and Littlewood for the definitions of the so-called "singular series" for Waring's problem (i.e., the representation of an integer as a sum of a fixed number of $k^{\text{th}}$ powers); see Hardy and Littlewood (1920, 1921, 1922a, b, d, 1925). These are the papers which are usually referred to by the title *Partitio Numerorum*. For other types of trigonometric sums see Vinogradow (1939, 1940).

## 17.7. Riemann's zeta function and the distribution of prime numbers

Let $s$ be a complex variable. Then, for Re $s > 1$, Riemann's zeta function

(1) $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

is an analytic function of $s$. As Euler has shown,

(2) $\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}$ Re $s > 1$

where the product is taken over all prime numbers $p = 2, 3, 5, 7, \ldots$. The integral representation

(3) $$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = -\frac{\Gamma(1-s)}{2\pi i} \int_\infty^{(0+)} \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

shows that $\zeta(s)$ can be continued analytically and is one-valued and regular everywhere with the exception of $s = 1$ where $\zeta(s)$ has a simple pole, with residue 1. Equation (3) also gives

(4) $\zeta(0) = -\tfrac{1}{2}, \quad \zeta(-2m) = 0, \quad \zeta(1-2m) = -B_m/(2m)$

where $m = 1, 2, 3, \ldots$, and where $B_m$ is the $m^{\text{th}}$ Bernoulli number (see sec. 1.13).

The Laurent series of $\zeta(s)$ for the neighborhood of $s = 1$ was given by Stieltjes. We have

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1 (s-1) + \gamma_2 (s-1)^2 + \cdots$$

where $\gamma$ denotes Euler's constant [see 1.1 (4)], and where for $k = 1, 2, 3, \ldots$

$$\gamma_k = \lim_{n\to\infty} \left\{ \sum_{\nu=1}^{n} \frac{(\log \nu)^k}{\nu} - \frac{1}{k+1} (\log n)^{k+1} \right\}$$

(see Hardy, 1912).

From (3) follows the functional equation

(5) $\zeta(s) = 2^s \pi^{s-1} \sin(\tfrac{1}{2}\pi s)\, \Gamma(1-s)\, \zeta(1-s)$,

(6) $\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\tfrac{1}{2}\pi s)\, \Gamma(s)\, \zeta(s)$.

The zeros of $\zeta(s)$ at $s = -2, -4, -6, \ldots$ are the only real zeros.

It can be shown that apart from these, $\zeta(s)$ has no zeros outside the strip $0 < \operatorname{Re} s < 1$, but that there are infinitely many complex zeros, $\rho$, within this strip, and that

(7) $$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(\tfrac{1}{2}s+1)} \prod_{\rho} \left[\left(1 - \frac{s}{\rho}\right) e^{s/\rho}\right]$$

where the product is taken over all complex zeros, $\rho$, and where

(8) $b = \log 2\pi - 1 - \tfrac{1}{2}\gamma$.

The definition of Euler's constant $\gamma$ is given in 1.1(4).

If $h$ is a positive constant, $s = \sigma + it$,

$$0 \le \sigma \le 1, \quad 2\pi xy = |t|, \quad x > h > 0, \quad y > h > 0$$

$$\chi(s) = 2^s \pi^{s-1} \sin(\tfrac{1}{2}\pi s)\, \Gamma(1-s) = \zeta(s)/\zeta(1-s),$$

then

(9) $$\zeta(s) = \sum_{n \le x} n^{-s} + \chi(s) \sum_{n \le y} n^{s-1} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1}).$$

This equation is called the *approximate functional equation* of the zeta function. The $O$-terms in (9) can be replaced by an asymptotic series which proceeds in powers of $|t|^{-\frac{1}{2}}$, and whose coefficients are trigonometric functions. See Siegel (1931) and Titchmarsh (1935, 1951).

The function

(10) $\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\, \Gamma(\tfrac{1}{2}s)\, \zeta(s)$

satisfies

(11) $\xi(1-s) = \xi(s)$,

and has the integral representation

(12) $$\xi(s) = \tfrac{1}{2}s(s-1) \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x}\right) x^{\frac{1}{2}s-1}\, dx.$$

With

(13) $s = \tfrac{1}{2} + it, \quad \xi(s) = \Xi(t)$,

equation (12) gives

(14) $$\Xi(t) = \tfrac{1}{2} - (t^2 + \tfrac{1}{4}) \int_1^\infty \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x}\right) x^{-3/2} \cos(\tfrac{1}{2}t \log x)\, dx.$$

For other results connected with $\zeta(s)$ consult sec. 1.12.

*Zeros of* $\zeta(s)$. Riemann conjectured that all the complex zeros of $\zeta(s)$ have the real part ½ (or that $\Xi(t)$ has only real zeros). Riemann's hypothesis has neither been proved nor disproved, although a great deal of relevant information has been obtained since Riemann's work. It is known that Riemann's hypothesis is true if and only if

$$\sum_{n=1}^{\infty} \mu(n)\, n^{-s}$$

converges for Re $s > ½$. [For $\mu(n)$ consult sections 17.2, 17.3]

The following are some of the known results about the complex zeros of $\zeta(s)$. Let $s = \sigma + it$, let $N_0(T)$ denote the number of those zeros of $\zeta(s)$ for which $\sigma = ½$ and $0 < t < T$, let $N(T)$ denote the number of those zeros for which $0 < \sigma < 1$ and $0 < t < T$, and let $N(\sigma', T)$ denote the number of those zeros for which $0 < t < T$ and $\sigma > \sigma'$. Selberg (1942) proved that there is a positive constant $A$ such that

(15) $N_0(T) > AT \log T$

for sufficiently large $T$. Also as $T \to \infty$ we have

(16) $2\pi N(T) = T \log T - (1 + \log 2\pi)\, T + O(\log T)$,

(17) $N(\sigma, T) = O[T^{3(1-\sigma)/(2-\sigma)} (\log T)^5]$.

The last result was obtained by Ingham (1940) and holds for any fixed $\sigma$ in $½ < \sigma < 1$. By taking $\sigma$ a function of $T$ such that $\sigma - ½$ is sufficiently small, Selberg (1946) obtained an improvement of (17).

Concerning numerical evidence in favor of the Riemann hypothesis, see Titchmarsh (1935, 1936). Titchmarsh uses the approximate functional equation (9) and replaces the $O$ terms by quantitative approximations. This enables him to compute the complex zeros of $\zeta(\sigma + it)$ as far as $t = 1468$ and he finds them all, 1041 in number, on the line $\sigma = ½$.

A large number of theorems has been proved about the distribution of values of $\zeta(s)$. For these see Titchmarsh (1930). For the zeros of

$$\sum_{n=1}^{\infty} (n + a)^{-s}$$

consult Davenport and Heilbronn (1936).

*Distribution of prime numbers.* Let $\pi(x)$ denote the number of primes $p$ not exceeding $x$. Then for $x \to \infty$

(18) $$\pi(x) = \int_2^x \frac{du}{\log u} + O\{x \exp[-a (\log x)^{½}]\}$$

where $a$ is a positive absolute constant. In particular

$$(19)\quad \lim_{x \to \infty} [x^{-1} \pi(x) \log x] = 1,$$

and this result is known as the prime number theorem. The function

$$(20)\quad \pi(x) - \int_2^x \frac{du}{\log u} = P(x)$$

changes its sign infinitely many times as $x \to \infty$. In fact, there exists a constant $a$ such that both of the inequalities

$$(21)\quad P(x) > a \frac{x^{1/2}}{\log x} \log \log \log x,$$

$$(22)\quad P(y) < -a \frac{y^{1/2}}{\log y} \log \log \log y$$

are true for certain arbitrarily large values of $x$, $y$. However, if $x > 10$ , $P(x) < 0$ for the range of any existing tables.

All of these results about $\pi(x)$ can be proved from theorems relating to the distribution of the zeros of $\zeta(s)$. If Riemann's hypothesis is true, then for $x \to \infty$,

$$(23)\quad P(x) = O(x^{1/2} \log x).$$

But this cannot be proved at present. On the other hand, if (23) could be proved, or even if it could be shown that for any $\epsilon > 0$

$$P(x) = O(x^{1/2 + \epsilon})$$

as $x \to \infty$, Riemann's hypothesis would be true.

Mills (1947) proved the existence of a real number $A > 1$ such that $[A^{3^n}]$ is a prime for all integers $n \geq 1$, deducing this in a simple way from a result due to Ingham (1937) namely, that for all large $x$ there is a prime between $x^3$ and $(x+1)^3$. See also Niven (1951).

*Generalizations.* Dedekind's zeta function is the analogue to Riemann's zeta function for an algebraic number field; $\zeta(s)$ may be considered as Dedekinds's zeta function for the field of rational numbers (Hasse, 1927, 1930; Brauer, 1947). For the definition of a zeta function in "fields of characteristic $p$" and in a "simple algebra" consult F. K. Schmidt (1931), Hasse (1933), Deuring (1935) and Eichler (1949). Other generalizations of Riemann's zeta function are the $L$-series of Dirichlet and their generalizations and the zeta function of P. Epstein. For these see sections 17.8 and 17.9.

### 17.8. Characters and $L$ - series

Let $n > 1$ be a fixed positive integer, and let $m$ be any integer. We shall consider functions $\chi(m)$ such that

(i) $\chi(m) = \chi(m')$ if $m \equiv m' \pmod{n}$,

(ii) $\chi(1) = 1$,

(iii) $\chi(m) = 0$ if $(m, n) \neq 1$,

(iv) $\chi(m)\chi(m') = \chi(mm')$.

A function with these four properties is called a *character* (mod $n$). The function

$$(1) \quad \chi_1(m) = \begin{cases} 1 & \text{if} \quad (m, n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

is called the *principal character* (mod $n$). The value of $\chi(m)$ is different from zero if and only if $(m, n) = 1$, and its $\phi(n)^{\text{th}}$ power is then equal to 1. Here $\phi(n)$ denotes Euler's function of sec. 17.1. A character is called real if all of its values are real. The real characters modulo $n$ are the principal character and the Legendre-Jacobi symbol $\left(\frac{m}{n}\right)$. A product $\chi_a(m)\chi_b(m)$ of two characters is again a character (mod $n$). There exist precisely $\phi(n)$ different characters (mod $n$). If we denote $\phi(n)$ by $h$ and the $h$ different characters by $\chi_1, \dots, \chi_h$, then

$$(2) \quad \sum_{m=1}^{n} \chi_\nu(m)\overline{\chi}_\mu(m) = \begin{cases} h & \text{if} \quad \nu = \mu \\ 0 & \text{if} \quad \nu \neq \mu \end{cases} \qquad \nu, \mu = 1, 2, \dots, h,$$

where a bar denotes the conjugate complex value. If $(m, n) = 1$, $(m', n) = 1$,

$$(3) \quad \sum_{\nu=1}^{h} \chi_\nu(m)\chi_\nu(m') = \begin{cases} h & \text{if} \quad mm' \equiv 1 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

If we take $\mu = 1$ in (2), we find that

$$\sum_{m=1}^{n} \chi(m) = 0$$

for all characters different from the principal character.

Let $n > 1$ be a fixed integer, and let $\chi$ be a character modulo $n$. Then

$$(4)\quad L(s, \chi) = \sum_{m=1}^{\infty} \chi(m)\, m^{-s} \qquad \text{Re } s > 1$$

is called an $L$-*series*. The $L$-series were introduced by Dirichlet. They have many properties in common with Riemann's zeta function. The analogue to Euler's product is

$$(5)\quad L(s, \chi) = \prod_{p} [1 - \chi(p)\, p^{-s}]^{-1} \qquad \text{Re } s > 1$$

where the product is taken over all prime numbers $p$. If $\chi_1$ denotes the principal character, then

$$(6)\quad L(s, \chi_1) = \zeta(s) \prod_{p | n} (1 - p^{-s})$$

where the product is taken over the finite number of primes which divide $n$. If $\chi \neq \chi_1$, then $L(s, \chi)$ is an entire function of $s$ which does not vanish at $s = 1$.

Let $\chi$ be a character modulo $n$. Suppose that for some fixed divisor, $N$, of $n$ $(N < n)$ and for all $m$ and $m'$ satisfying

$$m \equiv m' \pmod{N}, \qquad (m, n) = (m', n) = 1$$

we have

$$\chi(m) = \chi(m').$$

Then we say that the character $\chi$ is *imprimitive* (mod $n$). Otherwise we say $\chi$ is a *primitive* character (mod $n$). If $n > 1$ and we choose $N = 1$, then $\chi$ will be imprimitive (mod $n$) if $\chi(m) = \chi(m')$ for $(m, n) = (m', n) = 1$; since $(1, n) = 1$, and $\chi(1) = 1$, such a $\chi$ can only be the principal character (mod $n$). Hence the principal character (mod $n$) is primitive if and only if $n = 1$.

*Let* $\chi$ be a primitive character (mod $n$). Then $L(s, \chi)$ vanishes for $s = 0, -2, -4, \dots$, if $\chi(-1) = 1$ and for $s = -1, -3, -5, \dots$, if $\chi(-1) = -1$. If we introduce

$$(7)\quad a = \tfrac{1}{2} - \tfrac{1}{2}\chi(-1),$$

then for every primitive character $\chi$ and for $n > 2$

$$(8)\quad \xi(s, \chi) = \pi^{-\frac{1}{2}s - \frac{1}{2}a}\, n^{\frac{1}{2}s + \frac{1}{2}a}\, \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}a)\, L(s, \chi)$$

is an entire analytic function which does not vanish outside of the strip $0 < \text{Re } s < 1$. It has a representation as an infinite product analogous to 17.7.(7), and it satisfies the functional equation

$$(9)\quad \xi(s, \chi) = \epsilon(\chi)\, \xi(1 - s, \chi)$$

where

(10) $\epsilon(\chi) = - in^{-1/2} \sum_{m=1}^{n} \chi(m) \cos(2m\pi/n)$.

It can be shown that $|\epsilon(\chi)| = 1$.

The $L$-series are important for the investigation of the distribution of prime numbers in an arithmetic progression.

For the relation between (5) and (9) see Hecke (1944), Petersson (1948). The zeros of $\xi(s, \chi)$ show a behavior similar to that of the zeros of $\zeta(s)$; it has also been conjectured (but not proved) that their real part is always ½. For lower bounds for $L(1, \chi)$ and for applications to number theory consult Siegel (1935, 1943), Page (1935), Rosser (1949).

The $L$-series have been generalized by Artin (1924, 1931, 1932). Artin introduced into the coefficients the characters of other groups besides those of the multiplicative group of the residue classes which are coprime to $n$. (These are the coefficients of the ordinary $L$ series.)

### 17.9. Epstein's zeta function

Let $p$ be a positive integer, let

$$g = (g_1, \dots, g_p), \quad h = (h_1, \dots, h_p), \quad m = (m_1, \dots, m_p)$$

be vectors with $p$ real components (the components of $m$ will be integers), and let

(1) $$(g, h) = \sum_{\nu=1}^{p} g_\nu h_\nu$$

be the scalar product of $g$ and $h$, and similarly for other vectors. Let $[a_{\mu\nu}]$ be a non-singular symmetric $p \times p$ matrix, $[a^*_{\mu\nu}]$ the inverse (reciprocal) matrix,

(2) $$\phi(x) = \sum_{\mu=1}^{p} \sum_{\nu=1}^{p} a_{\mu\nu} x_\mu x_\nu$$

the quadratic form associated with $[a_{\mu\nu}]$, $\phi^*(x)$ the quadratic form associated with $[a^*_{\mu\nu}]$, and let $\Delta$ be the determinant of the $a_{\mu\nu}$. We assume that the real part of $\phi(x)$ is positive definite. Finally, let $s$ be a complex variable.

Epstein's zeta function of order $p$, associated with the quadratic form $\phi$ is defined by

(3) $$Z \begin{vmatrix} g \\ h \end{vmatrix} (s)_\phi = Z \begin{vmatrix} g_1, \dots, g_p \\ h_1, \dots, h_p \end{vmatrix} (s)_\phi$$

$$= \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_p=-\infty}^{\infty}{}' [\phi(m+g)]^{-\frac{1}{2}ps} \exp[2\pi i (m, h)].$$

The prime indicates that summation is over all integers $m_1, \dots, m_p$ except if all components of $g$ are integers when the term $m = -g$ is to be omitted. The series is absolutely convergent, and defines an analytic function of $s$, in the half-plane $\operatorname{Re} s > 1$.

The fundamental theorem in the theory of zeta functions is the *functional equation*

$$(4)\quad \pi^{-\frac{1}{2}ps}\,\Gamma(\tfrac{1}{2}ps)\, Z\begin{vmatrix} g \\ h \end{vmatrix}(s)_\phi$$
$$= \Lambda^{-\frac{1}{2}}\,\pi^{-\frac{1}{2}p(1-s)}\,\Gamma[\tfrac{1}{2}p(1-s)]\,e^{-2\pi i(g,h)}\,Z\begin{vmatrix} h \\ -g \end{vmatrix}(1-s)_{\phi^*}.$$

The function defined by (3) and its analytic continuation is an entire function of $s$ except when all components of $h$ are integers: in the latter case the zeta function has a simple pole at $s = 1$, and the residue at this pole is

$$(5)\quad \pi^{\frac{1}{2}p}\,\Lambda^{-\frac{1}{2}}/\Gamma(\tfrac{1}{2}p+1).$$

The zeta function vanishes at

$$(6)\quad s = -2k/p, \qquad k = 1, 2, 3, \dots .$$

It also vanishes at $s = 0$ unless all components of $g$ are integers when its value at $s = 0$ is

$$(7)\quad -\exp[-2\pi i(g, h)].$$

These results are due to P. Epstein (1903, 1907). Epstein has also investigated some special cases, for instance, the cases where $p = 1$ or $p = 2$ and where all components of $g$ and $h$ are zero. In particular, the constant $c_0$ in the Laurent expansion of

$$(8)\quad Z\begin{vmatrix} 0\ 0 \\ 0\ 0 \end{vmatrix}(s)_\phi = \frac{c}{s-1} + c_0 + c_1(s-1) + \cdots$$

has been determined by Epstein. He also showed that the results of Herglotz (1905) can be derived from his formulas. Herglotz had investigated sums of the type

$$(9)\quad \sum_{a=-\infty}^{\infty}\ \sum_{b=-\infty}^{\infty}{}'\ (a+ib)^n\,(a^2+b^2)^{-\frac{1}{2}n-s}$$

where $n = 0, 2, 4, \dots$ . Siegel (1943) has investigated and generalized Epstein's zeta function and has proved theorems about the zeros.

### 17.10. Lattice points

A lattice point in the $x$, $y$-plane is a point whose coordinates are integers. There exists a general theorem of van der Corput (1919) about the number of lattice points in certain domains, a special case of which will be stated below. We define a domain $D$ in the $x$, $y$-plane as follows. Let $w - \frac{1}{2}$ be a positive integer, and let $f(x)$ be defined and possess continuous and positive first and second derivatives in $\frac{1}{2} \leq x \leq w$. Let

(1) $f(\frac{1}{2}) > 2, \quad 0 < f'(x) < 1, \quad f''(x) > z^{-3}$

where $z > 1$ is independent of $x$. Let $D$ be the closed domain

(2) $\frac{1}{2} \leq x \leq w, \quad \frac{1}{2} \leq y \leq f(x),$

let

(3) $A(D) = \int_{\frac{1}{2}}^{w} [f(x) - \frac{1}{2}]\, dx$

be its area, and let $L(D)$ denote the number of lattice points in $D$. Then van der Corput's theorem states that

(4) $|L(D) - A(D)| < cz^2$

where $c$ is a constant. Jarník (1926) has proved that for certain curves $f(x)$, the exponent 2 on the right-hand side of (4) is the best possible in the sense that it cannot be replaced by any smaller exponent.

More detailed results have been obtained for domains enclosed by special curves, in particular by a circle. Let $A(u)$ denote the number of lattice points within the closed domain

(5) $x^2 + y^2 \leq u.$

With the notations of sec. 17.3 we may also write

(6) $A(u) = \sum_{n \leq u^{\frac{1}{2}}} r_2(n).$

Let $J_1(z)$ denote the Bessel function of the first kind of order one (see sec. 7.2.1). Then Hardy proved for all $u > 0$

(7) $$\lim_{\epsilon \to 0} [\tfrac{1}{2} A(u + \epsilon) + \tfrac{1}{2} A(u - \epsilon)]$$

$$= \pi u + u^{\frac{1}{2}} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} r_2(n) J_1[2\pi(nu)^{\frac{1}{2}}].$$

If $u$ is not an integer, the left-hand side of (7) is simply $A(u)$. It can be proved that

$$A(u) - \pi u = O(u^{\nu})$$

is true for every $\nu \geq 1/3$ and is not ture for any $\nu \leq 1/4$.

There exists a large number of papers on the theory of lattice points; in particular, the number of lattice points in an ellipsoid has been investigated by van der Corput.

## 17.11. Bessel function identities

Researches on the order of magnitude of various numerical functions have yielded a number of identities involving Bessel functions. The two examples

$$(1)\qquad \sum_{n\leq x}{}' \, r_2(n) = \pi x + x^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r_2(n)}{n^{\frac{1}{2}}} J_1[2\pi(nx)^{\frac{1}{2}}]$$

and

$$(2)\qquad \sum_{n\leq x}{}' \, \tau(n) = x^6 \sum_{n=1}^{\infty} \frac{\tau(n)}{n^6} J_{12}[4\pi(nx)^{\frac{1}{2}}]$$

have already been referred to. Other examples are

$$(3)\qquad \sum_{n\leq x}{}' \frac{\sigma(n)}{n} = \frac{\pi^2 x}{6} - \tfrac{1}{2}(\gamma + \log 2\pi x) + \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} \quad J_0[4\pi(nx)^{\frac{1}{2}}]$$

$$(4)\qquad \sum_{n\leq x}{}' \, d(n) = x \log x + (2\gamma - 1)\,x + \tfrac{1}{4}$$

$$- x^{\frac{1}{2}} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} d(n) \{ Y_1[4\pi(nx)^{\frac{1}{2}} + 2\pi^{-1} K_1[4\pi(nx)^{\frac{1}{2}}]\}$$

where $\gamma$ is Euler's constant and the prime indicates that the last term of the sum is to be multiplied by $\frac{1}{2}$ if $x$ is an integer. The infinite series of Bessel functions can be thought of as representing exact expressions for the error made in approximating the left-hand sides by the elementary functions on the right.

