Almost Sure Convergence to Zero in Stochastic Growth Models

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Abstract

This paper shows that in stochastic one-sector growth models, if the production function does not satisfy the Inada condition at zero, any feasible path converges to zero with probability one provided that the shocks are sufficiently volatile. This result seems significant since, as we argue, the Inada condition at zero is difficult to justify on economic grounds. Our convergence result is extended to the case of a nonconcave production function. The generalized result applies to a wide range of stochastic growth models, including stochastic endogenous growth models, overlapping generations models, and models with nonconcave production functions.

Keywords: Stochastic growth; Inada condition; convergence to zero.

1 Introduction

In a seminal paper, Brock and Mirman (1972) showed that the optimal paths of a stochastic one-sector growth model converges to a unique nondegenerate stationary distribution. While various cases are known in which their theorem can be extended,[1] it is not well understood when the theorem fails. Most of the extensions of the Brock-Mirman theorem assume that the production function satisfies the Inada condition at zero, i.e., that the marginal product of capital goes to infinity as capital goes to zero.[2]

Although the Inada condition at zero is almost ubiquitous in economics, the only justification for its use seems to be analytical simplicity. According to Barro and Sala-i-Martin (1995, p. 16), the Inada conditions $f'(0) = \infty$ and $f'(\infty) = 0$ are named after Inada (1963), who in fact used these conditions following Uzawa (1963). Neither Inada nor Uzawa, however, provided an economic justification for the conditions. We argue in Section 2 that the Inada condition at zero is difficult to justify on economic grounds since it has an unrealistic implication. More specifically, the condition is shown to imply that each unit of capital must be capable of producing any large amount of output with a sufficient amount of labor.

Given this unrealistic implication of the Inada condition at zero, it seems worthwhile to study the case in which the condition is not satisfied. We show in Section 3 that if the Inada condition at zero is not satisfied, i.e., if the marginal product of capital is finite at zero, then under an additional condition, any feasible path converges to zero with probability one. This statement itself is rather trivial since it is well-known that all feasible paths converge to zero in the deterministic case if the marginal product of capital at zero is less than one.

What we show, however, is that no matter how large the expected marginal product of capital at zero is, as long as it is finite, any feasible path converges to zero with probability one provided that the shocks are sufficiently volatile. To our knowledge, this result has not been documented in the literature.[3] The result is extended to the case of a nonconcave production function. The

[1] For example, see Stachurski (2002) and the references therein.

[2] An exception is a convergence result shown by Hopenhayn and Prescott (1992). We discuss their result in Section 3.

[3] Phelps (1962, p. 736) made the relevant observation that growth is not guaranteed by the condition that the expected marginal product of capital exceeds the reciprocal of the discount factor.

generalized result applies to a wide range of one-sector models, including stochastic endogenous growth models, overlapping generations models, and models with a nonconcave production function.

The rest of the paper is organized as follows. Section 2 argues that the Inada condition at zero has an unrealistic implication. Section 3 shows our main results. Section 4 concludes the paper.

2 An Unrealistic Implication of the Inada Condition at Zero

This section shows that the Inada condition at zero has an unrealistic implication. This well-known condition is that the marginal product of capital goes to infinity as capital goes to zero. The natural way to examine its implications would be to consider a situation in which capital is made arbitrarily small. However, since most capital goods in reality are more or less indivisible, it is not clear what an infinitestimal amount of capital represents. Thus in this section we regard capital as the capital-labor ratio and consider a situation in which labor is arbitrarily large.

