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ON CHOW-WEIGHT HOMOLOGY OF GEOMETRIC MOTIVES

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ABSTRACT. We describe new *Chow-weight* (co)homology theories on the category $DM_{gm}^{\text{eff}}(k,R)$ of effective geometric Voevodsky motives (R is the coefficient ring). These theories are interesting "modifications" of motivic homology; Chow-weight homology detects whether a motive $M \in \operatorname{Obj} DM_{gm}^{\text{eff}}(k,R)$ is *r-effective* (i.e., belongs to the rth Tate twist $DM_{gm}^{\text{eff}}(k,R)(r)$ of effective motives), bounds the weights of M (in the sense of the Chow weight structure defined by the first author), and detects the effectivity of "the lower weight pieces" of M. Moreover, we calculate the connectivity of M (in the sense of Voevodsky's homotopy t-structure, i.e., we study motivic homology) and prove that the exponents of the higher motivic homology groups (of an "integral" motive) are finite whenever these groups are torsion. We apply the latter statement to the study of higher Chow groups of arbitrary varieties.

These motivic properties of M have plenty of applications. They are closely related to the (co)homology of M; in particular, if the Chow groups of a variety X vanish up to dimension $r-1$ then the highest Deligne weight factors of the (singular or étale) cohomology of X with compact support are r-effective.

Our results yield vast generalizations of the so-called "decomposition of the diagonal" theorems, and we re-prove and extend some of earlier statements of this sort.

CONTENTS

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Introduction

The paper is dedicated to extending the well-known technique of *decomposition of the diagonal* (cf. Remark 0.5(1) below) to Voevodsky motives, and the application of the results to the study of arbitrary varieties and their cohomology. Our main tool are the completely new *Chow-weight homology* theories. They are closely related to motivic homology of Voevodsky motives; yet Chow-weight homology has several interesting properties that do not hold for motivic homology.

So, we consider Voevodsky's category $DM_{gm}^{\mathrm{eff}}(k,R)$ of R-linear effective geometric motives; here we assume the base field k is perfect and its characteristic p is invertible in the coefficient ring R whenever it is positive (this is equivalent to $1/e \in R$, where e is the *exponential characteristic* of k). Recall that $DM_{gm}^{\mathrm{eff}}(k,R)$ contains the category $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ of R-linear effective Chow motives over k. Now, the first author defined an exact (and conservative) *weight complex* functor $t_R : DM_{gm}^{\mathrm{eff}}(k,R) \to K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ whose restriction to $\mathrm{Chow}^{\mathrm{eff}}(k,R) \subset DM_{gm}^{\mathrm{eff}}(k,R)$ is the obvious embedding $\mathrm{Chow}^{\mathrm{eff}}(k,R) \to K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ (see Definition 1.4.1 and Remark 1.4.3 below). Then for $t_R(M) = (M^s)$ and a perfect field extension K/k we define the abelian group $\mathrm{CWH}_j^i(M_K,R)$ as the i-th homology of the complex $h_{2j,j}(M_K^s,R)$ obtained from $t_R(M)$; here $h_{2j,j} = \mathrm{CH}_j$ is the extension to Chow motives of the dimension j (R-linear) Chow group functor (whereas the notation originates from motivic homology), and the lower index K indicates that we extend the base field to K. Consequently, if M is a Chow motive then $\mathrm{CWH}_j^i(M_K) = \{0\}$ for $i \neq 0$ and $\mathrm{CWH}_j^0(M_K) = h_{2j,j}(M_K)$; thus one may say that Chow-weight homology is somewhat easier to compute than the motivic one (cf. Remark 0.5(2); in Remark 3.1.3 we recall that these restrictions of Chow-weight homology characterize it completely).

Next we recall that $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ contains the Lefschetz motive $\mathbb{L} = R\langle 1\rangle = R(1)[2]$; for $n \geq 0$ we say that a motive M is n-effective if it belongs to

$$DM_{gm}^{\mathrm{eff}}(k,R)\langle n\rangle = DM_{gm}^{\mathrm{eff}}(k,R) \otimes \mathbb{L}^{\otimes n} = DM_{gm}^{\mathrm{eff}}(k,R)(n).$$

The first statement that demonstrates the usefulness of Chow-weight homology is as follows.

Theorem 0.1. *Let M be an object of $DM_{gm}^{\mathrm{eff}}(k,R)$, $n > 0$, and K_0 is a universal domain containing k.*[1] *Denote the set $\mathbb{Z} \times [0, n-1] \subset \mathbb{Z} \times \mathbb{Z}$ by I.*

(1) *Then $M \in DM_{gm}^{\mathrm{eff}}(k,R)\langle n\rangle$ if and only if $\mathrm{CWH}_j^i(M_K) = \{0\}$ whenever $(i,j) \in I$ and K is the perfect closure of a finitely generated extension of k.*

(2) *If $R = \mathbb{Q}$ then $M \in DM_{gm}^{\mathrm{eff}}(k,R)\langle n\rangle$ if and only if $\mathrm{CWH}_j^i(M_{K_0}) = \{0\}$ for all $(i,j) \in I$.*

(3) *If $R \subset \mathbb{Q}$ and $\mathrm{CWH}_j^i(M_{K_0}) \otimes \mathbb{Q} = \{0\}$ for all $(i,j) \in I$ then there exists an integer $E > 0$ such that for any perfect field extension k'/k we have $E \cdot \mathrm{CWH}_j^i(M_{k'}) = \{0\}$ for all these (i,j).*

These statements can be vastly generalized; see Theorems 3.2.1, 3.3.3, 3.6.4, and 5.1.2, and §3.4 below. In particular (instead of effectivity) one can study weights and connectivity of M (that is, relate M to the filtrations induced by the Chow weight structure and the homotopy t-structure; see §2.1–2.2 below) and "measure effectivity" of the higher terms of the complex $t_R(M)$. For $R = \mathbb{Q}$ one can

[1] See Definition 2.3.1(2) and Proposition 5.2.3 below.

also study the case where the corresponding Chow-weight homology vector spaces are finite dimensional (over $\mathbb{Q}$). Moreover, we apply these results to study the Deligne weight filtration on singular and étale cohomology. Instead of formulating all motivic statements of this sort here (yet see the end of this introduction for a short plan of the paper), we will now describe one of their applications to motives with compact support of varieties.

Theorem 0.2. *Let $r>0$, X is a k-variety (that is, a reduced separated scheme of finite type over k), K_0 is a universal domain containing k, and $\mathrm{CH}_j(X_{K_0},\mathbb{Q})=\{0\}$ for $0\le j<r$. Then the following statements are valid.*

(1) *There exists $E>0$ such that the $\mathbb{Z}[1/e]$-linear Chow groups $\mathrm{CH}_j(X_{k'},\mathbb{Z}[1/e])$ are annihilated by the multiplication by E for all $0\le j<r$ and all field extensions k'/k (here $e=p$ if $p>0$ and $e=1$ if $p=0$).*
(2) *If k is a subfield of $\mathbb{C}$ and $q>0$ then the (highest) q-th weight factor of the mixed Hodge structure $H^q_c(X_{\mathbb{C}})$ (the singular cohomology of $X_{\mathbb{C}}$ with compact support) is r-effective (as a pure Hodge structure).*
   Moreover, the same property of the Deligne weight factors of $H^q_c(X_{k^{alg}})$ is fulfilled for étale cohomology with values in the category of $\mathbb{Q}_\ell[\mathrm{Gal}(k)]$-modules if k is the perfect closure of a finitely generated field.
   In particular, these factors are zero if $q<2r$.
(3) *The motive $\mathcal{M}^c_{\mathbb{Q}}(X)$ (see Definition 4.1.1(2)) is an extension of an element of $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}}\ge 1}$ (see §2.2) by an object of $\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle r\rangle$.*

Remark 0.3.

(1) The vanishing of lower Chow groups is quite "common" for non-proper varieties. In particular, it suffices to assume that X is an open subvariety of $X'\times\mathbb{A}^r$ for some k-variety X'; cf. Remark 4.1.7(2) below for more detail.
(2) These statements are completely new; yet they are easily seen to generalize the corresponding (rather well-known) properties of proper smooth varieties.
   Note also that the formulation of Theorem 4.2.3 below is somewhat similar to Theorem 0.2; yet it mentions higher motivic homology. Moreover, in Corollary 5.1.6(2) below parts (2) and (3) of our theorem are generalized to the case where the $\mathbb{Q}$-vector spaces $\mathrm{CH}_j(X_{K_0},\mathbb{Q})=\{0\}$ are finite dimensional if $0\le j<r$.
(3) By Proposition 3.5.5 below (see also Remark 4.2.2(2)), the combination of two of more or less "standard" motivic conjectures yields that the first implication in Theorem 0.2(2) is actually an equivalence.

Let us now recall some basics on ("classical") decomposition of the diagonal and relate it to our results. Decomposition of the diagonal (see Remark 0.5(1) below) was introduced by Bloch in §1A of [Blo80] (cf. also [BlS83]; a rich collection of recent results related to this notion can be found in [Voi14]). Let us recall some easily formulated motivic results obtained via this method (and essentially established in [Via17]). For simplicity, we will state them for motives and Chow groups with rational coefficients over a universal domain k; yet they also can be generalized similarly to Theorem 0.1.

Proposition 0.4.

(i) *Let O be an effective Chow motive over k. Then O is r-effective if and only if* $\mathrm{CH}_j(O) = \{0\}$ *for* $0 \le j < r$ *(see Remark 3.8 of* [Via17]*).*

(ii) *Let $h : N \to O$ be a morphism of effective Chow motives. Then* $\mathrm{CH}_0(h)$ *is surjective if and only if h "splits modulo* 1*-effective motives", i.e., if it corresponds to a presentation of O as a retract of $N \bigoplus (Q\langle 1\rangle)$ for some effective motive Q (cf. Proposition 3.5 of ibid. and Remark* 0.5*(1) below).*

(iii) *For $h : N \to O$ as above the homomorphisms* $\mathrm{CH}_j(h)$ *are surjective for all $j \ge 0$ if and only if h is split surjective (this is Theorem 3.18 of ibid.).*

Remark 0.5.

(1) In statements of this sort one usually takes O to be the motive of a smooth projective P/k, whereas N is obtained by resolving singularities of a closed subvariety P' of P (cf. Lemma 3 of [GoG13] and Proposition 3.5 of [Via17]). In this case, if $\mathrm{CH}_j(h)$ is surjective for all $j < c$ then the diagonal cycle Δ in $P \times P$ is rationally equivalent to the sum of a cycle supported on $P' \times P$ and one supported on $P \times W$ for some closed $W \subset P$ of codimension at least r. That is why one speaks about decomposing the diagonal; see Proposition 4.3.1 below for more detail.

One can usually reformulate these cycle-theoretic statements using the following trivial observation: if M is an object of an additive category $\underline{B}$, $\mathrm{id}_M = f_1 + f_2$ (for $f_1, f_2 \in \underline{B}(M, M)$), and f_i factor through some objects M_i of $\underline{B}$ (for $i = 1, 2$), then M is a retract of $M_1 \bigoplus M_2$. In particular, if $\underline{B}$ is idempotent complete (this is the case for all "standard" motivic categories) then M is a direct summand of $M_1 \bigoplus M_2$.

(2) Proposition 0.4(i) can easily be deduced from Theorem 0.1(2).

Moreover, we obtain that one cannot use motivic homology instead of the Chow-weight one in the theorem. Indeed, if $M = R\langle 1\rangle$ then

$$h_{10}(M_{K_0}) = DM^{\mathrm{eff}}_{gm}(K_0, R)(R[1], R(1)[2]) \cong K_0^* \otimes_{\mathbb{Z}} R \neq \{0\}$$

if R is not a torsion ring (see Definition 2.2.2(5) below for this notation). Note also that "classical" decomposition of the diagonal methods cannot yield Theorem 0.2 since one cannot avoid distinguished triangles and long exact sequences in the proof of this "mixed statement".

Thus the results of the current paper demonstrate that the language of Chow weight structures, weight complexes, and Chow-weight homology is appropriate for extending decomposition of the diagonal results to varieties that are either singular or non-proper, and to general Voevodsky motives. The main disadvantage of Chow-weight homology is that its values are often huge (since ordinary Chow groups are); cf. Remark 2.3.6(2) and Theorem 5.1.2 below.

(3) Proposition 0.4(ii,iii) follows from our general results as well; we demonstrate this in Corollary 3.3.9 and Remark 3.3.10(2) below.

To prove this corollary we will consider the motive $M = \mathrm{Cone}(h)$. Since the weight complex functor $t_{\mathbb{Q}}$ is exact, the Chow group assumptions in Propositions 0.4(ii,iii) are equivalent to the vanishing of $\mathrm{CWH}_0^0(M)$ and of $\mathrm{CWH}_j^0(M)$ for all $j \ge 0$, respectively. Moreover, $\mathrm{CWH}_j^i(M) = \{0\}$ for $i \neq -1, 0$ automatically.

For the sake of the readers scared of Voevodsky motives, we also note that our results can be applied to $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ (i.e., to complexes of R-linear Chow motives) instead of $DM_{gm}^{\mathrm{eff}}(k,R)$; see Remark 3.3.5 below. Yet even these more elementary versions of our results are "quite triangulated", and their proofs involve certain triangulated categories of birational motives.

Now let us describe the contents of the paper; some more information of this sort can be found at the beginnings of sections.

In §1 we recall some of the theory of weight structures.

In §2 we describe several properties of (various categories of) Chow and Voevodsky motives and of Chow weight structures for the latter. The most important (though somewhat technical) results of this section are Proposition 2.2.6 (3, 6) on morphisms between Chow motives inside $DM_{gm}^{\mathrm{eff}}(k,R)$. We also prove some auxiliary statements on the behaviour of complexes whose terms are certain (higher) Chow groups under morphisms of base fields; most of these results are more or less well-known.

In §3 we define Chow-weight homology theories and study the properties of Chow-weight homology of arbitrary objects of the Voevodsky category $DM_{gm}^{\mathrm{eff}}(k,R)$. In particular we express the weights of a motive $M \in \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,R)$ and its effectivity (i.e., whether it belongs to $\operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,R)\langle r\rangle$ for a given $r > 0$) in terms of its Chow-weight homology. We also relate the vanishing of the higher degree Chow-weight homology of M to that of its motivic homology (along with its motivic connectivity) and to the effectivity of the higher (Deligne) weight factors of cohomology. Moreover, the combination of two (of more or less "standard") motivic conjectures yields that the implications of the latter type are in fact equivalences (see Proposition 3.5.5). Furthermore, we prove that the vanishing of rational Chow-weight homology of M in a certain range is "almost equivalent" to M being an extension of a motive satisfying the integral Chow-weight homology vanishing in the same range by a torsion motive (see Theorem 3.6.4). This implies the following: if the higher motivic homology groups of a motive M are torsion, then their exponents are finite.

In §4 we apply our general results to motives with compact support of arbitrary k-varieties. We apply them to obtain Theorem 0.2 as well as several results related to it (see §4.2). Moreover, we re-prove and generalize certain decomposition of the diagonal results of [Par94] and [Lat96]; in the process we demonstrate the relation of our methods and results to the "usual" cycle-theoretic formulations of decompositions of the diagonal statements. We also recall that in the case where k is finite the effectivity conditions for motives are closely related to the number of rational points of k-varieties (taken modulo powers of $q = \#k$); see Proposition 4.2.4(2). Furthermore, we study tensor products of motives and relate them to varieties. In particular, we prove (roughly) that the aforementioned standard conjectures imply that the "effectivity and connectivity" of the tensor product of (geometric $\mathbb{Q}$-linear) motives over a characteristic 0 field cannot exceed the sums of the effectivities and connectivities of the multipliers, respectively.

In §5 we prove some more statements and discuss further developments of the theory. We study the finite-dimensionality of Chow-weight homology and of Chow groups in the case $R = \mathbb{Q}$; this gives a certain generalization of Theorems 3.3.3 and 0.2. We also dualize some of our results; this allows us to calculate the dimensions of motives and bound their weights (from above) in terms of their *Chow-weight cohomology*.

We also note that an alternative version of this text is available as [BoS14]; note however that some of the notation and the numeration of the statements in ibid. differs from the current text, and the exposition is less accurate. Moreover, some results of our paper are generalized in [BoK20]; see Remark 5.3.1(6) below.

List of main definitions and notation

For the convenience of the readers we list some of the terminology and notation used in this paper. The reader may certainly ignore this section.

- Karoubian categories, Karoubi envelopes, extension-closed and Karoubi-closed subcategories, extension-closures, Karoubi-closures, envelopes, $X \perp Y$, $D^{\perp}$, and ${}^{\perp}D$ are defined in §1.1.
- Weight structures (general and bounded ones), their hearts, the classes $\underline{C}_{w\geq i}$, $\underline{C}_{w\leq i}$, $\underline{C}_{w=i}$, $\underline{C}_{[i,j]}$, weight-exact functors, connective subcategories of triangulated categories, weight truncations $w_{\leq m}M$, $w_{\geq m}M$, and m-weight decompositions are recalled in §1.2.
- Weight complexes, weight filtrations, and weight spectral sequences are recalled in §1.4.
- The motivic categories $\mathrm{Chow}^{\mathrm{eff}}(k,R) \subset DM_{gm}^{\mathrm{eff}}(k,R) \subset DM_{-}^{\mathrm{eff}}(k,R) \subset DM^{\mathrm{eff}}(k,R)$ and $\mathrm{Chow}(k,R)$, the functor $\mathcal{M}_R$, (shifted) Tate twists $\langle r\rangle = -(r)[2r]$, the homotopy t-structure t_{hom}^R, and varieties (resp. motives) of the type X_K (resp. M_K) are introduced in §2.1.
- The Chow weight structures w_{Chow} on $DM_{gm}^{\mathrm{eff}}(k,R)$ and on its subcategories $d_{\leq m}DM_{gm}^{\mathrm{eff}}(k,R)$, along with r-effectivity and dimensions for motives and their motivic homology groups $h_{**}(-,R)$ are introduced in §2.2. We also define the functor $l^r : DM_{gm}^{\mathrm{eff}}(k,R) \to DM_{gm}^{r}(k,R)$, and introduce the Chow weight structure w_{Chow}^{r} on $DM_{gm}^{r}(k,R)$ for any $r \geq 0$.
- Essentially finitely generated fields, universal domains, fields of definition for motives, rational extensions, and function fields are defined in §2.3.
- Chow-weight homology functors $\mathrm{CWH}_*^*(-_K,R)$ and $\mathrm{CWH}_*^*(-_K,*,R)$ are introduced in §3.1 (whereas the "Poincare dual" Chow-weight cohomology functors $\mathrm{CWC}^{*,*}(-_K,R)$ and $\mathrm{CWC}^{*,*}(-_K,*,R)$ are defined in §5.2).
- Staircase sets $\mathcal{I} \subset \mathbb{Z} \times [0,+\infty)$ (this includes sets of the type $\mathcal{I}^{\langle c\rangle}$) are introduced in §3.3; an example is drawn in Corollary 3.4.2(3).
- Étale and singular cohomology functors $H_{et,\mathbb{Q}_\ell}$ and H_{sing}, and Deligne's weights $W_{D*}H^*$ on their values are considered in §3.5.
- Motives with compact support $\mathcal{M}_{\mathbb{Q}}^{c}(-)$ and $\mathcal{M}_{R}^{c}(-)$ are recalled in §4.1.