Voronoi (1904) stated (1) without proof and Hardy (1915) was the first to prove it rigorously. Formula (3) is due to Wigert (1917) and (4) to Voronoi (1904).

Delicate questions of convergence can be avoided by considering the 'integrated form' of such identities in which the left members assume the form

$$\sum_{n\leq x} a(n)\,(x-n)^q/q!$$

Oppenheim (1926) gave a general method for deriving most of the above and more general identities and discussed the summability by Rieszian means of the infinite series on the right in case of divergence. Apostol (1951) gave a short proof of a theorem of Landau (1915) which states that the general identity

$$(5)\quad \frac{1}{q!}\sum_{n=0}^{x} a(n)(x-n)^q = \rho\frac{\Gamma(k)}{\Gamma(k+q+1)}\, x^{k+q} + \gamma\frac{x^{\frac{1}{2}(q+k)}}{(2\pi/\lambda)^q}\sum_{n=1}^{\infty}\frac{a(n)}{n^{\frac{1}{2}(k+q)}}\, J_{k+q}\left[\frac{4\pi}{\lambda}(nx)^{\frac{1}{2}}\right]$$

holds if the numbers $a(n)$ are coefficients of a Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a(n)\, n^{-s}$$

converging absolutely for Re $s > k$, regular for all $s$ except for a possible pole at $s = k$ with residue $\rho$, and having functional equations of the form

$$\left(\frac{\lambda}{2\pi}\right)^s \Gamma(s)\,\phi(s) = \gamma\left(\frac{\lambda}{2\pi}\right)^{k-s}\Gamma(k-s)\,\phi(k-s).$$

Such Dirichlet series have been studied in detail by Hecke (1938). Examples of permissible coefficients $a(n)$ are Ramanujan's function $\tau(n)$ and the functions $r_k(n)$ of sec. 17.3. The series of Bessel functions on the right of (5) is absolutely convergent if $q > k - \frac{1}{2}$, but in special instances it may converge for smaller values of $q$.

An example of an identity of a different type is found in Hardy (1940):

$$\sum_{n=1}^{\infty} \tau(n)\, e^{-4\pi s n^{\frac{1}{2}}} = 2^{\frac{1}{2}}\, s\, \pi^{-25/2}\,\Gamma\left(\frac{25}{2}\right)\sum_{n=1}^{\infty}\frac{\tau(n)}{(s^2+n)^{25/2}},$$

this can be shown to be a special case of the Bessel function identity

$$2\sum_{n=0}^{\infty} a(n)\, K_\nu(4\pi s n^{\frac{1}{2}})\, n^{-\frac{1}{2}\nu} = (2\pi)^{\nu-k}\, s^{-\nu}\,\Gamma(k-\nu)\sum_{n=0}^{\infty}\frac{a(n)}{(s^2+n)^{k-\nu}}$$

the $a(n)$ satisfying the same conditions as in (5).

For related results in connection with "summation formulas" see Ferrar (1935, 1937).

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# CHAPTER XVIII

## MISCELLANEOUS FUNCTIONS

### 18.1. Mittag-Leffler's function $E_\alpha(z)$ and related functions

The function

$$(1) \quad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

was introduced by Mittag-Leffler (1903, 1904, 1905) and was investigated by several authors among whom we mention Wiman (1905), Pollard (1948), Humbert (1953). In this chapter $E$ will always stand for the function (1) which must not be confused with the physicists' notation for the incomplete gamma function mentioned in sec. 9.2.

$E_\alpha(z)$, for $\alpha > 0$, furnishes important examples of entire functions of any given finite order: in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order (Phragmén 1904). Mittag-Leffler's function also furnishes examples and counter-examples for the growth and other properties of entire functions of finite order, and has other applications (Buhl 1925).

We have

$$(2) \quad E_1(z) = e^z, \quad E_2(z^2) = \cosh z, \quad E_{1/2}(z^{1/2}) = 2\pi^{-1/2} e^{-z} \operatorname{Erfc}(-z^{1/2})$$

and $E_n(z^n)$ for positive integer $n$ is a generalized hyperbolic function (see also sec. 18.2).

Many of the most important properties of $E_\alpha(z)$ follow from Mittag-Leffler's integral representation

$$(3) \quad E_\alpha(z) = \frac{1}{2\pi i} \int_C \frac{t^{\alpha-1} e^t}{t^\alpha - z} dt$$

where the path of integration $C$ is a loop which starts and ends at $-\infty$, and encircles the circular disc $|t| \leq |z|^{1/\alpha}$ in the positive sense: $-\pi \leq \arg t \leq \pi$ on $C$. To prove (3), expand the integrand in powers of $z$, integrate term-by-term, and use Hankel's integral 1.6(2) for the reciprocal of the gamma function.

The integrand in (3) has a branch-point at $t = 0$. The complex $t$-plane is cut along the negative real axis, and in the cut plane the integrand is single-valued: the principal branch of $t^{\alpha}$ is taken in the cut plane. The integrand has poles at the points,

(4) $$t_m = z^{1/\alpha} e^{2\pi i m/\alpha} \qquad m \text{ integer}$$

but only those of the poles lie in the cut plane for which

(5) $$-\alpha\pi < \arg z + 2\pi m < \alpha\pi.$$

Thus, the number of the poles inside $C$ is either $[\alpha]$ or $[\alpha + 1]$, according to the value of $\arg z$.

Feller conjectured and Pollard (1948) proved that $E_{\alpha}(-x)$ is *completely monotonic* for $x \geq 0$ if $0 \leq \alpha \leq 1$, i.e., that

(6) $$(-1)^n \frac{d^n E_{\alpha}(-x)}{dx^n} \geq 0 \qquad x \geq 0, \quad 0 \leq \alpha \leq 1$$

The proof is based on (3).

To investigate the asymptotic behavior of $E_{\alpha}(z)$ as $z \to \infty$, first assume that $z \to \infty$ along a ray which is outside the sector $|\arg z| \leq \alpha\pi/2$ (there are such rays if $0 < \alpha < 2$). If there are any poles $t_m$ satisfying (5), they will lie in the half-plane $\operatorname{Re} t < 0$. Deform $C$ to consist of two rays in the half-plane $\operatorname{Re} t < 0$ so that the poles, if any, lie to the left of $C$, also set

$$\frac{t^{\alpha}}{t^{\alpha} - z} = -\sum_{n=1}^{N-1} \frac{t^{n\alpha}}{z^n} - \left(1 - \frac{t^{\alpha}}{z}\right)^{-1} \frac{t^{N\alpha}}{z^N}$$

in (3) and note that $(1 - t^{\alpha} z^{-1})^{-1}$ is bounded uniformly in $|z|$ and $t$ if $\arg z$ is constant and $t$ is on $C$. Using again 1.6(2), the result is

(7) $$E_{\alpha}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1 - \alpha n)} + O(|z|^{-N})$$

$$z \to \infty, \qquad |\arg(-z)| < (1 - \tfrac{1}{2}\alpha)\pi$$

The $O$-term is uniform in $\arg z$ if

$$|\arg(-z)| \leq (1 - \tfrac{1}{2}a - \epsilon)\pi, \qquad \epsilon > 0$$

The result is vacuous when $a \geq 2$.

Next assume that $z \to \infty$ along a ray, and $|\arg z| \leq a\pi/2$. Then there is at least one $t_m$ satisfying

$$(8) \quad -\tfrac{1}{2}a\pi \leq \arg z + 2\pi m \leq \tfrac{1}{2}a\pi,$$

and there may be several (if $a \geq 2$): these poles lie in the half-plane $\operatorname{Re} t \geq 0$. $C$ can now be deformed as before except that in the course of the deformation of $C$ the poles satisfying (8) are crossed and contribute residues. The result then is

$$(9) \quad E_a(z) = \frac{1}{a} \sum_m e^{t_m} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1-an)} + O(|z|^{-N})$$

$$z \to \infty, \quad |\arg z| \leq \tfrac{1}{2}a\pi$$

where $t_m$ is given by (4) and summation is over all those integers $m$ which satisfy (8). In particular, if $0 < a < 2$, $m = 0$is the only integer satisfying (8), and

$$(10) \quad E_a(z) = \frac{1}{a} \exp z^{1/a} + O(|z|^{-1}) \qquad 0 < a < 2, \quad |\arg z| \leq \tfrac{1}{2}a\pi, \quad z \to \infty$$

From (7), (9), (10), and the definition of the order of an entire function (see, for instance, Copson 1935, sec. 7.4) we infer that $E_a(z)$ is an *entire function of order* $1/a$ for $a > 0$. The asymptotic expansions (7), (9) were generalized to complex values of $a$ by Wiman (1905).

The zeros of $E_a(z)$ were investigated by Wiman (1905). For $a \geq 2$ Wiman proved that $E_a(z)$ has an infinity of zeros on the negative real axis, and it has no other zeros. If $n(r)$ is the number of zeros of $E_a(z)$ in $|z| < r$, Wiman proved

$$(11) \quad \left[\frac{r^{1/a}}{\pi} \sin\frac{\pi}{a}\right] \leq n(r) < \left[\frac{r^{1/a}}{\pi} \sin\frac{\pi}{a}\right] + 1 \qquad a \geq 2$$

where $[x]$ is the greatest integer $\leq x$. For $0 < a < 2$ the distribution of zeros is entirely different. Excluding the case $a = 1$ (when there are no zeros), Wiman shows that asymptotically the zeros lie on the curve

$$(12) \quad \operatorname{Re} z^{1/a} + \log|z| + \log|\Gamma(-a)| = 0$$

and also that

(13) $[\pi^{-1} r^{1/\alpha} - \tfrac{1}{2}\alpha] - 1 \leq n(r) \leq [\pi^{-1} r^{1/\alpha} - \tfrac{1}{2}\alpha] + 1 \qquad 0 < \alpha < 2, \quad \alpha \neq 1$

Moreover, for $1 < \alpha < 2$, there is an odd number of negative zeros. Wiman investigated the zeros of $E_\alpha(z)$ also for complex values of $\alpha$.

The functional relations

(14) $$\sum_{h=0}^{m-1} E_\alpha(z e^{2\pi i h/m}) = m E_{m\alpha}(z^m)$$

(15) $$\left(\frac{d}{dz}\right)^m E_m(z^m) = E_m(z^m)$$

(16) $$\left(\frac{d}{dz}\right)^m E_{m/n}(z^{m/n}) = \sum_{k=1}^{n-1} \frac{z^{-km/n}}{\Gamma(1-km/n)} + E_{m/n}(z^{m/n})$$

where $m$ and $n-1$ are positive integers, are immediate consequences of (1). From (16)

$$\frac{d}{dz}\left[e^{-z} E_{1/n}(z^{1/n})\right] = e^{-z} \sum_{k=1}^{n-1} \frac{z^{-k/n}}{\Gamma(1-k/n)}$$

and upon integration of this by means of 9.1(1)

(17) $$E_{1/n}(z^{1/n}) = e^z \left[1 + \sum_{k=1}^{n-1} \frac{\gamma(1-k/n, z)}{\Gamma(1-k/n)}\right] \qquad n = 2, 3, \ldots$$

An explicit expression for $E_{m/n}$ follows from (14) and (17). The third equation (2) follows from (17) for $n = 2$ by means of 9.9(1), (2).

The integral

(18) $$\int_0^\infty e^{-t} E_\alpha(t^\alpha z)\, dt = \frac{1}{1-z} \qquad \alpha \geq 0$$

was evaluated by Mittag-Leffler who showed that the region of convergence of (18) contains the unit circle and is bounded by the line $\operatorname{Re} z^{1/\alpha} = 1$. The Laplace transform of $E_\alpha(t^\alpha)$ may be obtained from (18), and was used by Humbert (1953) to obtain a number of functional relations satisfied by $E_\alpha(z)$.

The function

(19) $$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad \alpha,\ \beta>0$$

has properties very similar to those of Mittag-Leffler's function: see Wiman (1905), Agarwal (1953), Humbert and Agarwal (1953). The following formulas may be obtained precisely as their special cases $\beta = 1$ above.

(20) $$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_C \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}\,dt$$

(21) $$E_{\alpha,\beta}(z)=-\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-\alpha n)}+O(|z|^{-N})$$

$$z\to\infty, \qquad |\arg(-z)|<(1-\tfrac{1}{2}\alpha)\pi$$

(22) $$E_{\alpha,\beta}(z)=\frac{1}{\alpha}\sum_m t_m^{1-\beta}e^{t_m}-\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-\alpha n)}+O(|z|^{-N})$$

$$z\to\infty, \qquad |\arg z|\le \tfrac{1}{2}\alpha\pi$$

(23) $$E_{\alpha,1}(z)=E_\alpha(z)$$

$$E_{\alpha,\beta}(z)=\frac{1}{\Gamma(\beta)}+z\,E_{\alpha,\alpha+\beta}(z)$$

(24) $$\sum_{h=0}^{m-1}E_{\alpha,\beta}(ze^{2\pi ih/m})=m\,E_{m\alpha,\beta}(z^m)$$

(25) $$\left(\frac{d}{dz}\right)^m[z^{\beta-1}E_{\alpha,\beta}(z^\alpha)]=z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha)$$

(26) $$\int_0^\infty e^{-t}t^{\beta-1}E_\alpha(t^\alpha z)\,dt=\frac{1}{1-z} \qquad \alpha,\ \beta>0$$

In (20), $C$ is the same path as in (3). In (22), $t_m$ is given by (4) and $m$ runs over all integers satisfying (8). In (24) and (25) $m$ is any positive integer. The region of convergence of (26) is the same as that of (18). The Laplace transform of $t^{\beta-1} E_\alpha(t^\alpha)$ may be evaluated by means of (26) and was used by Agarwal (1953) and by Humbert and Agarwal (1953) to obtain further properties of $E_{\alpha,\beta}$.

A function of two variables resembling $E_{\alpha,\beta}$ was briefly discussed by Humbert and Delerue (1953).

The functions $E_\alpha$ and $E_{\alpha,\beta}$ increase indefinitely as $z \to \infty$ in a certain sector of angle $\alpha\pi$, and approach zero as $z \to \infty$ outside of this sector. Entire functions which increase indefinitely in a single direction, and approach zero in all other directions, are also known. Two such functions are

$$\sum_{k=2}^{\infty} \frac{z^k}{\Gamma[1 + k(\log k)^{-a}]} \qquad 0 < a < 1$$

$$\sum_{k=0}^{\infty} \left[\frac{z}{\log(k + 1/a)}\right]^k \qquad 0 < a < 1$$

They have been discussed, respectively, by Malmquist (1905) and Lindelöf (1903).

Barnes (1906) has investigated the asymptotic behavior of $E_\alpha(z)$, and also that of several similar functions, in particular of the functions

$$\sum_{k=0}^{\infty} \frac{z^k}{(k+\theta)^\beta\,\Gamma(1+ak)}, \qquad \sum_{k=0}^{\infty} \frac{z^k\,\Gamma(1+ak)}{k!},$$

$$\sum_{k=0}^{\infty} \frac{z^k\,\Gamma(1+ak)}{\Gamma(1+a+ak)}.$$

A function intimately connected with $E_{\alpha,\beta}$ is the entire function

$$(27)\quad \phi(a, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak+\beta)} \qquad a, \beta > 0$$

which was used by Wright (1934) in the asymptotic theory of partitions. The connection with $E_{\alpha,\beta}$ is given by

$$(28)\quad \int_0^\infty e^{-ts}\,\phi(a, \beta; t)\,dt = s^{-1} E_{\alpha,\beta}(s^{-1}) \qquad a > 1, \quad \beta > 0$$

$\phi(z)$ can be represented by the integral (Wright 1933)

$$(29)\quad \phi(a, \beta; z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} u^{-\beta}\exp(u + zu^{-a})\, du \qquad a > 0$$

To prove (29), expand the integrand in powers of $z$ and use 1.6(2). The asymptotic behavior of $\phi$ as $z \to \infty$ was also investigated by Wright (1934a, 1940). The relations

$$(30)\quad az\,\phi(a, a+\beta; z) = \phi(a, \beta-1; z) + (1-\beta)\,\phi(a,\beta; z)$$

$$(31)\quad \frac{d\phi(a, \beta; z)}{dz} = \phi(a, a+\beta; z)$$

$$(32)\quad az\,\frac{d\phi(a, a+\beta; z)}{dz} = \phi(a, \beta-1; z) + (1-\beta)\phi(a, \beta; z)$$

follow from (27). Since

$$(33)\quad J_\nu(z) = (\tfrac{1}{2}z)^\nu\, \phi\left(1, \nu+1; -\frac{z^2}{4}\right),$$

Wright's function may be regarded as a kind of generalized Bessel function. (30) is a generalization of the recurrence relation of Bessel functions, and (31), (32) are generalizations of the differentiation formulas. Some of the properties which $\phi$ shares with Bessel functions were enumerated by Wright. A generalized Hankel transformation with the kernel

$$(\tfrac{1}{2})^{\beta-1}(xy)^{\beta-\frac{1}{2}}\,\phi\left(a, \beta; -\frac{x^2y^2}{4}\right)$$

was discussed by Agarwal (1950, 1951, 1953a).

## 18.2. Trigonometric and hyperbolic functions of order $n$

In this section $n$ will be a positive integer and

$$(1)\quad \omega = \exp\left(\frac{2\pi i}{n}\right)$$

The $n$ functions

$$(2)\quad h_i(x, n) = \frac{1}{n}\sum_{m=1}^{n}\omega^{(1-i)m}\exp(\omega^m x) \qquad i = 1, 2, \ldots, n$$

are sometimes called *hyperbolic functions of order n*. They reduce to hyperbolic functions when $n = 2$.

(3) $h_1(x, 1) = e^x, \quad h_1(x, 2) = \cosh x, \quad h_2(x, 2) = \sinh x$

In general, $n$ will be a fixed positive integer and will, as a rule, not be indicated. It will also be convenient to extend the definition (2) to all (positive, zero, or negative) integers $i$ which is tantamount to setting

(4) $h_{i+n}(x, n) = h_i(x, n)$ $\qquad i$ integer

This will often simplify the writing of formulas.

Since $\omega^n = 1$, all $h_i$ satisfy the differential equation

(5) $$\frac{d^n y}{dx^n} - y = 0$$

and since

(6) $$\sum_{m=1}^{n} \omega^{rm} = 0 \quad \text{for integers } r \text{ not divisible by } n$$
$$= n \quad \text{for integers } r \text{ divisible by } n,$$

the $h_i$ also satisfy the initial conditions

(7) $$\frac{d^{j-1} h_i}{dx^{j-1}}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$
$i, j = 1, 2, \dots, n$

Thus, $h_1, \dots, h_n$ form a linearly independent set of solutions of (5), and their Wronskian is equal to unity.

The power series expansion

(8) $$h_i(x, n) = \sum_{r=0}^{\infty} \frac{x^{nr+i-1}}{(nr+i-1)!} \qquad i = 1, 2, \dots, n$$

is obtained by expanding the exponential functions in (2) and using (6); the integral representation

(9) $$h_i(x, n) = \frac{1}{2\pi i} \int_C \frac{t^{n-i} e^{xt}}{t^n - 1}\, dt \qquad i = 1, \dots, n$$

where $C$ is a simple closed curve encircling the unit circle once in the positive sense, is proved by the remark that the evaluation of the integral as a sum of residues leads to (2); and the relation

(10) $\exp(\omega^m x) = \sum_{i=1}^{n} \omega^{(i-1)m} h_i(x, n)$ $\qquad m$ integer

follows from (8).

Some of the basic formulas for hyperbolic functions of order $n$ are

(11) $h_i(\omega^m x) = \omega^{(i-1)m} h_i(x)$

(12) $\dfrac{d^j h_i(x)}{dx^j} = h_{i-j}(x)$

(13) $h_i(x+y) = \sum_{j=1}^{n} h_j(x) h_{i-j+1}(y)$

(14) $$\begin{vmatrix} h_1 & h_2 & \dots & h_n \\ h_n & h_1 & \dots & h_{n-1} \\ \dots & \dots & \dots & \dots \\ h_2 & h_3 & \dots & h_1 \end{vmatrix} = \prod_{m=1}^{n} \left( \sum_{i=1}^{n} \omega^{(i-1)m} h_i(x) \right) = 1$$

(15) $\int_0^\infty e^{-st} h_i(t)\, dt = \dfrac{s^{n-i}}{s^n - 1}$ $\qquad \operatorname{Re} s > 1, \quad i = 1, 2, \dots, n$

Here $i$, $j$, $m$ are any integers [except in (15) where $i$ is restricted]. (11) and (12) follow from (2), (13) follows from (5) since $h_i(x+a)$ is that solution of the differential equation (5) whose $j$-th derivative is $h_{i-j}(a)$ when $x = 0$, (14) is the Wronskian of $h_1, \dots, h_n$ which is a *circulant* (see Aitken 1939, sec. 51) and can be evaluated explicitly, and (15) is the Laplace transform of $h_i(t)$ and follows likewise from (2) or (8).

For these and other formulas see Poli (1940, 1949a, the latter with a detailed bibliography), Oniga (1948), Bruwier (1949, 1949a), and Silverman (1953). Poli (1949a) indicates some relations which hold when $n$ is a composite number, gives expansions in terms of the $h_i$, and some applications. Bruwier (1949b) considers $1, \omega, \omega^2, \dots, \omega^{n-1}$ as the units of a linear algebra, the multiplication table being specified by

$$\omega^i \cdot \omega^j = \omega^{i+j}$$

(hypercomplex numbers). $e^{\omega x}$ is a hypercomplex number, and (10) shows that the $h_i$ are the components of $e^{\omega x}$. This fact is used by Bruwier to prove the properties of the $h_i(x)$. Matrices whose elements in the $i$-th row and $j$-th column are $a_i h_{j-i}(x, n)/a_j$, where $i, j = 1, 2, \dots, n$ and $a_1, \dots, a_n$ is a given set of constants, were considered by Lehrer (1954).

From (8) and 18.1 (19),

$$(16)\quad h_i(x) = x^{i-1} E_{n,i}(x^n) \qquad i = 1, 2, \dots, n$$

and in particular

$$(17)\quad h_1(x) = E_n(x^n)$$

giving the connection with Mittag-Leffler's function.