As in most neoclassical models, suppose output is produced by a linearly homogenous production function $F(K, L)$ using capital K and labor L. Define $f(k) = F(k, 1)$. We assume that F and f are concave and continuously differentiable. The Inada condition at zero is

$$f'(0) \equiv \lim_{k \downarrow 0} f'(k) = \infty. \tag{2.1}$$

To discuss an implication of this condition, it is useful to obtain the following equivalent expression for $f'(0)$:

$$\forall K \geq 0, \quad f'(0) = \lim_{L \uparrow \infty} [F(K+1, L) - F(K, L)]. \tag{2.2}$$

To see this, note that by linear homogeneity,

$$F(K+1, L) - F(K, L) = L \left[f\left(\frac{K+1}{L}\right) - f\left(\frac{K}{L}\right) \right] \tag{2.3}$$

$$= L f'(\tilde{k}) \frac{1}{L} = f'(\tilde{k}), \tag{2.4}$$

where the first equality in (2.4) holds for some $\tilde{k} \in [K/L, (K+1)/L]$ by the mean value theorem. As $L \uparrow \infty$, $\tilde{k} \to 0$. Hence (2.2) holds.

By (2.2), the Inada condition at zero implies that an additional unit of capital is capable of producing any large amount of output as long as a sufficient amount of labor is available. Since this is true for each additional unit, it follows that every unit of capital must be unboundedly productive. This, however, seems unrealistic.

For example, consider a closed economy in which there is one machine that must be operated by one worker. For simplicity we assume that this machine is the only capital in the economy.[4] For the moment, assume

$$\forall L \geq 0, \quad F(0, L) = 0. \tag{2.5}$$

This means that workers with no access to the machine produce nothing. If there are three workers in the economy, then the machine can run 24 hours a day with each worker working eight hours a day. If there are six workers, output may rise since each worker, working only four hours a day, may be less tired and more efficient in handling the machine. If there are twelve workers, output may rise for the same reason, but perhaps to a lesser degree. Adding more workers probably will not make much difference though they can cooperate in various ways to increase efficiency. In any case, since workers produce nothing when they are not using the machine, it seems reasonable to assume that output will reach its upper bound fairly soon as more and more workers are added. But if this is the case, the Inada condition at zero is violated since, as shown above, it together with (2.2) implies that the machine must be capable of producing any large amount if a sufficiently large number of workers are available.

Now suppose (2.5) is not satisfied, i.e., capital is not an essential factor of production. In this case, total output grows unboundedly if more and more workers become available. But the above example works with appropriate modifications. Note that if we define $G(K, L) = F(K, L) - F(0, L)$, then $F(K, L) = G(K, L) + F(0, L)$, i.e., G is the relevant part of the production function. It is easy to see that G satisfies (2.2) and (2.5) with G replacing F. Thus we may apply the above argument to G to draw the same conclusion.

[4] By linear homogeneity, it does not matter how many machines there are in the economy. Though we consider here the case in which (2.2) holds with $K = 0$ for simplicity, the following argument is valid for any $K \geq 0$, where K is the number of machines in the economy less one.

Hence, even without (2.5), the Inada condition at zero is violated as long as the machine's capacity is limited. In view of this result, it seems difficult to conceive of a real-life situation in which the Inada condition at zero is satisfied. Note that producing more machines is a part of capital accumulation, so is repairing the existing machine.

3 Main Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{\mathcal{F}_t\}_{t=0}^{\infty} \subset \mathcal{F}$ be a filtration. From here on, any sequence of the form $\{x_t\}_{t=0}^{\infty}$ is assumed to be a stochastic process adapted to this filtration. Consider an infinite horizon economy in which the resource constraint in period $t \in \mathbb{Z}_+$ is given by

$$c_t + k_{t+1} = s_t f(k_t), \tag{3.1}$$

where c_t is consumption in period t, k_t is the capital stock at the beginning of period t, and s_t is the productivity shock in period t. We say that a nonnegative stochastic process $\{k_t\}_{t=0}^{\infty}$ is a *feasible path* if it satisfies (3.1) for all $t \in \mathbb{Z}_+$ for some nonnegative stochastic process $\{c_t\}$.

Since our results apply to any feasible path, no further structure is required. No matter what objective function is specified, optimal paths are required to be feasible. Therefore our results apply to optimal paths as well.

We state our results after discussing our assumptions.

Assumption 3.1. (i) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is C^1 on $\mathbb{R}_{++}$, concave, and strictly increasing. (ii) $f(0) = 0$. (iii) $m \equiv \lim_{k \downarrow 0} f'(k) < \infty$.