We will treat both the characteristic 0 and the positive characteristic case below. Yet the reader may certainly assume that the characteristic of k is 0 throughout the paper; clearly, in this case one does not have to think about perfectness and the assumption $1/e \in R$, and can use singular cohomology.

1. Some preliminaries on weight structures

This section is dedicated to recalling the theory of weight structures in triangulated categories.

In §1.1 we introduce some notation and conventions for (mostly, triangulated) categories; we also prove two simple lemmas.

In §1.2 we recall the definition and basic properties of weight structures.

In §1.3 we relate weight structures to localizations.

In §1.4 we recall several properties of weight complexes and weight spectral sequences.

1.1. **Some (categorical) notation and lemmas.**

- For $a \leq b \in \mathbb{Z}$ we will write $[a, b]$ (resp. $[a, +\infty)$, resp. $[a, +\infty]$) for the set $\{i \in \mathbb{Z} : \ a \leq i \leq b\}$ (resp. $\{i \in \mathbb{Z} : \ i \geq a\}$, resp. $[a, +\infty) \cup \{+\infty\} \subset \mathbb{Z} \cup \{+\infty\}$); we will never consider real line segments in this paper. Respectively, when we write $i \geq c$ (for $c \in \mathbb{Z}$) we mean that i is an integer satisfying this inequality.
- Given a category C and $X, Y \in \operatorname{Obj} C$ we write $C(X, Y)$ for the set of morphisms from X to Y in C.
- For categories C', C we write $C' \subset C$ if C' is a full subcategory of C.
- Given a category C and $X, Y \in \operatorname{Obj} C$, we say that X is a *retract* of Y if id_X can be factored through Y.
- An additive subcategory $\underline{H}$ of an additive category C is said to be *Karoubi-closed* in C if it contains all retracts of its objects in C. The full subcategory $\operatorname{Kar}_C(\underline{H})$ of additive category C whose objects are all the retracts of objects of a subcategory $\underline{H}$ (in C) will be called the *Karoubi-closure* of $\underline{H}$ in C.
- The *Karoubi envelope* $\operatorname{Kar}(\underline{B})$ (no lower index) of an additive category $\underline{B}$ is the category of "formal images" of idempotents in $\underline{B}$. Consequently, its objects are the pairs (A, p) for $A \in \operatorname{Obj} \underline{B}, \ p \in \underline{B}(A, A), \ p^2 = p$, and the morphisms are given by the formula
$$\operatorname{Kar}(\underline{B})((X, p), (X', p')) = \{f \in \underline{B}(X, X') : \ p' \circ f = f \circ p = f\}.$$
The correspondence $A \mapsto (A, \mathrm{id}_A)$ (for $A \in \operatorname{Obj} \underline{B}$) fully embeds $\underline{B}$ into $\operatorname{Kar}(\underline{B})$. Moreover, $\operatorname{Kar}(\underline{B})$ is *Karoubian*, i.e., any idempotent morphism yields a direct sum decomposition in $\operatorname{Kar}(\underline{B})$. Recall also that $\operatorname{Kar}(\underline{B})$ is triangulated if $\underline{B}$ is (see [BaS01]).
- The symbol $\underline{C}$ below will always denote some triangulated category; usually it will be endowed with a weight structure w.
- For any $A, B, C \in \operatorname{Obj} \underline{C}$ we say that C is an *extension* of B by A if there exists a distinguished triangle $A \to C \to B \to A[1]$.
- A class $D \subset \operatorname{Obj} \underline{C}$ is said to be *extension-closed* if it is closed with respect to extensions and contains 0. We call the smallest extension-closed subclass of objects of $\underline{C}$ that contains a given class $B \subset \operatorname{Obj} \underline{C}$ the *extension-closure* of B.

  Moreover, we call the smallest extension-closed Karoubi-closed subclass of objects of $\underline{C}$ that contains B the *envelope* of B.
- Given a class D of objects of $\underline{C}$ we write $\langle D \rangle$ or $\langle D \rangle_{\underline{C}}$ for the smallest full Karoubi-closed triangulated subcategory of $\underline{C}$ containing D. We call $\langle D \rangle$ the triangulated category *densely generated* by D.
- For $X, Y \in \operatorname{Obj} \underline{C}$ we write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset \operatorname{Obj} \underline{C}$ we write $D \perp E$ if $X \perp Y$ for all $X \in D, \ Y \in E$. Given $D \subset \operatorname{Obj} \underline{C}$ we will write $D^{\perp}$ for the class
$$\{Y \in \operatorname{Obj} \underline{C} : \ X \perp Y \ \forall X \in D\}.$$

Dually, ${}^{\perp}D$ is the class $\{Y \in \operatorname{Obj} \underline{C} :\ Y \perp X\ \forall X \in D\}$.

- Given $f \in \underline{C}(X, Y)$, where $X, Y \in \operatorname{Obj} \underline{C}$, we call the third vertex of (any) distinguished triangle $X \stackrel{f}{\to} Y \to Z$ a *cone* of f.
- For an additive category $\underline{B}$ we write $K(\underline{B})$ for the homotopy category of (cohomological) complexes over $\underline{B}$. Its full subcategory of bounded complexes will be denoted by $K^b(\underline{B})$. We will write $M = (M^i)$ if M^i are the terms of the complex M.

1.2. **Weight structures: Basics.** Let us recall the definition of the notion that is central for this paper.

Definition 1.2.1.

(I) A couple of subclasses $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset \operatorname{Obj} \underline{C}$ will be said to define a weight structure w on a triangulated category $\underline{C}$ if they satisfy the following conditions.

(i) $\underline{C}_{w\ge 0}$ and $\underline{C}_{w\le 0}$ are Karoubi-closed in $\underline{C}$ (i.e., contain all $\underline{C}$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**
$\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$, $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.

(iii) **Orthogonality.**
$\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.

(iv) **Weight decompositions**.
For any $M \in \operatorname{Obj} \underline{C}$ there exists a distinguished triangle

$$X \to M \to Y {\to} X[1]$$

such that $X \in \underline{C}_{w\le 0},\ Y \in \underline{C}_{w\ge 0}[1]$.

We will also need the following definitions.

Definition 1.2.2. Let $i, j \in \mathbb{Z}$; assume that a triangulated category $\underline{C}$ is endowed with a weight structure w.

(1) The full subcategory $\underline{Hw}$ of $\underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w\ge 0} \cap \underline{C}_{w\le 0}$ is called the *heart* of w.

(2) $\underline{C}_{w\ge i}$ (resp. $\underline{C}_{w\le i}$, resp. $\underline{C}_{w=i}$) will denote $\underline{C}_{w\ge 0}[i]$ (resp. $\underline{C}_{w\le 0}[i]$, resp. $\underline{C}_{w=0}[i]$).

(3) $\underline{C}_{[i,j]}$ denotes $\underline{C}_{w\ge i} \cap \underline{C}_{w\le j}$; hence this class equals $\{0\}$ if $i > j$.
$\underline{C}^b \subset \underline{C}$ will be the category whose object class is $\cup_{i,j\in\mathbb{Z}} \underline{C}_{[i,j]}$.

(4) We say that $(\underline{C}, w)$ is *bounded* if $\underline{C}^b = \underline{C}$ (i.e., if $\cup_{i\in\mathbb{Z}} \underline{C}_{w\le i} = \operatorname{Obj} \underline{C} = \cup_{i\in\mathbb{Z}} \underline{C}_{w\ge i}$).

(5) Let $\underline{C}'$ be a triangulated category endowed with a weight structure w'; let $F : \underline{C} \to \underline{C}'$ be an exact functor.
F is said to be *weight-exact* (with respect to w, w') if it maps $\underline{C}_{w\le 0}$ into $\underline{C}'_{w'\le 0}$ and sends $\underline{C}_{w\ge 0}$ into $\underline{C}'_{w'\ge 0}$.

(6) Let $\underline{D}$ be a full triangulated subcategory of $\underline{C}$.
We say that w *restricts* to $\underline{D}$ whenever the couple $(\underline{C}_{w\le 0} \cap \operatorname{Obj} \underline{D},\ \underline{C}_{w\ge 0} \cap \operatorname{Obj} \underline{D})$ is a weight structure on $\underline{D}$.

(7) Let $\underline{H}$ be a full subcategory of a triangulated category $\underline{C}$.
We say that $\underline{H}$ is *connective* if $\operatorname{Obj} \underline{H} \perp (\cup_{i>0} \operatorname{Obj}(\underline{H}[i]))$.

Remark 1.2.3.

(1) A simple (and yet quite useful) example of a weight structure comes from the stupid filtration on $K^b(\underline{B})$ (or on $K(\underline{B})$) for an arbitrary additive category $\underline{B}$. In this case $K^b(\underline{B})_{w\le 0}$ (resp. $K^b(\underline{B})_{w\ge 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees ≥ 0 (resp. ≤ 0); see [BoS18b, Remark 1.2.3(1)].

The heart of this weight structure is the Karoubi-closure of $\underline{B}$ in $K^b(\underline{B})$ (or in $K(\underline{B})$, respectively).

(2) A weight decomposition (of any $M \in \operatorname{Obj}\underline{C}$) is almost never canonical.

Still for any $m \in \mathbb{Z}$ the axiom (iv) gives the existence of a distinguished triangle

$$w_{\le m}M \to M \to w_{\ge m+1}M \tag{1.1}$$

with some $w_{\ge m+1}M \in \underline{C}_{w\ge m+1}$ and $w_{\le m}M \in \underline{C}_{w\le m}$; we call it an *m-weight decomposition* of M.

We will often use this notation below (even though $w_{\ge m+1}M$ and $w_{\le m}M$ are not canonically determined by M); we will call any possible choice either of $w_{\ge m+1}M$ or of $w_{\le m}M$ (for any $m \in \mathbb{Z}$) a *weight truncation* of M. Moreover, when we write arrows of the type $w_{\le m}M \to M$ or $M \to w_{\ge m+1}M$ we will always assume that they come from some m-weight decomposition of M.

(3) In the current paper we use the "homological convention" for weight structures; it was previously used in [Wil09], [Bon18a], [BoI15], [BoS18b], [BoK18], [Bon18b], [Bon21], and [Bon19], whereas in [Bon10a] and in [Bon10b] the "cohomological convention" was used. In the latter convention the roles of $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ are interchanged, i.e., one considers $\underline{C}^{w\le 0} = \underline{C}_{w\ge 0}$ and $\underline{C}^{w\ge 0} = \underline{C}_{w\le 0}$. Consequently, a complex $X \in \operatorname{Obj} K(\underline{B})$ whose only non-zero term is the fifth one (i.e., $X^5 \neq 0$) has weight -5 in the homological convention, and has weight 5 in the cohomological convention. Thus the conventions differ by "signs of weights"; $K(\underline{B})_{[i,j]}$ is the class of retracts of complexes concentrated in degrees $[-j, -i]$.

We also recall that D. Pauksztello has introduced weight structures independently in [Pau08]; he called them co-t-structures.

(4) The orthogonality axiom (iii) in Definition 1.2.1 immediately yields that $\underline{Hw}$ is connective in $\underline{C}$. We will formulate a certain converse to this statement below.

Let us recall some basic properties of weight structures. Starting from this moment we will assume that all the weight structures we consider are bounded (unless specified otherwise; this is quite sufficient for our purposes everywhere except in the proof of Lemma 3.1.4(1).

Proposition 1.2.4. *Let $\underline{C}$ be a triangulated category, $n \ge 0$; we will assume that w is a fixed (bounded) weight structure on $\underline{C}$ everywhere except in assertion* (8).

(1) *The axiomatics of weight structures is self-dual, i.e., for $\underline{C}' = \underline{C}^{op}$ (consequently,* $\operatorname{Obj}\underline{C}' = \operatorname{Obj}\underline{C}$*) there exists the (opposite) weight structure w' for which $\underline{C}'_{w'\le 0} = \underline{C}_{w\ge 0}$ and $\underline{C}'_{w'\ge 0} = \underline{C}_{w\le 0}$.*

(2) *$\underline{C}_{w\le 0}$ is the extension-closure of $\cup_{i\le 0}\underline{C}_{w=i}$ in $\underline{C}$; $\underline{C}_{w\ge 0}$ is the extension-closure of $\cup_{i\ge 0}\underline{C}_{w=i}$ in $\underline{C}$.*

(3) $\underline{C}_{w\ge 0}=(\underline{C}_{w\le -1})^{\perp}$ *and* $\underline{C}_{w\le 0}={}^{\perp}\underline{C}_{w\ge 1}$.

(4) *Let* $m\le l\in\mathbb{Z}$, $X,X'\in \operatorname{Obj}\underline{C}$*; fix certain weight decompositions of* $X[-m]$ *and* $X'[-l]$. *Then any morphism* $g:X\to X'$ *can be extended to a commutative diagram of the corresponding distinguished triangles (see Remark* 1.2.3(2)*):*

$$\begin{CD} w_{\le m}X @>>> X @>>> w_{\ge m+1}X\\ @VVV @VV{g}V @VVV\\ w_{\le l}X' @>>> X' @>>> w_{\ge l+1}X' \end{CD}$$

Moreover, if $m<l$ *then this extension is unique (provided that the rows are fixed).*

(5) *Assume that* w' *is a weight structure for a triangulated category* $\underline{C}'$. *Then an exact functor* $F:\underline{C}\to\underline{C}'$ *is weight-exact if and only if* $F(\underline{C}_{w=0})\subset \underline{C}'_{w'=0}$.

(6) *If* M *belongs to* $\underline{C}_{w\ge -n}$ *then* $w_{\le 0}M$ *belongs to* $\underline{C}_{[-n,0]}$.

(7) *If* $m<l\in\mathbb{Z}$ *and* $M\in\operatorname{Obj}\underline{C}$ *then for any choice of arrows* $w_{\le l}M\to M$ *and* $w_{\le m}(w_{\le l}M)\to w_{\le l}M$ *that can be completed to an* l*-weight decomposition and an* m*-weight decomposition triangle (see Remark* 1.2.3(2)*) respectively, the composition morphism* $w_{\le m}(w_{\le l}M)\to M$ *can be completed to an* m*-weight decomposition of* M.

(8) *Let* $D\subset\operatorname{Obj}\underline{C}$ *be a connective additive subcategory. Then there exists a unique weight structure* w_T *on* $T=\langle D\rangle_{\underline{C}}$ *such that* $D\subset T_{w_T=0}$. *It is bounded; its heart equals the Karoubi-closure of* D *in* $\underline{C}$. *Moreover,* T *is Karoubian whenever* D *is.*

Furthermore, if there exists a weight structure w *on* $\underline{C}$ *such that* $D\subset \underline{Hw}$, *then the embedding* $T\to\underline{C}$ *is strictly weight-exact, i.e.,* $T_{w_T\le 0}=\operatorname{Obj}T\cap\underline{C}_{w\le 0}$ *and* $T_{w_T\ge 0}=\operatorname{Obj}T\cap\underline{C}_{w\ge 0}$.

(9) *For any* $M,N\in\operatorname{Obj}\underline{C}$ *and* $f\in\underline{C}(N,M)$ *if* M *belongs to* $\underline{C}_{w\ge 0}$, *then* f *factors through (any possible choice of)* $w_{\ge 0}N$. *Dually, if* N *belongs to* $\underline{C}_{w\le 0}$ *then* f *factors through* $w_{\le 0}M$.

(10) *Let* $\underline{D}$ *be a (full) triangulated subcategory of* $\underline{C}$ *such that* w *restricts to* $\underline{D}$*; let* $M\in\underline{C}_{w\le 0}$, $N\in\underline{C}_{w\ge -n}$, *and* $f\in\underline{C}(M,N)$. *Suppose that* f *factors through an object* P *of* $\underline{D}$, *i.e., there exist* $u_1\in\underline{C}(M,P)$ *and* $u_2\in\underline{C}(P,N)$ *such that* $f=u_2\circ u_1$. *Then* f *factors through an element of* $\underline{D}_{[-n,0]}$.

Proof. Assertions (1)–(4) were proved in [Bon10a] (pay attention to Remark 1.2.3(3)!). Assertion (5) follows immediately from Lemma 2.7.5 of [Bon10b].

Assertion (6) follows immediately from the fact that the classes $\underline{C}_{w\ge -n}$ and $\underline{C}_{w\le 0}$ are extension-closed (cf. assertion (2)).

(7). The octahedral axiom of triangulated categories implies that the object $C=\operatorname{Cone}(w_{\le m}(w_{\le l}M)\to M)$ is an extension of (the corresponding) $w_{\ge l+1}M$ by $w_{\ge m+1}(w_{\le l}M)$. Hence C belongs to $\underline{C}_{w\ge m+1}$ (cf. assertion (2) once again); thus $w_{\le m}(w_{\le l}M)\to M\to C$ is an m-weight decomposition triangle.

Assertion (8) is given by Remark 2.1.2 of [BoS18b].

Assertion (9) is an easy consequence of assertion (4).

(10). Assertion (9) yields that u_2 factors through $w_{\ge -n}P$; thus we can assume that P belongs to $\underline{D}_{w\ge -n}$. Next, the dual to assertion 9 (see assertion (1)) yields that u_1 factors through $w_{\le 0}P$. It remains to note that we can choose $w_{\le 0}P$ that belongs to $\underline{D}_{[-n,0]}$ (see assertion (6)). $\square$

1.3. **Weight structures on localizations.**

Definition 1.3.1. We call a category $\frac{A}{B}$ the *factor* of an additive category A by its full additive subcategory B if $\operatorname{Obj}(\frac{A}{B}) = \operatorname{Obj} A$ and

$$(\frac{A}{B})(X,Y) = A(X,Y)/(\sum_{Z\in \operatorname{Obj} B} A(Z,Y)\circ A(X,Z)).$$

Proposition 1.3.2. *Let $\underline{D}\subset \underline{C}$ be a triangulated subcategory of $\underline{C}$; suppose that w restricts to a weight structure $w_{\underline{D}}$ on $\underline{D}$ (see Definition 1.2.2(6)). Denote by l the localization functor $\underline{C}\to \underline{C}/\underline{D}$ (the latter category is the Verdier quotient of $\underline{C}$ by $\underline{D}$) .*

Then the following statements are valid.