The $n$ functions

$$(18)\quad k_i(x, n) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{nr+i-1}}{(nr+i-1)!} \qquad i = 1, 2, \dots, n$$

are sometimes called *trigonometric functions of order n*; they are solutions of the differential equation

$$(19)\quad \frac{d^n y}{dx^n} + y = 0$$

and satisfy the initial conditions

$$(20)\quad \frac{d^{j-1} k_i}{dx^{j-1}}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \qquad i, j = 1, 2, \dots, n$$

Here again we extend the definition to all integers $i$ by setting

$$(21)\quad k_{i+n}(x, n) = -k_i(x, n).$$

These functions have been investigated by the above-mentioned authors and also by Mikusiński (1948). With

$$(22)\quad \lambda = \exp\left(\frac{\pi i}{n}\right)$$

so that $\lambda$ is an $n$-th root of $-1$, we have

(23) $k_i(x) = \lambda^{1-i} h_i(\lambda x, n)$

and the properties of the $k_i$ follow easily from those of the $h_i$. The principal formulas are

(24) $k_i(\lambda x) = \lambda^{i-1} h_i(x)$

(25) $k_i(\omega^m x) = \omega^{(i-1)m} k_i(x)$

(26) $\dfrac{d^j k_i(x)}{dx^j} = k_{i-j}(x)$

(27) $k_i(x) = \dfrac{1}{n} \sum_{m=1}^{n} \lambda^{(1-i)(2m+1)} \exp(\lambda^{2m+1} x)$

(28) $\exp(\lambda^{2m+1} x) = \sum_{i=1}^{n} \lambda^{(i-1)(2m+1)} k_i(x)$

(29) $k_i(x) = \dfrac{1}{2\pi i} \int_C \dfrac{t^{n-i} e^{xt}}{t^n + 1} \, dt$

(30) $k_i(x+y) = \sum_{j=1}^{n} k_j(x) k_{i-j+1}(y)$

(31) $\prod_{m=1}^{n} \left( \sum_{i=1}^{n} \lambda^{(i-1)(2m+1)} k_i(x) \right) = 1$

(32) $\int_0^\infty e^{-st} k_i(t) \, dt = \dfrac{s^{n-i}}{s^n + 1}$ $\qquad$ $\operatorname{Re} s > 1, \quad i = 1, \ldots, n$

(33) $h_i(x, n) + k_i(x, n) = 2h_i(x, 2n)$

$h_i(x, n) - k_i(x, n) = 2h_{n+i}(x, 2n)$

It can be seen from (27) that $k_i(x, n)$ is not a periodic function except for $n = 1, 2$. The zeros of $k_i(x)$ have been investigated by Poli (1949 a) for $n = 3$ and by Mikusiński (1948) for any $n > 1$. Mikusiński's investigations are based on the system of linear differential equations satisfied by $k_1(x), \dots, k_n(x)$ and lead to the following conclusions. Each $k_i(x, n)$ has an infinity of simple positive zeros: the zeros of $k_i(x, n)$ and $k_j(x,n)$, $i \not\equiv j \pmod n$ interlace. The least positive zero of $k_i(x, n)$ is between

$$\left[\frac{(i+n-1)!}{(i-1)!}\right]^{1/n} \quad \text{and} \quad \left[\frac{2(i+n-1)!}{(i-1)!}\right]^{1/n}$$

The large positive zeros of $k_i(x, n)$ are approximately equally spaced, the distance between two consecutive zeros of $k_i(x, n)$ approaches $\pi \csc(\pi/n)$.

Quotients like $k_i(x, n)/k_j(x, n)$ may be regarded as generalizations of $\tan x$ and $\operatorname{ctn} x$: for these generalizations see Oniga (1948), Poli (1949).

An entirely different generalization of trigonometric functions was given by Grammel (1948, 1948a, 1950).

### 18.3. The function $\nu(x)$ and related functions

The functions to be considered in this section are

$$(1) \quad \nu(x) = \int_0^\infty \frac{x^t\,dt}{\Gamma(t+1)}, \quad \nu(x, a) = \int_0^\infty \frac{x^{a+t}\,dt}{\Gamma(a+t+1)}$$

$$(2) \quad \mu(x, \beta) = \int_0^\infty \frac{x^t t^\beta\,dt}{\Gamma(\beta+1)\,\Gamma(t+1)}$$

$$\mu(x, \beta, a) = \int_0^\infty \frac{x^{a+t} t^\beta\,dt}{\Gamma(\beta+1)\,\Gamma(a+t+1)}$$

The first of these functions was encountered by Volterra in his theory of convolution-logarithms (Volterra 1916, Chapter VI, Volterra and Pérès 1924, Chapter X): Volterra denoted $\nu(y - x)$ by $\lambda(x, y)$, and $\nu(y - x, a)$ by $\lambda(x, y; a)$ or $\lambda(x, y|a)$. These functions also occur in connection with operational calculus, appear in an inversion formula of the Laplace transformation, and are of interest in connection with certain integral equations. It may be noted that (2) is the definition of $\mu$ adopted in recent papers; some of the older papers write $\mu$ for a function which differs from (2) by a factor $\Gamma(\beta + 1)$.

Between the four functions defined by (1), (2) we have the following relations

$$(3)\quad \nu(x) = \nu(x, 0) = \mu(x, 0) = \mu(x, 0, 0)$$
$$\nu(x, a) = \mu(x, 0, a), \qquad \mu(x, \beta) = \mu(x, \beta, 0) = x\,\mu(x, \beta - 1, -1)$$
$$x\,\nu(x, a-1) - a\,\nu(x, a) = \mu(x, 1, a)$$

All integrals in (1), (2) converge if $x \neq 0$, $a$ is arbitrary, and $\operatorname{Re}\beta > -1$. All four functions are analytic functions of $x$ with branch-points at $x = 0$ and $\infty$, and no other singularity; $\nu(x, a)$ and $\mu(x, \beta, a)$ are entire functions of $a$. The definition of $\mu$ can be extended to the entire $\beta$-plane by repeated integrations by parts. From (2) it follows that

$$(4)\quad \mu(x, \beta, a) = \int_0^\infty \frac{x^{a+t}}{\Gamma(a+t+1)}\, d\left[\frac{t^{\beta+1}}{\Gamma(\beta+2)}\right]$$
$$= -\frac{1}{\Gamma(\beta+2)} \int_0^\infty t^{\beta+1} \frac{d}{dt}\left[\frac{x^{a+t}}{\Gamma(a+t+1)}\right] dt$$
$$= \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$= \frac{(-1)^m}{\Gamma(\beta+m+1)} \int_0^\infty t^{\beta+m} \frac{d^m}{dt^m}\left[\frac{x^{a+t}}{\Gamma(a+t+1)}\right] dt$$

and the last expression may be regarded as the definition of $\mu(x, \beta, a)$ for $\operatorname{Re}\beta > -m - 1$. The so extended functions $\mu(x, \beta, a)$ and $\mu(x, \beta) = \mu(x, \beta, 0)$ are entire functions of $\beta$, and they are analytic functions of $x$, and $\mu(x, \beta, a)$ is also an entire function of $a$.

From (4) it follows that

$$(5)\quad \mu(x, -m, a) = (-1)^{m-1} \frac{d^{m-1}}{da^{m-1}}\left[\frac{x^a}{\Gamma(a+1)}\right] \qquad m = 1, 2, \dots$$

and since $x^a/\Gamma(a+1)$ is an entire function of $a$, we have by Taylor's expansion

$$(6)\quad \frac{x^{a+t}}{\Gamma(a+t+1)} = \sum_{n=0}^{\infty} \mu(x, -n-1, a) \frac{(-t)^m}{m!}$$

In order to investigate the behavior of $\mu(x, \beta, a)$ as $x \to 0$, we rewrite the second formula (2) as

$$(7) \quad \Gamma(\beta+1)\,\mu(x, \beta, a) = x^{a} \int_0^\infty \exp\left(-t \log\frac{1}{x}\right) \frac{t^{\beta}\,dt}{\Gamma(a+t+1)}$$

From (6) we have

$$(8) \quad \frac{1}{\Gamma(a+t+1)} = \sum_{n=0}^{\infty} \mu(1, -n-1, a)\, \frac{(-t)^{m}}{m\,!}$$

and it is known from Watson's lemma (Copson 1935, sec. 9.52) that substitution of (8) in (7) and integration term-by-term will give the asymptotic expansion of the integral in descending powers of $\log(1/x)$. Thus,

$$(9) \quad \mu(x, \beta, a) = x^{a}\left(\log\frac{1}{x}\right)^{-\beta-1}\left[\sum_{n=0}^{N-1} \frac{(-1)^{m}\,(\beta+1)_{m}}{m!} \times \mu(1, -n-1, a)\left(\log\frac{1}{x}\right)^{-n} + O\left(\left|\log\frac{1}{x}\right|^{-N}\right)\right]$$

$$\operatorname{Re}\beta > -1, \quad x \to 0, \quad \left|\arg\left(\log\frac{1}{x}\right)\right| < \pi$$

The asymptotic expansions of the other three functions in descending powers of $\log(1/x)$ follow by (3). The first terms of the asymptotic expansions of $\nu(x)$ and of $\nu(x, a)$ were obtained by Volterra.

The behavior of $\nu(x)$ as $\operatorname{Re} x \to \infty$ can be seen from Ramanujan's integral (Hardy 1940, p. 196)

$$(10) \quad \nu(x) = e^{x} - \int_0^\infty \frac{e^{-xt}\,dt}{t[\pi^2 + (\log t)^2]} \qquad \operatorname{Re} x > 0$$

A thorough investigation of the asymptotic behavior of $\nu(x)$ was undertaken by Ford (1936). Briefly, Ford's method is as follows. Let us integrate

$$H(x, w) = \frac{1}{[\sin(\pi w)]^2} \int_0^w \frac{x^{a+t}\,dt}{\Gamma(a+t+1)}$$

around a rectangle in the $w$-plane whose corners are $-N-\frac{1}{2}-ic$, $k+\frac{1}{2}-ic$, $k+\frac{1}{2}+ic$, $-N-\frac{1}{2}+ic$ where $k$ and $N$ are integers, $k+N \geq 0$, and $c$ is a positive number. $H(x, w)$ is a meromorphic function, and its poles inside the rectangle are at $w = n$, $n = -N, -N+1, \ldots, k-1, k$. The residue of $H$ at $w = n$ is $\pi^{-2} x^{a+n}/\Gamma(a+n+1)$. If $c \to \infty$, the integrals along the horizontal lines of the rectangle vanish so that

$$\frac{1}{2\pi i}\int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} H\,dw - \frac{1}{2\pi i}\int_{-N-\frac{1}{2}-i\infty}^{-N-\frac{1}{2}+i\infty} H\,dw = \sum_{n=-N}^{k} \frac{\pi^{-2} x^{a+n}}{\Gamma(a+n+1)}$$

Clearly, the second integral is $O(|x|^{a-N-\frac{1}{2}})$. In the first integral we set

$$H(x, w) = H_1 + H_2 = \frac{1}{[\sin(\pi w)]^2}\left(\int_0^{k+\frac{1}{2}} + \int_{k+\frac{1}{2}}^{w}\right)$$

It can then be shown that

$$\frac{1}{2\pi i}\int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} H_1\,dw = \frac{1}{2\pi i}\int_0^{k+\frac{1}{2}} \frac{x^{a+t}dt}{\Gamma(a+t+1)} \int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} [\sin(\pi w)]^{-2} dw$$

$$= \pi^{-2}\int_0^{k+\frac{1}{2}} \frac{x^{a+t}\,dt}{\Gamma(a+t+1)}$$

$$\frac{1}{2\pi i}\int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} H_2\,dw \to 0 \qquad \text{as} \qquad k \to \infty,$$

and hence, making $k \to \infty$,

$$\nu(x, a) - \sum_{n=-N}^{\infty} \frac{x^{a+n}}{\Gamma(a+n+1)} = -\tfrac{1}{2}\pi i \int_{-N-\frac{1}{2}-i\infty}^{-N-\frac{1}{2}+i\infty} H\,dw$$

$$= O(|x|^{a-N-\frac{1}{2}}) \qquad |x| \to \infty$$

Combining this result with 18.1(21), (22),

$$\nu(x,\ a) = e^{x} + O(|x|^{a-N}) \qquad x \to \infty, \quad |\arg x| \leq \tfrac{1}{2}\pi$$
$$= O(|x|^{a-N}) \qquad x \to \infty, \quad \tfrac{1}{2}\pi < |\arg x| \leq \pi$$

for any integer $N$.

For $\mu(x,\ \beta,\ a)$ a somewhat less complete result can similarly be derived. Because of the branch-point of

$$H(x,\ w,\ \beta) = \frac{1}{[\sin(\pi w)]^2} \int_0^w \frac{x^{a+t}\, t^{\beta}\, dt}{\Gamma(a+t+1)}$$

at $w = 0$, one is forced to take $N = -1$ and obtains, as above,

$$\mu(x,\ \beta,\ a) - \sum_{n=1}^{\infty} \frac{x^{a+n}\, n^{\beta}}{\Gamma(a+n+1)} = -\tfrac{1}{2}\pi i \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(x,\ w,\ \beta)\, dw$$

Further progress then would seem to depend on the asymptotic expansion of the entire function

$$\sum_{n=1}^{\infty} \frac{x^{n}\, n^{\beta}}{\Gamma(a+n+1)}$$

The following recurrence formula, differentiation formulas, series, and integral are easy consequences of (1) and (2).

(11) $\mu(x,\ \beta+1,\ a) = x\,\mu(x,\ \beta,\ a-1) - a\,\mu(x,\ \beta,\ a)$

(12) $\dfrac{d^n\, \nu(x)}{dx^n} = \nu(x,\ -n), \qquad \dfrac{d^n\, \nu(x,\ a)}{dx^n} = \nu(x,\ a-n)$

(13) $\dfrac{d^n \mu(x,\ \beta)}{dx^n} = \mu(x,\ \beta,\ -n), \qquad \dfrac{d^n\, \mu(x,\ \beta,\ a)}{dx^n} = \mu(x,\ \beta,\ a-n)$

(14) $\sum_{n=0}^{\infty} u^n\, \mu(x,\ n) = e^{-u}\, \nu(xe^{u}), \qquad \sum_{n=0}^{\infty} u^n\, \mu(x,\ n,\ a) = e^{-(a+1)u}\, \nu(xe^{u},\ a)$

$$\sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n!}\, u^n\, \mu(x,\ \beta+n,\ a) = e^{-(a+1)u}\, \mu(xe^{u},\ \beta,\ a)$$

(15) $\int_0^\infty e^{au} u^{\gamma-1} \mu(xe^{-u}, \beta, a)\, du = \dfrac{\Gamma(\gamma)\Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \mu(x, \beta-\gamma, a)$

$\mathrm{Re}\,\beta > -1, \quad \mathrm{Re}\,\gamma > 0$

For numerous other formulas regarding these functions see in particular Barrucand (1951), Colombo (1950, 1953), Humbert and Poli (1944).

The occurrence of the functions $\nu$ and $\mu$ in operational calculus is due on the one hand to the formulas

$$(16) \quad \int_0^\infty \frac{e^{-st}}{\Gamma(t+1)}\, dt = \nu(e^{-s}), \qquad \int_0^\infty \frac{e^{-st}}{\Gamma(a+t+1)}\, dt = e^{as}\,\nu(e^{-s}, a)$$

$$(17) \quad \int_0^\infty \frac{t^\beta e^{-st}}{\Gamma(t+1)}\, dt = \mu(e^{-s}, \beta) \qquad \mathrm{Re}\,\beta > -1$$

$$\int_0^\infty \frac{t^\beta e^{-st}}{\Gamma(a+t+1)}\, dt = e^{as}\,\mu(e^{-s}, \beta, a) \qquad \mathrm{Re}\,\beta > -1$$

which are equivalent to (1), (2) and show that the functions $\nu$, $\mu$ are Laplace transforms of simple functions; and on the other hand to the formulas

$$(18) \quad \int_0^\infty e^{-st}\nu(t)\, dt = (s \log s)^{-1} \qquad \mathrm{Re}\, s > 1$$

$$\int_0^\infty e^{-st}\nu(t, a)\, dt = s^{-a-1} (\log s)^{-1} \qquad \mathrm{Re}\, a > -1, \quad \mathrm{Re}\, s > 1$$

$$(19) \quad \int_0^\infty e^{-st}\mu(t, \beta)\, dt = s^{-1} (\log s)^{-\beta-1} \qquad \mathrm{Re}\, s > 1$$

$$\int_0^\infty e^{-st}\mu(t, \beta, a)\, dt = s^{-a-1} (\log s)^{-\beta-1} \qquad \mathrm{Re}\, a > -1, \ \mathrm{Re}\, s > 1$$

which may be established by means of (1), (2), (4) and show that $\nu$ and $\mu$ have very simple Laplace transforms. For derivations of many properties of the functions $\nu$ and $\mu$ by means of operational calculus, and for the application of these functions in operational calculus, see Barrucand and Colombo (1950), Colombo (1943, 1943a, 1948), Humbert (1944, 1950), Humbert and Poli (1944), Parodi (1945, 1947, 1948), and Poli (1946). Moreover, one of the numerous inversion formulas for the Laplace transformation

(20) $f(s) = \int_0^\infty e^{-st} F(t)\, dt,$

viz. the formula (Paley and Wiener 1934, p. 39, Doetsch 1937)

(21) $$F(t) = \lim_{\lambda \to \infty} \frac{1}{2\pi i} \int_0^\infty f(s)[\nu(st, -\tfrac{1}{2} + \lambda i) - \nu(st, -\tfrac{1}{2} - \lambda i)]\, ds$$

involves $\nu(x, \alpha)$.

The integral formulas

(22) $$\int_0^\infty \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta, \alpha)\, dx = 2^{\beta+1} y^{1/2} \pi^{1/2} \mu(y, \beta, \tfrac{1}{2}\alpha)$$

$\operatorname{Re} \alpha > -1, \quad \operatorname{Re} y > 0$

(23) $$\int_0^\infty x \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta, \alpha)\, dx = 2^{\beta+2} \pi^{1/2} y^{3/2} \mu(y, \beta, \tfrac{1}{2}\alpha - \tfrac{1}{2})$$

$\operatorname{Re} \alpha > -2, \quad \operatorname{Re} y > 0$

(24) $$\int_0^\infty \exp\left(-\frac{x^2}{8y}\right) D_\nu\left(\frac{x}{2^{1/2} y^{1/2}}\right) \mu(x, \beta, \alpha)\, dx$$

$$= 2^{\beta + \frac{1}{2}\nu + 1} \pi^{1/2} y^{\frac{1}{2}\nu + \frac{1}{2}} \mu(y, \beta, \tfrac{1}{2}\alpha - \tfrac{1}{2}\nu)$$

$\operatorname{Re} \alpha > -1, \quad \operatorname{Re} y > 0$

may be established by substituting (4) in the integrands: in the last case, (24), use 8.3(20). These formulas show, in particular, that the functions $\nu$, $\mu$ satisfy the following integral equations

(25) $$\tfrac{1}{2}\pi^{-1/2} y^{-1/2} \int_0^\infty \exp\left(-\frac{x^2}{4y}\right) \nu(x)\, dx = \nu(y)$$

$$\tfrac{1}{2}\pi^{-1/2} y^{-1/2} \int_0^\infty \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta)\, dx = 2^\beta \mu(y, \beta)$$

(26) $$\tfrac{1}{4}\pi^{-1/2}\, y^{-3/2} \int_0^\infty x \exp\left(-\frac{x^2}{4y}\right) \nu(x, -1)\, dx = \nu(y, -1)$$

$$\tfrac{1}{4}\pi^{-1/2}\, y^{-3/2} \int_0^\infty x \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta, -1)\, dx = 2^{\beta} \mu(y, \beta, -1)$$

(27) $$2^{-\frac{1}{2}\nu-1}\, \pi^{-\frac{1}{2}}\, y^{-\frac{1}{2}\nu-\frac{1}{2}} \int_0^\infty \exp\left(-\frac{x^2}{8y}\right) D_{-\alpha}\left(\frac{x}{2^{\frac{1}{2}} y^{\frac{1}{2}}}\right) \nu(x, \alpha)\, dx$$

$$= \nu(y, \alpha) \qquad \text{Re}\, \alpha > -1$$

$$2^{-\frac{1}{2}\nu-1}\, \pi^{-\frac{1}{2}}\, y^{-\frac{1}{2}\nu-\frac{1}{2}} \int_0^\infty \exp\left(-\frac{x^2}{8y}\right) D_{-\alpha}\left(\frac{x}{2^{\frac{1}{2}} y^{\frac{1}{2}}}\right) \mu(x, \beta, \alpha)\, dx$$

$$= 2^{\beta} \mu(y, \beta, \alpha) \qquad \text{Re}\, \alpha > -1$$

In the case of the integral equation with the nucleus

$$\frac{1}{2\pi^{\frac{1}{2}} y^{\frac{1}{2}}} \exp\left(-\frac{x^2}{4y}\right)$$

it is known (Stanković 1953) that (25) gives all characteristic functions which, in a certain sense, are of regular growth; a similar statement is likely to be true in the case of (26) and (27). For other integral equations whose solutions involve the functions $\nu$ and $\mu$ see Colombo (1943 a, 1952) and Parodi (1948).

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# CHAPTER XIX

## GENERATING FUNCTIONS[1]

### FIRST PART: GENERAL SURVEY

### 19.1. Introduction

If a sequence of numbers $g_1, g_2, \dots$, is determined as the sequence of coefficients in the expansion into an infinite series of a certain function, this function is called the *generating function* of the numbers $g_n$.

The most frequent type of infinite series to occur in this connection is a *power series*

$$G(t) = \sum_{n=0}^{\infty} g_n t^n.$$

Often the $g_n$ are functions of one or several variables, $x_1, x_2, \dots, x_p$, say, and we have a relation of the form

$$(1) \quad G(x_1, \dots, x_p; t) = \sum_{n=0}^{\infty} g_n(x_1, \dots, x_p)\, t^n.$$

Here $G$ is called the generating function of the functions $g_n(x_1, \dots, x_p)$, and $x_1, \dots, x_p$, $t$ are regarded as $p+1$ independent variables. With the exception of a few particularly important examples, $p$ will always be unity in this chapter, and accordingly we write

$$G(x,t) = \sum_{n=0}^{\infty} g_n(x) t^n$$

for the generating function $G(x,t)$ of the functions $g_n(x)$ of a single variable.

As a rule, the power series occurring as generating functions have a positive radius of convergence. Sometimes, however, it is useful to

(1) This chapter is based on an extensive list of generating functions compiled by the late Professor Harry Bateman.