Parts (i) and (ii) are standard. Part (iii) means that f does not satisfy the Inada condition at zero. Though m is required to be finite, it is allowed to be arbitrarily large. Part (iii) seems reasonable given our discussion in Section 2.

Assumption 3.2. $\forall t \in \mathbb{Z}_+$, (i) $s_t > 0$ *a.s.* and (ii) $Es_t = 1$.

Part (i) is standard. Part (ii) is only a normalization, implying that m as given in Assumption 3.1 is the expected marginal product of capital at zero.

Assumption 3.3. $\exists \nu \in (-\infty, \infty], \forall t \in \mathbb{Z}_+, E \ln s_t = -\nu$.

This assumption means only that $E \ln s_t$ does not depend on t. By Jensen's inequality and Assumption 3.2,

$$-\nu = E \ln s_t \leq \ln E s_t = 0. \tag{3.2}$$

Thus ν is in fact nonnegative.

Assumption 3.4. We have

$$\lim_{T \uparrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln s_t = -\nu \quad a.s., \tag{3.3}$$

where ν is given by Assumption 3.3.

Assumptions 3.3 and 3.4 mean that $\{\ln s_t\}$ has constant mean and satisfies the law of large numbers. These assumptions hold if $\{\ln s_t\}$ is stationary and ergodic with $E|\ln s_t| < \infty$ (e.g., White, 2000, Theorem 3.34). For example, $\{\ln s_t\}$ may be an i.i.d. process, as typically assumed in the stochastic growth literature; more generally, it may be a stationary ARMA process.

We are now ready to state the following.

Theorem 3.1. *Suppose Assumptions 3.1–3.4 hold. Suppose*

$$\ln m < \nu. \tag{3.4}$$

Then any feasible path converges to zero a.s.

Proof. Let $\{k_t\}$ be any feasible path. Let

$$\Omega_1 = \{\omega \in \Omega \,|\, \forall t \in \mathbb{Z}_+, s_t(\omega) > 0, k_t(\omega) > 0\}, \tag{3.5}$$

$$\Omega_2 = \left\{ \omega \in \Omega \,\middle|\, \lim_{T \uparrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln s_t(\omega) = -\nu \right\}. \tag{3.6}$$

Note that $\forall \omega \in \Omega \setminus \Omega_1, k_t(\omega) = 0$ for t sufficiently large by (3.1) and Assumption 3.1(ii). By Assumption 3.4, $P(\Omega_2) = 1$. Thus to show the conclusion, it suffices to verify that $\forall \omega \in \Omega_1 \cap \Omega_2, \lim_{t \uparrow \infty} k_t(\omega) = 0$. For the rest of the proof, we fix $\omega \in \Omega_1 \cap \Omega_2$. But we write k_t instead of $k_t(\omega)$, etc., for notational simplicity.

By Assumption 3.1, $\forall k \geq 0, f(k) \leq mk$. Thus

$$\forall t \in \mathbb{Z}_+, \quad k_{t+1} \leq s_t f(k_t) \leq s_t m k_t. \tag{3.7}$$

Since $\omega \in \Omega_1$, it follows that

$$\forall t \in \mathbb{Z}_+, \quad \ln k_{t+1} \leq \ln s_t + \ln m + \ln k_t. \tag{3.8}$$

Hence

$$\forall T \in \mathbb{N}, \quad \ln k_T \leq \sum_{t=0}^{T-1} \ln s_t + T \ln m + \ln k_0. \tag{3.9}$$

Dividing through by T, we get

$$\forall T \in \mathbb{N}, \quad \frac{\ln k_T}{T} \leq \frac{\sum_{t=0}^{T-1} \ln s_t}{T} + \ln m + \frac{\ln k_0}{T}. \tag{3.10}$$

The right-hand side converges to $\ln(m) - \nu$ as $T \uparrow \infty$ since $\omega \in \Omega_2$. By (3.4), $\ln(m) - \nu < 0$. Let $b \in (\ln(m) - \nu, 0)$. Then for T sufficiently large, $\ln(k_T)/T \leq b$, i.e., $\ln k_T \leq bT$. Thus $\lim_{T\uparrow\infty} \ln k_T \leq \lim_{T\uparrow\infty} bT = -\infty$; i.e., $\lim_{T\uparrow\infty} k_T = 0$. □