1. *w induces a weight structure on $\underline{C}/\underline{D}$, i.e., the Karoubi-closures of $l(\underline{C}_{w\le 0})$ and $l(\underline{C}_{w\ge 0})$ in $\underline{C}/\underline{D}$ give a weight structure on this category.*
2. *Suppose $(\underline{C},w)$ is bounded, and for $X\in \operatorname{Obj}\underline{C}$ assume $l(X)\in \underline{C}/\underline{D}_{w_{\underline{C}/\underline{D}}\ge 0}$.*
   Then X is an extension of some element of $\underline{C}_{w\ge 0}$ by an element of $\underline{D}_{w_{\underline{D}}\le -1}$ (see §1.1).
3. *The obvious functor $\frac{\underline{Hw}}{\underline{Hw}_{\underline{D}}}\to \underline{C}/\underline{D}$ is a full embedding, and the heart $\underline{Hw}_{\underline{C}/\underline{D}}$ of the weight structure $w_{\underline{C}/\underline{D}}$ given by assertion 1 is the Karoubi-closure of the image of $\frac{\underline{Hw}}{\underline{Hw}_{\underline{D}}}$ in $\underline{C}/\underline{D}$.*
4. *If $(\underline{C},w)$ is bounded, then $\underline{C}/\underline{D}$ also is.*

Proof. Assertions (1), (3), and (4) were proved in §8.1 of [Bon10a]; assertion (2) is an easy consequence of Theorem 3.3.1 of [BoS18c] (as demonstrated by Remark 3.3.2(1) of ibid.). $\square$

Remark 1.3.3.

1. Part (2) of our proposition gives the existence of a distinguished triangle $D\to X\to C\to D[1]$ for some $C\in \underline{C}_{w\ge 0}$ and $D\in \underline{D}_{w\le -1}$. Clearly, this triangle is just a -1-weight decomposition of X. In particular, Proposition 1.2.4(2) (or part 6 of that proposition along with its dual) easily yields the following: if we also have $X\in \underline{C}_{[r,m]}$ for $r\le 0\le m$ then $C\in \underline{C}_{[0,m]}$ and $D\in \underline{C}_{[r,-1]}$.
2. If w is bounded then all weight structures compatible with it (for $\underline{D}\subset \underline{C}$) come from additive subcategories of $\underline{Hw}$ (see Proposition 1.2.4 (8, 5)). Moreover, in this case the heart $\underline{Hw}_{\underline{C}/\underline{D}}$ actually equals the essential image of $\frac{\underline{Hw}}{\underline{Hw}_{\underline{D}}}$ in $\underline{C}/\underline{D}$ (see Proposition 3.3.3(1) of [BoS18c]).

   On the other hand, to ensure that there exists a weight structure for $\underline{C}/\underline{D}$ such that the localization functor is weight-exact it actually suffices to assume that $\underline{D}$ is densely generated by some set of elements of $\underline{C}_{[0,1]}$; see Theorem 3.2.2 of [BoS19] for a more general statement.

1.4. **On weight complexes and weight spectral sequences.** We will need certain *weight complexes* below. We define weight complexes of objects here only; however, we will discuss certain extensions of this definition in Remark 1.4.3(3,4) below.

Definition 1.4.1. For an object M of $\underline{C}$ (where $\underline{C}$ is endowed with a weight structure w) choose some $w_{\le l}M$ (see Remark 1.2.3(2)) for all $l \in \mathbb{Z}$; then connect $w_{\le l-1}M$ with $w_{\le l}M$ using Proposition 1.2.4(4) (i.e., we consider those unique connecting morphisms that are compatible with id_M). Next, take the corresponding triangles

$$w_{\le l-1}M \to w_{\le l}M \to M^{-l}[l] \to (w_{\le l-1}M)[1] \tag{1.2}$$

(so, we just introduce the notation for the corresponding cones). All of these triangles along with the corresponding morphisms $w_{\le l}M \to M$ are called a choice of a *weight Postnikov tower* for M, whereas the objects M^i along with the morphisms connecting them (obtained by composing the morphisms $M^{-l} \to (w_{\le l-1}M)[1-l] \to M^{-l+1}$ that come from two consecutive triangles of the type (1.2)) will be denoted by $t(M)$ and said to be a choice of a *weight complex* for M.

Let us recall some basic properties of weight complexes. Note that the boundedness of w is only needed in assertions (5) and (3) below; moreover, a much weaker restriction on w is sufficient for the latter statement according to Proposition 3.1.6(2) and Theorem 2.3.4(I.1) of [Bon19].

Proposition 1.4.2. *Let $M \in \operatorname{Obj}\underline{C}$, where $\underline{C}$ is endowed with a weight structure w.*

Then the following statements are valid.

1. *Any choice of $t(M) = (M^i)$ is a complex indeed (i.e., the square of the boundary is zero); all M^i belong to $\underline{C}_{w=0}$.*
2. *M determines its weight complex $t(M)$ up to a homotopy equivalence. In particular, if $M \in \underline{C}_{w\ge 0}$ (resp. $M \in \underline{C}_{w\le 0}$) then any choice of $t(M)$ is $K(\underline{Hw})$-isomorphic to a complex with non-zero terms in non-positive (resp. non-negative) degrees only.*
3. *If $t(M)$ is homotopy equivalent to 0, then $M = 0$.*
4. *If $M_0 \xrightarrow{f} M_1 \to M_2$ is a distinguished triangle in $\underline{C}$ then for any possible choice of $t(M_0)$ and $t(M_1)$ there exists a choice of $t(M_2)$ that completes them to a distinguished triangle.*

   Moreover, if $M_0 \in \underline{C}_{w\ge 0}$ and $M_1 \in \underline{C}_{w\le 0}$ then there exists $t(M_2)$ of the form $\cdots \to M_0^{-2} \to M_0^{-1} \to M_0^0 \xrightarrow{f_0} M_1^0 \to M_1^1 \to \dots$. That is, one can take any choice of $t(M_1)$ that is concentrated in non-negative degrees and put it in the same degrees of $t(M_2)$, take a "dual choice" of $t(M_0)$, shift it by $[1]$, and put it inside $t(M_2)$ also, whereas f_0 is the composed morphism $M_0^0 \to M_0 \xrightarrow{f} M_1 \to M_1^0$ (the unlabeled morphisms in this row are provided by our construction).
5. *If $t(M)$ is homotopy equivalent to a bounded complex (M'^i) then M belongs to the extension-closure of the set $\{M'^{-i}[i]\}$.*
6. *Let $N \in \underline{C}_{w=0}$, $M \in \underline{C}_{w\ge 0}$; assume that a $\underline{C}$-morphism $f : N \to M$ factors through some $L \in \operatorname{Obj}\underline{C}$. Then for any possible choice of L^0 (i.e., of the zeroth term of $t(L)$) f can be factored through L^0.*
7. *Let $H : \underline{Hw} \to \underline{A}$ be an additive functor, where $\underline{A}$ is an abelian category. Choose a weight complex $t(M) = (M^j)$ for each object M of $\underline{C}$, and denote by $\tilde{H}(M)$ the zeroth homology of the complex $H(M^i)$. Then $\tilde{H}(-)$ yields a homological functor from $\underline{C}$ to $\underline{A}$ (that does not depend on the choices of*

weight complexes for objects); we call a functor of this type a w-pure one (cf. Remark 3.1.3 *below).*

(8) *Let $\underline{C}'$ be a triangulated category endowed with a weight structure w'; let $F : \underline{C} \to \underline{C}'$ be a weight-exact functor. Then for any choice of $t(M)$ the complex $(F(M^i))$ yields a weight complex of $F(M)$ with respect to w'. Moreover, this observation is "compatible with the construction of functors" mentioned in the previous assertion, and is natural with respect to transformations of (weight-exact) functors.*

Proof. Assertions (1)–(4) easily follow from Theorem 3.3.1 of [Bon10a]. Moreover, Proposition 1.3.4 and Appendices A–B of [Bon21] give some more detail for the proofs.

Next, assertions (8), (7), and (5) are given by Proposition 1.3.4(12), Theorem 2.1.2(1), and Corollary 3.3.3(2) of ibid., respectively.

Assertion (6) was essentially established in the course of proving Proposition 1.2.4(10). □

Remark 1.4.3.

(1) Moreover, Theorem 3.3.1(VI) of [Bon10a] easily yields that t induces a bijection between the class of isomorphism classes of elements of $\underline{C}_{[0,1]}$ and the corresponding class for $K(\underline{Hw})$ (i.e., with the class of homotopy equivalence classes of complexes that have non-zero terms in degrees -1 and 0 only).

(2) The term "weight complex" originates from [GiS96], where a certain complex of Chow motives $W(X)$ was constructed for a variety X over a characteristic 0 field. The weight complex functor of Gillet and Soulé can essentially be obtained by composing the "triangulated motivic" weight complex functor $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Z}) \to K^b(\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Z}))$ (or $DM_{gm}(k,\mathbb{Z}) \to K^b(\mathrm{Chow}(k,\mathbb{Z}))$; cf. Definition 3.1.1 below) with the functor $\mathcal{M}^c$ of *motive with compact support* (see Propositions 6.3.1 and 6.6.2 and Remark 6.3.2(2) of [Bon09]; cf. also Definition 4.1.1(2) and the proof of Proposition 4.1.8(2) below). Note however that in [GiS96] the so-called contravariant category of Chow motives is considered, i.e., all arrows point in the opposite direction.

Certainly, our notion of weight complex is much more general.

(3) The basics of our weight complex theory was developed in §3 of [Bon10a]; in §1.3 of [Bon21] the theory was exposed more carefully (via extending Definition 1.4.1). In ibid. a (canonical) *weak weight complex functor* $t : \underline{C}_w \to K_{\mathfrak{w}}(\underline{Hw})$ was defined; here $\underline{C}_w$ is a (triangulated) category canonically equivalent to $\underline{C}$, and $K_{\mathfrak{w}}(\underline{Hw})$ is a certain "weak homotopy" category of $\underline{Hw}$-complexes (and there exists a natural conservative functor $K(\underline{Hw}) \to K_{\mathfrak{w}}(\underline{Hw})$).

Moreover, throughout this paper one can actually assume that all the weight complexes we need are given by "compatible" exact *strong* weight complex functors whose targets are the corresponding $K^b(\underline{Hw})$; see Corollary 3.5 of [Sos19], Remark 1.3.5(3) of [Bon21], and Proposition 1.3.1 of [Bon20b]. This approach is also applied in (§1.5 of) [BoK20].

(4) All the weight complexes in this paper can be assumed to be bounded, since for any (w-bounded) object M of $\underline{C}$ one can take $w_{\leq l}M = 0$ for l small enough and $= M$ for l large enough. Moreover, we can assume that for any

object M of $\underline{C}$ a canonical choice $t_0(M)$ of its weight complex is fixed; cf. part (3) of this remark.

Now, the possible choices of bounded weight complexes for M are precisely the bounded $\underline{Hw}$-complexes homotopy equivalent to $t_0(M)$; see Corollary 3.3.3(1) of [Bon21].

Let us now recall some of the properties of weight spectral sequences established in §2 of [Bon10a].

Let $\underline{A}$ be an abelian category. In §2 of [Bon10a] for $H : \underline{C} \to \underline{A}$ that is either cohomological or homological (i.e., it is either covariant or contravariant, and converts distinguished triangles into long exact sequences) certain *weight filtrations* and *weight spectral sequences* (corresponding to w) were introduced.

Definition 1.4.4. Let $\underline{A}$ be a an abelian category, $i \in \mathbb{Z}$, and $M \in \operatorname{Obj} \underline{C}$.

(1) If $H : \underline{C} \to \underline{A}$ is a (covariant) functor then we will write H_i for the functor $H \circ [-i] : \underline{C} \to \underline{A}$.

(2) If H is a contravariant functor from $\underline{C}$ into $\underline{A}$ then we write H^i for the composed functor $H^i = H \circ [-i]$.

Moreover, we fix a choice of $w_{\geq i}M$ and define the *weight filtration* on $H(M)$ as $W^i(H)(M) = \operatorname{Im}(H(w_{\geq i}M) \to H(M))$. Recall that $W^iH(M)$ is functorial in M (in particular, it does not depend on the choice of $w_{\geq i}M$); see Proposition 2.1.2(2) of ibid. We will use the notation $Gr^i_W H(M)$ for the quotient object $W^i(H)(M)/W^{i+1}(H)(M)$.

Proposition 1.4.5.

(1) *For a homological functor $H : \underline{C} \to \underline{A}$ and any $M \in \operatorname{Obj} \underline{C}$ there exists a spectral sequence $T = T_w(H, M)$ with $E_1^{pq}(T) = H_{-q}(M^p)$, such that the objects M^i and the boundary morphisms of $E_1(T)$ come from any choice of $t(M)$. $T_w(H, M)$ is $\underline{C}$-functorial in M starting from E_2.*

It converges to $E_\infty^{p+q} = H_{-p-q}(M)$ (at least) if M is w-bounded.

(2) *Dually, if H is a cohomological functor from $\underline{C}$ into $\underline{A}$ then for any $M \in \operatorname{Obj} \underline{C}$ there exists a spectral sequence $T = T_w(H, M)$ with $E_1^{pq} = H^q(M^{-p})$, for M^i and the boundary morphisms of $E_1(T)$ coming from $t(M)$. $T_w(H, M)$ converges to $H^{p+q}(M)$ whenever M is w-bounded; it is $\underline{C}$-functorial in M starting from E_2, and also functorial with respect to composition of H with exact functors between abelian categories.*

The step of the filtration given by $(E_\infty^{l,m-l} : l \geq n)$ on $H^m(M)$ (for some $n, m \in \mathbb{Z}$) equals $(W^n H^m)(M)$.

Proof. These statements are essentially contained in Theorems 2.3.2 and 2.4.2 of [Bon10a], respectively (yet take into account Remark 1.2.3(3)!). □

Corollary 1.4.6. *Let $M \in \underline{C}_{w\geq 0}$, $N \in \underline{C}_{w=0}$. Then the following statements are valid.*

(1) *Choose some $t(M) = (M^i)$. Then $\underline{C}(N, M)$ is isomorphic to the zeroth homology of the complex $(\underline{Hw}(N, M^i))$.*

(2) *Let $\underline{D} \subset \underline{C}$ be a triangulated subcategory of $\underline{C}$; suppose that w restricts to a weight structure $w_{\underline{D}}$ on $\underline{D}$ (see Definition 1.2.2(6)). Assume that a morphism $f \in \underline{C}(N, M)$ vanishes in the Verdier quotient $\underline{C}/\underline{D}$. Then f factors through some object of $\underline{Hw}_{\underline{D}}$.*

Proof.

(1) We may assume that $M^i = 0$ for $i > 0$ (see Proposition 1.4.2(2); note that making a choice here does not affect the homology of the complex $(\underline{Hw}(N, M^*))$). Hence we have a weight spectral sequence for the homological functor $\underline{C}(N, -) : \underline{C} \to Ab$ that starts from $E_1^{pq} = \underline{C}(N, M^p[q])$ and converges to $\underline{C}(N, M[p+q])$. Since $N \perp M^i[-i]$ for all $i < 0$ and $N \perp M^i[-i-1]$ whenever $i < -1$, this spectral sequence gives the result.

(2) The Verdier localization theory yields that f factors through an object of $\underline{D}$. Hence the assertion follows from Proposition 1.4.2(6).

□

2. On motives, their weights, and various (complexes of) Chow groups

In this section we study several motivic categories, Chow weight structures on them, and certain (complexes of) Chow groups.

In §2.1 we recall some basics on Voevodsky motives with coefficients in a $\mathbb{Z}[1/e]$-algebra R and introduce some notation.

In §2.2 we introduce and study in detail Chow weight structures on various versions of $DM_{gm}^{\text{eff}}(k, R)$.

In §2.3 we associate to extensions of k and complexes of Chow motives the homology of complexes consisting of their Chow groups (of fixed dimension and "highness"). We prove several properties of these homology theories (and of motivic homology); however, most of them appear to be standard.

2.1. **Some notation and basics on Voevodsky motives.** Below k will denote a perfect base field of characteristic p. We set $e = 1$ if $p = 0$ and $e = p$ otherwise; that is, e is the *exponential characteristic* of k.

We will use the term *k-variety* for reduced separated (possibly, reducible) schemes of finite type over $\operatorname{Spec} k$; we write Var for the set of all k-varieties. Respectively, the set of smooth varieties (resp. of smooth projective varieties) over k will be denoted by SmVar (resp. by SmPrVar), and we do not assume these schemes to be connected.

Recall that (as was shown in [MVW06] and [BeV08]; cf. also [CiD15] and [BoK18]) one can do the theory of motives with coefficients in an arbitrary commutative associative ring with a unit R. One obtains a tensor triangulated category $DM_{gm}^{\text{eff}}(k, R)$ (along with its embeddings into $DM_{gm}(k, R)$ and into $DM_{-}^{\text{eff}}(k, R)$; see below) that satisfies all the basic properties of the usual Voevodsky's motives (i.e., of those with integral coefficients for $p = 0$). Moreover, we recall that all of the results that were stated in [Voe00] in this case are currently known for $\mathbb{Z}[1/e]$-motives (also if) $p > 0$; see [Kel17], [Deg08], and [Bon11]. Consequently, these properties are also valid for R-linear motives whenever R is a $\mathbb{Z}[1/e]$-algebra (see [BeV08] and [BoK18]), and we will apply some statements of this sort below without further mention. We will mostly be interested in the cases $R = \mathbb{Z}[1/e]$ and $R = \mathbb{Q}$.

An important part of the construction of motives is a functor $\mathcal{M}_R$ (R-motive) from the category of smooth k-varieties into $DM_{gm}^{\text{eff}}(k, R)$. Actually, $\mathcal{M}_R$ extends to the category of all k-varieties (see [Voe00] and [Kel17]); yet we will mention this extension just a few times.

We will write pt for the point $\operatorname{Spec} k$ (considered as a k-variety); we write just R for $\mathcal{M}_R(\mathrm{pt})$.

We write $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ for the Karoubi-closure in $DM^{\mathrm{eff}}_{gm}(k,R)$ of the subcategory whose objects are R-motives of smooth projective varieties; $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ will be called the category of R-linear effective homological Chow motives (see Proposition 2.2.6(1) below or Remark 1.3.2(4) of [BoK18] for a justification of this terminology).

For $c \geq 0$ and $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)$ we write $M\langle c\rangle$ for the tensor product of M by the cth tensor power of the Lefschetz motive $\mathbb{L}$ (recall that the latter is characterized by the condition $\mathcal{M}_R(\mathbb{P}^1) \cong \mathbb{L} \bigoplus R$). The relation of this notation to the notation for twists in [Voe00] is as follows: $M\langle c\rangle = M(c)[2c]$ and $M(c) = M\langle c\rangle[-2c]$.