Professor E. D. Rainville kindly contributed a supplementary list of generating functions, and assisted us in the preparation of this chapter by very helpful discussions and suggestions.

consider also power series which have zero radius of convergence, that is to say, are divergent except for $t = 0$. If questions of convergence do not matter, we speak of *formal* power series, write

$$(2) \quad G(x,t) \sim \sum_{n=0}^{\infty} g_n(x)\, t^n,$$

and say that, $G(x,t)$ is *equivalent to* or *associated with* the formal power series on the right-hand side of (2).

Occasionally Laurent series, that is expansions of the form

$$(3) \quad G(x,t) = \sum_{n=-\infty}^{\infty} g_n(x)\, t^n,$$

will also be considered.

Power series and Laurent series are not the only expansions which occur as generating functions. Another type of series, which is of frequent occurrence in number theory, can be exemplified by the generating functions of sec. 17.12. Yet another type, factorial series, is frequently met with for instance in combinatorial analysis.

The name "generating function" was introduced by Laplace in 1812. A brief discussion of Laplace's work on generating functions is found in Doetsch (1937). Laplace used not only generating series, but also generating *integrals*. The most important integral of this kind is now known as Laplace's integral and usually written as

$$f(s) = \int_0^{\infty} e^{-su} g(u)\, du.$$

The connection with generating power series is more easily seen after a change of variable, $t = e^{-s}$. Actually, both series and integrals may be replaced by the Laplace-Stieltjes integral

$$(4) \quad f(s) = \int_0^{\infty} e^{-su}\, d\alpha(u)$$

where $\alpha(u)$ is a function of bounded variation, and the right-hand side is a Stieltjes integral. Many modern authors, for instance Widder (1936), use the term "generating function" in the sense of (4). Anyone familiar with Stieltjes integrals will see at once that generating power series, Dirichlet series, and Laplace integrals are particular cases of (4).

## 19.2. Typical examples for the application of generating functions

Often the generating function of a sequence $\{g_n\}$ of numbers or functions is constructed in order to investigate the properties of the $g_n$. We shall give a typical example from combinatorial analysis.

In ordinary algebra multiplication is *associative*, i.e., $(ab)c = a(bc)$, and similarly for any number of factors. A product of $n$ factors is determined by the succession of these factors, and is independent of the grouping of the factors for purposes of multiplying two factors at a time. Even in some algebras in which the commutative law of multiplication, $ab = ba$, no longer holds, multiplication is still associative. The algebra of matrices is such an algebra. Nevertheless, there are algebras in which the associative law of multiplication does not hold; they are called *non-associative* algebras. In such algebras it may happen that $(ab)c$ and $a(bc)$ are different, so that the product $abc$ may have $p_3 = 2$ different values according as we multiply the product $ab$ by $c$, or else $a$ by the product $bc$. Note that the order of the factors has not been changed, and the difference in the result is due entirely to the non-associative character of multiplication. Given $n$ factors, in a pre-assigned order, we may insert parantheses in several ways, so as to reduce the multiplication of $n$ factors to $n-1$ multiplications of two factors at a time. With four factors, $a$, $b$, $c$, $d$, we have the possibilites

$$((ab)(cd)), \quad (a(b(cd))), \quad (((ab)c)d), \quad (a((bc)d)), \quad ((a(bc))d).$$

Let $p_n$ be the number of ways of inserting parentheses in a product of $n$ factors. Clearly, $p_1 = p_2 = 1$, $p_3 = 2$ and $p_4 = 5$.

The last step in forming the product of $n$ factors is the multiplication of a product of the first $m$ factors by a product of the last $n-m$ factors. There are $p_m$ different products of the first $m$ factors and $p_{n-m}$ different products of the last $n-m$ factors, and $m$ may take any of the values $1, 2, \cdots, n-1$. Thus we have

$$(1) \quad p_n = p_1 p_{n-1} + p_2 p_{n-2} + \cdots + p_{n-1} p_1.$$

With $n = 4$, we obtain $p_4 = 1\cdot2 + 1\cdot1 + 2\cdot1 = 5$;
with $n = 5$, we obtain $p_5 = 1\cdot5 + 1\cdot2 + 2\cdot1 + 5\cdot1 = 14$, etc.

Let us now form the generating function

$$(2) \quad G(t) = \sum_{n=1}^{\infty} p_n t^n,$$

and observe that on account of (1), the coefficient of $t^n$, for $n \geq 2$, in

$$(3) \quad [G(t)]^2 = p_1^2 t^2 + (p_2 p_1 + p_1 p_2) t^3 + (p_3 p_1 + p_2 p_2 + p_1 p_3) t^4 + \cdots$$

is again $p_n$. There is no linear term in (3). Therefore, we have

$$(4) \quad [G(t)]^2 + t = G(t).$$

This is a quadratic equation for $G(t)$, and $G(t)$ is that root of this equation which vanishes when $t = 0$. We assume $|4t| < 1$, take $(1-4t)^{1/2}$ to mean that value of the square root which has a positive real part, and obtain

$$(5) \quad G(t) = \tfrac{1}{2} - \tfrac{1}{2}(1-4t)^{1/2}.$$

Expanding the right-hand side of (5) in a binomial series we thus find

$$(6) \quad G(t) = -\tfrac{1}{2} \sum_{n=1}^{\infty} (-4)^n \binom{1/2}{n} t^n$$

and, therefore,

$$(7) \quad p_n = (-1)^{n-1} 2^{2n-1} \binom{1/2}{n} = \frac{1}{n-1} \binom{2n-2}{n} \qquad n \geq 2.$$

Apart from giving a simple formula for a computation of $p_n$, which is independent of the computation of $p_{n-1}, p_{n-2}, \ldots$, formula (7) can be used to investigate the asymptotic behavior of $p_n$ as $n \to \infty$. From 1.18 (4) we derive that

$$(8) \quad p_n = \pi^{-1/2}\, 2^{2n-2} n^{-3/2} [1 + O(n^{-1})] \qquad n \to \infty.$$

Generating functions are also a powerful tool in the investigation of systems of polynomials. As an example we shall investigate Tchebycheff polynomials, $T_n(x)$, defined by the generating function

$$(9) \quad G(x,t) \equiv \frac{1-t^2}{1-2xt+t^2} = \sum_{n=0}^{\infty} \epsilon_n T_n(x) t^n,$$

where $\epsilon_0 = 1$ and $\epsilon_n = 2$ if $n = 1, 2, 3, 4, \ldots$ . The properties of the $T_n(x)$ have already been reviewed in Chap. 10. An expansion of $G$ in a geometric series,

$$(10) \quad G(x,t) = (1-t^2) \sum_{n=0}^{\infty} (-t^2 + 2xt)^n,$$

shows that the coefficient of $t^n$ on the right-hand side is a polynomial in $x$, that the highest power of $x$ in the coefficient of $t^n$ is exactly $x^n$, and that the coefficient of $x^n t^n$ is $2^n$. We thus see that $T_n(x)$ must be a polynomial of $x$ which is of degree $n$, and that the coefficient of $x^n$ in $T_n(x)$ for $n \geq 1$ is $2^{n-1}$.

Multiplying (9) by $1-2tx+t^2$ and collecting the terms involving $t^n$ on both sides, we find

$$\epsilon_n T_n - 2x\, \epsilon_{n-1} T_{n-1} + \epsilon_{n-2} T_{n-2} = \begin{cases} 0 & n > 2 \\ -1 & n = 2 \end{cases}$$

Since (10) gives $T_0 = 1$, $T_1 = x$, we find that

$$(11)\quad T_n - 2x\, T_{n-1} + T_{n-2} = 0 \qquad n = 2, 3, 4, \dots .$$

Let $x$ be real and let $-1 < x < 1$. Then the series on the right-hand side in (9) converges absolutely for all complex values of $t$ for which $|t| < 1$, since the singularities of $G(x, t)$ as a function of $t$ are at $t = t_1$ and $t = t_2$, where

$$(12)\quad t_1 = x + (x^2-1)^{1/2}, \quad t_2 = x - (x^2-1)^{1/2}, \quad |t_1| = |t_2| = 1.$$

Cauchy's formula then gives

$$(13)\quad \epsilon_n T_n(x) = \frac{1}{2\pi i} \int_C t^{-n-1}\, G(x, t)\, dt,$$

where $C$ denotes any simple closed circuit surrounding $t = 0$, and such that $|t| < 1$ on $C$. Integral representations such as (13) may be used to estimate the functions represented by them. In the particular case under consideration, it is even possible to evaluate the integral in (13) explicitly. We put

$$x = \cos\phi, \quad t_1 = e^{i\phi}, \quad t_2 = e^{-i\phi},$$

so that

$$G(x, t) = (1-t^2)(t - e^{i\phi})^{-1}\, (t - e^{-i\phi})^{-1},$$

and from (13)

$$(14)\quad \epsilon_n T_n(x) = (2\pi i)^{-1} \int_C t^{-n-1}(1-t^2)(t - e^{i\phi})^{-1}(t-e^{-i\phi})^{-1}\, dt.$$

If $n \geq 1$, there is no singularity at infinity, and the evaluation of the integral in terms of the residues at the poles $t_1$ and $t_2$ gives

$$(15)\quad T_n(x) = \cos n\phi = \cos(n \cos^{-1} x).$$

This expression is also valid when $n = 0$.

If we introduce in (9)

$$(16)\quad x = \cos\phi, \quad t = e^{i\omega},$$

we find

$$(17)\quad G(x, t) = G^*(\phi, \omega) = (1-e^{i\omega+i\phi})^{-1} + (1 - e^{i\omega-i\phi})^{-1} - 1.$$

Thus $G^*$ is a sum of functions depending on $\phi + \omega$ and $\phi - \omega$ only and therefore

$$(18)\quad \frac{\partial^2 G^*}{\partial \omega^2} = \frac{\partial^2 G^*}{\partial \phi^2}.$$

Now

$$(19)\quad \frac{\partial}{\partial \phi} = \frac{dx}{d\phi}\frac{\partial}{\partial x} = -(1-x^2)^{1/2}\frac{\partial}{\partial x}$$

$$(20)\quad \frac{\partial}{\partial \omega} = \frac{dt}{d\omega}\frac{\partial}{\partial t} = it\frac{\partial}{\partial t}.$$

By substituting (19), (20) in (18) we find from (17)

$$(21)\quad \left[(1-x^2)^{1/2}\frac{\partial}{\partial x}\right]\left[(1-x^2)^{1/2}\frac{\partial}{\partial x}G\right] + \left(t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t}G\right) = 0.$$

By expanding the left-hand side of (21) in a power series in $t$ we find that $T_n(x) = y$ satisfies the differential equation

$$(22)\quad (1-x^2)y'' - xy' + n^2 y = 0.$$

The orthogonality relations of the $T_n(x)$ can be obtained by computing

$$(23)\quad \int_{-1}^{1} \frac{1-t^2}{1-2xt+t^2}\,\frac{1-s^2}{1-2xs+s^2}\,(1-x^2)^{-1/2}\,dx.$$

This integral is an elementary integral which can be evaluated explicitly. The result is

$$2\pi\left(\frac{1}{1-st} - \frac{1}{2}\right).$$

Expanding in powers of $s$ and $t$, and comparing coefficients of $s^m t^n$, we see that

$$(24)\quad \int_{-1}^{1} T_n(x)\,T_m(x)\,(1-x^2)^{-1/2}\,dx = \begin{cases} 0 & n \neq m \\ \pi/\epsilon_n & n = m. \end{cases}$$

Although this is a somewhat laborious proof of (24), the method deserves to be mentioned since it can be applied in many instances.

The proofs of formulas (13), (22), (24) are to some extent typical examples for the way in which the generating function can be used in order to obtain integral representations, differential equations, or integral relations for the generated functions. In general, a combination of recurrence relations and differential equations is obtained if it is possible to find a relation between $G$ and the derivatives of $G$ with respect to $t$ and to $x$. For instance, if

(25) $G(x,t) = (1 - 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n,$

where $P_n(x)$ is the Legendre polynomial of degree $n$ (see Chap. 3), the identity

(26) $t\dfrac{\partial G}{\partial t} = (x - t)\dfrac{\partial G}{\partial x}$

gives

(27) $n P_n(x) = x P_n'(x) - P_{n-1}'(x),$

and the relation,

(28) $(1 - 2tx + t^2)\dfrac{\partial G}{\partial x} = tG$

gives

(29) $P_n' - 2x P_{n-1}' + P_{n-2}' = P_{n-1}.$

If we eliminate $P_{n-2}'$ by using (27) with $n - 1$ instead of $n$, we obtain

(30) $n P_{n-1} = P_n' - x P_{n-1}'.$

From (27) and (30) we find

(31) $(1 - x^2) P_n' = -nx P_n + n P_{n-1}.$

Differentiating (31) with respect to $x$, and combining the result with (27), we find Legendre's differential equation

(32) $(1 - x^2) P_n'' - 2x P_n' + n(n + 1) P_n = 0.$

Difference equations can be obtained in many cases where the generating function involves an exponential function. The generating function for the Bernoulli polynomials $B_n(x)$ (see Chap. 1),

(33) $te^{xt}(e^t - 1)^{-1} = \sum_{n=0}^{\infty} B_n(x)\, t^n/n!\,,$

gives

(34) $t(e^t - 1)^{-1}[e^{(x+1)t} - e^{xt}] = \sum_{n=0}^{\infty} [B_n(x+1) - B_n(x)] t^n/n!$.

Since the left-hand side in (34) is $t \exp(xt)$, we have

(35) $B_n(x+1) - B_n(x) = nx^{n-1}$.

Other types of functional equations for generated functions may be obtained in a similar manner.

Finally, the existence of a generating function for a sequence $g_n$ of numbers or functions may be useful for determining

(36) $\sum_{n=0}^{\infty} g_n$

by Abel's or Cesàro's summation method. If

(37) $G(t) = \sum_{n=0}^{\infty} g_n t^n$

and if

(38) $\Lambda(t) = \sum_{n=0}^{\infty} \lambda_n t^n$,

then

(39) $\Lambda(t) G(t) = \sum_{n=0}^{\infty} \gamma_n t^n$

where

(40) $\gamma_n = \lambda_n g_0 + \lambda_{n-1} g_1 + \cdots + \lambda_0 g_n$.

**19.3. General theorems**

For each $n = 0, 1, 2, \ldots$, let $g_n(x)$ be a polynomial in $x$ which is of exact degree $n$. If

$$\frac{d}{dx} g_n(x) \equiv g_n'(x) = g_{n-1}(x) \qquad n = 1, 2, 3, \ldots$$

the $g_n(x)$ are said to form an *Appell set* of polynomials. In this case there exists a formal power series

(1) $A(t) \sim \sum_{n=0}^{\infty} a_n t^n \qquad a_0 \neq 0$

such that

(2) $A(t)\, e^{tx} \sim \sum_{n=0}^{\infty} g_n(x)\, t^n.$

Thorne (1945) showed: A set of polynomials $g_n(x)$ is an Appell set if and only if there exists a function $\alpha(x)$ which is of bounded variation in $(0, \infty)$ such that the Stieltjes integrals

$$\mu_n = \int_0^\infty x^n \, d\alpha(x) \qquad n = 0, 1, 2, \ldots$$

exist,

$$\mu_0 \neq 0,$$

and

$$\int_0^\infty g_n^{(r)}(x)\, d\alpha(x) = \begin{cases} 0 & n \neq r \\ 1 & n = r. \end{cases}$$

Then the formal power series $A(t)$ is defined by

$$A(t) \sim \Big(\sum_{n=0}^{\infty} \mu_n t^n/n!\Big)^{-1} \sim \Big[\int_0^\infty e^{xt}\, d\alpha(x)\Big]^{-1}.$$

Scheffer (1945) proved that the set $g_n(x)$ is an Appell set if and only if there exists a function $\beta(x)$ of bounded variation in $(0, \infty)$ such that

$$b_n = \int_0^\infty x^n \, d\beta(x) \qquad n = 0, 1, 2, \ldots$$

exists,

$$b_0 \neq 0,$$

and

(3) $$g_n(x) = \int_0^\infty \frac{(x+t)^n}{n!}\, d\beta(t) \qquad n = 0, 1, 2, \ldots .$$

Varma (1951) showed that then, with the same $\beta(t)$, the polynomials

(4) $$g_n^*(x) = \int_0^\infty \frac{x^n}{n!}\, {}_3F_2(-n, a, b; c, d; -t/x)\, d\beta(t)$$

also form an Appell set. Here ${}_3F_2$ denotes a generalized hypergeometric series (see 4.1). The generating function associated with the $g_n^*$ becomes

(5) $$A^*(u)\, e^{ux} \sim \sum_{n=0}^{\infty} g_n^*(x)\, u^n$$

where

(6) $A^*(u) \sim \int_0^\infty {}_2F_2(a, b; c, d; ut)\, d\beta(t).$

For examples of Appell sets see formulas 19.7(1), 19.7(2), 19.7(23) and 19.7(34).

Expansions of the type

(7) $$\frac{e^{xt}}{f(t)} = \sum_{n=-\infty}^{\infty} g_n(x)\, t^n$$

were studied by Halphén (1881) and Bird (1934).

Scheffer (1939) used the notion of an Appell set as a basis for a classification of polynomial sets. For each $n = 0, 1, 2, \dots$, let $g_n(x)$ be a polynomial which is precisely of degree $n$ in $x$. Then there exists an operator $J$ which is uniquely determined by the $g_n(x)$ and which has the following properties:

$J$ is a linear operator acting on $x^n$ (and hence on any polynomial in $x$). Let $y \equiv y(x)$ be any polynomial in $x$. Let $J[y]$ denote the polynomial upon which $y$ is mapped by $J$. Let $J$ be such that, for $n = 1, 2, 3, \dots$, $J[x^n]$ is precisely of degree $n-1$ and that $J[x^0]$ is zero. Then it can be shown that for all $y$

(8) $$J[y] = \sum_{m=1}^{\infty} L_m(x)\, y^{(m)}(x),$$

where $y^{(m)}$ is the $m$-th derivative of $y$, and where

(9) $$L_m(x) = l_{m,0} + l_{m,1}\, x + \cdots + l_{m,m-1}\, x^{m-1}$$

is a polynomial in $x$ of degree $\le m-1$ such that

(10) $$\lambda_m = m l_{1,0} + m(m-1)\, l_{2,1} + \cdots + m!\, l_{m,m-1} \ne 0 \qquad m = 1, 2, 3, \dots .$$

Now the $L_m(x)$ (and therefore $J$) are uniquely determined by

(11) $$J[g_n] = g_{n-1} \qquad n = 1, 2, 3, \dots .$$

Let $k$ be the maximum degree of the $L_m(x)$. (If the degrees of the $L_m$ are not bounded, $k = \infty$.) Then the set of polynomials $g_n(x)$ is said to be of *A-type* $k$. The Appell sets are special sets of $A$-type zero. For these,

$$L_m(x) = c_m \qquad c_1 \ne 0, \quad m = 1, 2, 3, \dots ,$$

and the $c_m$ are constants. If we associate with $J$ the formal power series

$$J(t) \sim c_1 t + c_2 t^2 + c_3 t^3 + \cdots ,$$

we may define another formal power series $H(t)$ by

(12) $J[H(t)] = t.$

Then all sets $g_n(x)$ satisfying (11) can be constructed by choosing an arbitrary set of constants $a_n (n = 0, 1, 2, \ldots)$, with $a_0 \neq 0$, putting

$$A(t) \sim \sum_{n=0}^{\infty} a_n t^n,$$

and

(13) $$A(t)\, e^{xH(t)} \sim \sum_{n=0}^{\infty} g_n(x)\, t^n.$$

All orthogonal sets $g_n(x)$ defined by a generating function of this type have been determined by Meixner (1934) (see sec. 19.12).

The case where

(14) $$(1-t)^{\beta}\, \Phi(t) \exp\left(\frac{x}{1-t}\right) = \sum_{n=0}^{\infty} g_n(x)\, t^n$$

and $\Phi(t)$ is regular for $|t| \leq 1$ has been studied by Wright (1932), who obtained results on the asymptotic behavior of the $g_n(x)$ for $n \to \infty$.

Huff (1947) and Huff and Rainville (1952), proved: if

(15) $$f(z) = \sum_{n=0}^{\infty} a_n z^n/n!$$

and

(16) $$\phi(t) = \sum_{n=0}^{\infty} b_n t^n,$$

then

(17) $$\phi(t)\, f(xt) = \sum_{n=0}^{\infty} g_n(x)\, t^n$$

defines a set $g_n(x)$ of $A$-type $k$ if and only if

(18) $$f(z) = {}_0F_k(\beta_1, \beta_2, \ldots, \beta_k;\ \sigma z),$$

where ${}_0F_k$ denotes a generalized hypergeometric series (see Chap. 4). and $\beta_1, \ldots, \beta_k$, $\sigma$ are arbitrary constants. For numerous other results on generating functions of the type (17) consult Huff (1947), and Brenke (1945). Rainville (1947) (unpublished) observed that in the particular case where $\phi(t)$ in (16) is $\exp t$, the $g_n(x)$ in (17) satisfy

$$(19)\quad (1-t)^{-c} \sum_{n=0}^{\infty} \frac{(c)_n a_n}{n!} \left(\frac{xt}{1-t}\right)^n = \sum_{n=0}^{\infty} (c)_n g_n(x) t^n.$$

For applications see 19.10(15) and 19.10(16).

Rainville (1945) proved: if

$$(20)\quad H(x) = \sum_{n=0}^{\infty} a_n x^n/n!$$

and

$$(21)\quad G(x,t) = e^t H(xt) = \sum_{n=0}^{\infty} g_n(x) t^n/n!,$$

then

$$(22)\quad g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$$

$$(23)\quad x g_n'(x) = n[g_n(x) - g_{n-1}(x)] \qquad n \geq 1$$

$$(24)\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} g_k(x) = (-1)^n a_n x^n$$

$$(25)\quad g_n(xy) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} g_k(x).$$

Fasenmyer (1947) showed: if

$$(26)\quad H(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$(27)\quad \frac{1}{1-t} H\left[\frac{-4tx}{(1-t)^2}\right] = \sum_{n=0}^{\infty} g_n(x) t^n$$

where

$$(28)\quad g_n(x) = \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k a_k}{(\tfrac{1}{2})_k\, k!} x^k.$$

In the case where $H(x)$ is a generalized hypergeometric series ${}_pF_q$ (see Chap. 4) each $g_n$ becomes a generalized hypergeometric series ${}_{p+2}F_{q+2}$.