The basic idea of the above proof is that (3.4) and (3.8) imply that k_t converges to zero *a.s.* A similar argument was used by Kelly (1992, Proposition 1) to show the convergence of output paths with different initial capital stocks in an endogenous growth model. Another similar argument was used by De Hek and Roy (2001, Lemma 1) to obtain a sufficient condition for sustained growth in a stochastic growth model. Our contribution here is to point out that without the Inada condition at zero, it is possible that every feasible path converges to zero *a.s.* even if the marginal product of capital at zero is arbitrarily large.

If $\nu = 0$, i.e., if s_t is non-random, then (3.4) reduces to

$$m < 1, \tag{3.11}$$

which is exactly the condition under which all feasible paths converge to zero in the deterministic case (for an arbitrary production function satisfying Assumption 3.1). Since $\nu \geq 0$ by (3.2), (3.11) implies (3.4) even in the stochastic case. In the stochastic case, however, (3.4) holds even if m is arbitrarily large, provided that ν is sufficiently large. Roughly speaking, the more volatile s_t is, the larger ν $(= \ln Es_t - E \ln s_t)$ is. Thus without the Inada condition at zero, almost sure convergence to zero occurs if the shocks are sufficiently volatile. A simple example illustrates this point.

Suppose s_t is unconditionally log-normal.[5] Then by Assumption 3.2(i) and log-normality,

$$1 = Es_t = E\exp(\ln s_t) = \exp\left(E\ln s_t + \frac{Var(\ln s_t)}{2}\right). \tag{3.12}$$

Recalling that $\nu = -E\ln s_t$, we obtain

$$\nu = \frac{Var(\ln s_t)}{2}. \tag{3.13}$$

Hence (3.4) holds as long as $Var(\ln s_t)$ is sufficiently large. In other words, as long as the production function violates the Inada condition at zero, any feasible path converges to zero *a.s.* provided that the shocks are log-normal with sufficiently large variance.

It is easy to see that such shocks can be constructed even if their support is required to be bounded and bounded away from zero, as in the original Brock-Mirman (1972) model. One way to do this is by approximating a sufficiently volatile log-normal shock by a random variable whose support is bounded and bounded away from zero. As a simpler example, suppose $\{s_t\}$ is i.i.d., and $s_t = 1 - r$ with probability 1/2, and $1 + r$ with probability 1/2, where $r \in (0, 1)$. Then $Es_t = 1$ and

$$-\nu = E\ln s_t = \frac{1}{2}[\ln(1-r) + \ln(1+r)]. \tag{3.14}$$

Thus $\nu \to \infty$ as $r \uparrow 1$, so that (3.4) holds for r sufficiently close to one.

Hopenhayn and Prescott (1992, Sec. 6.B(i)) extended the Brock-Mirman theorem to the case in which the Inada condition at zero need not hold. Let us clarify the relationship between their result and ours. They assumed that the production function $\tilde{f}$ satisfies $\beta\tilde{f}'(0) > 1$ for some $\beta \in (0, 1)$ and that the shocks $\tilde{s}_t$ are i.i.d. with $\tilde{s}_t \in [1, \overline{\alpha}]$ for some $\overline{\alpha} > 1$. Under our normalization, i.e., Assumption 3.2(ii), their assumptions are expressed as follows:

$$\text{(i) } f'(0) > \theta/\beta, \quad \text{(ii) } s_t \in [1/\theta, \overline{\alpha}/\theta], \tag{3.15}$$

where $\overline{\alpha} > 1, \theta \in (1, \overline{\alpha})$, and $\beta \in (0, 1)$.[6] Since $m = f'(0)$ and $\nu = -\ln Es_t$, by (3.15),

$$\ln(m) - \nu > \ln(\theta/\beta) + \ln(1/\theta) \geq -\ln\beta > 0. \tag{3.16}$$

[5]This is true, for example, if $\{\ln s_t\}$ is i.i.d. normal or a stationary AR process with normal innovations.