Next, recall that the twist functor $-\langle 1\rangle$ is a full embedding of $DM^{\mathrm{eff}}_{gm}(k,R)$ into itself (this fact is often called the Cancellation theorem) that restricts to an embedding of $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ into itself. $-\langle 1\rangle$ extends to an autoequivalence of the corresponding category $DM_{gm}(k,R) = DM^{\mathrm{eff}}_{gm}(k,R)[\langle -1\rangle]$ (i.e., we invert the functor $-\langle 1\rangle = - \otimes \mathbb{L}$); note that this category contains $DM^{\mathrm{eff}}_{gm}(k,R)$ together with $\mathrm{Chow}(k,R) = \mathrm{Chow}^{\mathrm{eff}}(k,R)[\langle -1\rangle]$. Moreover, $DM_{gm}(k,R)$ is equipped with an exact Poincaré duality functor $\widehat{-} : DM_{gm}(k,R) \to DM_{gm}(k,R)^{op}$ (constructed in [Voe00] for $p = 0$; see Theorem 5.3.18 of [Kel17] or [Bon11] for the positive characteristic case) that sends $\mathcal{M}_R(P)$ into $\mathcal{M}_R(P)\langle -d\rangle$ if P is smooth projective everywhere of dimension d. It restricts to the "usual" Poincaré duality for $\mathrm{Chow}(k,R)$.

Both $DM^{\mathrm{eff}}_{gm}(k,R)$ and $DM_{gm}(k,R)$ are Karoubian by definition.

An important property of motives is the Gysin distinguished triangle (see Proposition 4.3 of [Deg08] that establishes its existence in the case of an arbitrary characteristic p). For a closed embedding $Z \to X$ of smooth varieties with Z is everywhere of codimension c in X, it has the following form:

$$\mathcal{M}_R(X \setminus Z) \to \mathcal{M}_R(X) \to \mathcal{M}_R(Z)\langle c\rangle \to \mathcal{M}_R(X \setminus Z)[1]. \tag{2.1}$$

Remark 2.1.1. Some of our formulations below will use the homotopy t-structure for the Voevodsky motivic complexes. We recall that the methods of [Voe00] yield an embedding $DM^{\mathrm{eff}}_{gm}(k,R)$ into a certain category $DM^{\mathrm{eff}}_{-}(k,R)$, and the latter can be endowed with the so-called homotopy t-structure t^R_{hom} (which gives a filtration on $DM^{\mathrm{eff}}_{gm}(k,R) \subset DM^{\mathrm{eff}}_{-}(k,R)$ that we will sometimes call the *motivic connectivity* one). Furthermore, $DM^{\mathrm{eff}}_{-}(k,R)$ is a full subcategory of the triangulated category $DM^{\mathrm{eff}}(k,R)$ of unbounded motivic complexes that is closed with respect to arbitrary coproducts. The t-structure t^R_{hom} can be extended to $DM^{\mathrm{eff}}(k,R)$ (see §4 of [BeV08] or Corollary 5.2 of [Deg11]), and the corresponding class $DM^{\mathrm{eff}}(k,R)^{t^R_{hom}\leq 0}$ equals $DM^{\mathrm{eff}}_{-}(k,R)^{t^R_{hom}\leq 0}$; it also equals the smallest extension-closed class of objects of $DM^{\mathrm{eff}}(k,R)$ that is closed with respect to coproducts and contains $\mathcal{M}_R(X)$ for all smooth X/k.

We will often have to mention base fields distinct from k. It will be convenient for us to use the following notation.

Definition 2.1.2. Assume that K/k be a field extension, X is a k-variety, and M an object of $DM_{gm}(k, R)$.

(1) We will write X_K for the K-variety $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$.
(2) K^{perf} will denote the perfect closure of K.
(3) We will use the notation M_K for the image of M with respect to the extension of base field functor $DM_{gm}(k, R) \to DM_{gm}(K^{perf}, R)$ (cf. Remark 2.2.3 below); see Appendix A of [BoK20] for some information on functors of this type.

Let us recall a well-known statement related to this convention. Below we will only apply it for X that is smooth over K (yet cf. Proposition 4.1.2(2) and Remark 4.1.3 below); in this case it is essentially true by definition.

Lemma 2.1.3. *For a variety X over k we have $\mathcal{M}_R(X)_K \cong \mathcal{M}_R(X_{K^{perf}})$ (in the category $DM^{\text{eff}}_{gm}(K^{perf}, R)$).*

Proof. The statement is given by Proposition A.1(1) of [BoK20]. Alternatively, it easily follows from (8.7.1), Corollary 3.2, and Theorem 3.1 of [CiD15] along with Proposition 4.3.13 of [CiD19]. $\square$

2.2. **Chow weight structures on various motivic categories.** Now we note that the arguments used in the construction of the *Chow weight structures* in [Bon10a] and [Bon11] can be easily applied to R-motives (for any $\mathbb{Z}[1/e]$-algebra R).

Proposition 2.2.1.

(1) *There exists a bounded weight structure w_{Chow} on $DM^{\text{eff}}_{gm}(k, R)$ (resp. on $DM_{gm}(k, R)$) whose heart equals* $\text{Chow}^{\text{eff}}(k, R)$ *(resp.* $\text{Chow}(k, R)$*; we assume these subcategories of $DM_{gm}(k, R)$ to be strict). These weight structures on $DM^{\text{eff}}_{gm}(k, R)$ and $DM_{gm}(k, R)$ are compatible (i.e., the embedding $DM^{\text{eff}}_{gm}(k, R) \to DM_{gm}(k, R)$ is weight-exact).*

   Moreover, $DM^{\text{eff}}_{gm}(k, R)_{w_{\text{Chow}} \le 0}$ (resp. $DM_{gm}(k, R)_{w_{\text{Chow}} \le 0}$) is equal to the extension-closure of the class $\cup_{i \le 0} \operatorname{Obj} \text{Chow}^{\text{eff}}(k, R)[i]$ in $DM^{\text{eff}}_{gm}(k, R)$ (resp. of the class $\cup_{i \le 0} \operatorname{Obj} \text{Chow}(k, R)[i]$ in $DM_{gm}(k, R)$); analogously, $DM^{\text{eff}}_{gm}(k, R)_{w_{\text{Chow}} \ge 0}$ (resp. $DM_{gm}(k, R)_{w_{\text{Chow}} \ge 0}$) is the extension-closure of $\cup_{i \ge 0} \operatorname{Obj} \text{Chow}^{\text{eff}}(k, R)[i]$ in $DM^{\text{eff}}_{gm}(k, R)$ (resp. of $\cup_{i \ge 0} \operatorname{Obj} \text{Chow}(k, R)[i]$ in $DM_{gm}(k, R)$).
(2) *If $U \in \text{SmVar}$ and $\dim U \le m$ then $\mathcal{M}_R(U) \in DM^{\text{eff}}_{gm}(k, R)_{[-m,0]}$.*
(3) *If $U \to V$ is an open dense embedding of smooth varieties then the motive* $\operatorname{Cone}(\mathcal{M}_R(U) \to \mathcal{M}_R(V))$ *belongs to $DM^{\text{eff}}_{gm}(k, R)_{w_{\text{Chow}} \le 0}$.*
(4) *Let k'/k be a field extension. Then the extension of scalars functors $-_{k'}$: $DM^{\text{eff}}_{gm}(k, R) \to DM^{\text{eff}}_{gm}(k'^{perf}, R)$ and $DM_{gm}(k, R) \to DM_{gm}(k'^{perf}, R)$ (see Definition* 2.1.2*) are weight-exact with respect to the corresponding Chow weight structures.*
(5) *For any $n \in \mathbb{Z}$ the functor $-\langle n \rangle$ is weight-exact on $DM_{gm}(k, R)$; the same is true for $DM^{\text{eff}}_{gm}(k, R)$ if $n \ge 0$.*
(6) *If $M \in \operatorname{Obj} DM^{\text{eff}}_{gm}(k, R)\langle n \rangle$, $n \in \mathbb{Z}$, then there exists a choice of its weight complex $t(M) = (M^i)$ (with respect to the Chow weight structure for $DM^{\text{eff}}_{gm}(k, R)$; see Definition* 1.4.1 *and Remark* 1.4.3(4)*) with $M^i \in$* $\operatorname{Obj} \text{Chow}^{\text{eff}}(k, R)\langle n \rangle$.

Proof. The first three assertions were stated in Theorem 2.2.1 of [Bon11] in the case $R = \mathbb{Z}[1/e]$. The proof carries over to the case of a general R without any difficulty; see Remark 2.1.3(1) of [BoK18] or Proposition 2.3.2 of [BoI15].

The remaining statements are simple as well. Assertions (4) and (5) easily follow from Proposition 1.2.4(5) along with Lemma 2.1.3, whereas assertion (6) follows from the previous one by Proposition 1.4.2(8). □

Now we deduce some simple corollaries from this proposition. Their formulation requires the following definition, that will be important for us below.

Definition 2.2.2.

(1) For $M \in \operatorname{Obj} DM_{gm}^{\text{eff}}(k,R)$ and a non-negative integer r we say that M is r-*effective* if it has the form $N\langle r\rangle$ for some $N \in \operatorname{Obj} DM_{gm}^{\text{eff}}(k,R)$.

(2) We say that the *dimension* of M is not greater than an integer m if M belongs to $\langle \mathcal{M}_R(P) :\ P \in \operatorname{SmPrVar},\ \dim P \le m\rangle$.

The (full) subcategory of $DM_{gm}^{\text{eff}}(k,R)$ (resp. of $\operatorname{Chow}^{\text{eff}}(k,R)$) of motives of dimension at most m is denoted by $d_{\le m}DM_{gm}^{\text{eff}}(k,R)$ (resp., by $d_{\le m}\operatorname{Chow}^{\text{eff}}(k,R)$; consequently, $d_{\le m}DM_{gm}^{\text{eff}}(k,R) = d_{\le m}\operatorname{Chow}^{\text{eff}}(k,R) = \{0\}$ if $m<0$).

(3) For $r \ge -1$ we define the functor $l^r : DM_{gm}^{\text{eff}}(k,R) \to DM_{gm}^{r}(k,R)$ as the Verdier localization of $DM_{gm}^{\text{eff}}(k,R)$ by $DM_{gm}^{\text{eff}}(k,R)\langle r+1\rangle$.

(4) We also use the following extension of this notation: $\operatorname{Chow}^{\text{eff}}(k,R)\langle +\infty\rangle = DM_{gm}^{\text{eff}}(k,R)\langle +\infty\rangle = \{0\}$, $l^{+\infty} = l^{+\infty-1}$ will denote the identity functor for $DM_{gm}^{\text{eff}}(k,R)$. Respectively, $DM_{gm}(k,R)^{+\infty}(k,R) = DM_{gm}^{\text{eff}}(k,R)$, and any subclass of objects of $DM_{gm}^{\text{eff}}(k,R)\langle +\infty\rangle$ is zero.

(5) If M is an object of $\operatorname{Chow}^{\text{eff}}(k,R)$ or of $DM_{gm}^{\text{eff}}(k,R)$ and $j,l \in \mathbb{Z}$ then we define $h_{2j+l,j}(M,R)$ as $DM_{gm}(k,R)(R\langle j\rangle[l],M) = DM_{gm}(k,R)(R(j)[2j+l],M)$ (cf. Theorem 5.3.14 of [Kel17] or Proposition 4.1.2(3) below where these groups are related to the corresponding Chow-Bloch groups of varieties).

More generally, for an extension K/k we write $h_{2j+l,j}(M_K,R)$ for the group $DM_{gm}(K^{perf},R)(R\langle j\rangle[l],M_K)$ (see Definition 2.1.2(1,2)).

The last part of this definition can be naturally extended to $DM_{-}^{\text{eff}}(k,R)$. When we will use this notation for general (l,M), we will usually take $j=0$ in it.

Remark 2.2.3. We will sometimes mention "ordinary" Chow groups of varieties over fields that are not necessarily perfect. One can define them in the usual way (in spite of the conventions described in Definition 2.1.2; cf. also Proposition 4.1.2(3) below) since for any variety X over k and any extension K/k we have the following isomorphism of Chow groups of cycles of dimension $j \ge 0$:

$$\operatorname{CH}_j(X_K,R) \cong \operatorname{CH}_j(X_{K^{perf}},R).$$

This fact is probably well-known; it can either be proved similarly to Lemma 1.2 of [Via17] (where the case $R = \mathbb{Q}$ is considered; recall that we always assume that R is a $\mathbb{Z}[1/e]$-algebra) or deduced from Proposition 8.1 of [CiD15].

Corollary 2.2.4. *Let $c \geq 1$, $m \geq 0$.*

(1) *The Chow weight structure restricts to a weight structure w_c on the category $DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ (see Definition 1.2.2(6)). Moreover, $DM_{gm}^{\mathrm{eff}}(k,R)_{w_c\leq 0} = DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}\leq 0}\langle c\rangle$ and $DM_{gm}^{\mathrm{eff}}(k,R)_{w_c\geq 0} = DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}\geq 0}\langle c\rangle$.*

(2) *An object M of $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ is c-effective (as an object of $DM_{gm}^{\mathrm{eff}}(k,R)$) if and only if it can be presented as $N\langle c\rangle$ for $N \in DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}=0}$.*

(3) *The Chow weight structure also restricts to a weight structure on the category $d_{\leq m}DM_{gm}^{\mathrm{eff}}(k,R)$ (that will also be denoted by w_{Chow}). The heart of the latter consists of all objects of $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ inside $d_{\leq m}DM_{gm}^{\mathrm{eff}}(k,R)$; these motives are exactly the retracts of $\mathcal{M}_R(P)$ for smooth projective P/k of dimension at most m.*

(4) *If $U \to V$ is an open embedding of smooth varieties such that $V \setminus U$ is everywhere of codimension c in V, $\dim V \leq m$, then $\mathrm{Cone}(\mathcal{M}_R(U) \to \mathcal{M}_R(V)) \in (d_{\leq m-c}DM_{gm}^{\mathrm{eff}}(k,R))_{w_{\mathrm{Chow}}\leq 0}\langle c\rangle$.*

(5) *If V is a smooth k-variety of dimension at most m then $\mathcal{M}_R(V)$ is an object of $d_{\leq m}DM_{gm}^{\mathrm{eff}}(k,R)$.*

(6) *The Karoubi-closures of the classes*

$$l^{c-1}(DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}\leq 0}) \text{ and } l^{c-1}(DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}\geq 0})$$

in $DM_{gm}^{c-1}(k,R)$ give a bounded weight structure w_{Chow}^{c-1} on this category.

Proof. (1) Note that $DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ is precisely the subcategory of $DM_{gm}^{\mathrm{eff}}(k,R)$ densely generated by $\mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle c\rangle$. Hence Proposition 1.2.4(8), (2) yields the result immediately.

(2) This is an immediate consequence of the "moreover" part of the previous assertion (since $-\langle c\rangle$ gives an equivalence of $DM_{gm}^{\mathrm{eff}}(k,R)$ with $DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$).

(3) The statement immediately follows from Proposition 1.2.4(8) (once again).

(4) There clearly exists a chain of open embeddings $U = U_0 \to U_1 \to U_2 \to \dots \to U_m = V$ (for some $m \geq 1$) such that $U_i \setminus U_{i-1}$ are smooth for all $1 \leq i \leq m$. Hence the distinguished triangles (2.1) along with Corollary 2.2.4(5) imply (by induction on m) that $\mathrm{Cone}(\mathcal{M}_R(U) \to \mathcal{M}_R(V)) \in \mathrm{Obj}(d_{\leq m-c}DM_{gm}^{\mathrm{eff}}(k,R))\langle c\rangle$. Thus it remains to combine the equality

$$((d_{\leq m-c}DM_{gm}^{\mathrm{eff}}(k,R))\langle c\rangle)_{w_c\leq 0} = ((d_{\leq m-c}DM_{gm}^{\mathrm{eff}}(k,R))_{w_{\mathrm{Chow}}\leq 0})\langle c\rangle$$

(cf. assertion (1) and its proof) with Proposition 2.2.1(3).

(5) The arguments used for the proof of [Bon11, Theorem 2.2.1(1)] give the result without any difficulty (cf. Corollary 1.2.2 of ibid. and Lemma 4.1.4(4) below).

(6) According to assertion (1), we can apply Proposition 1.3.2(1,4) to obtain the result in question. □

Remark 2.2.5. Let $l \in \mathbb{Z}$ and $c \geq 1$, and assume that there exists a choice of $w_{\mathrm{Chow}\leq l}M$ that belongs to $\mathrm{Obj}\,DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$.

(1) Proposition 1.2.4(7) implies that we can choose $w_{\mathrm{Chow}\leq l-1}M$ that belongs to $\mathrm{Obj}\,DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$. Then the corresponding choice (see (1.2)) of M^{-l} belongs to $DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}=0}$ as well as to $\mathrm{Obj}\,DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ (since $DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ is a full triangulated subcategory of $DM_{gm}^{\mathrm{eff}}(k,R)$; see Proposition 1.2.4(2)). Thus $M^{-l} \in DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}}=0}\langle c\rangle$.

(2) Now suppose $M \in DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}} \le l}$. Then M is a retract of $w_{\mathrm{Chow}\le l}M$ (since id_M factors through $w_{\mathrm{Chow}\le l}M$ by Proposition 1.2.4(9)). Thus M is an object of $DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ as well.

(3) It is easily seen that the weight structure w^r_{Chow} can be extended to the category $\mathrm{Kar}(DM^r_{gm}(k,R)) \supset DM^r_{gm}(k,R)$ (see Theorem 2.2.2(II.2) of [BoS18b] or Remark 1.2.3(4) and Proposition 1.2.4(8 above). Moreover, $\mathrm{Kar}(DM^0_{gm}(k,\mathbb{Z}))$ is easily seen to be equivalent to the (geometric) birational motivic category DM^o_{gm} introduced in Definition 4.2.1 of [KaS17].

Let us prove some more lemmas that will be very important for us below.

Proposition 2.2.6. *Let $m, j \ge 0$, $c \ge 1$, $U, V \in \mathrm{SmVar}$, $Q \in \mathrm{SmPrVar}$, $M \in \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}(k,R)$.*

(1) *If U is of constant dimension d then $DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(U)\langle j\rangle, \mathcal{M}_R(Q))$ is naturally isomorphic to the group $\mathrm{CH}_{d+j}(U \times Q, R)$ of R-linear cycles of dimension $d+j$ modulo rational equivalence.*

(2) *Let $u : U \to V$ be an open embedding such that $V \setminus U$ is everywhere of codimension at least c in V and $\dim V \le m$. Let $N \in DM_{gm}(k,R)_{w_{\mathrm{Chow}} \ge 0}$, and assume that a morphism $g \in DM_{gm}(k,R)(\mathcal{M}_R(V)\langle j\rangle, N)$ vanishes when composed with $\mathcal{M}_R(u)\langle j\rangle$. Then there exists a smooth projective P/k of dimension at most $m-c$ such that g factors through $\mathcal{M}_R(P)\langle j+c\rangle$.*

(3) *If Q is of dimension at most m then any morphism $q : \mathcal{M}_R(Q) \to M\langle c\rangle$ can be factored through $\mathcal{M}_R(P)\langle c\rangle$ for some smooth projective P/k of dimension at most $m-c$. Moreover, there exists an open embedding $w : W \to Q$ such that $Q \setminus W$ is (everywhere) of codimension at least c in Q and the composition $q \circ \mathcal{M}_R(w)$ vanishes.*

(4) $\mathrm{Obj}\, d_{\le m} DM_{gm}^{\mathrm{eff}}(k,R) \cap \mathrm{Obj}\, DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle = \mathrm{Obj}(d_{\le m-c} DM_{gm}^{\mathrm{eff}}(k,R))\langle c\rangle.$
In particular, if $M\langle c\rangle$ is of dimension at most m (in $DM_{gm}^{\mathrm{eff}}(k,R)$), then M is of dimension at most $m-c$ (thus it is zero if $c > m$).