K. P. Williams (1924) studied generating functions $\Phi(2xt + t^2)$ where $\Phi(z)$ is a power series in $z$, and used his results for a characterization of the Legendre and Hermite polynomials.

Truesdell (1948) studied generating functions $F(z, a)$ which satisfy the equation

$$(29) \quad \frac{\partial}{\partial z} F(z, a) = F(z, a+1).$$

In particular, Truesdell proved the following theorem.

If $F(z + t, a)$ possesses a Taylor series in powers of $t$, then

$$(30) \quad F(z + t, a) = \sum_{n=0}^{\infty} F(z, a + n)\, t^n/n\,!.$$

If for fixed values of $a$ and $z = z_0$

$$(31) \quad \sup_{n\to\infty} \frac{F(z_0, a + n + 1)}{F(z_0, a + n)} = \frac{1}{k} \qquad k \neq 0$$

and if there exists a real number $h < 1$ such that for certain values of $w$ for which $|w| < k$

$$(32) \quad |F(z + tw, a)| < e^{ht} \qquad t > t_0$$

then, for the same values of $w$

$$(33) \quad \sum_{n=0}^{\infty} F(z, a + n)\, w^n = \int_0^{\infty} e^{-t} F(z + tw, a)\, dt,$$

provided that the series converges uniformly in $z$ in a domain including the fixed point $z_0$.

Other theorems of Truesdell deal with the series

$$(34) \quad \sum_{n=0}^{\infty} F(z, a - n)\, w^n.$$

Various applications will be listed in the table of generating functions, in particular, see sections 19.9 and 19.10.

### 19.4. Symbolic relations

In the older literature symbolic relations were often used in order to express certain identities in a concise form, and also in order to abbreviate proofs. In contemporary literature the use of symbolic relations is rarer. We shall give two examples.

Following Rainville (1946), we shall use the convention that the notation $\doteq$ used in place of = means that exponents shall be lowered to subscripts on any symbol such as $B$, $P$, $H$, $L$ which is undefined here except with subscripts. Thus, if $B_n$ denotes the Bernoulli numbers which can be defined by the generating function

(1) $$t(e^t - 1)^{-1} = \sum_{n=0}^{\infty} B_n t^n/n!$$

we write

(2) $$B_n(x) \doteq (x + B)^n$$

to indicate that the Bernoulli polynomial $B_n(x)$ of 19.2(33) can be expressed explicitly as

(3) $$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_r x^{n-r}.$$

The symbolic derivation of this expression is as follows. The generating functions (1) and 19.2(33) are in symbolic form

$$t(e^t - 1)^{-1} \doteq e^{Bt}, \quad te^{xt}(e^t - 1)^{-1} \doteq e^{B(x)t}.$$

By comparison,

$$e^{B(x)t} \doteq e^{xt} e^{Bt} \doteq e^{(x+B)t},$$

and (2) follows on comparing coefficients of $t^n$.

Similarly, if $L_n(x)$ is the Laguerre polynomial of degree $n$,

(4) $$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!\,k!\,(n-k)!} x^k,$$

and if $P_k$ is the Legendre polynomial of degree $k$, the relation

(5) $$2^n L_n[P(x)] \doteq [L(x-1) + L(x+1)]^n$$

means that

(6) $$2^n \sum_{k=0}^{n} \frac{(-1)^k n!}{k!\,k!\,(n-k)!} P_k(z) = \sum_{k=0}^{n} \binom{n}{k} L_k(x-1) L_{n-k}(x+1).$$

The relations (5), (6) were proved by Rainville (1946), who gave a large number of similar relations between the Hermite Legendre and Laguerre polynomials. The proofs use generating functions.

In the calculus of finite differences the symbol $E$ is often used for the *shift operator* which increases subscripts (or any other specified variable) by unity. Thus

$$(7)\quad E g_n = g_{n+1}, \quad E^k g_n = g_{n+k} \qquad k, n = 0, 1, 2, \dots .$$

Using this operator, the generating function of Hermite polynomials

$$(8)\quad e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\, t^n/n!$$

may be written as

$$(9)\quad e^{2xt-t^2} = e^{Et} H_0(x).$$

The operator $E$, as defined above, acts on the (discrete) variable $n$. Friedman (1952), extends its definition so as to act also on the variable $x$. Given any function of $x$, expand it in a series of Hermite polynomials, and apply $E$ to the Hermite polynomials. That is to say, if

$$(10)\quad f(x) = a_0 H_0(x) + a_1 H_1(x) + \cdots$$

define

$$(11)\quad E f(x) = a_0 H_1(x) + a_1 H_2(x) + \cdots .$$

All other variables ($s$, $t$, $y$, ... ) are unaffected by, and hence commute with, $E$. Multiplication by the variable $x$ also defines an operator acting on any function $f(x)$. From the recurrence relation

$$(12)\quad H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$$

we find

$$x H_n(x) = \tfrac{1}{2} H_{n+1}(x) + n H_{n-1}(x).$$

Therefore, multiplication by $x$ maps the function $f(x)$ in (10) upon

$$(13)\quad x f(x) = a_1 H_0(x) + \sum_{n=1}^{\infty} [\tfrac{1}{2} a_{n-1} + (n+1)\, a_{n+1}]\, H_n(x).$$

From (11) and (13) we find

$$(14)\quad x E f(x) - E x f(x) = f(x).$$

Relations of the type (14) between two operators play a role in quantum theory. Equation (14) illustrates that $E$ does not commute with $x$. However we may multiply any expressions involving $E$ by quantities not involving $x$ and add. For instance, from

(15) $e^{iEt} H_0(x) = \sum_{n=0}^{\infty} i^n H_n(x)\, t^n/n! = e^{2ixt + t^2}$

and

(16) $\int_{-\infty}^{\infty} \exp(ist - \tfrac{1}{4} t^2 y^{-2})\, dt = 2\pi^{1/2} y \exp(-y^2 s^2)$

we find by substituting $E$ for $s$ that

(17) $2y\pi^{1/2} e^{-E^2 y^2} H_0(x) = 2y\, \pi^{1/2} \sum_{n=0}^{\infty} (-1)^n H_{2n}(x)\, y^{2n}/n\,!$

(18) $\qquad = \int_{-\infty}^{\infty} \exp(2ixt + t^2 - \tfrac{1}{4} t^2 y^{-2})\, dt$

(19) $\qquad = \dfrac{2\pi^{1/2} y}{(1-4y^2)^{1/2}} \exp\left(-\dfrac{x^2 y^2}{1-4y^2}\right).$

Comparing (17) and (19) gives

(20) $w^{-1} \exp(-x^2 y^2 w^{-2}) = \sum_{n=0}^{\infty} (-1)^n H_{2n}(x)\, y^{2n}/n\,!$

where $w^2 = 1 - 4y^2$.

### 19.5. Asymptotic representations

Generating functions may be used with good effect for the determination of the asymptotic behavior of the generated numbers (or functions) as $n \to \infty$. If

(1) $G(t) = \sum_{n=0}^{\infty} g_n t^n$

has a finite radius of convergence, then $G(t)$ has one or several singularities on the circle of convergence, and the location and nature of these singularities determines the behavior of $g_n$ for large $n$. If (1) converges everywhere, then $G(t)$ is an entire function, and the behavior of $G(t)$ for large $|t|$ determines the behavior of $g_n$ for large $n$.

The case of a finite radius of convergence of (1) was first investigated by Darboux (1878), and later by many authors. Darboux's method leads to the following general theorem formulated by Szegö (1939, theorem 8.4).

Let $G(t)$ be regular for $|t| < 1$, and let it have a finite number of distinct singularities

(2) $e^{i\phi_1}, e^{i\phi_2}, \dots, e^{i\phi_r}$

on the unit circle $|t| = 1$. Let

(3) $G(t) = \sum_{\nu=0}^{\infty} c_\nu^{(k)} (1 - te^{-i\phi_k})^{a_k + \nu b_k}, \qquad k = 1, 2, \dots, r$

in the vicinity of $e^{i\phi_k}$, where $b_k > 0$. Then the expression

(4) $\sum_{\nu=0}^{\infty} \sum_{k=1}^{r} c_\nu^{(k)} \binom{a_k + \nu b_k}{n} (-e^{-i\phi_k})^n$

furnishes an asymptotic expansion of $g_n$ in the following sense: if $Q$ is an arbitrary positive number, and if a sufficiently large number of terms is taken in (4), then we obtain an expression which approximates $g_n$ with an error which is $O(n^{-Q})$ as $n \to \infty$.

Any finite radius, $R$, of convergence can be reduced to unity by the transformation $t = Ru$. Darboux's method can also be adapted to cover the case of (proper) logarithmic singularities. The case of exponential singularities on the circle of convergence is more difficult. It has been investigated by Perron, Faber, Häusler, and, more recently by Wright (1932, 1933, 1949), who gives references to earlier literature.

Darboux's method has been applied successfully to the investigation of the asymptotic behavior of the classical orthogonal polynomials, and of certain arithmetical functions. In the case of a generating function which is entire, in many cases it is possible to find an alternative generating series with a finite radius of convergence. Hermite polynomials (of even degree), for instance, may be generated either by 19.4 (8), or by 19.4 (20), and Darboux's method applies to the latter but not to the former, generating function.

The case of a generating function which is entire has been investigated by many authors. Among earlier writers the most important papers are due to Barnes, Lindelöf, and Watson. Ford (1936) gives a summary of the results and references to most of the literature before 1936, and Wright (1948) gives references to more recent literature.

SECOND PART: FORMULAS

No completeness has been attempted in the following list. The generating functions are listed in increasing order of complexity. A "hierarchy" of functions has been laid down, and is indicated in the section headings. Each generating function is listed in the section corresponding to its "highest" function. No lexicographical order was developed, but the following principles were used as guides in compiling the list, and may help the user in finding any desired result. A function of a parameter is considered more elementary than a similar function of the principal variables $x$ or $t$. Thus, $(1+t)^x$ appears later than $(1-2xt+t^2)^{-\nu}$. A product of an algebraic function and of an exponential function is considered more elementary than an exponential function of an algebraic function.

Almost every entry is accompanied by references to the literature. These references have been selected as convenient, and they do not necessarily indicate that the generating function was introduced in the paper referred to, nor do they give the newest or most comprehensive source.

The generating functions of number theory have not been included here. For these see Chap. 17. The generating functions of combinatorial analysis have likewise been excluded.

### 19.6. Rational and algebraic functions. General powers

(1) $$\frac{1-t^2}{1-2tx+t^2} = 1 + 2\sum_{n=1}^{\infty} T_n(x)\, t^n.$$

The $T_n(x)$ are the Tchebycheff polynomials of Chap. 10.

(2) $$(1-t)^{-k-1}(1-xt)^{-1} = \sum_{n=0}^{\infty} g_n^{(k)}(x)\, t^n \qquad k = 0, 1, 2, \dots$$

(3) $$g_n^{(k)}(x) = \sum_{m=0}^{n} (-1)^{n-m} \binom{k-1}{n-m} x^m.$$

Here $g_n^{(k)}(x)$ is the $k$-th Cesáro mean of the first $n$ partial sums of the series

$$1 + x + x^2 + \cdots.$$

(For applications see Obrechkoff, 1934).

(4) $$(1-2tx+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$

The $P_n(x)$ are the Legendre polynomials, (see Chap. 10).

(5) $$\frac{(1-x^2)^{1/2}}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_{n+1}(x)\, t^n.$$

The $U_{n+1}$ are the Tchebycheff polynomials of the second kind (see Chap. 10).

(6) $$\frac{1-t^2}{(1-2tx+t^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)\, P_n(x)\, t^n.$$

The $P_n(x)$ are the Legendre polynomials of Chap. 10.

(7) $$(1-3xt+t^3)^{-1/2} = \sum_{n=0}^{\infty} g_n(x)\, t^n.$$

Recurrence relations and a linear homogeneous differential equation of the third degree for $g_n$ were derived by Pincherle (1889). The polynomials generated by

$$(1-3xt+t^3)^{-\nu}$$

have been investigated by P. Humbert (1920).

(8) $$\frac{1+t}{(1-t)^k\,(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} g_n(x)\, t^n$$

(9) $$g_n(x) = \sum_{\nu=0}^{n} \frac{\Gamma(k+n-\nu+1)\,(2\nu+1)}{\Gamma(k+1)\,\Gamma(n-\nu+1)}\, P_\nu(x)$$

where $P_\nu(x)$ is the Legendre polynomial of degree $\nu$ and $\mathrm{Re}\, k > -1$. Applications to the problem of summability of the series of Laplace and Legendre are given by Gronwall (1914).

(10) $$t^{-1}\,(1-t)^{-k}\left[\frac{1+t}{(1-2xt+t^2)^{1/2}} - 1\right] = \sum_{n=0}^{\infty} g_n(x)\, t^n,$$

(11) $$g_n(x) = \sum_{\nu=0}^{n} \frac{\Gamma(k+n-\nu)}{\Gamma(k)\,\Gamma(n+1-\nu)}\,[P_\nu(x) + P_{\nu+1}(x)].$$

For applications consult Gronwall (1914); also compare with (8).

(12) $$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)\, t^n.$$

The $C_n^\nu$ are Gegenbauer's polynomials. Consult Chap. 10, sec. 11.1.2, and also Gegenbauer (1874).

Let $w = (1 - 2xt + t^2)^{1/2}$, then

$$(13)\quad \frac{2^\alpha}{w(1 - xt + w)^\alpha} = \sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{(2\alpha+1)_n} C_n^{1/2+\alpha}(x)\, t^n$$

where the $C_n^\nu$ are the Gegenbauer polynomials of Chap. 10; compare also (12) and Szegö (1939).

$$(14)\quad (1 - 3xt + 3yt^2 - t^3)^{-\nu} = \sum_{n=0}^{\infty} H_n^\nu(x, y)\, t^n.$$

Ordinary and partial differential equations for the $H_n^\nu$ have been derived by Devisme (1932, 1933).

$$(15)\quad [1 - x^m + (x - t)^m]^{-\nu} = \sum_{n=0}^{\infty} {}_m c_n^\nu(x)\, t^n.$$

For an investigation of the ${}_m c_n^\nu$ consult Devisme (1936).

$$(16)\quad (1-t)^{b-c}(1 - t + xt)^{-b} = \sum_{n=0}^{\infty} (c)_n F(-n, b; c; x)\, t^n/n!$$

The notations are in those of sec. 2.1. For applications (in physics) see Gordon (1929).

Let $w = (1 - 2tx + t^2)^{1/2}$, then

$$(17)\quad 2^{\alpha+\beta} w^{-1}(1 - t + w)^{-\alpha}(1 + t + w)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n$$

$$(18)\quad = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{n!} F(\alpha + \beta + n + 1, -n; \alpha + 1; \tfrac{1}{2} - \tfrac{1}{2}x)\, t^n,$$

where the $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials (compare Chap. 10 and sec. 2.5.1 where a proof is given).

$$(19)\quad (1+t)^x = \sum_{n=0}^{\infty} \binom{x}{n} t^n$$

where

$$(20)\quad \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \qquad n = 1, 2, 3, \ldots$$

(21) $\binom{x}{0} = 1,$

are the *binomial polynomials* of $x$. Equation (19) is the well-known binomial theorem which was rigorously proved by Abel in 1826.

(22) $\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} g_n(x)\, t^n.$

(23) $g_n(x) = \frac{(x)_n}{n!} F(-n, -x;\ 1-n-x; -1)$

(24) $\quad = 2x\, F(1-n,\ 1-x;\ 2;\ 2) \qquad n \geq 1.$

The notations are as in sec. 2.1. References: Mittag-Leffler (1891), Bateman (1940). The generating function is of the generalized Appell's type 19.3(13) with $A(t) = 1$,

(25) $\left(\frac{1+t}{1-t}\right)^x (1-t)^{-1} = \sum_{n=0}^{\infty} g_n(x)\, t^n.$

An explicit expression for $g_n(x)$ can be found from (22) and 19.2(37) to 19.2(40) with $\Lambda(t) = (1-t)^{-1}$. For applications consult Pidduck (1910, 1912).

(26) $(2xt)^{-q}\left[\left(\frac{1+t}{1-t}\right)^x - 1\right]^q = \sum_{n=0}^{\infty} g_n(x)\, t^n$

[see Mittag-Leffler (1891).]

(27) $[1+\beta t(a_0 + a_1 t + \cdots + a_k t^k)]^{x/\beta} = \sum_{n=0}^{\infty} g_n(x) t^n.$

The $g_n(x)$ satisfy the functional equation

(28) $g_n(x+y) = \sum_{r=0}^{n} g_r(x)\, g_{n-r}(y),$

and every solution of (28) in terms of polynomials can be obtained from a generating function of the form

$$(1 + \beta t \sum_{m=0}^{\infty} a_m t^m)^{x/\beta}$$

by an appropriate choice of the constants $\beta$, $a_0$, $a_1$, ... . Reference: René Lagrange (1928). The generating function is of the generalized Appell type 19.3 (13).

Let $G = G(x, t)$ be that root of the equation

(29) $1 + xG - (1 + G)^x = xt^2$,

which possesses an expansion

(30) $G(x, t) = \sum_{n=1}^{\infty} g_n(x)\, t^n/n!$.

Then

(31) $$g_n(x) = \left\{ \frac{\partial^{n-1}}{\partial G^{n-1}} \left[ \frac{1 + xG - (G+1)^x}{xG^2} \right]^{-\frac{1}{2}n} \right\}_{G=0}$$

and $g_1 = 2^{\frac{1}{2}} (1 - x)^{-\frac{1}{2}}$. The $g_n(x)$ were used by Barnes (1906) for the investigation of the asymptotic behavior (for $z \to \infty$) of

$$\sum_{n=0}^{\infty} \frac{\Gamma(1 + nx)}{n!} z^n.$$

## 19.7. Exponential functions

(1) $(t - 1)^m e^{xt} = \sum_{n=-m}^{\infty} x^m L_m^n(x)\, m!\, t^{n+m}/(m + n)!$.

The $L_m^n$ are the Laguerre polynomials of Chap. 10; see also Truesdell (1948).

(2) $\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\, t^n/n!$.

The $H_n$ are the Hermite polynomials of Chap. 10.

(3) $$(1 - t)^{-1} \exp \frac{x^2 t(t-2)}{(1-t)^2} = \sum_{n=0}^{\infty} g_{2n}(x)\, t^n$$

(4) $$g_{2n}(x) = \frac{e^{x^2}}{n!} \frac{d^n}{dx^n} (x^n e^{-x^2})$$

(see Humbert, 1923). The $g_{2n}(x)$ have the property that

$$\int_0^{\infty} e^{-x^2} x^s g_{2n}(x)\, dx = \int_{-\infty}^{0} e^{-x^2} x^s g_{2n}(x)\, dx = 0 \qquad s = 0, 1, 2, \ldots, n-1$$

(5) $\exp[\tfrac{1}{2}x(t-t^{-1})] = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n.$

The $J_n(x)$ are Bessel functions of the first kind (see Chap. 7).

(6) $\exp\{x[u+t-(ut)^{-1}]/3\} = \sum_{n,m=-\infty}^{\infty} J_{n,m}(x)u^m t^n.$

(7) $$J_{n,m}(x) = \frac{x^{n+m}}{3^{n+m}\Gamma(n+1)\Gamma(m+1)}\,{}_0F_2(m+1, n+1; -x^3/27)$$

where ${}_0F_2$ is a generalized hypergeometric series (compare sec. 4.1). For negative values of $n$, $m$ the right-hand side in (7) means

(8) $$(x/3)^{n+m}\sum_{l=0}^{\infty}\frac{(-x/3)^{3l}}{\Gamma(l+1+n)\Gamma(l+1+m)}.$$

For proofs and applications to the equation

(9) $$\frac{\partial^3 U}{\partial x^3}+\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial z^3}-3\frac{\partial^3 U}{\partial x\,\partial y\,\partial z}+U=0$$

consult P. Humbert (1930).

(10) $\exp[(t^2-uv)x-t^2/3] = \sum_{l,m,n=0}^{\infty} t^l u^m v^n P_{l,m,n}(x)$

See Devisme (1932, 1933).

(11) $$(1+4t^2)^{-3/2}(1+2xt+4t^2)\exp\left(\frac{4x^2t^2}{1+4t^2}\right) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{l!}$$

where $l = \tfrac{1}{2}n$ if $n$ is even and $l = \tfrac{1}{2}n - \tfrac{1}{2}$ if $n$ is odd. The $H_n$ are the Hermite polynomials of Chap. 10; see also (2) and Szegö (1939).

(12) $$(1-t^2)^{-\frac{1}{2}}\exp\left(\frac{2x^2t}{1+t}\right) = \sum_{n=0}^{\infty}[H_n(x)]^2\frac{(\tfrac{1}{2}t)^n}{n!}$$

The $H_n$ are the Hermite polynomials of (2) [consult also Chap. 10 and Tchebycheff (1889)].