[6]Hopenhayn and Prescott's assumptions can be recovered by setting $\tilde{s}_t = \theta s_t$ and $\tilde{f} = f/\theta$. Note that $\theta = E\tilde{s}_t$.

Thus in this case, (3.4) does not hold.

The critical assumption here is that $s_t \geq 1/\theta$. Without this lower bound, Hopenhayn and Prescott's result may fail. For example, let $\underline{\alpha} \in (0, 1)$ and suppose $s_t \in [\underline{\alpha}/\theta, \overline{\alpha}/\theta]$ instead of (3.15)(ii). Suppose $s_t = \underline{\alpha}/\theta$ with probability $p(\underline{\alpha})$, and $\overline{\alpha}/\theta$ with probability with $1 - p(\underline{\alpha})$, where $p(\underline{\alpha}) = (\overline{\alpha} - \theta)/(\overline{\alpha} - \underline{\alpha})$. Then $Es_t = 1$ and

$$-\nu = E \ln s_t = p(\underline{\alpha}) \ln(\underline{\alpha}/\theta) + (1 - p(\underline{\alpha})) \ln(\overline{\alpha}/\theta). \tag{3.17}$$

Since $p(0) > 0$, we have $\nu \to \infty$ as $\underline{\alpha} \downarrow 0$. Hence (3.4) holds for $\underline{\alpha}$ sufficiently close to zero, in which case every feasible path converges to zero *a.s.* by Theorem 3.1.

Theorem 3.1 can easily be extended to the case of a nonconcave production function. In the proof of Theorem 3.1, the production function $f(x)$ is estimated above by mk. Hence the proof goes through as long as this estimation is valid. The following result does not assume Assumption 3.1.

Theorem 3.2. *Suppose Assumptions 3.2–3.4 hold. Suppose there exists $m > 0$ satisfying (3.4) such that*

$$\forall k \geq 0, \quad f(k) \leq mk. \tag{3.18}$$

Then every feasible path converges to zero a.s.

Proof. Identical to the proof of Theorem 3.1 except that (3.18) replaces Assumption 3.1 here. □

Theorem 3.2 covers various cases. For example, if $f(k) = mk$ for some $m > 0$, then the hypotheses of Theorem 3.2 obviously hold except for (3.4). Hence in a stochastic "AK" model, under (3.4), every feasible path converges to zero *a.s.* regardless of the objective function.[7] More generally, Theorem 3.2 applies to any one-sector stochastic growth model with bounded marginal product. It applies to nonconvex stochastic growth models of the type studied by Majumdar, Mitra, and Nyarko (1989) as well as stochastic overlapping generations models of the type studied by Wang (1993).[8]

[7] Phelps (1962) was the first to study a stochastic "AK" model. The analysis was extended by Levhari and Srinivasan (1969). These articles studied mainly the properties of the consumption policy function rather than the asymptotic properties of optimal paths.

[8] In overlapping generations models, c_t in (3.1) represents aggregate consumption.

4 Conclusion

This paper has shown that in one-sector stochastic growth models, if the production function does not satisfy the Inada condition at zero, then any feasible path converges to zero with probability one provided that the shocks are sufficiently volatile. This result seems significant since, as we have argued, the Inada condition at zero is difficult to justify on economic grounds. Our convergence result has been extended to the case of a nonconcave production function. The generalized result applies to a wide range of one-sector models, including stochastic endogenous growth models, overlapping generations models, and models with a nonconcave production function.

Dealing with feasible paths, our results of course apply to optimal models. Optimal paths, however, converge to zero with probability one under a weaker condition.[9] For example, if there is an upper bound on the marginal propensity to save, then it can be used to estimate optimal paths (rather than feasible paths) from above as in the proof of Theorem 3.1. If optimal paths are characterized by a stationary policy function, then our argument can be applied to the policy function instead of the production function. Likewise it can be applied to equilibrium models in which there is an upper bound on the aggregate marginal propensity to save, or in which the equilibria are characterized by a stationary function. Such extensions are left for future research.

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