(5) *Let $g \in DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(V)\langle j\rangle, M)$. Assume that V is connected and the obvious image of g in the group $h_{2j,j}(M_{k(V)})$ (see Definition 2.1.2(3) and the proof of this assertion) is zero. Then the morphism g can be factored through an object of $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j+1\rangle$.*

(6) *If Q is connected then $DM^j_{gm}(k,R)(\mathcal{M}_R(Q)\langle j\rangle, M) \cong h_{2j,j}(M_{k(Q)}, R)$.*

(7) *Assume that $\dim(Q) + j \le r$ and that the dimension of M (see Definition 2.2.2(2)) is not greater than r. Then the group $h_{2j,j}(M_{k(Q)}, R)$ is isomorphic to the group of morphisms from $\mathcal{M}_R(Q)\langle j\rangle$ into M in the localization $d_{\le r} DM_{gm}^{\mathrm{eff}}(k,R)/((d_{\le r-j-1} DM_{gm}^{\mathrm{eff}}(k,R))\langle j+1\rangle)$ (as well).*

Proof. (1). This statement was established in [Voe00] in the case $p = 0$; in the general case it follows immediately from the formulae (6.4.2) and (6.7.1) of [BeV08]; cf. Corollary 6.7.3 of ibid.

(2). Clearly, g can be factored through $\mathrm{Cone}(\mathcal{M}_R(u))\langle j\rangle$. Next, Corollary 2.2.4(4) implies that $\mathrm{Cone}(\mathcal{M}_R(u))\langle j\rangle$ belongs to $DM_{gm}^{\mathrm{eff}}(k,R)_{w_{\mathrm{Chow}} \le 0}\langle j+c\rangle$. Hence for $\mathrm{Cone}(\mathcal{M}_R(u)) = N'\langle c\rangle$ we can take $w_{\mathrm{Chow}\ge 0}(\mathrm{Cone}(\mathcal{M}_R(u))\langle j\rangle)$ to be equal to $(w_{\mathrm{Chow}\ge 0}N')\langle j+c\rangle \in \mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j+c\rangle$ (see Proposition 1.2.4(6)). Hence applying part (9) of that proposition we conclude the proof.

(3). Let $Q = \sqcup Q_i$ be the decomposition of Q into connected components, whose dimensions will be denoted by m_i; clearly, $m_i \le m$. Assume that M is a retract of

$\mathcal{M}_R(S)$ for some smooth projective S/k. By the classical theory of Chow motives (cf. assertion (1)), the morphism q is supported on subvarieties of dimension $m_i - c$ in $Q_i \times S$. Hence there exists an open $W \subset Q$ such that $Q \setminus W$ is everywhere of codimension at least c in Q and the "restriction" of q to W vanishes. Hence $q \circ \mathcal{M}_R(w) = 0$ according to assertion (1), and assertion (2) implies that q factors through some $\mathcal{M}_R(P)\langle c\rangle$ for a smooth projective P/k of dimension at most $m - c$.

(4). The first part of the assertion follows immediately from Theorem 2.2 of [Bon18a] (see also Remark 2.3(2) of ibid.).

To prove the second part it suffices to recall that any motive in the heart of $d_{\leq m-c} DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ is a retract of $\mathcal{M}_R(P)\langle c\rangle$ for some smooth projective P/k of dimension at most $m - c$ (see Corollary 2.2.4(1,3)), and apply the Cancellation theorem.

(5). Clearly, we can assume that $M = \mathcal{M}_R(Q)$, Q is (smooth projective and connected) of dimension $d_Q \geq 0$, and V is of dimension d. Then assertion (1) says that $DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(V)\langle j\rangle, M) \cong \mathrm{CH}^{d_Q - j}(V \times Q, R)$ (the R-linear Chow group of codimension $d_Q - j$ cycles).

Next, we recall that Chow functors of this type are well-known to be "continuous"; thus we have $\mathrm{CH}^{d_Q-j}(Q_{k(V)}, R) = \varinjlim_W \mathrm{CH}^{d_Q-j}(W \times Q, R)$; here W runs through open subvarieties of V (cf. Lemma 3.4 of [Via17] and its proof). Moreover, it is easily seen that $\mathrm{CH}^{d_Q-j}(Q_{k(V)}, R) \cong \mathrm{CH}_j(Q_{k(V)}, R) \cong \mathrm{CH}_j(Q_{k(V)^{perf}}, R)$; see Remark 2.2.3. Thus there exists an open embedding $w : W \to Q$ such that $g \circ \mathcal{M}_R(w)\langle j\rangle = 0$; hence we can apply assertion (2).

(6). Denote $\dim Q$ by d. Similarly to the proof of the previous assertion, we have $DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(Q)\langle j\rangle, M) \cong h_{2j+2d,j+d}(\mathcal{M}_R(Q) \otimes M, R)$, and there is a natural surjective homomorphism

$$DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(Q)\langle j\rangle, M) \cong h_{2j+2d,j+d}(\mathcal{M}_R(Q) \otimes M, R) \to h_{2j,j}(M_{k(Q)}, R).$$

By Proposition 1.3.2(3), the natural homomorphism

$$DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(Q)\langle j\rangle, M) \to DM_{gm}^{j}(k,R)(\mathcal{M}_R(Q)\langle j\rangle, M)$$

is surjective as well. Thus we should compare the kernels.

According to Proposition 1.3.2(3), the second of these kernels consists exactly of morphisms that can be factored through $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j+1\rangle$. Hence we should prove that the first kernel can be described by this criterion as well. Now, (the rational equivalence class of cycles representing) any morphism that factors through $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j+1\rangle$ vanishes in $h_{2j,j}(M_{k(Q)}, R)$ for simple dimension reasons (cf. Proposition 2.3.3(2) below). Conversely, any morphism that belongs to

$$\mathrm{Ker}(DM_{gm}^{\mathrm{eff}}(k,R)(\mathcal{M}_R(Q)\langle j\rangle, M) \to h_{2j,j}(M_{k(Q)}, R))$$

can be factored through an object of $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j+1\rangle$ according to the previous assertion.

(7). The chain or arguments used for the proof of the previous assertion can easily be adjusted to yield the result. □

Remark 2.2.7.

(1) The proof of (part (5)) of the proposition uses an abstract version of the well-known *decomposition of the diagonal* arguments (cf. Proposition 1 of [BlS83]). The "usual" way to construct the factorization in question (see Proposition 3.5 of [Via17] and Lemma 3 of [GoG13]) is to resolve the

singularities of $V \setminus W$. Yet it is somewhat difficult to apply this more explicit method if $p > 0$ (at least, for $\mathbb{Z}[1/e]$-coefficients). Moreover, our reasoning is somewhat shorter than the one of loc. cit. (given the properties of Chow weight structures that are absolutely necessary for this paper anyway).

(2) In the case $R = \mathbb{Q}$ the "in particular" part of Proposition 2.2.6(4) was established in §3 of [Via17] (see Remark 3.11 of ibid.). The general case of the assertion is completely new.

(3) The idea of studying $DM^j_{gm}(k, R)$ and the formulation of part (5) of the proposition was inspired by Theorem 4.2.2(f) of [KaS17] (where our assertion was established in the case $j = 0$).

2.3. **On complexes of Chow groups over various fields.** We start with some simple definitions.

Definition 2.3.1. Let K be a field.

(1) We say that K is *essentially finitely generated* if it is the perfect closure of a field that is finitely generated over its prime subfield.

(2) We call K a *universal domain* if it is algebraically closed and of infinite transcendence degree over its prime subfield.

(3) We say that a field F_0 is a *field of definition* for an object M of $DM^{\mathrm{eff}}_{gm}(k, R)$ (resp. of $K^b(\mathrm{Chow}(k, R))$) if it is a part of a quintuple $(F_0,\ k_0,\ i,\ M_0,\ f)$ where k_0 is a perfect subfield of F_0, i is an embedding $k_0 \to k$, $M_0 \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k_0, R)$ (resp. $M_0 \in \operatorname{Obj} K^b(\mathrm{Chow}(k_0, R))$), and f is an isomorphism $M_k \to M$ (cf. Definition 2.1.2).

(4) We call K a *rational extension* of k if $K \cong k(t_1, \dots, t_n)$ for some $n \geq 0$.

(5) We say that K is a *function field* over k if K is finitely generated over k.

Remark 2.3.2.

(1) Fields of definition for M obviously form a category if we define a morphism from $(F_0,\ k_0,\ i,\ M_0,\ f)$ into $(F'_0,\ k'_0,\ i',\ M'_0,\ f')$ to be a couple as follows: a field embedding $F_0 \to F'_0$ that induces an embedding $k_0 \to k'_0$ that is compatible with i and i', and an isomorphism $M'_0 \cong M_{0,k'_0}$ that is compatible with (f, f').

Clearly, for any field of definition of M as above any field embedding $F_0 \to F'_0$ makes F'_0 a field of definition of M (with $k'_0 = k_0$) and also gives a morphism of these fields of definition. Consequently, it is usually sufficient to specify F_0 only.

(2) Clearly, any function field is a finite extension of a rational extension of k. Moreover, since k is perfect by our convention, any function field over it is the function field of some smooth connected variety V/k; recall here that varieties over perfect fields are generically smooth.

Proposition 2.3.3. *Let $j, l \in \mathbb{Z}$ and $r \geq 0$. Then the following statements are valid.*

(1) *Let $N \in \operatorname{Obj} \mathrm{Chow}(k, R)$. Then*

$$h_{2j+l,j}(N_K, R) \cong DM_{gm}(K, R)(\widehat{N}_K, R\langle j \rangle[-l])$$

(see Definition 2.2.2(5)*) for any field extension K/k, where $\widehat{N}$ is the Poincaré dual of N (in* $\mathrm{Chow}(k, R) \subset DM_{gm}(k, R)$*).*

(2) *For any* $N \in \operatorname{Obj} \mathrm{Chow}^{\mathrm{eff}}(k, R)$ *and any field extension* K/k *we have*

$$h_{2j+l,j}(N_K\langle r\rangle, R) = \{0\}$$

if $j - r + l < 0$.

(3) *An object* N *of* $DM^{\mathrm{eff}}_{gm}(k, R)$ *(or* $DM^{\mathrm{eff}}_{-}(k, R)$*) belongs to* $DM^{\mathrm{eff}}_{-}(k, R)^{t^R_{hom} \le 0}$ *(see Remark* 2.1.1*) if and only if* $h_{l,0}(N_K, R) = \{0\}$ *for all* $l < 0$ *and all function fields* K/k.

Moreover, these conditions are equivalent to the vanishing of $h_{l-r,-r}(N_K, R)$ *for all* $l < 0$, $r \ge 0$, *and all function fields* K/k.

(4) *Any object either of* $DM^{\mathrm{eff}}_{gm}(k, R)$ *or of* $K^b(\mathrm{Chow}(k, R))$ *possesses an essentially finitely generated field of definition.*

Proof. (1) This is an immediate consequence of Poincaré duality for Voevodsky motives; see Theorem 5.23 of [Deg08].

(2) Obviously, it suffices to establish the statement for $N = \mathcal{M}_R(P)$, where P is as in the previous assertion; consequently, we will now treat this particular case. Next, recall that motivic cohomology of smooth varieties can be computed as the (co)homology of certain (Suslin or Bloch) cycle complexes; see Theorem 5.3.14 of [Kel17] (cf. Proposition 4.1.2(3) below). Therefore the group in question is a subquotient of a certain group of cycles of K^{perf}-dimension $j - r + l$. The result follows immediately.

(3) See the (probably, well-known) Proposition A.1(3) of [BoK20].

(4) This fact appears to be well-known; its proof can easily be obtained using *continuity* arguments as in Remark 1.3.3 of [Bon20a] (that relies on §4.3 of [CiD19]). □

Now let us prove some facts relating (complexes of) higher Chow groups over various base fields.

Our first statement is rather "classical" (cf. Lemma IA.3 of [Blo80] and §3 of [Via17]; one can also apply the more advanced formalism of [CiD15] to prove it).

Proposition 2.3.4. *Let* $j, l \in \mathbb{Z}$.

Fix an object (M^i) *of* $K^b(\mathrm{Chow}(k, R))$*; for a field of definition* F_0 *of* (M^i) *denote by* $G(F_0)$ *the zeroth homology of the complex* $h_{2j+l,j}(M^i_{F_0}, R)$ *(clearly,* G *is functorial with respect to morphisms of fields of definition for* (M^i)*; see Remark* 2.3.2(1)*).*

I. *The following statements are valid.*

(1) *Let* $F_0 \subset F_0'$ *be fields of definition for* M. *Then* $G(F_0')$ *is the (filtered) direct limit of* $G(K)$ *if we take* K *running through all finitely generated extensions of* F_0 *inside* F_0'*; here all these extensions as well as* F_0' *are endowed with the structure of fields of definition for* M *that "comes from* F_0*" (see Remark* 2.3.2(1) *once again).*

(2) *Let* F_1/k_0^1 *and* F_2/k_0^2 *be fields of definition for* M*; let* $s : F_1 \to F_2$ *be an embedding of fields such that* $(M^1_{0\,F_1})_{F_2} \cong M^2_{0\,F_2}$ *(yet we do not require* s *to extend to a morphism of fields of definition). Then* s *induces a homomorphism* $G(F_1) \to G(F_2)$ *that is an isomorphism if* $s(F_1) = F_2$, *and is injective if* F_1 *is algebraically closed.*

II. *Let* $R = \mathbb{Q}$. *Then the following conditions are equivalent.*

(1) $G(K) = \{0\}$ *for any function field* K/k.

(2) $G(F_0) = \{0\}$ *for some universal domain of definition for* M.

(3) $G(F_0) = \{0\}$ *for any algebraically closed field of definition for* M.
(4) $G(F_0) = \{0\}$ *for any field of definition for* M.

III. *All the statements above remain valid if we define* $G(K)$ *as* $h_{2j,j}(M_K, R)$ *for a fixed* $M \in \operatorname{Obj} DM_{gm}(k, R)$.

Proof. We note (for convenience) that we can pass to the Poincaré duals in all of these statements (see Proposition 2.3.3(1)). Thus one can express $G(K)$ in terms of motivic cohomology instead of motivic homology. We obviously do not have to track the indices involved.

I. Recall that the motivic cohomology of Chow motives over F_0 can be (functorially) computed using certain complexes whose components are expressed in terms of algebraic cycles in fixed F_0-varieties. This fact easily yields all our assertions except the (very) last injectivity one.

In order to verify the remaining statement we note that, for a (Voevodsky) motive N defined over a perfect field L, the motivic cohomology of $N_{L'}$ (for a perfect field extension L'/L) can be (functorially in N) expressed as the filtered direct limit of the corresponding cohomology of $N \otimes \mathcal{M}_L^R(V_a)$ for certain smooth varieties V_a over L. Next, if L is algebraically closed, then the $DM_{gm}(L, R)$-morphism $R \to \mathcal{M}_R(V_a)$ possesses a splitting given by any L-point of V_a. Hence the homomorphism in question is injective since it can be presented as the direct limit of a system of (split) injections.

One may also apply ("explicitly") the continuity arguments mentioned in the proof of Proposition 2.3.3(4) in these proofs.

II. The existence of trace maps for higher Chow groups (with respect to finite extensions of base fields; see Lemma 1.2 of [Via17]) yields the following: if F_0'/F_0 is an algebraic extension and $G(F_0') = \{0\}$, then $G(F_0) = \{0\}$ as well. Along with Proposition 2.3.3(4) and assertion I, this observation easily yields our claim.

III. Note that the motivic (co)homology of any Voevodsky motive can be computed using certain complexes of algebraic cycles. The existence of these complexes is immediate from (the R-module analogue of) Theorem 3.1.1 of [Bon09] (note that this result is valid for any p; this is a consequence of Proposition 5.3.12(iv) of [Kel17]). Hence the arguments above carry over to this setting without any difficulty. □

The following statement appears to be new; yet it will be somewhat less important for us below.

Proposition 2.3.5. *Once again, assume that* $j, l, r \in \mathbb{Z}$, $r > 0$, (M^i) *is an object of* $K^b(\mathrm{Chow}(k, R))$; *let* F_1 *and* F_2 *be function fields over* k. *Suppose that there exists a geometric* k-*valuation of rank* r *for* F_2 *such that the corresponding residue field is isomorphic to* F_1. *Then there exists a split injection of the complex* $h_{2j+l,j}(M^i_{F_1}, R)$ *into the complex* $h_{2j+l-r,j-r}(M^*_{F_2}, R)$.

Proof. Clearly it suffices to verify this statement in the case $j = 0$.

Once again, we apply Proposition 2.3.3(1) and reduce our assertion to the following statement: for a complex (N^i), where $N^i \in \operatorname{Obj} \mathrm{Chow}(k, R)$, there exists a split injection of the complex

$$(DM_{gm}(k, R)(N^*_{F_1{}^{perf}}, R[-l])) \text{ into } (DM_{gm}(k, R)(N^*_{F_2{}^{perf}}, R\langle r \rangle[-r - l])).$$

Note also that if the schemes $\operatorname{Spec} F_b$ (for $i=1,2$) are the inverse (filtered) limits of some systems of smooth varieties X_n^b/k (cf. Remark 2.3.2(2)) and $O \in \operatorname{Obj}\operatorname{Chow}(k,R)$, then

$$DM_{gm}(k,R)(O_{F_b{}^{perf}}, R[s]) \cong \varinjlim DM_{gm}(k,R)(\mathcal{M}_R(X_n^b)\otimes O, R[s])$$

for any $s\in\mathbb{Z}$; here we apply the well-known "continuity" of Chow groups similar to that discussed in the proof of Proposition 2.2.6(5) (cf. also Remark 2.2.3 and Proposition 8.1 of [CiD15]).

Hence the statement would be proved if we had a motivic category $\mathfrak{D}^R \supset DM_{gm}(k,R)$ that contains certain homotopy limits $\varprojlim \mathcal{M}_R(X_n^b)$ for $b=1,2$ (that can be denoted as $\mathcal{M}(\operatorname{Spec} K^b)$), is equipped with a bi-additive tensor product bifunctor $DM_{gm}(k,R)\times\mathfrak{D}^R \to \mathfrak{D}^R$ such that the groups $\mathfrak{D}^R((\varprojlim \mathcal{M}_R(X_n^b))\otimes O, R[s])$ are functorially isomorphic to $\varinjlim(\mathcal{M}_R(X_n^b)\otimes O, R[s])$, and such that there exists a split $\mathfrak{D}^R$-morphism $\varprojlim(\mathcal{M}_R(X_n^1))\langle r\rangle[-r] \to \varprojlim(\mathcal{M}_R(X_n^2))$.