(13) $$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)\,t^n.$$

The $L_n^{\alpha}(x)$ are the generalized Laguerre polynomials of Chap. 10; see also Szegö (1939).

$$(14)\quad \exp\left[x\,\frac{(1-t^2)^{1/2}-1}{t}\right]=\sum_{n=0}^{\infty} g_n(x)t^n$$

$$(15)\quad g_n(x)=(-\tfrac{1}{2}x)^n\,(n-1)!\sum_{l=0}^{[\frac{1}{2}n]}\frac{x^{-2l}}{l!\,(n-l)!\,(n-2l-1)!}$$

$$(16)\qquad =(n!)^{-1}(-\tfrac{1}{2}x)^n\;{}_3F_0(-n,\tfrac{1}{2}-\tfrac{1}{2}n,1-\tfrac{1}{2}n;-4x^{-2})$$

where $[\frac{1}{2}n]=\frac{1}{2}n$ if $n$ is even and $[\frac{1}{2}n]=\frac{1}{2}n-\frac{1}{2}$ if $n$ is odd, and ${}_3F_0$ is a generalized hypergeometric series; notations are as in sec. 4.1. For applications to a problem in the theory of electrons see Mott (1932).

$$(17)\quad (1-2xt)^{-1/2}\exp\{x^{-1}[1-(1-2xt)^{1/2}]\}=\sum_{n=0}^{\infty}{}_2F_0(-n,n+1;-\tfrac{1}{2}x)\,t^n/n!\,.$$

The ${}_2F_0$ are Bessel polynomials. See (18), (19) and Krall and Frink (1949), Burchnall (1951).

$$(18)\quad (1-2xt)^{-1/2}\,[\tfrac{1}{2}+\tfrac{1}{2}(1-2xt)^{1/2}]^{2-a}\exp\{\tfrac{1}{2}bx^{-1}[1-(1-2tx)^{1/2}]\}$$

$$=\sum_{n=0}^{\infty} y_n(x,a,b)\,t^n/n!\,.$$

The $y_n(x,\ a,\ b)$ are called *generalized Bessel polynomials* by Krall and Frink (1949), and satisfy orthogonality relations on the unit circle of the complex $x$-plane. For a proof of (18) see Burchnall (1951). Explicit expressions are:

$$(19)\quad y_n(x,a,b)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k+a-2}{k}k!\left(\frac{x}{b}\right)^k$$

$$={}_2F_0(-n,a-1+n;-x/b)$$

$$(20)\quad y_n(bx,a,b)=x^{1-\frac{1}{2}a}\,e^{1/(2x)}\,W_{1-\frac{1}{2}a,\,n-\frac{1}{2}+\frac{1}{2}a}(x^{-1}).$$

Notations: ${}_2F_0$ is as in Chap. 4; $W$ is as in Chap. 6; compare also sec. 4.7 and (17).

$$(21)\quad (1-t)^{\beta}\exp\left(\frac{x}{1-t}\right)=\sum_{n=0}^{\infty} g_n(x)\,t^n,$$

(22) $$g_n(x) = \sum_{m=0}^{\infty} \frac{(m-\beta)_n}{n!\,m!} x^m = e^x L_n^{-\beta-1}(-x)$$

see (13) and sections 2.1 and 6.9.2 for the notation. The $L_n^{-\beta-1}$ are the generalized Laguerre polynomials. Reference: Wright (1932).

(23) $$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\, t^n/n!$$

(24) $$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\, t^n/n!\,.$$

Here $B_n(x)$ is the *Bernoulli polynomial* and $E_n(x)$ is the *Euler polynomial* of degree $n$. Let

(25) $$B_n = B_n(0)$$

(26) $$E_n = 2^n E_n(\tfrac{1}{2}).$$

The $B_n$ are the *Bernoulli numbers* (cf. Chapter 1) and the $E_n$ are the *Euler numbers*.

(27) $$B_n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_\nu x^{n-\nu}$$

(28) $$E_n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{-\nu} E_\nu (x-\tfrac{1}{2})^{n-\nu}.$$

For a report on the extensive literature and for numerous results and applications consult Fort (1948) and Nörlund (1924). For generalizations compare (30), (34) to (37) and (57).

(29) $$\frac{e^{tx}-1}{e^t-1} = \sum_{n=0}^{\infty} g_n(x)\, t^n.$$

The $g_n(x)$ are closely related to the Bernoulli polynomials [see (23), and also Hermite (1878) and Berger (1888) for generalizations and applications].

(30) $$\frac{t^l e^{xt}}{(e^t-1)^l} = \sum_{n=0}^{\infty} B_n^{(l)}(x)\, t^n/n!\,.$$

The $B_n^{(l)}(x)$ are called *generalized Bernoulli polynomials*. Compare (23) and see also Nörlund (1920, 1924). Some special cases are:

$$(31)\quad B_n^{(n)}(x) = \int_x^{x+1} (s-1)(s-2)\cdots(s-n)\,ds,$$

$$(32)\quad B_n^{(l+1)}(x) = \frac{n!}{l!}\frac{d^{\,l-n}}{dx^{\,l-n}}(x-1)(x-2)\cdots(x-n) \qquad l \geq n$$

$$B_n^{(l)}(x+y) = \sum_{r=0}^{n}\binom{n}{r} x(x-1)\cdots(x-r+1)\,B_{n-r}^{(l-r)}(y)$$

$$(33)\quad 2te^{tx}\left(\frac{p+t}{p-t}e^{2t}-1\right)^{-1} = \frac{p}{p+1} + \sum_{n=1}^{\infty} \omega_n^{(p)}(x)\,t^n/n!\,.$$

The $\omega_n^{(p)}$ are polynomials in $x$ of degree $n$ if $p \neq 0$. If $p \to \infty$,

$$\omega_n^{(p)}(x) \to 2^n B_n(\tfrac{1}{2}x)$$

where $B_n(x)$ is the Bernoulli polynomial [cf. 19.4(3), (23)]. The $\omega_n^{(p)}(x)$ can be expanded into a series of functions

$$\sin \mu_l x, \qquad \cos \mu_l x \qquad\qquad l = 1, 2, 3, \ldots$$

where $\mu_l$ is the $l$-th real root of

$$\mu \cos \mu + p \sin \mu = 0.$$

For this and other results and for applications to a problem in the conduction of heat compare Koshliakov (1935).

$$(34)\quad \frac{(e^{\omega_1 t}-1)\cdots(e^{\omega_l t}-1)}{\omega_1\cdots\omega_l t^l}e^{xt} = \sum_{n=0}^{\infty} B_n^{(-l)}(x|\omega_1,\ldots,\omega_l)\,t^n/n!\,.$$

$$(35)\quad 2^{-l}(e^{\omega_1 t}+1)\cdots(e^{\omega_l t}+1)\,e^{xt} = \sum_{n=0}^{\infty} E_n^{(-l)}(x|\omega_1,\ldots,\omega_l)\,t^n/n!$$

$$(36)\quad \frac{\omega_1\cdots\omega_l t^l}{(e^{\omega_1 t}-1)\cdots(e^{\omega_l t}-1)}e^{xt} = \sum_{n=0}^{\infty} B_n^{(l)}(x|\omega_1,\ldots,\omega_l)\,t^n/n!$$

$$(37)\quad \frac{2^l e^x}{(e^{\omega_1 t}+1)\cdots(e^{\omega_l t}+1)} = \sum_{n=0}^{\infty} E_n^{(l)}(x|\omega_1,\ldots,\omega_l)\,t^n/n!$$

$$l = 1, 2, 3, \ldots\,.$$

The $B_n^{(-l)}$, $B_n^{(l)}$ are the *Bernoulli polynomials of order* $-l$ *and* $l$, respectively. The $E_n^{(-l)}$, $E_n^{(l)}$ are the corresponding *Euler polynomials of higher order.* For results and applications of these polynomials see Nörlund (1920, 1924).

(38) $$(e^{xt}-1)\,t^{\nu}(e^{t}-1)^{-\nu}=\sum_{n=0}^{\infty}\Phi_{\nu,n+1}(x)\,t^{n},$$

(39) $$(e^{xt}-1)\,t^{-\nu-2}(e^{t}-1)^{\nu+1}=\sum_{n=0}^{\infty}\Psi_{\nu,n+1}(x)\,t^{n}.$$

Imschenetzky (1884) investigated the $\Phi_{\nu,n+1}$, $\Psi_{\nu,n+1}$ for $\nu=0,1,2,\ldots$. They are closely related to the generalized Bernoulli and Euler polynomials of (34) to (37); see also Nörlund (1924).

(40) $$\exp[x(1+t-e^{t})]=\sum_{n=0}^{\infty}g_n(x)\,t^n/n!$$

Mahler (1930) introduced the $g_n(x)$ for the investigation of the zeros of the incomplete gamma function [see Chap. 9 and also (41), (46)].

(41) $$\exp[at+x(1-e^{t})]=\sum_{n=0}^{\infty}g_n^{(a)}(x)\,t^n/n!\,.$$

The $g_n^{(a)}$ have been investigated by Toscano (1950). Related functions are those studied by Hilb (1922), compare (46), and Mahler (1930), compare (40). Toscano (1930) gives references to the older literature where the $g_n^{(a)}$ have been introduced in connection with problems of actuarial mathematics. Some of the results of Toscano (1930) are:

(42) $$g_n^{(a+1)}(a)-g_n^{(a)}(x)=-\frac{d}{dx}\,g_n^{(a)}(x).$$

Equation (42) relates the $g_n^{(a)}$ with the functional equation studied by Truesdell (1948). The $g_n^{(a)}(x)$ is a polynomial of degree $n$ both in $x$ and $a$

(43) $$g_n^{(a)}(x)=x^{-a}e^{x}\left(x\frac{d}{dx}\right)^{n}x^{a}e^{-x}.$$

If $\Delta_a$ is the difference operator defined by

(44) $$\Delta_a f(a)=f(a+1)-f(a),$$

then

(45) $$g_n^{(a)}(x)=[\exp(-x\,\Delta_a)]a^n=\sum_{m=0}^{n}\frac{(-1)^m}{m!}x^m\,\Delta_a^m\,a^n.$$

Toscano gives expansions of $g_n^{(\alpha)}(x)$ in a series of Laguerre polynomials $L_m^{\beta}$ (consult Chap. 10) and proves the following integral relation

$$\int_0^\infty e^{-x} x^{\frac{1}{2}\alpha} J_\alpha[2(xu)^{\frac{1}{2}}]\, g_n^{(\beta)}(x)\, dx = (-1)^n u^{\frac{1}{2}\alpha} e^{-u} g_n^{(\alpha-\beta+1)}(u).$$

The relation

$$e^x g_n^{(\alpha)}(-x) = \sum_{m=0}^{\infty} (\alpha+m)^n x^m/m!$$

has been stated by Whittaker and Watson (1935), page 336.

(46) $\exp(e^t - tx)\int_t^\infty \exp(sx - e^s)\, ds = \sum_{n=0}^{\infty} g_n(x) t^n.$

The $g_n(x)$ were used by Hilb (1922) to construct a solution of the functional equation

(47) $u(x+1) - xu(x) = h(x)$

where $h(x)$ is given. Hilb shows that under certain conditions $h(x)$,

(48) $\sum_{n=0}^{\infty} g_n(x) h^{(n)}(x) = u(x)$

is a solution of (47). Here $h^{(0)} \equiv h(x)$ and $h^{(n)}$ is the $n$-th derivative of $h(x)$.

(49) $e^{-t}(1+a^{-1}t)^x = \sum_{n=0}^{\infty} a^{-\frac{1}{2}n}(n!)^{-\frac{1}{2}} p_n(x)\, t^n.$

The $p_n(x)$ are the Poisson-Charlier polynomials of Chap. 10; see also Szegö (1939).

(50) $(1-t)^{-x} e^{tx} \equiv \exp[x(2t + t^2/2 + t^3/3 + \cdots)]$

$$= \sum_{n=0}^{\infty} g_n(x)\, t^n.$$

The $g_n(x)$ are a set of polynomials of the generalized Appell's type 19.3(13). With the notation of sec. 2.1,

(51) $g_n(x) = \sum_{l=0}^{n} \dfrac{x^{n-l}(x)_l}{(n-l)!\, l!} = \dfrac{x^n}{n!}\, {}_2F_0(-n, x; -x^{-1}).$

Sylvester (1879) investigated the $g_n(x)$ and showed that the numbers $g_n(1/4)$ can be used for the computation of the number of different terms in the determinant of a skew-symmetric matrix of degree $2n$. Similarly,

$g_n(1/8)$ is significant for the computation of the number of different terms in a determinant of degree $4n$ which is skew-symmetric with respect to both diagonals.

$$(52)\quad (1+t)^{-x} e^{x(t-\frac{1}{2}t^2)} = \exp[x(-t^3/3 + t^4/4 - \cdots)]$$
$$= \sum_{n=0}^{\infty} g_n(x)\, t^n.$$

The $g_n(x)$ are a generalized Appell set of type 19.3(13) which is related to the Hermite polynomials (see Chap. 10). Consult van Veen (1931) for results and applications to the problem of asymptotic behavior of Hermite polynomials.

$$(53)\quad (1-t^2)^{-\frac{1}{2}c} \left(\frac{1+t}{1-t}\right)^{x} e^{-2xt}$$
$$\equiv (1-t^2)^{-\frac{1}{2}c} \exp[x(t^2/2 + t^3/3 + \cdots)] = \sum_{n=0}^{\infty} g_n(x)\, t^n.$$

The $g_n(x)$ are a generalized Appell set of polynomials of type 19.3(13). They have been introduced by Tricomi (1949) for the investigation of the asymptotic behavior of Laguerre polynomials (see Chap. 10). The fundamental recurrence relation is

$$(54)\quad (n+1)\, g_{n+1} = (n+c-1)\, g_{n-1} + 2x\, g_{n-2}$$

$$(55)\quad e^{-x}(1+xt)^{1/t} = \sum_{n=0}^{\infty} (-1)^n A_n(x)\, t^n.$$

The $A_n(x)$ are sometimes called Appell's polynomials. They are related to a special case of the $g_n(x)$ defined by 19.3(13). This can be seen by writing

$$e^{-x}(1+xt)^{1/t} = \exp\{x[s^{-1}\log(1+s) - 1]\} \qquad s = xt.$$

We have

$$(56)\quad \frac{dA_n}{dx} = x A_{n-1} + x^2 A_{n-2} + \cdots, \qquad A_n(x) = x^{n+1} \sum_{m=1}^{n} P_m\, x^{m-1},$$

$$\lim_{n\to\infty} \sum_{m=1}^{n} P_m = e^{-1}.$$

The numbers $P_m$ are used for the computation of a number-theoretical function [see Appell (1880)].

(57) $\left(\frac{t}{e^t - 1}\right)^x = \sum_{n=0}^{\infty} \frac{B_n^{(x)}}{n!} t^n.$

The $B_n^{(x)}$ generalize the Bernoulli numbers [see Chap. 1 and compare with (25)]; $B_n^{(x)}$ is a polynomial in $x$ of degree $n$ and is a special case of the polynomials of R. Lagrange (1928) [cf. 19.6(27)]. For another generating function see 19.8(6); theory and applications in Nörlund (1920, 1924). By a slight modification [see (58)] the Stirling polynomials are obtained from the $B_n^{(x)}$.

(58) $\left(\frac{1-e^{-t}}{t}\right)^{-x-1} = 1 + (x+1) \sum_{n=0}^{\infty} \psi_n(x)\, t^{n+1}.$

The $\psi_n(x)$ are called *Stirling polynomials*. They are connected with the *Stirling numbers* $C_n^{(r)}$ and $\mathfrak{C}_n^{(r)}$ by the relations

(59) $C_{n+1}^{(r)} = \frac{(n+1)!}{(n-r)!} \psi_{r-1}(n) \qquad r = 1, 2, 3, \ldots$

(60) $\mathfrak{C}_{n+1}^{(r)} = \frac{(-1)^{r+1}(n+r)!}{(n-1)!} \psi_{r-1}(-n-1).$

Here, $\psi_0$ is defined to be ½ and the Stirling numbers are defined independently by

(61) $(t)_n = \sum_{r=0}^{n-1} C_n^{(r)} t^{n-r} \qquad C_n^{(0)} = 1$

(62) $\frac{1}{(t)_n} = \sum_{r=0}^{\infty} \frac{(-1)^r \mathfrak{C}_n^{(r)}}{t^{n+r}} \qquad \mathfrak{C}_n^{(0)} = 1$

(63) $t^n = -\sum_{r=0}^{n-1} \mathfrak{C}_{n-r+1}^{(r)} (-t)_r$

Definition of $(t)_n$ is as in sec. 2.1. References: N. Nielsen (1906); Nörlund (1924). See also (57) and 19.8(7).

(64) $(1-t)^{-1/2} \exp\{x[(1-t)^{-1/2} - 1]\} = \sum_{n=0}^{\infty} g_n(x) t^n$

(65) $g_n(x) = (n!)^{-1} x e^{-x} \left[\frac{d}{d(x^2)}\right]^n (x^{2n-1} e^x)$

(66) $(1-t)^{-3/2}\exp\{x[(1-t)^{-1/2}-1]\} = \sum_{n=0}^{\infty} (2n+1)!p_n(x)t^n/(2^n n!)^2$

(67) $p_n(x) = \dfrac{\pi^{1/2} e^{-x}}{2x\,\Gamma(n+3/2)} \left[\dfrac{d}{d(x^2)}\right]^n (x^{2n+1} e^x).$

For applications of the $g_n$, $p_n$ in (65), (67) to the theory of hyperbolic differential equations, see Courant and Hilbert (1937) pp. 391-398.

## 19.8. Logarithms, trigonometric and inverse trigonometric functions. Other elementary functions and their integrals

(1) $\dfrac{1-(1-t)^x}{\log(1-t)} = \sum_{n=0}^{\infty} g_{n+1}(x)\, t^n$

(2) $g_{n+1}(x) = (n!)^{-1} \int_0^x u(1-u) \cdots (n-1-u)\, du.$

See Appell (1929), Jordan (1929) and compare with 19.6(19) and 19.10(14). Applications to the computation of $\Sigma\, n^{-\lambda}$.

(3) $[-\log(1-t)]^{\kappa}(1-t)^{-x} = \sum_{n=0}^{\infty} A_n^{(\kappa)}(x)\, t^n/n! \qquad \kappa = 1, 2, 3, \ldots .$

The asymptotic behavior of the $A_n^{(\kappa)}$ for $n \to \infty$ was studied by Narumi (1929). Here $A_n^{(\kappa)}(x)$ is the coefficient of $t^{\kappa}/\kappa!$ in the expansion of

$$\frac{\Gamma(n+t+x)}{\Gamma(n+t)}$$

in powers of $t$. Application to the proof of theorems about functions which are regular in the unit circle $|z| < 1$ and have exactly one singularity with a prescribed location $(z=1)$ and type on $|z| = 1$.

(4) $[1 - x\log(1+t)]^{-\nu} = \sum_{n=0}^{\infty} g_n(x) t^n$

(5) $g_n(x) = \dfrac{1}{\Gamma(\nu)} \int_0^{\infty} \binom{\lambda x}{n} e^{-\lambda} \lambda^{\nu-1}\, d\lambda \qquad \operatorname{Re} \nu > 0.$

Here $\binom{\lambda x}{n}$ is the binomial polynomial 19.6(19) of degree $n$ in $\lambda x$ [see Lerch (1905)].

(6) $$[t^{-1}\log(1+t)]^x = x\sum_{n=0}^{\infty}\frac{B_n^{(x+n)}}{x+n}\frac{t^n}{n!}$$

where the $B_n^{(x)}$ are defined in 19.7(57); see Nörlund (1920, 1924).

(7) $$[-t^{-1}\log(1-t)]^x = 1 + xt\sum_{n=0}^{\infty}\psi_n(x+n)\,t^n.$$

The $\psi_n$ are the Stirling polynomials; see 19.7(58) and Nielsen (1906).

(8) $$kt\,e^{xt}\csc(kt) = \sum_{n=0}^{\infty} g_n(x,k)\,t^n.$$

Let $[\frac{1}{2}n] = \frac{1}{2}n$ if $n$ is even and $[\frac{1}{2}n] = \frac{1}{2}n - \frac{1}{2}$ if $n$ is odd. Let the constants $b_{2n}$ be defined by

(9) $$\sum_{n=0}^{\infty} b_{2n}\,t^{2n} = t\,\operatorname{sech} t.$$

Then

(10) $$g_n(x,k) = \sum_{m=0}^{[\frac{1}{2}n]} b_{2m}\frac{k^{2m}x^{n-2m}}{(n-2m)!}.$$

For this result and for applications to the problem of approximating a function for which the mean value of the function and its derivatives are given see Léauté (1881). Appell (1897) showed that for $-k < x < k$

$$g_{2m}(x,k) = 2(-1)^m k^{2m}\pi^{-2m}\sum_{l=1}^{\infty}(-1)^{l-1}l^{-2m}\cos(l\pi x/k) \qquad m > 0$$

$$g_{2m+1}(x,k) = 2(-1)^m k^{2m+1}\pi^{-2m-1}\sum_{l=1}^{\infty}(-1)^{l-1}l^{-2m-1}\sin(l\pi x/k).$$

The $g_n$ are connected with the Bernoulli polynomials of 19.7(23).

$$g_n(x,k) = \frac{(2k)^n}{n!}B_n\left(\frac{x+k}{2k}\right)$$

(11) $$\frac{\sinh xt}{\sinh t} = \sum_{n=0}^{\infty} g_n(x)\,t^n$$

can be reduced to 19.7(23) (Bernoulli polynomials). Applications to two-point expansions of analytic functions by Whittaker (1933).

$$g_n(x) = \frac{2^n}{(n+1)!}\left[B_n\left(\frac{1+x}{2}\right) - B_n\left(\frac{1-x}{2}\right)\right].$$

(12) $$\frac{\cosh(xt)}{\cosh t} = \sum_{n=0}^{\infty} g_n(x) t^n.$$

The $g_n$ are related to Euler polynomials [see 19.7(24) and also Whittaker (1933)].

(13) $$(\tfrac{1}{2}\pi x)^{-\frac{1}{2}} \cos(x^2 - 2xt)^{\frac{1}{2}} = \sum_{n=0}^{\infty} J_{n-\frac{1}{2}}(x) t^n/n!$$

(14) $$(\tfrac{1}{2}\pi x)^{-\frac{1}{2}} \sin(x^2 - 2xt)^{\frac{1}{2}} = \sum_{n=0}^{\infty} J_{\frac{1}{2}-n}(x) t^n/n!$$

where $J_\nu(x)$ is the Bessel function of the first kind of order $\nu$ (see Chap. 7 for the notations). Reference: Glaisher (1873).

(15) $$(\cos t)^x = \sum_{n=0}^{\infty} c_n(x)\, t^n$$

(16) $$(t^{-1} \sin t)^x = \sum_{n=0}^{\infty} s_r(x)\, t^n.$$

For applications to the theory of Bernoulli numbers and other properties of the $c_n$, $s_n$ see Nielsen (1914) and compare with Nörlund (1920, 1924) and 19.7(57).

(17) $$\exp(x \tan^{-1} t) = \left(\frac{1+it}{1-it}\right)^{-\frac{1}{2}ix}$$

See 19.6(22).

(18) $$\exp(x \sin^{-1} t) = \sum_{n=0}^{\infty} g_n(x)\, t^n$$

(19) $$g_0(x) = 1, \quad g_1(x) = x$$

(20) $$g_{2k}(x) = \frac{1}{(2k)!} x^2(x^2+2^2)(x^2+4^2) \cdots [x^2 + (2k-2)^2]$$

(21) $$g_{2k+1}(x) = \frac{1}{(2k+1)!} x(x^2+1^2)(x^2+3^2) \cdots [x^2+(2k-1)^2].$$

This is a generalized Appell set of type 19.3(13) with $A(t) \equiv 1$. The explicit form of the $g_n(x)$ can be obtained by introducing $\sin \phi = t$ and differentiating (18) with respect to $\phi$.