Luckily, the results of previous papers yield the existence of $\mathfrak{D}^R$ having all these properties. Indeed, for $R=\mathbb{Z}$ a certain category of this sort was constructed in [Bon10b]. It has suffered from two drawbacks: it only contained $DM_{gm}^{\text{eff}}(k,\mathbb{Z})$ instead of $DM_{gm}(k,\mathbb{Z})$, and the splitting in question was established (see Corollary 4.2.2(2) of ibid.) only in the case where k is countable. Yet one can easily "correct" that category so that it would contain $DM_{gm}(k,R)$, and Proposition 5.2.6(8) of [Bon18b] implies that the desired splitting exists for any perfect k (see Remark 5.2.7(7) of ibid.). $\square$

Remark 2.3.6.

(1) Since a function field can be presented as a finite separable extension of $k(t_1,\ldots,t_d)$ (see Remark 2.3.2(2)), it is also a residue field for a (rank 1) geometric valuation of $k(t_1,t_2,\ldots,t_{d+1})$. Thus one may say that it suffices to compute these stalks at rational extensions of k only!

(2) One can also verify that $h_{2j+l-r,j-r}(M_K^*,R)$ contains (as a retract) the sum of any finite number of $h_{2j+l,j}(M_{k_m}^*,R)$, where k_m are residue fields for distinct geometric valuations of K of rank r. Hence the homology groups of $h_{2j-l-r,j-r}(M_K^*,R)$ can be quite huge. Consequently, we will not try to calculate them in general (at least, in the current paper; yet §5.1); we will rather be interested in their vanishing.

3. On Chow-weight homology of "general" motives

In this section we prove the central motivic results of this paper; their applications to (motives and cohomology with compact support of) varieties will be described later. The main results of this section are Theorems 3.2.1, 3.3.3, and 3.6.4, and Corollary 3.4.2, whereas the relation to cohomology is discussed in §3.5. Most of the results of this section will be illustrated by Theorems 4.2.1 and 4.2.3 below.

In §3.1 we introduce (using the weight complex functor) the main homology theories of this paper and prove several of their properties.

In §3.2 we relate Chow-weight homology to the c-effectivity of motives and their weights. A very particular case of these result yields: a cone of a morphism h of Chow motives is c-effective if and only if h induces isomorphisms on Chow groups of dimension less than c.

In §3.3 we generalize the aforementioned results to obtain equivalent criteria for the vanishing of Chow-weight homology in a certain "range" (we introduce the term "staircase set" for this purpose); we also note that the corresponding "decompositions" of motives can be assumed not to increase their dimension. We demonstrate the utility of our Theorem 3.3.3 by applying it to morphisms of Chow motives.

In §3.4 we prove that the properties of motives studied in the previous subsection can also be "detected" through higher Chow-weight homology. As a consequence, we relate the vanishing of Chow-weight homology of a motive M to that for its higher degree (zero-dimensional) motivic homology.

In §3.5 we relate the vanishing properties of the Chow-weight homology of M to the weight factors of the cohomology $H^*(M)$ (for various cohomology theories). The fact that "motivic effectivity" conditions imply the corresponding effectivity of the factors of the weight filtration on $H^*(M)$ is immediate from the general theory of weight spectral sequences. We also prove that a pair of (more or less) "standard" motivic conjectures gives the converse implication for singular cohomology (of motives with rational coefficients).

In §3.6 we study in detail the question when the higher $\mathbb{Q}$-linear Chow-weight homology of an "integral" motive M vanishes (using the results of [BoS18c]). In particular, we prove that if the Chow-weight homology (or motivic homology; see Corollary 3.6.5(II)) groups of M are torsion in higher degrees then their exponents are finite.

3.1. **Chow-weight homology: Definition and basic properties.** Let us define the main homology theories of this paper; see Definition 2.2.2(5) for the notation that we use here.

Definition 3.1.1. Let M be an object of $DM_{gm}(k,R)$.

(1) We write $t_R(M)$ for a choice of a weight complex for M; recall that one can assume t_R to be an exact functor $DM_{gm}(k,R) \to K^b(\mathrm{Chow}(k,R))$.

In the case $M \in \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,R)$ we will always assume that $t_R(M)$ is an object of $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$.

(2) Let $j,l,i \in \mathbb{Z}$; let K be a field extension of k.

For $t_R(M) = (M^s)$ we define the group $\mathrm{CWH}^i_j(M_K,R)$ (resp. $\mathrm{CWH}^i_j(M_K,l,R)$) as the 0-th homology of the complex $h_{2j,j}(M_K^{s+i},R)$ (resp. of $h_{2j+l,j}(M_K^{s+i},R)$) obtained from $t_R(M)$.[2] We will often omit R in this notation when its choice is clear.

Let us prove some basic properties of these functors.

Proposition 3.1.2. *Let l,i,j,K be as above, $r \geq 0$.*

(1) *Then $\mathrm{CWH}^i_j(-_K,l,R)$ yields a homological functor on $DM_{gm}^{\mathrm{eff}}(k,R)$ (that does not depend on any choices). Moreover, this functor factors through the base field change functor $DM_{gm}^{\mathrm{eff}}(k,R) \to DM_{gm}^{\mathrm{eff}}(K^{perf},R)$.*

(2) *Assume $r \geq j+l$. Then $\mathrm{CWH}^i_j(-_K,l,R)$ kills $DM_{gm}^{\mathrm{eff}}(k,R)\langle r+1\rangle$; consequently, $\mathrm{CWH}^i_j(-_K,l,R)$ induces a well-defined functor $DM_{gm}^r(k,R) \to Ab$ (see Definition 2.2.2(3)).*

[2]Consequently, $\mathrm{CWH}^i_j(-_K,R) = \mathrm{CWH}^0_j(-_K,R) \circ [i]$. Note that we do not follow the convention introduced in Definition 1.4.4(1) here; yet this should not cause any confusion since we write i as an upper index.

(3) *Suppose* $N \in DM^r_{gm}(k,R)_{w_{\mathrm{Chow}}\ge 0}$. *Then for any smooth projective connected variety* P/k *the group* $DM^r_{gm}(k,R)(l^r(\mathcal{M}_R(P)\langle r\rangle), N)$ *is isomorphic to*

$$\mathrm{CWH}^0_r(N_{k(P)}, R);$$

note that the latter group is well-defined according to the previous assertion.

(4) *Assume* $N \in DM^r_{gm}(k,R)_{w^r_{\mathrm{Chow}}\ge -n}$ *(see Corollary* 2.2.4(6) *for the notation) for some* $n \in \mathbb{Z}$. *Then* $\mathrm{CWH}^i_j(N_K, l) = \{0\}$ *for all* $i > n$, $j \le r-l$.

(5) *Assume* $0 \le m \le r$; *let* N *be an element of* $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\ge -i}$ *(resp. of* $DM^r_{gm}(k,R)_{w^r_{\mathrm{Chow}}\ge -i}$*) and assume* $\mathrm{CWH}^i_j(N_K) = \{0\}$ *for all* $0 \le j \le m$ *and all function fields* K/k. *Then for any fixed choice of a* $-i$*-weight decomposition* $w_{\mathrm{Chow}\le -i}N \xrightarrow{g} N \to w_{\mathrm{Chow}\ge 1-i}N$ *(resp.* $w^r_{\mathrm{Chow}\le -i}N \xrightarrow{g} N \to w^r_{\mathrm{Chow}\ge 1-i}N$*) of* N *(see Remark* 1.2.3(2)*) the morphism* $g[i]$ *can be factored through an object of* $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle m+1\rangle$ *(resp. through an image of an object of this sort in* $DM^r_{gm}(k,R)$*).*

Proof. (1) The first part of the assertion is just a particular case of Proposition 1.4.2(7). The second part follows immediately from the weight-exactness of this base field change functor (provided by part (4) of that proposition along with Lemma 2.1.3) along with Proposition 1.4.2(8).

(2) Recall that $DM^{\mathrm{eff}}_{gm}(k,R)\langle r\rangle$ is densely generated by $\mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle r\rangle$ (as a triangulated subcategory of $DM^{\mathrm{eff}}_{gm}(k,R)$). Hence the statement follows immediately from Proposition 2.3.3(2).

(3) By Proposition 2.2.6(6), $\mathrm{CWH}^0_r(N_{k(P)})$ is isomorphic to the zeroth homology of the complex $DM^r_{gm}(k,R)(l^r(\mathcal{M}_R(P)\langle r\rangle), N^*)$ (where N^* are the terms of a weight complex for N). Hence it remains to apply Corollary 1.4.6(1).

(4) Clearly, we can assume that the weight complex of N is concentrated in degrees at most n (see Proposition 1.4.2(2)). Next, recall that any object of $\underline{Hw}^r_{\mathrm{Chow}}$ is a retract of a one coming from $\mathrm{Chow}^{\mathrm{eff}}(k,R)(\subset DM^{\mathrm{eff}}_{gm}(k,R))$; see Proposition 1.3.2(3). Hence the statement follows from Proposition 2.3.3(2).

(5) Obviously, we can assume $i = 0$.

The motive $w_{\mathrm{Chow}\le 0}N$ belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}=0}$ (resp. $w^r_{\mathrm{Chow}\le 0}N \in DM^r_{gm}(k,R)_{w^r_{\mathrm{Chow}}=0}$); consequently, this motive is a retract of $\mathcal{M}_R(P)$ (resp. of $l^r(\mathcal{M}_R(P))$) for some $P \in \mathrm{SmPrVar}$.

It suffices to check the following for any $0 \le j \le m$ and $P^j \in \mathrm{SmPrVar}$: any morphism $g_j \in DM^{\mathrm{eff}}_{gm}(k,R)(\mathcal{M}_R(P^j)\langle j\rangle, N)$ (resp. $DM^r_{gm}(k,R)(l^r(\mathcal{M}_R(P^j)\langle j\rangle), N)$) can be factored through $\mathcal{M}_R(P^{j+1})\langle j+1\rangle$ (resp. through $l^r(\mathcal{M}_R(P^{j+1})\langle j+1\rangle)$) for some $P^{j+1} \in \mathrm{SmPrVar}$.

By Corollary 1.4.6(2) applied to l^j (resp. to the functor $l^j_r : DM^r_{gm}(k,R) \to DM^j_{gm}(k,R)$), to achieve the goal it suffices to verify that the image of g_j in $DM^j_{gm}(k,R)$ is 0. It remains to note that $l^j(g_i)$ is an element of

$$DM^j_{gm}(k,R)(l^j(\mathcal{M}_R(P^j)\langle j\rangle), l^j(N))$$

(resp. $DM^j_{gm}(k,R)(l^j(\mathcal{M}_R(P^j)\langle j\rangle), l^j_r(N))$), which is zero according to assertion (3) along with our assumptions on $\mathrm{CWH}^*_j(N_{k(P_j)})$. □

Remark 3.1.3. Recall that functors constructed by means of Proposition 1.4.2(7) are called pure ones. The reason for this is their relation to Deligne's purity of singular and étale cohomology; see Remark 2.1.3(3) of [Bon21]. It is easily seen (from Proposition 1.4.5(1); see Theorem 2.1.2 of ibid.) that a homological functor on $DM^{\mathrm{eff}}_{gm}(k,R)$ is pure with respect to w_{Chow} if and only if it annihilates $\mathrm{Chow}^{\mathrm{eff}}(k,R)[i]$ for all $i \neq 0$.

Other interesting functors that are pure with respect to Chow weight structures were considered in [KeS17] and [Bac17]. Note also that the "purification" of the zeroth homotopy functor on SH with respect to the *spherical* weight structure on it (see [Bon21, §4.2]) is isomorphic to the (zeroth) singular homology functor on this category; see Theorem 4.2.1(2) of loc. cit.

For some of the less important statements below we will also need the following assertions.

Lemma 3.1.4. *Let K be an extension of k, and $j, l \geq 0$.*

(1) *If $N \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k,R) \cap \mathrm{Obj}\, DM^{\mathrm{eff}}_{-}(k,R)^{t^R_{hom} \leq 0}$ (see Remark 2.1.1) and $i > j+l$, then $\mathrm{CWH}^i_j(N_K, l) = \{0\}$.*

(2) *If $N \in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} \geq 0}$ then $\mathrm{CWH}^0_j(N_K, R) \cong h_{2j,j}(N_K, R)$.*

Proof. (1) Clearly, $\mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k,R) \cap DM^{\mathrm{eff}}_{-}(k,R)^{t^R_{hom} \leq 0} = \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k,R) \cap DM^{\mathrm{eff}}(k,R)^{t^R_{hom} \leq 0}$ (see Remark 2.1.1).

Now, in [BoD17] the following statement was proved (see Theorem 2.4.3 and Example 2.3.5(1) of ibid.): $DM^{\mathrm{eff}}(k,R)^{t^R_{hom} \leq 0}$ is the smallest extension-closed subclass of $\mathrm{Obj}\, DM^{\mathrm{eff}}(k,R)$ that is closed with respect to coproducts and contains $\mathrm{Obj}\, \mathrm{Chow}^{\mathrm{eff}}(k,R)\langle a \rangle[b-a]$ for all $a, b \geq 0$.

Next recall that w_{Chow} can be extended (from $DM^{\mathrm{eff}}_{gm}(k,R)$) to $DM^{\mathrm{eff}}(k,R)$ in a way that "respects coproducts" (weight structures of this type are called *smashing*; see Theorem 3.2.3 of [Bon21] or Proposition 1.7(1) of [Bon18a]). Hence Chow-weight homology (as well as any other w_{Chow}-pure homology theory whose target is an AB4 abelian category) can be extended to a homological functor $DM^{\mathrm{eff}}(k,R) \to Ab$ that respects coproducts (see Proposition 2.3.2(6) of [Bon21]).

It suffices to verify the vanishing in question for N from $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle a \rangle[b-a]$ (for some $a, b \geq 0$). This follows from Proposition 2.3.3(2).

More detail for this argument can be found in the proof of [BoK20, Proposition 2.1.2(6)] (along with the pre-requisites to loc. cit.).

(2) Proposition 2.2.1(4) (combined with Proposition 1.4.2(8)) allows us to assume that $K = k$. Thus it remains to apply Corollary 1.4.6(1) (once again). □

3.2. **Relating Chow-weight homology to c-effectivity and weights.** Now we start proving the central results of this paper; consult §2.1, Proposition 2.2.1(1), and Definition 3.1.1 (along with Definition 2.2.2(5)) for the notation.

Theorem 3.2.1. *Let $M \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k,R)$, $c > 0$, $n \in \mathbb{Z}$.*

Then the following statements are valid.

(1) *M belongs to $DM^{\mathrm{eff}}_{gm}(k,R)\langle c \rangle$ (i.e., M is c-effective) if and only if*

$$\mathrm{CWH}^i_j(M_K) = \{0\}$$

for all $i \in \mathbb{Z}$, $0 \leq j < c$, and all function fields K/k.

(2) *More generally, $\mathrm{CWH}^i_j(M_K) = \{0\}$ for all $0 \leq j < c$, $n < i$, and all function fields K/k if and only if there exists a choice of $w_{\mathrm{Chow} \leq -n-1} M$ (see Remark 1.2.3(2)) that belongs to $DM^{\mathrm{eff}}_{gm}(k,R)\langle c \rangle$.*

(3) $\mathrm{CWH}^i_j(M_K) = \{0\}$ *for all* $j \geq 0$, $i > n$, *and all function fields* K/k, *if and only if* M *belongs to* $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} \geq -n}$.

Proof. (1) If M is an object of $DM^{\mathrm{eff}}_{gm}(k,R)\langle c\rangle$ then $\mathrm{CWH}^i_j(M_K) = \{0\}$ for all j, i, and K as in the assertion by Proposition 3.1.2(2).

Conversely, assume that M satisfies the corresponding Chow-weight homology vanishing assumptions. Since the weight structure w^{c-1}_{Chow} is bounded (see Corollary 2.2.4(6)), it suffices to prove that $l^{c-1}(M)$ belongs to $DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq r}$ for any $r \in \mathbb{Z}$. Hence this assertion reduces to the next one.

(2) Assume there exists a choice of $w_{\mathrm{Chow} \leq n-1}M$ that belongs to $DM^{\mathrm{eff}}_{gm}(k,R)\langle c\rangle$. Then the object $l^{c-1}(M)$ clearly belongs to $DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq -n}$. Hence the vanishing of Chow-weight homology groups in question is immediate from Proposition 3.1.2(2,4).

Conversely, assume that our Chow-weight homology vanishing assumptions are fulfilled. Clearly, there exists an integer q such that

$$l^{c-1}(M) \in DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq q}.$$

By Proposition 1.3.2(2), it suffices to verify the following: if $l^{c-1}(M)$ belongs to $DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq t}$ for some $t < -n$, then it belongs to $DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq t+1}$ as well.

Let us take a t-weight decomposition

$$w^{c-1}_{\mathrm{Chow} \leq t} l^{c-1}(M) \xrightarrow{g} l^{c-1}(M) \to w^{c-1}_{\mathrm{Chow} \geq t+1} l^{c-1}(M)$$

of $l^{c-1}(M)$. Proposition 3.1.2(5) implies $g = 0$. Hence $l^{c-1}(M)$ is a retract of an element of $DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq t+1}$; thus it belongs to $DM^{c-1}_{gm}(k,R)_{w^{c-1}_{\mathrm{Chow}} \geq t+1}$ itself.

(3) If M belongs to $DM_{gm}(k,R)_{w_{\mathrm{Chow}} \geq -n}$ then the previous assertion yields the vanishing of $\mathrm{CWH}^i_j(M_K) = \{0\}$ for all $j \geq 0$, $i > n$, and all function fields K/k.

Conversely, it suffices (similarly to the previous argument) to check the following: if M belongs to $DM_{gm}(k,R)_{w_{\mathrm{Chow}} \geq t}$ for some $t < -n$ then

$$M \in DM_{gm}(k,R)_{w_{\mathrm{Chow}} \geq t+1}.$$

Again, we can fix a t-weight decomposition $w_{\mathrm{Chow} \leq t}M \xrightarrow{g} M \to w_{\geq t+1}M$ and check that $g = 0$. Assume that $w_{\mathrm{Chow} \leq t}M[-t]$ is (a Chow motive) of dimension at most s for some $s \geq 0$. By Proposition 3.1.2(5), our Chow-weight homology assumptions yield that $g[-t]$ can be factored through $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle s+1\rangle$. Hence Proposition 2.2.6(3) implies that $g = 0$. □

Remark 3.2.2. We make some simple remarks.

(1) In the case $R = \mathbb{Q}$ Proposition 2.3.4(II) implies that, instead of checking whether the corresponding $\mathrm{CWH}^i_j(M_K) = \{0\}$ for all function fields K/k, it suffices to take K to be a single universal domain containing k; see Proposition 3.4.1(3) below.

Moreover, in all the statements of this paper where it is said $R = \mathbb{Q}$ (and no realizations of motives are mentioned) it suffices to assume that R is a $\mathbb{Q}$-algebra. This generalization may be relevant for studying motives similar to those considered in [Wil09].

(2) As a very particular case of the theorem, we obtain the following fact: for a morphism h of effective Chow motives the complex $\mathrm{Cone}(h)$ is c-effective

(i.e., it is homotopy equivalent to a cone of a morphism of c-effective Chow motives) if and only if h induces isomorphisms on the corresponding Chow groups of dimension less than c; cf. Remark 3.3.5 below. Another equivalent condition is that "h possesses an inverse modulo cycles supported in codimension c"; see Corollary 3.3.9 and Remark 3.3.10 below for more detail.

We will prove an extension of this equivalence statement in Corollary 3.3.9 below. Even for $R = \mathbb{Q}$ these particular cases of the theorem haven't been previously stated in the literature.

(3) The Chow-weight homology groups are rather difficult to calculate (and they tend to be huge; cf. Remark 2.3.6(2) and §5.1). Still they are somewhat easier to treat than the (ordinary) motivic homology groups. In particular, CWH^*_* can be (more or less) explicitly computed for any motive that belongs to the subcategory of $DM^{\mathrm{eff}}_{gm}(k,R)$ densely generated by the class $\cup_{j\geq 0}((d_{\leq 1}DM^{\mathrm{eff}}_{gm}(k,R))\langle j\rangle)$, whereas the 0-dimensional motivic homology is very difficult to compute already for $\mathbb{CP}^2$. We will say more on the comparison of Chow-weight homology with motivic one in §3.4 below.

(4) According to Proposition 1.3.2(2), the (equivalent) conditions of Theorem 3.2.1(2) are fulfilled if and only if M is an extension of an element of $(DM^{\mathrm{eff}}_{gm}(k,R))_{w_{\mathrm{Chow}}\geq -n}$ by an element of $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\leq -n-1}\langle c\rangle$; cf. Theorem 0.2(3).

3.3. **A generalization (in terms of staircase sets).** To generalize Theorem 3.2.1 we need the following technical definition.

Definition 3.3.1. Let $\mathcal{I}$ be a subset of $\mathbb{Z}\times[0,+\infty)$ (see §1.1).

We call it a *staircase* set if for any $(i,j)\in\mathcal{I}$ and $(i',j')\in\mathbb{Z}\times[0,+\infty)$ such that $i'\geq i$ and $j'\leq j$ we have $(i',j')\in\mathcal{I}$.

For $i\in\mathbb{Z}$ the minimum of $j\in[0,+\infty]$ such that $(i,j)\notin\mathcal{I}$ will be denoted by $a_{\mathcal{I},i}$.

Remark 3.3.2.

(1) Obviously, $\mathcal{I}\subset\mathbb{Z}\times[0,+\infty)$ is a staircase set if and only if it equals the union of the strips $\bigcup_{(i_0,j_0)\in\mathcal{I}}\mathcal{I}_{i_0,j_0}$, where $\mathcal{I}_{(i_0,j_0)}=[i_0,+\infty)\times[0,j_0]$ (see §1.1).

(2) Let us give a simple illustration for these sets. If $\mathcal{I}=\mathcal{I}_{(2,2)}$ then $\mathrm{CWH}^i_j(M_K) = \{0\}$ for all $(i,j)\in\mathcal{I}$ and all function fields K/k if and only if M is an extension of an element of $(DM^{\mathrm{eff}}_{gm}(k,R))_{w_{\mathrm{Chow}}\geq -1}$ by an element of $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\leq -2}\langle 2\rangle$; see Theorem 3.2.1(2).

Similarly, for $\mathcal{I}=\mathbb{Z}\times[0,c-1]$ the vanishing of the corresponding Chow-weight homology of a motive $M\in DM^{\mathrm{eff}}_{gm}(k,R)$ means that M is in $DM^{\mathrm{eff}}_{gm}(k,R)\langle c\rangle$ (see part (1) of the theorem). The vanishing for $\mathcal{I}=[n+1,+\infty)\times[0,+\infty)$ means that $M\in(DM^{\mathrm{eff}}_{gm}(k,R))_{w_{\mathrm{Chow}}\geq -n}$. Other relevant staircase sets are introduced in Definition 3.3.6 and Corollary 3.4.2 below; the picture in the latter corollary illustrates our term.

Now we prove a generalization of Theorem 3.2.1; consequently, the reader may consult §2.1, Proposition 2.2.1(1), and Definitions 3.1.1 and 2.2.2(5) for the notation used in the formulation.

Theorem 3.3.3. *Let $\mathcal{I}\subset\mathbb{Z}\times[0,+\infty)$, $M\in\operatorname{Obj}DM^{\mathrm{eff}}_{gm}(k,R)$. Then the following statements are valid.*

(1) *The vanishing of $\mathrm{CWH}^i_j(M_K)$ for all function fields K/k and all $(i,j)\in\mathcal{I}$ is equivalent to the same vanishing for all field extensions K/k.*

(2) *Suppose that $\mathcal{I}$ is a staircase set. Then the following conditions are equivalent.*

A. $\mathrm{CWH}^i_j(M_K)=\{0\}$ *for all function fields K/k and all $(i,j)\in\mathcal{I}$.*

B. *The object $l^j(M)$ (see Definition 2.2.2(3)) belongs to*

$$DM^j_{gm}(k,R)_{w^j_{\mathrm{Chow}}\geq -i+1}$$

whenever $(i,j)\in\mathcal{I}$.

C. *For any $i\in\mathbb{Z}$ there exists a choice of $w_{\mathrm{Chow}\leq -i}M$ (see (1.1)) that belongs to $DM^{\mathrm{eff}}_{gm}(k,R)\langle a_{\mathcal{I},i}\rangle$.*

D. *M belongs to the extension-closure of*

$$\cup_{i\in\mathbb{Z}}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)[-i]\langle a_{\mathcal{I},i}\rangle).^3$$

E. *There exists a choice of a weight complex (see §1.4) for M such that its i-th term is $j+1$-effective whenever $(i,j)\in\mathcal{I}$.*

(3) *For any staircase set $\mathcal{I}$ and $M\in DM^{\mathrm{eff}}_{gm}(k,R)_{[a,b]}$ (for some $a\leq b\in\mathbb{Z}$) the (equivalent) conditions of the previous assertion are fulfilled if and only if M belongs to the extension-closure of $\cup_{-b\leq i\leq -a}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)[-i]\langle a_{\mathcal{I},i}\rangle)$.*

Proof. Assertion (1) follows from Proposition 2.3.4 immediately.

(2), (3). We apply Remark 3.3.2(1). According to Theorem 3.2.1(2) (cf. also its proof), the vanishing of $\mathrm{CWH}^i_j(M_K)$ for all function fields K/k and $(i,j)\in\mathcal{I}_{(i_0,j_0)}$ is equivalent to $l^{j_0}(M)\in DM^{j_0}_{gm}(k,R)_{w^{j_0}_{\mathrm{Chow}}\geq -i_0+1}$. The combination of these equivalences for all $(i_0,j_0)\in\mathcal{I}$ yields the equivalence of Conditions A and B in assertion (2).

Next, Condition B implies Condition C for a fixed $i\in\mathbb{Z}$ if $a_{\mathcal{I},i}<+\infty$ according to Theorem 3.2.1(2) (since $(i,a_{\mathcal{I},i}-1)\in\mathcal{I}$; cf. also Proposition 4.2.1 of [BoS18c]). If $a_{\mathcal{I},i}=+\infty$ then one should apply Theorem 3.2.1(3) instead.

Now assume that M satisfies Condition C and belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{[a,b]}$ for some $a\leq b\in\mathbb{Z}$. Then M is also an object of $DM^{\mathrm{eff}}_{gm}(k,R)\langle a_{\mathcal{I},b}\rangle$ (see Remark 2.2.5(2)). Thus we can modify the choices of $w_{\mathrm{Chow}\leq -i}M$ coming from Condition C (for $-i\notin[a,b-1]$) by setting $w_{\mathrm{Chow}\leq -i}M=0$ for $-i<a$ and $w_{\mathrm{Chow}\leq -i}M=M$ for $-i\geq b$. Then the corresponding triangles (1.2) yield that (for the motives M^i coming from this choice of a Chow-weight Postnikov tower for M) we have $M^i\in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}=0}\langle a_{\mathcal{I},i}\rangle$ (see Remark 2.2.5(1)), and we obtain Condition E. Next, Proposition 1.4.2(5) yields that M belongs to the extension-closure of $\cup_{-b\leq i\leq -a}\mathrm{Chow}^{\mathrm{eff}}(k,R)[-i]\langle a_{\mathcal{I},i}\rangle$ (i.e., we have proved the corresponding implication from assertion (3); we clearly also obtain Condition D.

Finally, assume that $t_R(M)=(M^i)$ for M^i as in Condition E (i.e., $M^i\in\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle a_{\mathcal{I},i}\rangle$). Since (for any (i,j)) the group $\mathrm{CWH}^i_j(M_K)$ is a subquotient of $h_{2j,j}(M^i_K,R)$, and the latter group vanishes whenever $(i,j)\in\mathcal{I}$ (by Proposition 2.3.3(2)), we obtain Condition A.

This finishes the proof. □

[3]In this theorem we use the convention of Definition 2.2.2(4) in the case $a_{\mathcal{I},i}=+\infty$.

Let us also verify that one can "bound dimensions" in our theorem.

Proposition 3.3.4. *Assume that M is of dimension at most $r \geq 0$ (see Definition 2.2.2(2)) and that $\mathcal{I}$ is a staircase set. Then the (equivalent) conditions of Theorem 3.3.3(2) are also equivalent to the following modifications of Condition* C (resp. D): *there is a choice of $w_{\mathrm{Chow} \leq -i} M$ that belongs to $\mathrm{Obj}(d_{\leq r - a_{\mathcal{I},i}} DM^{\mathrm{eff}}_{gm}(k,R))\langle a_{\mathcal{I},i}\rangle$ (resp. M is in the extension-closure of $\cup_{i\in\mathbb{Z}}(\mathrm{Obj}\, d_{\leq r-a_{\mathcal{I},i}} \mathrm{Chow}^{\mathrm{eff}}(k,R)[-i]\langle a_{\mathcal{I},i}\rangle))$.*

Moreover, a similar modification can also be made in Theorem 3.3.3(3).

Proof. According to Proposition 2.2.6(4), it suffices to verify that in the conditions listed in Theorem 3.3.3(2,3) one may replace the classes $\mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k,R)\langle a_{\mathcal{I},i}\rangle$ and $\mathrm{Obj}\,\mathrm{Chow}^{\mathrm{eff}}(k,R)[-i]\langle a_{\mathcal{I},i}\rangle$ by their intersections with $\mathrm{Obj}\, d_{\leq r} DM^{\mathrm{eff}}_{gm}(k,R)$.

As can be easily seen from the proof of these two assertions, to establish the resulting statement it suffices to verify the corresponding versions of Theorem 3.2.1(2,3). The latter can be easily achieved via replacing the usage of Proposition 2.2.6(4) in their proofs (thus actually the corresponding modification should be made for Proposition 3.1.2(5)) by the application of Proposition 2.2.6(7). □

Remark 3.3.5. The reader can easily check that everywhere in the proofs of Theorems 3.2.1, 3.3.3, and Proposition 3.3.4 (and in the prerequisites to them) we could have replaced $DM^{\mathrm{eff}}_{gm}(k,R)$ by $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$. Certainly, then we would have to replace $DM^{j}_{gm}(k,R)$ by the localization

$$K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))/(K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))\langle j+1\rangle),$$

whereas the Chow weight structure for $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ is just the stupid weight structure mentioned in Remark 1.2.3(1). The main observation here is that the heart of the corresponding weight structure on this localization is equivalent to $\underline{Hw}^{j}_{\mathrm{Chow}}$ (see Proposition 1.3.2(3) above and Theorem 4.1 of [BoV20]); thus the corresponding version of Proposition 2.2.6(6) is valid.

The resulting statements may be said to be more general than their $DM^{\mathrm{eff}}_{gm}(k,R)$-versions since there can exist objects of $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ that cannot be presented as weight complexes of motives. Besides, these results are easier to understand for the readers that are not well-acquainted with Voevodsky motives. Their disadvantage is that they hardly can be used for controlling "substantially mixed" motivic phenomena; this includes motivic homology (cf. Corollary 3.4.2 below).

We will apply the $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$-version of Theorem 3.3.3 to complexes of length 1. Note that we could have considered these complexes as objects of $DM^{\mathrm{eff}}_{gm}(k,R)$ (see Remark 1.4.3(1)); yet looking at $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ instead makes our argument somewhat "more elementary".

Now we consider two relevant particular cases of our theorem, and deduce a nice general corollary from it.

We will look at a certain filtration on the class $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\geq 0}$ (each of whose steps contains $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\geq 1}$).

Definition 3.3.6. For any $c \geq 0$ we will use the notation $DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c\rangle}_{\geq 0}$ for the $DM^{\mathrm{eff}}_{gm}(k,R)$-envelope (see §1.1) of the set $(\cup_{i>0}\mathrm{Chow}^{\mathrm{eff}}(k,R)[i])\cup \mathrm{Chow}^{\mathrm{eff}}(k,R)\langle c\rangle$.

Respectively (cf. Corollary 3.3.7(I)) we write $\mathcal{I}^{\langle c\rangle}_0$ for the staircase set $[1,+\infty)\times[0,+\infty)\cup\{0\}\times[0,c-1]$.

Corollary 3.3.7.

I. *For $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)$ and $c \geq 0$ the following conditions are equivalent.*
   (1) *M belongs to $DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c\rangle}_{\geq 0}$.*
   (2) *$\mathrm{CWH}^i_j(M_K) = \{0\}$ for all function fields K/k and $(i,j) \in \mathcal{I}^{\langle c\rangle}_0$.*
   (3) *M is an extension of an element of $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\geq 1}$ by an object of $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle c\rangle$.*
   (4) *M belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\geq 0}$ and $h_{2j,j}(M_K) = \{0\}$ (see Definition 2.2.2(5) for this notation) for all function fields K/k and $0 \leq j < c$.*

II. *If $c_1, c_2 \geq 0$ then $DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c_1\rangle}_{\geq 0} \otimes DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c_2\rangle}_{\geq 0} \subset DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c_1+c_2\rangle}_{\geq 0}$.*

III. *Assume that $\mathcal{I}_j$ are staircase sets for j running through some index set J. Then for a fixed M the (equivalent) conditions of Theorem 3.3.3(2) are fulfilled for $\mathcal{I} = \mathcal{I}_j$ (for all $j \in J$) if and only if they are fulfilled for $\mathcal{I} = \cup_j \mathcal{I}_j$.*

Proof. I. The equivalence of conditions (I1) and (I2) is immediate from Theorem 3.3.3(2) (see conditions (2)(A) and (2)(D) of the theorem). Furthermore, these conditions are equivalent to the assumption that we can take $w_{\mathrm{Chow}\leq -1}M = 0$ and $w_{\mathrm{Chow}\leq 0}M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)\langle c\rangle$. Thus M belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\geq 0}$; hence Proposition 1.2.4(6) implies that the aforementioned choice of $w_{\mathrm{Chow}\leq 0}M$ belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}=0} \cap \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)\langle c\rangle = DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}=0}\langle c\rangle$ (see Corollary 2.2.4(1)). Therefore the corresponding choice of weight decomposition of M gives condition (I3) for M. Next, condition (I3) clearly implies condition (I1).

Now, we have just checked that M belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\geq 0}$ whenever it belongs to $DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c\rangle}_{\geq 0}$. Thus $\mathrm{CWH}^0_j(M_K) = h_{2j,j}(M_K)$ for all K/k and $j \geq 0$ (see Lemma 3.1.4(2)); hence conditions (I1) and (I2) together imply condition (I4). Conversely, if condition (I4) is fulfilled then $\mathrm{CWH}^i_j(M_K) = \{0\}$ for all K/k and all $(i,j) \in [1,+\infty) \times [0,+\infty)$ according to Theorem 3.2.1(3) and it remains to apply Lemma 3.1.4(2) once again to obtain condition (I4).

II. Obvious from our definitions.

III. Obviously, $\cup_{j\in J}\mathcal{I}_j$ is a staircase set. Thus it suffices to note that the equivalence statement in question is obviously fulfilled for condition A in Theorem 3.3.3(2). $\square$

Remark 3.3.8. Part III of our Corollary says that the intersections of subclasses of $\operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)$ corresponding to the staircase sets $\mathcal{I}_j$ is "as small as possible". This statement appears to be interesting and quite non-trivial if one describes these subclasses using condition D in Theorem 3.3.3(2). The authors have no idea how to prove it avoiding our results.

Next we apply Remark 3.3.5 to cones of morphisms of Chow motives.

Corollary 3.3.9. *Let $h : N \to O$ be a $\mathrm{Chow}^{\mathrm{eff}}(k,R)$-morphism and $0 \leq r_1 \leq r_2 \in \mathbb{Z}$. Then the following conditions are equivalent.*

(1) *$h_{2j,j}(-_K,R)(h)$ is a bijection for $j \in [0, r_1-1]$ and is a surjection for $j \in [r_1, r_2-1]$ for all function fields K/k.*
(2) *The complex $N \to O$ is homotopy equivalent to a complex $N'\langle r_1\rangle \to O'\langle r_2\rangle$ for some $N', O' \in \operatorname{Obj} \mathrm{Chow}^{\mathrm{eff}}(k,R)$.*

(3) *There exists* $h' \in \mathrm{Chow}^{\mathrm{eff}}(k,R)(O,N)$ *such that the morphism* $\mathrm{id}_O - h\circ h'$ *factors through* $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle r_2\rangle$, *and* $\mathrm{id}_N - h'\circ h$ *factors through* $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle r_1\rangle$.

Proof. (1) $\iff$ (2). We take $M = \mathrm{Cone}\, h \in \mathrm{Obj}\, K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ (or in $DM^{\mathrm{eff}}_{gm}(k,R)$; we put N in degree -1 and put O in degree 0), and consider the index set $\mathcal{I} = [-1,+\infty)\times[0,r_1-1]\cup[0,+\infty)\times[r_1,r_2-1]$ (see §1.1).

We immediately obtain the equivalence of our condition (1) to the vanishing of $\mathrm{CWH}^i_j(M_K)$ for $i\in\mathcal{I}$. Combining the equivalence of Conditions A and D in Theorem 3.3.3(2) (in the version mentioned in Remark 3.3.5) with Remark 1.3.3(1), we obtain the result.