(22) $\exp[\int_1^t s^{-1}(1+s)^x(1-s)^{-x}ds] = \sum_{n=0}^{\infty} g_n(x)\, t^n.$

See Mittag-Leffler (1901) and compare with 19.6 (22).

(23) $\exp\left\{m\int_0^t \left[\left(\frac{1+s}{1-s}\right)^x - 1\right]\frac{ds}{s}\right\} = \sum_{n=0}^{\infty} g_n^m(x)t^n$

See Mittag-Leffler (1901).

(24) $e^x \int_{1-t}^1 e^{-x/u} u^{1/2}\, du = -t \sum_{n=0}^{\infty} g_n(x)t^n.$

See Rogowski (1932).

(25) $\prod_{l=1}^{\infty} (1+tx^l) = \sum_{n=0}^{\infty} g_n(x)\, t^n \qquad |x| < 1$

(26) $g_n(x) = x^{1/2(n+1)n} \prod_{l=1}^{n} (1-x^l)^{-1}.$

For results and applications to the theory of probability see Oettinger (1867).

## 19.9. Bessel functions. Confluent hypergeometric functions (including special cases such as functions of the parabolic cylinder)

In this section the notations of Chap. 7 for Bessel functions and of Chapters 6 and 8 for confluent hypergeometric functions and their special cases have been used.

(1) $J_0\{[(x^2-2xt)]^{1/2}\} = \sum_{n=0}^{\infty} J_n(x)\, t^n/n!$

See Chap. 7 and Truesdell (1948).

(2) $(x+t)^{-1/2\alpha} J_\alpha[2(x+t)^{1/2}] = \sum_{n=0}^{\infty} x^{-1/2\alpha-1/2n} J_{\alpha+n}(x)\,(-t)^n/n!$

See Truesdell (1948).

(3) $(x+t)^{1/2\alpha} J_\alpha[2(x+t)^{1/2}] = \sum_{n=0}^{\infty} x^{1/2\alpha-1/2n} J_{\alpha-n}(x)\, t^n/n!$

See Truesdell (1948).

(4) $e^{xt} J_0[t(1-x^2)^{1/2}] = \sum_{n=0}^{\infty} P_n(x)\, t^n/n!$

(5) $e^{t} I_0(tx) = \sum_{n=0}^{\infty} (1-x^2)^{1/2 n} P_n[(1-x^2)^{-1/2}]\, t^n/n\,!$

(6) $e^{t} J_0[2(tx)^{1/2}] = \sum_{n=0}^{\infty} L_n(x) t^n/n\,!$

(7) $I_0[2t(x-1)^{1/2}]\, I_0[2t(x+1)^{1/2}] = \sum_{n=0}^{\infty} (n!)^{-2} P_n(x) t^n.$

Here $P_n$, $L_n$ are the Legendre and Laguerre polynomials (see Chap. 10). References and applications: for (4), (5), (6), Rainville (1945); for (7), Bateman (1905).

(8) $e^{t}\ {}_0F_1(1+a; -xt) = \sum_{n=0}^{\infty} \frac{L_n^{a}(x)}{(1+a)_n} t^n.$

The $L_n^{a}$ are the generalized Laguerre polynomials of Chap. 10; see also Szegö (1939).

(9) $e^{xt} t^{-1/2 a} J_a(2t^{1/2}) = \sum_{n=0}^{\infty} x^n L_n^{a}(x^{-1})\, t^n/\Gamma(a+n+1).$

The $L_n^{a}$ are the Laguerre polynomials of Chap. 10; see also Truesdell (1948) p. 2.

(10) $e^{xt}\ {}_0F_1[1+a; \tfrac{1}{4}t^2(x^2-1)] \equiv e^{xt}[\tfrac{1}{2}t^2(1-x^2)]^{-a} J_a[t(1-x^2)^{1/2}]$

$$= \sum_{n=0}^{\infty} [(2a+1)_n]^{-1}\, C_n^{a+1/2}(x)\, t^n$$

where $C_n^{\nu}$ is the Gegenbauer polynomial of Chap. 10; see Truesdell (1948).

(11) $e^{t}(xt)^{-1/2 a} J_a[2(xt)^{1/2}] = \sum_{n=0}^{\infty} \frac{L_n^{a}(x)\, t^n}{\Gamma(n+a+1)}.$

The $L_n^{a}$ are the Laguerre polynomials of Chap. 10; see Szegö (1939).

(12) ${}_0F_1[1+a; \tfrac{1}{2}t(x-1)]\,{}_0F_1[1+\beta; \tfrac{1}{2}t(x+1)]$

$$\equiv \Gamma(a+1)\Gamma(\beta+1)(\tfrac{1}{2}t)^{-1/2 a - 1/2 \beta}(1-x)^{-1/2 a}(1+x)^{-1/2 \beta}$$

$$\times J_\nu\{[2t(1-x)]^{1/2}\}\, I_\nu\{[2t(x+1)]^{1/2}\}$$

$$= \sum_{n=0}^{\infty} \frac{P_n^{(a,\beta)}(x) t^n}{(1+a)_n (1+\beta)_n}.$$

The $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials. Consult Chapters 10, 7 and 2 for the notations and Bateman (1905) for a proof. In the case $\alpha = \beta$, the right-hand side involves Gegenbauer polynomials; for $\alpha = \beta = 0$, the $P_n^{(\alpha,\beta)}$ are the Legendre polynomials (compare Chap. 10) and (12) becomes (7).

$$(13)\quad D_m(x+t)\exp[\tfrac{1}{4}(2xt+t^2)] = \sum_{n=0}^{m} \binom{m}{n} D_n(x)\, t^{m-n}.$$

The $D_m$ ($m = 0, 1, 2, \dots$) are parabolic cylinder functions [compare sec. 8.2 and Prasad (1926)].

$$(14)\quad (1-t)^{-p}\, {}_1F_1\left(p; 1+\alpha; -\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} \frac{(p)_n L_n^{\alpha}(x)}{(1+\alpha)_n}\, t^n.$$

Here $p$ is arbitrary; the $L_n^{\alpha}$ are the generalized Laguerre polynomials of Chap. 10. Reference: Chaundy (1943).

$$(15)\quad (1+4t^2)^{-c}\, {}_1F_1\left(c; \tfrac{1}{2}; \frac{4x^2t^2}{1+4t^2}\right) + \frac{32ct^3x^3}{3(1+4t^2)^{c+2}}\, {}_1F_1\left(c+1; \frac{5}{2}; \frac{4x^2t^2}{1+4t^2}\right)$$

$$+ \frac{2xt(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}}\, {}_1F_1\left(c; \frac{3}{2}; \frac{4x^2t^2}{1+4t^2}\right) = \sum_{n=0}^{\infty} \frac{(c)_l}{(2l)!}\, 2^{2l} H_n(x) t^n$$

where $l = \tfrac{1}{2}n$ if $n$ is even and $l = \tfrac{1}{2}n - \tfrac{1}{2}$ if $n$ is odd. The $H_n$ are the Hermite polynomials of Chap. 10. Reference: Brafman (1951).

$$(16)\quad e^{-t}\, {}_1F_1(-b, a+1; x+t)$$

$$= \sum_{n=0}^{\infty} \frac{(a+b+1)_n}{(a+1)_n}\, {}_1F_1(-b, a+n+1; x)\, \frac{(-t)^n}{n!}$$

See Truesdell (1948).

## 19.10. Gamma function. Legendre functions and Gauss' hypergeometric function. Generalized hypergeometric functions

The notations in this section are: for $\Gamma$, $(a)_n$ see Chap. 1; for $F$, ${}_2F_1$ see Chap. 2; for $P_\nu^\mu$ see Chap. 3; for ${}_pF_q$ see Chap. 4.

$$(1)\quad \frac{\Gamma(m+x+t)}{\Gamma(m+t)} = \sum_{n=0}^{\infty} A_m^{(n)}(x)\, t^n/n!.$$

The $A_m^{(n)}$ are the functions defined in 19.8(3); see also Narumi (1929).

(2) $$(1-2tx+t^2)^{-\frac{1}{2}-\frac{1}{2}\nu} P_\nu^\mu\left[\frac{x-t}{(1-2xt+t^2)^{\frac{1}{2}}}\right] = \sum_{n=0}^{\infty} \binom{\nu-\mu+n}{n} P_{\nu+n}^\mu(x) t^n$$

(3) $$(1-2xt+t^2)^{\frac{1}{2}\nu} P_\nu^\mu\left[\frac{x-t}{(1-2xt+t^2)^{\frac{1}{2}}}\right] = \sum_{n=0}^{\infty} \binom{\nu+\mu}{n} P_{\nu-n}^\mu(x)(-t)^n$$

(4) $$[1-t^2-2(1-x^2)^{-\frac{1}{2}}xt]^{-\frac{1}{2}\mu} P_\nu^\mu[x+t(1-x^2)^{\frac{1}{2}}] = \sum_{n=0}^{\infty} P_\nu^{\mu+n}(x)\, t^n/n\,!$$

(5) $$[1-2t(1-x^2)^{\frac{1}{2}}]^{-\frac{1}{2}-\frac{1}{2}\nu} P_\nu\left\{\frac{x}{[1-2t(1-x^2)^{\frac{1}{2}}]^{\frac{1}{2}}}\right\} = \sum_{n=0}^{\infty} P_{\nu+n}^n(x)\, t^n/n!$$

See Truesdell (1948).

(6) $$R^{-1} P_\nu\left(\frac{1+t}{R}\right) P_\nu\left(\frac{1-t}{R}\right) = \sum_{n=0}^{\infty} P_n(\cos x) F_\nu(-2n-1)\, t^n$$

where

(7) $$R = (1-2t\cos x+t^2)^{\frac{1}{2}}, \quad F_\nu(z) = {}_3F_2(-\nu, \nu+1, \tfrac{1}{2}+\tfrac{1}{2}z; 1, 1; 1).$$

Here $P_\nu$ is Legendre's function; $P_n$ ($n = 0, 1, 2, \ldots$) Legendre's polynomial (see Chapters 3 and 10); ${}_3F_2$ is a generalized hypergeometric series (see sec. 4.1.1). References: Bateman (1934), Rice (1940).

(8) $$\frac{1}{1-t}\, {}_2F_1[\zeta, \tfrac{1}{2}; p; -4xt(1-t)^{-2}] = \sum_{n=0}^{\infty} {}_3F_2(-n, n+1, \zeta; 1, p; x) t^n$$

Here ${}_2F_1$, ${}_3F_2$ denote hypergeometric and generalized hypergeometric series. References: Rice (1940), Fasenmyer (1947); see also 19.3(27), 19.3(28) and sec. 4.7.

(9) $$(1-t)^{-1-\alpha-\beta}\, {}_2F_1[\tfrac{1}{2}+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, 1+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta; 1+\alpha; 2t(x-1)(1-t)^{-2}]$$

$$= \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(x)\, t^n$$

(10) $$(1+t)^{-1-\alpha-\beta}\, {}_2F_1[\tfrac{1}{2}+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, 1+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta; 1+\beta; 2t(x+1)(1+t)^{-2}]$$

$$= \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\beta)_n} P_n^{(\alpha,\beta)}(x)\, t^n.$$

The $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials; the ${}_2F_1$ is the hypergeometric series. Consult Chapters 2 and 10 for the notations and Watson (1939) for a proof.

(11) $$(1-xt)^{-p}\ {}_2F_1[\tfrac{1}{2}p, \tfrac{1}{2}p+\tfrac{1}{2}; 1+\alpha; t^2(x^2-1)(1-xt)^{-2}]$$

$$= \sum_{n=0}^{\infty} \frac{(p)_n}{(2\alpha+1)_n} C_n^{\alpha+\frac{1}{2}}(x)\, t^n$$

where the $C_n^{\nu}$ are the Gegenbauer polynomials of Chap. 10. Reference: Brafman (1951). The parameter $p$ is arbitrary. For $\alpha = 0$, the $C_n^{\nu}$ become Legendre polynomials.

(12) $${}_2F_1(p, 1+\alpha+\beta-p; 1+\alpha; \tfrac{1}{2}-\tfrac{1}{2}t-\tfrac{1}{2}w)$$

$$\times\ {}_2F_1(p, 1+\alpha+\beta-p; 1+\beta; \tfrac{1}{2}+\tfrac{1}{2}t-\tfrac{1}{2}w)$$

$$= \sum_{n=0}^{\infty} \frac{(p)_n (1+\alpha+\beta-p)_n}{(1+\alpha)_n (1+\beta)_n} P_n^{(\alpha,\beta)}(x)\, t^n,$$

where $w = (1 - 2xt + t^2)^{\frac{1}{2}}$, $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of Chap. 10, and $p$ denotes an arbitrary parameter. Reference: Brafman (1951). The special cases $\alpha = \beta$ and $\alpha = \beta = 0$ give generated functions which are multiples of the ultraspherical or Gegenbauer polynomials and of the Legendre polynomials (cf. Chap. 10).

(13) $$[F(a,b;c;-t)]^2\, e^{xt} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ {}_4F_2\left[\begin{matrix} 2a, 2b, a+b, -n; \\ c, 2c-1; \end{matrix}\ x^{-1}\right] t^n.$$

Here $F$ is the hypergeometric series as in sec. 2.1; ${}_4F_2$ is a generalized hypergeometric series as in sec. 4.1. Reference: P. Humbert (1924).

(14) $$\int_0^{-x} F(\alpha, \beta; \gamma; t)\, d\alpha = \sum_{n=0}^{\infty} g_{n+1}(x) \frac{(\beta)_n}{(\gamma)_n} t^n.$$

Here $F$ is the hypergeometric series. Consult sec. 2.1 for the notations and see Appell (1929) for applications. The $g_{n+1}$ are the functions 19.8(2). Let

(15) $$e^t\ {}_1F_2(a; b_1, b_2; -x^2t^2) = \sum_{n=0}^{\infty} g_n(x)\, t^n/n!\, .$$

Then for any $c$

(16) $$(1-t)^{-c} {}_3F_3[\tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, a; \tfrac{1}{2}, b_1, b_2; -x^2t^2(1-t)^{-2}]$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n}{n!} g_n(x)\, t^n.$$

see Rainville (1947). Compare with 19.3 (19).

(17) $$e^{xt}(1-x^2)^{\frac{1}{2}m} {}_2F_3[m+\tfrac{1}{2}, m+\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}m+\tfrac{1}{2}, \tfrac{1}{2}m+1; -\tfrac{1}{4}t^2(1-x^2)]$$

$$= \frac{\pi 2^{-m}\, m!}{\Gamma(m+\tfrac{1}{2})\,\Gamma(m+\tfrac{1}{2})} \sum_{n=0}^{\infty} P^m_{m+n}(x) \frac{t^n}{(m+n)!}$$

The $P_n^m$ are the Legendre functions of Chap. 3; see Truesdell (1948).

(18) $$(1-t)^{a-1} {}_2F_0[\tfrac{1}{2}a-\tfrac{1}{2}; \tfrac{1}{2}a; 4xtb^{-1}(1-t)^{-2}]$$

$$\sim \sum_{n=0}^{\infty} y_n(x,a,b)\,(a-1)_n\, t^n/n!$$

The $y_n(x, a, b)$ are the generalized Bessel polynomials. Compare 19.7 (18) (Rainville, unpublished). Equation (18) is a special case of (23).

(19) $$(1-2xt)^{-1} {}_2F_0[1, \tfrac{1}{2}; -4t^2(1-2xt)^{-2}] \sim \sum_{n=0}^{\infty} H_n(x)\, t^n.$$

The $H_n$ are the Hermite polynomials of Chap. 10. Reference: Rainville (1947).

(20) $$(1-2xt)^{-a} {}_2F_0[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; -4t^2(1-2tx)^{-2}] \sim \sum_{n=0}^{\infty} \frac{(a)_n}{n!} H_n(x)\, t^n.$$

The $H_n$ are the Hermite polynomials of Chap. 10. Reference: Brafman (1951).

(21) $$\frac{1}{1-t}\, {}_pF_q\left[\begin{matrix} a_1, \dots, a_p; \\ b_1, \dots, b_q; \end{matrix}\; -\frac{4xt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1, \dots, a_p; \\ \tfrac{1}{2}, 1, b_1, \dots, b_q; \end{matrix}\; x\right] t^n$$

Notations are as in sec. 4.1.1. See also 19.3 (27), 19.3 (28) and Fasenmyer (1947).

(22) $$(1-t)^{-\lambda}\,{}_{p+1}F_q\left[\begin{matrix}\lambda, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix}\; -\frac{xt}{1-t}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\;{}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix}\; x\right] t^n$$

Chaundy (1943).

(23) $$(1-4xt)^{-1/2}\, 2^{c-1}[1+(1-4xt)^{1/2}]^{1-c}\;{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\; \frac{1-(1-4xt)^{1/2}}{2x}\right]$$

$$= \sum_{n=0}^{\infty} {}_{q+2}F_p\left[\begin{matrix}-n, c+n, 1-\beta_1-n, \ldots, 1-\beta_q-n; \\ 1-\alpha_1-n, \ldots, 1-\alpha_p-n;\end{matrix}\; (-1)^{p+q+1}x\right]\lambda_n t^n$$

where

(24) $$\lambda_n = \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\;\frac{1}{n!}$$

Rainville (1947).

(25) $$e^t\,{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\; -xt\right] = \sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\; x\right]\frac{t^n}{n!}$$

Rainville (1947).

(26) $$F_4[\gamma, \delta; 1+\alpha, 1+\beta; \tfrac{1}{2}t(x-1), \tfrac{1}{2}t(x+1)] = \sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n}{(1+\alpha)_n(1+\beta)_n} P_n^{(\alpha,\beta)}(x)t^n$$

where $F_4$ is Appell's hypergeometric function of two variables (see Chap. 5); $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of Chap. 10.

## 19.11. Generated functions of several variables

(1) $$(1-xt)^{-\alpha}(1-yt)^{-\beta} = \sum_{n=0}^{\infty} g_n(x,y)\,t^n.$$

The $g_n(x,y)$ are called *Lagrange's polynomials.*

(2) $$g_n(x,y) = \sum_{r=0}^{n} \frac{(\alpha)_r(\beta)_{n-r}}{r!\,(n-r)!}\,x^r y^{n-r}.$$

Applications to statistics and general reference: J. L. Lagrange (1867).

(3) $$(1+t)^{\lambda}(1+xt)^{\mu}(1+yt)^{\nu} = \sum_{n=0}^{\infty} \binom{\lambda}{n} F_1(-n,-\mu,-\nu,\lambda-n+1;x,y)t^n$$

where $F_1$ is Appell's hypergeometric series in two variables (see Chap. 5). Reference: Devisme (1932, 1933).

(4) $$\exp(xt - yt^2 + t^3/3) = \sum_{n=0}^{\infty} U_n(x,y)\,t^n.$$

Explicit (but complicated) expressions for the polynomials $U_n$ were given by Devisme (1932, 1933); also given are applications to

$$\frac{\partial^3 U}{\partial x^3} + \frac{\partial^3 U}{\partial y^3} + \frac{\partial^3 U}{\partial z^3} - 3\,\frac{\partial^3 U}{\partial x\,\partial y\,\partial z} = 0,$$

and related partial differential equations.

(5) $$\exp\{i[x(1+t^2)^{1/2} - yt]\} = \sum_{n=0}^{\infty} g_n(x,y)\,t^n$$

(6) $$g_n(x,y) = (-iy)^n(\tfrac{1}{2}\pi x)^{1/2} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\tfrac{1}{2}x/y^2)^k H^{(1)}_{k-\frac{1}{2}}(x)}{\Gamma(n-2k+1)\,k\,!}$$

where $[\tfrac{1}{2}n] = \tfrac{1}{2}n$ if $n$ is even and $[\tfrac{1}{2}n] = \tfrac{1}{2}n - \tfrac{1}{2}$ if $n$ is odd, and where $H^{(1)}_{k-\frac{1}{2}}$ is the first Hankel function of order $k - \tfrac{1}{2}$. See Hall (1936) and, for applications to a problem in the theory of conduction of heat see Green (1934).

(7) $$[(1-xt-ys)^2+(t^2+s^2)(1-x^2-y^2)]^{-\frac{1}{2}-\alpha} = \sum_{m,n=0}^{\infty} g_{m,n}(x,y)t^m s^n.$$

Let $p = 1 - x^2 - y^2$ and $\alpha > -\tfrac{1}{2}$. Then

(8) $$g_{m,n} = \frac{(-1)^{n+m}}{2^{m+n}\,m!\,n!}\,\frac{\Gamma(\alpha+1)\Gamma(2\alpha+m+n+1)}{\Gamma(2\alpha+1)\,\Gamma(\alpha+m+n+1)}\,p^{-\alpha}\,\frac{\partial^{m+n}p^{\alpha+m+n}}{\partial x^m\,\partial y^n}$$

consult Koschmieder (1924).

Put

(9) $$\phi(x,y) = ax^2+2bxy+cy^2, \quad a>0, \quad \Lambda = ac - b^2 > 0,$$

$$\Lambda\psi(x,y) = cx^2 - 2bxy + ay^2$$

$$\xi = ax + by, \quad \eta = bx + cy.$$

The polynomials generated by

$$(10)\quad \exp[t\xi + s\eta - \tfrac{1}{2}\phi(t,s)] = \sum_{m,n=0}^{\infty} H_{m,n}(x,y)\frac{t^m}{m!}\frac{s^n}{n!}$$

$$(11)\quad \exp[tx + sy - \tfrac{1}{2}\psi(t,s)] = \sum_{m,n=0}^{\infty} G_{m,n}(x,y)\frac{t^m}{m!}\frac{s^n}{n!}$$

are Hermite polynomials in two variables. For their properties, and for generalizations to several variables, see Appell and Kampé de Fériet (1926). For generating functions of products of such polynomials see Koschmieder (1937, 1938) and Erdélyi (1938).

SOME GENERATING FUNCTIONS OF SEVERAL VARIABLES

Let $x_1, \dots, x_l$ be variables and let

$$(12)\quad G_0(t) \equiv \prod_{r=1}^{l}(1-tx_r) = \sum_{r=0}^{l}(-1)^r s_r t^r.$$

Then $s_0 = 1$, $s_1 = x_1 + x_2 + \cdots + x_l$ and $s_r$ is the $r$-th elementary symmetric function of $x_1, \dots, x_l$. Let $k = 0, 1, 2, \dots$, and let

$$(13)\quad p_k = x_1^k + x_2^k + \cdots + x_l^k$$

be the sum of the $k$-th powers of the variables. Then we have

$$(14)\quad -\frac{\partial}{\partial t}(\log G_0) = \sum_{k=1}^{\infty} p_k t^{k-1}.$$

Multiplying both sides in (14) by $G_0$ and comparing the coefficients of the powers of $t$ on both sides gives Newton's recurrence formulas from which expressions for the $p_k$ in terms of the $s_r$ can be obtained. Let $y_k$ $(k = 1, 2, 3, \dots)$ be variables and let

$$(15)\quad \exp[\sum_{k=1}^{\infty}(y_k/k)\, t^k] = 1 + \sum_{n=1}^{\infty} B_n t^n.$$

Then

$$(16)\quad B_n(y_1, \dots, y_n) = \sum \frac{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n}}{\alpha_1!\,\alpha_2!\cdots\alpha_n!\,1^{\alpha_1}\,2^{\alpha_2}\cdots n^{\alpha_n}}$$

where the sum is taken over all non-negative integers $a_1, \dots, a_n$ for which

(17) $a_1 + 2a_2 + \cdots + n a_n = n.$

Now let $G_0$ and $s_r$ be defined by (12), and let $p_k$ be defined by (13). Then

$$(18)\quad [G_0(t)]^{-1} = \sum_{n=0}^{\infty} h_n(x_1, \dots, x_l)\, t^n$$

where

$$(19)\quad h_n(x_1, \dots, x_l) = B_n(p_1, p_2, \dots, p_n)$$

$$(20)\quad s_r(x_1, \dots, x_l) = (-1)^r B_r(-p_1, -p_2, \dots, -p_r).$$

For $r > l$, the left-hand side in (20) is identically zero which means that then, the right-hand side gives an algebraic relation between the sums of powers of $x_1, \dots, x_l$. The proof of these formulas follows from the remark that

$$(21)\quad \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k} t^k\right) = \exp(-\log G_0) = \frac{1}{G_0(t)}$$

$$(22)\quad \exp\left(-\sum_{k=1}^{\infty} \frac{p^k}{k} t^k\right) = G_0(t).$$

The functions $B_n$ are used in the theory of group characters. See Littlewood (1940) for other explicit expressions for the $B_n$. With a slight change in the definition, the $B_n$ have been thoroughly investigated by Bell (1934).

For generating functions in several variables, consult also sections 11.5, 11.6, and 11.8 where generating functions for spherical and hyperspherical harmonic polynomials are given. See Appell and Kampé de Fériet (1926) for the harmonic polynomials investigated by these authors.

### 19.12. Some generating functions connected with orthogonal polynomials

In this section, two sets of generating functions are given which were constructed from the point of view of the theory of orthogonal polynomials.

Let $g_n(x)$ $(n = 0, 1, 2, \dots)$ be a sequence of polynomials, such that $g_n(x)$ is of degree $n$ and let $\alpha(x)$ be a function of bounded variation such that the Stieltjes integrals

(1) $\int_{-\infty}^{\infty} g_n(x)\, g_m(x)\, d\alpha(x) = \lambda_{n,m}$

exist for $n$, $m = 0, 1, 2, \ldots$ . If $\lambda_{n,m} = 0$ for $n \neq m$, the $g_n(x)$ form an orthogonal system; if also $\lambda_{n,n} = 1$ for $n = 0, 1, 2, \ldots$ , the system is called orthonormal (see Chap. 10). If $d\alpha/dx = w(x)$ exists, it is called the weight function associated with the $g_n$. *If* $w$ is zero outside an interval $a \leq x \leq b$, we shall write an integral from $a$ to $b$ in (1) and we shall call the $g_n$ an orthogonal or orthonormal system for $(a, b)$. Watson (1933, 1934) has found explicit expressions for the bilinear generating functions

(2) $\sum_{n=0}^{\infty} g_n(x)\, g_n(y) t^n$

where the $g_n$ are the orthonormal systems derived from the Legendre, Gegenbauer, Jacobi, Laguerre and Hermite polynomials of Chap. 10. Using the notations of Chap. 10, Watson's results can be summarized as follows:

(3) $$\sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, P_n(x)\, P_n(y)\, t^n$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{(1-t^2)\, dw}{\{1-2t\,[xy+(1-x^2)^{1/2}(1-y^2)^{1/2}\cos w]+t^2\}^{3/2}}.$$

For explicit expressions for

(4) $\sum_{n=0}^{\infty} P_n(x)\, P_n(y)\, t^n$

see Watson (1933).

(5) $$2^{2\nu-1}\,[\Gamma(\nu)]^2\,(1-x^2)^{\frac{1}{2}\nu}\,(1-y^2)^{\frac{1}{2}\nu} \sum_{n=0}^{\infty} \frac{(n+\nu)\, n!}{\Gamma(n+2\nu)}\, C_n^{(\nu)}(x)\, C_n^{(\nu)}(y)\, t^n$$

$$= \frac{(1-x^2)^{\frac{1}{2}\nu}(1-y^2)^{\frac{1}{2}\nu}}{\pi} \int_0^{\pi} \frac{\nu(1-t^2)\,(\sin w)^{2\nu-1}\, dw}{\{1-2t\,[xy+(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}\cos w]+t^2\}^{\nu+1}}.$$

Let

(6) $$\theta_n = (2n+\alpha+\beta+1)\, \frac{n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}\, 2^{-\alpha-\beta-1},$$

that is,

(7) $\theta_n^{-1} = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2\,dx.$

Let

(8) $u = \tfrac{1}{2}(1-x)^{\frac{1}{2}}(1-y)^{\frac{1}{2}}, \quad v = \tfrac{1}{2}(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}.$

(9) $k = \tfrac{1}{2}(t^{\frac{1}{2}} + t^{-\frac{1}{2}}).$

(10) $y = \{[(k \sec w)^2 - u^2 - v^2]^2 - 4u^2 v^2\}^{\frac{1}{2}}$

(11) $z_1 = (k \sec w)^2 + u^2 - v^2 + y$

(12) $z_2 = (k \sec w)^2 - u^2 + v^2 + y.$

Then

(13) $[(1-x)(1-y)]^{\frac{1}{2}\alpha}[(1+x)(1+y)]^{\frac{1}{2}\beta} \sum_{n=0}^{\infty} \theta_n P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) t^n$

$$= t^{\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta} \frac{d}{dt} \left\{ u^{\alpha} v^{\beta} k \times \int_0^{\frac{1}{2}\pi} \left(\frac{2k \sec w}{z_1}\right)^{\alpha} \left(\frac{2k \sec w}{z_2}\right)^{\beta} \frac{\cos[(\alpha-\beta)w]\,dw}{y \cos^2 w} \right\}$$

(14) $\pi^{-\frac{1}{2}} e^{-\frac{1}{2}x^2-\frac{1}{2}y^2} \sum_{n=0}^{\infty} 2^{-n} H_n(x) H_n(y) t^n/n!$

$$= \pi^{-\frac{1}{2}}(1-t^2)^{-\frac{1}{2}} \exp\left[\frac{4xyt - (x^2+y^2)(1+t^2)}{2(1-t^2)}\right].$$

This formula was already derived by Mehler (1866); see also Erdélyi (1938).

(15) $(xy)^{\frac{1}{2}\alpha} e^{-\frac{1}{2}x-\frac{1}{2}y} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) t^n$

$$= t^{-\frac{1}{2}\alpha}(1-t)^{-1} \exp\left[-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right] I_{\alpha}\left[\frac{2(xy)^{\frac{1}{2}}}{1-t}\right]$$

where $I_{\alpha}$ is the modified Bessel function of Chap. 7. This is the Hille-Hardy formula; see also Myller-Lebedeff (1907).

Meixner (1934) determined all orthogonal polynomials $g_n(x)$ which possess a generating function of the form

(16) $f(t)\exp[xu(t)] = \sum_{n=0}^{\infty} g_n(x)\, t^n/n!\,.$

He shows that there are only five possibilities:

(i) Polynomials expressible in terms of Hermite polynomials

(17) $f(t) = \exp(-\tfrac{1}{2}kt^2), \quad u(t) = t, \quad \dfrac{d\alpha}{dx} = \exp\left(\dfrac{-x^2}{2k}\right).$

(ii) Polynomials expressible in terms of (generalized) Laguerre polynomials

(18) $f(t) = (1-\lambda t)^{-k/\lambda^2} \exp\left[\dfrac{kt}{\lambda(\lambda t - 1)}\right], \quad u(t) = \dfrac{t}{(1-\lambda t)}$

(19) $\dfrac{d\alpha}{dx} = 0 \qquad x > k/\lambda$

(20) $\dfrac{d\alpha}{dx} = (-x + k/\lambda)^{-1+k/\lambda^2}\, e^{x/\alpha} \qquad -\infty < x < k/\lambda.$

(iii) Polynomials expressible in terms of Poisson-Charlier polynomials

(21) $f(t) = (1-\lambda t)^{k/\lambda^2}\, e^{kt/\lambda}$

(22) $u(t) = -\lambda^{-1}\log(1-\lambda t).$

Here $\alpha(x)$ is constant except for

(23) $x = x_n = \lambda^{-1}k - \lambda n \qquad n = 0, 1, 2, \ldots$

where $\alpha(x)$ has a jump defined by

(24) $\alpha(x_n + 0) - \alpha(x_n - 0) = \dfrac{1}{n!}\left(\dfrac{k}{\lambda^2}\right)^n.$

(iv) Hypergeometric polynomials; discrete variable

(25) $f(t) = [(1-\mu t)^{-\mu^{-1}}(1-\lambda t)^{-\lambda^{-1}}]^{k/(\mu-\lambda)},$

$u(t) = (\lambda-\mu)^{-1}[\log(1-\mu t) - \log(1-\lambda t)]$

where $\lambda$ and $\mu$ are real and $\alpha(x)$ is constant except for

(26) $x = x_n = k/\lambda - (\lambda - \mu)\, n \qquad n = 0, 1, 2, \dots$

where $\alpha(x)$ has a jump such that

(27) $$\alpha(x_n + 0) - \alpha(x_n - 0) = \left(-\frac{\mu}{\lambda}\right)^n \binom{-k/(\lambda\mu)}{n}.$$

(v) Hypergeometric polynomials; continuous variable; we have again equations (25) with $\lambda$ and $\mu$ conjugate complex and

(28) $\operatorname{Im} \lambda > \operatorname{Im} \mu.$

Then, for $-\infty < x < \infty$

(29) $$\frac{d\alpha}{dx} = \left(-\frac{\mu}{\lambda}\right)^{x/(\mu-\lambda)} \Gamma(\omega)\, \Gamma(\phi)$$

where

(30) $$\omega = \frac{x}{\mu - \lambda} + \frac{k}{\mu(\lambda - \mu)}, \qquad \phi = \frac{x}{\lambda - \mu} + \frac{k}{\lambda(\mu - \lambda)},$$

and where

(31) $$\left| \arg\left(-\frac{\mu}{\lambda}\right) \right| < \pi.$$

In all cases, differential equations or difference equations for the $g_n(x)$ can be established.

For references to other cases in which the generated functions involve orthogonal functions compare the end of sec. 19.11.

## 19.13. Generating functions of certain continuous orthogonal systems

The Hermite, Laguerre, Legendre, Gegenbauer and Jacobi polynomials arise from the investigation of certain linear differential equations of the Sturm-Liouville type. After multiplication by a weight function, the orthogonal functions thus obtained are the eigenfunctions of a Sturm-Liouville problem which in these cases has a discrete spectrum. For the linear and bilinear generating functions of these systems see sec. 19.12.

For another range of the variable, the same differential equation may have a continuous spectrum. Let $f_\nu(x)$ be the corresponding system of eigenfunctions. Then the integrals

$$\int t^{\nu} f_{\nu}(x)\, d\nu, \qquad \int t^{\nu} f_{\nu}(x)\, f_{\nu}(y)\, d\nu,$$

taken over an appropriate range of values of $\nu$, may be called the linear and bilinear generating functions for the $f_{\nu}(x)$; this agrees with Laplace's original definition of a generating function (cf. sec. 19.1).

In this section, linear and bilinear generating functions are given for the parabolic cylinder functions $D_{\nu}$ of Chap. 8; the confluent hypergeometric functions $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ of Chap. 6; the Gegenbauer functions $C_{\mu}^{\nu}$ and the hypergeometric functions corresponding to the Jacobi polynomials (cf. Chapters 2 and 10).

For proofs and for references to applications of continuous orthogonal systems to boundary value problems consult Erdélyi (1941).

(1) $$\exp(-\tfrac14 x^2 - xt - \tfrac12 t^2) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} t^{\nu}\, \Gamma(-\nu)\, D_{\nu}(x)\, d\nu$$

$$c < 0, \qquad |\arg t| < \tfrac14 \pi$$

(2) $$(1+t^2)^{-\frac12} \exp\left[\frac14 \frac{1-t^2}{1+t^2}(x^2+y^2) + \frac{ixyt}{1+t^2}\right]$$

$$= \frac{(\tfrac12 \pi)^{\frac12}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{t^{-\nu-1}}{\sin(-\nu\pi)} \left[D_{\nu}(x)\, D_{-\nu-1}(iy) + D_{\nu}(-x)\, D_{-\nu-1}(-iy)\right] d\nu$$

$$-1 < c < 0, \qquad |\arg t| < \tfrac12 \pi$$

(3) $$\Gamma(2\mu+1)(1+t)^{-2\mu-1} x^{\mu+\frac12} \exp\left(\tfrac12 x\, \frac{t-1}{t+1}\right)$$

$$= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} t^{-\frac12+\kappa-\mu}\, \Gamma(\tfrac12+\kappa+\mu)\, \Gamma(\tfrac12-\kappa+\mu)\, M_{\kappa,\mu}(x)\, d\kappa$$

$$|c| < \tfrac12 + \mathrm{Re}\,\mu, \qquad |\arg t| < \pi$$

(4) $$\frac{(txy)^{\frac12}}{1+t} \exp\left(-\frac{x+y}{2}\, \frac{1-t}{1+t}\right) J_{2\mu}\left[\frac{2(txy)^{\frac12}}{1+t}\right]$$

$$= \frac{1}{2\pi i} \int_L t^{\kappa}\, \frac{\Gamma(\tfrac12-\kappa+\mu)\, \Gamma(\tfrac12+\kappa+\mu)}{[\Gamma(2\mu+1)]^2}\, M_{\kappa,\mu}(x)\, M_{\kappa,\mu}(y)\, d\kappa$$

$$|\arg t| < \pi$$

where $J_{2\mu}$ denotes the Bessel function of the first kind of order $2\mu$ (cf.

Chap. 7), and where $L$ is a path from $-i\infty$ to $i\infty$ separating the poles of $\Gamma(\frac{1}{2}-\kappa+\mu)$ from those of $\Gamma(\frac{1}{2}+\kappa+\mu)$.

With the Hankel function $H^{(1)}_{2\mu}$ of the first kind instead of $J_{2\mu}$, (4) becomes

$$(5)\quad \frac{(txy)^{\frac{1}{2}}}{1+t}\exp\left(-\frac{x+y}{2}\,\frac{1-t}{1+t}\right)H^{(1)}_{2\mu}\left[\frac{2(txy)^{\frac{1}{2}}}{1+t}\right]$$
$$= (2\pi i)^{-1}\int_L t^{\kappa}e^{i\pi(\kappa-\mu)}[U(\kappa)W_{\kappa,\mu}(x)W_{\kappa,\mu}(y)$$
$$+\,U(-\kappa)W_{-\kappa,\mu}(-x)W_{-\kappa,\mu}(-y)]\,d\kappa$$

where

$$U(\kappa)=\Gamma(\tfrac{1}{2}-\kappa-\mu)\,\Gamma(\tfrac{1}{2}-\kappa+\mu).$$

The Gegenbauer functions $C^{\nu}_{\mu}$ can be defined by

$$(6)\quad C^{\nu}_{\mu}(x)=\frac{\Gamma(\mu+2\nu)}{\Gamma(\mu+1)\,\Gamma(2\nu)}\;F(\mu+2\nu,-\mu;\nu+\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}x)$$

where $F$ denotes the hypergeometric series of 2.1. For $\mu=0,1,2,\ldots$, $C^{\nu}_{\mu}$ is the Gegenbauer or ultraspherical polynomial of sec. 11.1.2. The linear generating function is

$$(7)\quad (1+2tx+x^2)^{-\nu}=-\frac{1}{2i}\int_{c-i\infty}^{c+i\infty}t^{\mu}\frac{C^{\nu}_{\mu}(x)}{\sin(\mu\pi)}\,d\mu\qquad -2\operatorname{Re}\nu<c<0.$$

Evaluation of the integral in (7) by means of residues gives 11.1(16).

The most significant case in which Gegenbauer functions with non-integer subscript $\mu$ appear in mathematical physics is $\mu=-\frac{1}{2}+i\sigma$, $\sigma$ real; in this case the $C^{\frac{1}{2}-l}_{\mu+l}$ are involved in the definition of the associated conal harmonics. The normalized form of these as introduced by Weyl (1910) is

$$(8)\quad \psi^{\pm l}_{\mu}(x,\phi)=N^{\frac{1}{2}}\,\frac{(x^2-1)^{\frac{1}{2}l}}{2^l l!}F(l-\mu,l+\mu+1;\,l+1;\tfrac{1}{2}-\tfrac{1}{2}x)e^{\pm il\phi}$$
$$l=0,1,2,\ldots$$

where

$$(9)\quad N=(-1)^l(\mu+\tfrac{1}{2})\operatorname{ctn}(-\mu\pi)\,\frac{\Gamma(1+\mu+l)}{\Gamma(1+\mu-l)}.$$

Let

(10) $\omega = xy - (x^2-1)^{1/2}\,(y^2-1)^{1/2}\cos(\phi - \theta).$

Then

(11) $(t^2-1)\,(1+2t\,\omega + t^2)^{-3/2}$

$$= i\int_{c-i\infty}^{c+i\infty} \frac{t^{\mu}}{\cos(\mu\pi)} \sum_{l=-\infty}^{\infty} \psi_{\mu}^{\,l}(x,\phi)\,\psi_{\mu}^{-l}(y,\theta)\,d\mu \qquad -1 < c < 0.$$

For the generalization of the Jacobi polynomials we have the following results. Let

(12) $S = [1+2(1-2x)\,ut^{1/2}+u^2t]^{1/2}$

(13) $T = [1+2(1-2y)\,ut^{-1/2} + u^2\,t^{-1}]^{1/2}$

(14) $V = [1+2(1-2x)t + t^2]^{1/2}.$

Then

(15) $$\frac{\Gamma(\gamma)}{V}\left(\frac{V-t-1}{-2\,tx}\right)^{\gamma-1}\left(\frac{V-t+1}{2}\right)^{\gamma-\alpha}$$

$$= (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\Gamma(-\nu)\,\Gamma(\gamma+\nu)\,t^{\nu}\,F(-\nu,\alpha+\nu;\gamma;x)\,d\nu$$

$$0 < -c < \operatorname{Re}\gamma$$

and

(16) $$t^{-\frac{1}{2}\alpha}\int_0^{\infty} u^{\alpha-1}\left(\frac{S-ut^{1/2}-1}{-2ux}\;\frac{T-ut^{-1/2}-1}{-2uy}\right)^{\gamma-1}$$

$$\times\left(\frac{S-ut^{1/2}+1}{2}\;\frac{T-ut^{-1/2}+1}{2}\right)^{\gamma-\alpha}\frac{du}{ST}$$

$$= (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\Phi(\nu)\,t^{\nu}\,F(-\nu,\alpha+\nu;\gamma;x)\,F(-\nu,\alpha+\nu;\gamma;y)\,d\nu$$

$$0 < -c < \operatorname{Re}\alpha, \quad \operatorname{Re}(\alpha-\gamma) < -c < \operatorname{Re}\gamma$$

where

(17) $\Phi(\nu) = \Gamma(-\nu)\,\Gamma(\alpha+\nu)\,\Gamma(\gamma+\nu)\,\Gamma(\gamma-\alpha-\nu).$

If the parameter $\nu$ in (7), or the parameters $\alpha$, $\gamma$ in (15), (16) do not satisfy the respective inequalities, the path of integration must be indented so as to separate the different groups of poles of the integrands. These curved paths may be deformed so as to coincide with the straight line from $c - i\infty$ to $c + i\infty$. If we do so, a number of poles are crossed, contributing a sum of residues. Our generating functions are now a sum plus an integral and represent the eigenfunctions of a "mixed" spectrum.

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M

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W

$W_{\kappa,\mu}(z)$ confluent hypergeometric function (*see* vol. I, 264)

Z

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GREEK LETTERS

$\gamma$ Euler's constant (*see* vol. I, p. 1)
$\Delta$ discriminant of Weierstrass' canonical form, 17
$\Delta$ Laplace's operator, 45
$\epsilon_0 = 1$, $\epsilon_n = 2$, $n = 1, 2, \ldots$
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$\nu(x)$, $\nu(x, \alpha)$, 217
$\pi(x)$ number of primes, 191
$\sigma(n)$, $\sigma_k(n)$, 168
$\tau(n)$ Ramanujan's function, 184 ff.
$\tau_k(n)$ $k$th totient, 168
$\phi(n)$ Euler s function, 168
$\phi(\alpha, \beta; z)$ E.M. Wright's generalized Bessel function, 211
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## MISCELLANEOUS NOTATIONS

arg $z$ argument (or phase) of complex number $z$
Im $z$ imaginary part of $z$ (complex)
Re $z$ real part of $z$ (complex)
$a \equiv b \pmod{n}$, 175
$(a)_n = \Gamma(a+n)/\Gamma(a)$
$\left(\frac{k}{p}\right)$ Legendre-Jacobi symbol, 186
$m|n$, $m \nmid n$, 167
$(m, n)$, 168
$\binom{x}{n} = \dfrac{x(x-1)\cdots(x-n+1)}{n!}$
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$[x]$ largest integer $\leq x$
$\sum_p$, $\sum_{d|n}$, $\sum_{(m,n)=1}$, $\prod_p$, $\prod_{d|n}$, 168
$\sim$ approximate or asymptotic equality
$⨍$ Cauchy principal value of an integral
$\int_\infty^{(0+)}$ loop integral