(2) $\Longrightarrow$ (3). We have $l^{r_2-1}(M) \cong l^{r_2-1}(N'\langle r_1\rangle[1])$. Next, this isomorphism clearly gives a similar isomorphism in the category $K^b(\underline{Hw}_{\mathrm{Chow}^{r_2-1}})$. Hence M (considered as a $\underline{Hw}_{\mathrm{Chow}^{r_2-1}}$-complex) is homotopy equivalent to $N'\langle r_1\rangle[1]$; denote the corresponding morphisms $M\to N'\langle r_1\rangle[1]\to M$ by f and g, respectively. Since id_M is $\underline{Hw}_{\mathrm{Chow}^{r_2-1}}$-homotopic to $g\circ f$, there exists $h''\in \underline{Hw}_{\mathrm{Chow}^{r_2-1}}(O,N)$ such that $\mathrm{id}_N - g\circ f = h''\circ h$ and $h\circ h'' = \mathrm{id}_O$. Lifting h'' to a morphism $h'\in \mathrm{Chow}^{\mathrm{eff}}(k,R)(O,N)$ (see Proposition 1.3.2(3)), we obtain the desired implication.

(3) $\Longrightarrow$ (1). Arguing as above, we see that in the category $K^b(\underline{Hw}_{\mathrm{Chow}^{r_2-1}})$ the morphism id_M factors through an object of $\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle r_1\rangle[1]$. The desired Chow-weight homology vanishing conditions follow immediately (cf. the proof of Theorem 3.2.1(2)). □

Remark 3.3.10.

(1) If $N = \mathcal{M}_R(Q)$ and $O=\mathcal{M}_R(P)$ for some $P,Q\in\mathrm{SmPrVar}$ then condition (3) of the corollary can be easily translated into the following assumption: the cycle $\mathrm{id}_O - h\circ h'$ in $P\times P$ (here clearly id_O is represented by the diagonal) is rationally equivalent to a cycle supported on $P'\times P$, and $\mathrm{id}_N - h'\circ h$ is rationally equivalent to a cycle supported on $Q'\times Q$, where $P'\subset P$ and $Q'\subset Q$ are some closed subvarieties of codimensions r_2 and r_1, respectively (see Proposition 2.2.6(1)–(3) and its proof).

Moreover, if h comes from a morphism $Q\to P$ then the cycle class $h\circ h'$ is clearly supported on the product of P by the image of h.

(2) Assume that M belongs to $d_{\le m}K^b(\mathrm{Chow}^{\mathrm{eff}}(k,R))$ (for some $m\ge 0$; this is certainly the case if N and O are of dimension at most m). Then $\mathrm{CWH}^i_j(M_K) = \{0\}$ for j greater than m (and all $i\in\mathbb{Z}$). Thus if r_2 is greater than m then our result yields that h splits; if $r_1>m$ then h is an isomorphism. The first of these observations generalizes Theorem 3.18 of [Via17] (where the case $R=\mathbb{Q}$ was considered).

3.4. **Higher Chow-weight homology criteria and motivic homology.** Now we invoke Proposition 2.3.5.

Proposition 3.4.1. *For a subset* $\mathcal{I}$ *of* $\mathbb{Z}\times[0,+\infty)$ *consider the following assumptions on an object* M *of* $DM^{\mathrm{eff}}_{gm}(k,R)$.

(1) *For a function* $f_M:\mathcal{I}\to[0,+\infty)$ *we have* $\mathrm{CWH}^i_{j-f_M(i,j)}(M_K, f_M(i,j), R) = \{0\}$ *for all* $(i,j)\in\mathcal{I}$ *and all function fields* K/k.

(2) $\mathrm{CWH}^i_j(M_K,R)=\{0\}$ *for all* $(i,j)\in\mathcal{I}$ *and all function fields* K/k.

(3) *For all rational extensions* K/k *and all* $(i,j)\in\mathcal{I}$ *we have* $\mathrm{CWH}^i_{j-1}(M_K,1) = \{0\}$.

(4) $\mathrm{CWH}^i_0(M_K, j) = \{0\}$ *for all* $(i,j) \in \mathcal{I}$ *and all function fields* K/k.
(5) $\mathrm{CWH}^i_a(M_K, j-a) = \{0\}$ *for all* $(i,j) \in \mathcal{I}$, $a \in \mathbb{Z}$, *and all field extensions* K/k.

Then the following statements are valid.

(1) *Condition* (5) *implies conditions* (4) *and* (3), *either of the latter two conditions implies condition* (2), *whereas the first two conditions are equivalent.*
(2) *Suppose that* $\mathcal{I}$ *is a staircase set (in the sense of Definition* 3.3.1*). Then our conditions* (1)–(5) *are equivalent.*
(3) *Assume* $R = \mathbb{Q}$. *Then our conditions are also equivalent to the vanishing of* $\mathrm{CWH}^i_j(M_{K_0})$ *for a single universal domain* K_0 *containing* k *and all* $(i,j) \in \mathcal{I}$.

Proof. (1) Clearly, condition (5) is the strongest of the five, whereas condition (1) follows from condition (2) and 4. The remaining implications are given by Proposition 2.3.5 (see also Remark 2.3.6(1)).

(2) Since the first two conditions are equivalent, it suffices to verify that condition (2) implies condition (5).

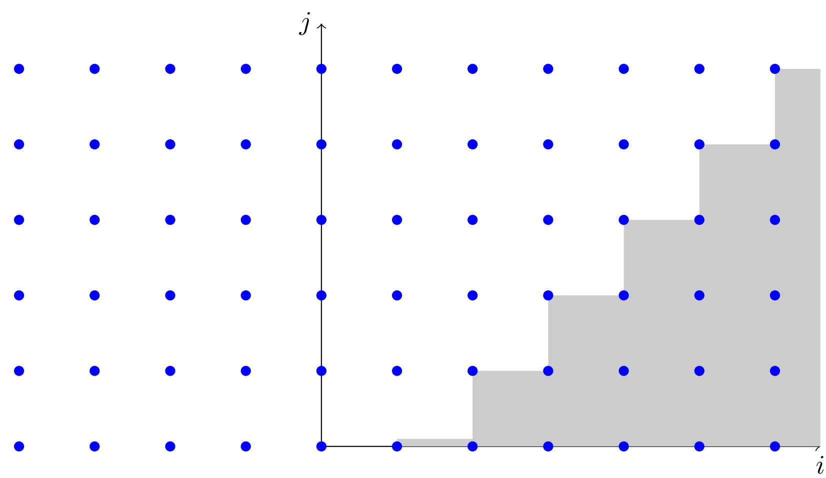
By Theorem 3.3.3(2), M satisfies Condition D of this theorem. Hence Proposition 3.1.2(4) yields the implication in question (cf. the proof of Theorem 3.3.3(2), D $\Longrightarrow$ A).

(3) This is an easy combination of assertion (2) with Proposition 2.3.4. □

Now we describe an interesting particular case of the proposition; recall that the homotopy t-structure t^R_{hom} was mentioned in Remark 2.1.1.

Corollary 3.4.2. *Let* $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)$. *Then the following conditions are equivalent.*

(1) M *belongs to* $DM^{\mathrm{eff}}_{-}(k,R)^{t^R_{hom} \le 0}$ $(= DM^{\mathrm{eff}}(k,R)^{t^R_{hom} \le 0}$; *one may say that* M *is motivically connective).*
(2) $h_{l,0}(M_K, R) = \{0\}$ *for all* $l < 0$ *and all function fields* K/k.
(3) *Conditions* (1)–(5) *of the previous proposition for* $\mathcal{I} = \{(i,j) : \ i > j \ge 0\}$ *are fulfilled (note that it suffices to verify only one of these conditions); the points of* $\mathcal{I}$ *are marked in grey on the following picture:*



(4) *M belongs to the extension-closure E of* $(\cup_{a>0} DM^{\text{eff}}_{gm}(k,R)_{w_{\text{Chow}}=-a}\langle a\rangle) \cup DM^{\text{eff}}_{gm}(k,R)_{w_{\text{Chow}}\geq 0}$ *(in* $\operatorname{Obj} DM^{\text{eff}}_{gm}(k,R)$*).*

Proof. The first condition is equivalent to the second one by Proposition 2.3.3(3). (Each of) these two conditions also imply the third condition (i.e., all of the equivalent conditions from Proposition 3.4.1) by Lemma 3.1.4(1). Next, our condition (2) is the corresponding case of condition (2) of Proposition 3.4.1. Hence it yields our condition (4) by Theorem 3.3.3(2) (see Condition D in that theorem; note that $a_{\mathcal{I},i}$ for $i \in \mathbb{Z}$ equals $\max(i,0)$ in this case).

Finally, our assumption (4) implies assumption (1) since for any $a \geq 0$ the classes $DM^{\text{eff}}_{gm}(k,R)_{w_{\text{Chow}}=-a}\langle a\rangle$ and $DM^{\text{eff}}_{gm}(k,R)_{w_{\text{Chow}}=a}$ lie in $DM^{\text{eff}}_{-}(k,R)^{t^R_{hom}\leq 0}$ (cf. the proof of Lemma 3.1.4(1)). □

3.5. **Relation of effectivity conditions to cohomology.** Now we relate our effectivity conditions on motives to the properties of Chow-weight filtrations and spectral sequences $T_{w_{\text{Chow}}}(H,M)$.

Proposition 3.5.1. *Let H be a cohomological functor from $DM^{\text{eff}}_{gm}(k,R)$ into an abelian category $\underline{A}$, M is an object of $DM^{\text{eff}}_{gm}(k,R)$, and $l,m \in \mathbb{Z}$.*

(1) *Then $(Gr^{-l}_W H^{m-l})(M)$ (see Definition* 1.4.4(2)*) is a subquotient of $E_2^{-l,m}T(M)$ for $T(M) = T_{w_{\text{Chow}}}(H,M)$, and isomorphic to it if $T(M)$ degenerates at E_2.*

(2) *Assume that M satisfies the equivalent conditions of Theorem* 3.3.3(2) *(for some staircase set $\mathcal{I}$; see Definition* 3.3.1*). Then $E_2^{-l,m}T(M)$ and $(Gr^{-l}_W H^{m-l})(M)$ are subquotients of $H^m(\mathcal{M}_R(P)\langle a_{\mathcal{I},l}\rangle)$ for some $P \in$* SmPrVar *whenever $a_{\mathcal{I},l} < +\infty$; these two objects vanish if $a_{\mathcal{I},l} = +\infty$.*

Moreover, if M is of dimension at most $r \in \mathbb{Z}$ (see Definition 2.2.2(2)*) then we can assume here that* $\dim P \leq r - a_{\mathcal{I},l}$.

Proof. (1) Immediate from Proposition 1.4.5(2).

(2) According to Theorem 3.3.3(2), we may assume that the lth term M^l of $t(M)$ belongs to $\operatorname{Obj} \operatorname{Chow}^{\text{eff}}(k,R)\langle a_{\mathcal{I},l}\rangle$ for the first part of the assertion and to $\operatorname{Obj}(d_{\leq r-a_{\mathcal{I},l}} \operatorname{Chow}^{\text{eff}}(k,R))\langle a_{\mathcal{I},l}\rangle$ for its "moreover" part (recall that this means $M^l = 0$ if $a_{\mathcal{I},l} = +\infty$). Thus it remains to apply assertion (1). □

Remark 3.5.2.

(1) Clearly, here and in Theorem 3.5.4 and Proposition 3.5.5 below one may consider homology instead of cohomology; see Proposition 1.4.5(1). We chose to concentrate on cohomology here due to the occurrence of cohomology with compact support in §4.

(2) We obtain that the study of the weight filtration on the (co)homology of M can yield the non-vanishing of certain Chow-weight and motivic homology groups (see Corollary 3.4.2 for the latter); cf. Theorem 3.5.4 below. This is quite remarkable since the corresponding cycle class maps (see Remark 5.1.3 of [BoS14]) are far from being surjective in general.

Let us now discuss concrete (Weil) cohomology theories.

We need some definitions.

Definition 3.5.3. Let $c \in [0, +\infty]$, ℓ be a prime distinct from p, and denote the absolute Galois group of k by G.

(1) Then we say that a mixed Hodge structure V (we will consider $\mathbb{Q}$-linear Hodge structures only in this paper; thus one should take $R = \mathbb{Q}$ in Definition 3.1 of [PeS08]) is c-effective and write $V \in \operatorname{Obj} MHS^c_{eff}$ whenever either $c \in \mathbb{Z}$ and $F^c V_{\mathbb{C}} = V_{\mathbb{C}}$ or if $c = +\infty$ and $V = 0$.

(2) Let k be an essentially finitely generated field (see Definition 2.3.1(1)). Then we will say that a (finite dimensional) mixed $\mathbb{Q}_\ell$-Galois representation V (certainly, V is a finite dimensional space over $\mathbb{Q}_\ell$ endowed with an action of G) is c-effective whenever either $c \in \mathbb{Z}$ and any (geometric) Frobenius eigenvalue coming from a residue field isomorphic to $\mathbb{F}_q$ (see Example 6.8 of [Jan90] and §1.2 of [Del80]; cf. the proof of Proposition 4.2.4(1) below) is divisible by $q^{c/2}$ as an algebraic integer, or if $c = +\infty$ and $V = 0$.

(3) For V of any of these two types and $m \in \mathbb{Z}$ we write $W_{Dm}V$ for the mth step of the (Deligne's) weight filtration, and $Gr^{W_D}_m V = W_{Dm}V/W_{Dm-1}V$.

(4) For $\ell \neq p$ we write $H_{et,\mathbb{Q}_\ell}$ for the restriction to $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})$ of the functor $H^0_{et}(-_{k^{alg}}, \mathbb{Q}_\ell) : DM_{gm}(k,\mathbb{Q})^{op} \to \mathbb{Q}_\ell[G] - \mathrm{Mod}$ of the zeroth 'etale $\mathbb{Q}_\ell$-cohomology (of $-_{k^{alg}}$). Here we define $H^0_{et}(-_{k^{alg}}, \mathbb{Q}_\ell)$ as the composition of the exact realization functor $RH^{et}(-_{k^{alg}}, \mathbb{Q}_\ell) : DM_{gm}(k,\mathbb{Q}) \to D^b(\mathbb{Q}_\ell[G]-$ Mod) provided by Theorem 7.2.24 and Proposition 7.2.21 of [CiD16] with the Poincare duality on $DM_{gm}(k,\mathbb{Q})$ and the zeroth homology functor on $D^b(\mathbb{Q}_\ell[G] - \mathrm{Mod})$.

Moreover, if k is a subfield of $\mathbb{C}$ then we write $H = H_{\mathrm{sing}} : DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{op} \to MHS_{eff}$ for the (zeroth) singular cohomology functor provided by Theorem 2.3.3 of [Hub00].

Theorem 3.5.4. *Assume* $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})$ *and* $l, m \in \mathbb{Z}$.

Moreover, suppose that either k *is a subfield of* $\mathbb{C}$ *and* $H = H_{\mathrm{sing}}$ *or that* k *is an essentially finitely generated field,* $\operatorname{char} k \neq \ell$, *and* $H = H_{et,\mathbb{Q}_\ell}$.

Then the following statements are valid.

(1) *The spectral sequence* $T(M) = T_{w_{\mathrm{Chow}}}(H, M)$ *degenerates at* E_2.

(2) *The subobject* $(W^l H^m)(M) \subset H^m(M)$ *equals* $W_{Dm-l}H^m(M)$ *and*

$$(Gr^l_W H^m)(M) = Gr^{W_D}_{m-l} H^m(M) \cong E_2^{l,m-l}T(M).$$

Here $(Gr^l_W H^m)(M) = (W^l H^m)(M)/(W^{l-1}H^m)$ *(note here that Proposition* 3.5.1 *easily implies that the values of* $H_{et,\mathbb{Q}_\ell}$ *are mixed* $\mathbb{Q}_\ell$*-Galois representations in our case).*

(3) *Consequently, if* M *satisfies the equivalent conditions of Theorem* 3.3.3(2) *(for some staircase set* $\mathcal{I}$*) then* $Gr^{W_D}_{m+l}H^m(M)$ *and*

$$H^m(M)/W_{Dm+l-1}H^m(M)$$

are $a_{ii,l}$*-effective.*

Thus if M *belongs to* $DM^{\mathrm{eff}}_{gm}(k,R)^{\langle c\rangle}_{\geq 0}$ *(see Definition* 3.3.6*) then* $H^m(M) = W_{Dm}H^m(M)$ *and* $H^m(M)/W_{Dm-1}H^m(M)$ *is c-effective.*

Proof. (1), (2). This is a standard weight argument. Recall that effective Chow motives are retracts of motives of smooth projective varieties, and that the object $H^q(\mathcal{M}_R(P))$ is (pure) of Deligne weight q for both of these cohomology theories, any $P \in \mathrm{SmPrVar}$, and $q \in \mathbb{Z}$ (cf. Proposition 4.1.8 below). Hence the object $E^{sq}_r T(M)$

is of Deligne weight q in both cases, for any $r > 0$ and $s, q \in \mathbb{Z}$. Since there are no morphisms between objects of distinct weights, we obtain the degeneration at E_2 (compare the weights of the domains and the targets of boundaries). Moreover, assertion (2) follows the definition of convergence of spectral sequences easily.

(3) Proposition 3.5.1(2) implies that the object $Gr_r^{W_D} H^m(M)$ is $a_{\mathcal{I},r-m}$-effective for any $r \in \mathbb{Z}$ (note here that the conventions of $+\infty$-effectivity in Definitions 2.2.2(4) and 3.5.3(1, 2) are compatible). Since $\mathcal{I}$ is a staircase set, $a_{\mathcal{I},s} \geq a_{\mathcal{I},l}$ if $s \geq l$. Thus the objects $Gr_r^{W_D} H^m(M)$ are $a_{\mathcal{I},l}$-effective if $r \geq m+l$. Since weight filtrations are bounded both on mixed Hodge structures and on mixed Galois representation, and the $a_{\mathcal{I},l}$-effective subcategories are extension-closed in the corresponding "mixed" categories, we obtain the first part of the assertion.

Lastly, it remains to recall that for the set $\mathcal{I}_0^{\langle c\rangle}$ in Definition 3.3.6 we have $a_{\mathcal{I}_0^{\langle c\rangle},l} = +\infty$ if $l > 0$ and $a_{\mathcal{I}_0^{\langle c\rangle},0} = c$. $\square$

Now we will study the question whether the c-effectivity restrictions on $H^*(M)$ as in Theorem 3.5.4(3) are equivalent to the conditions of Theorem 3.3.3(2).

Proposition 3.5.5. *Assume $k \subset \mathbb{C}$ and that the following conjectures hold.*

(A) *The Hodge conjecture.*

(B) *Any morphism of Chow motives (over $\mathbb{C}$) that induces an isomorphism on their singular cohomology is an isomorphism.*

Suppose also that for some staircase set $\mathcal{I}$ and an object M of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})$ one of the following conditions is fulfilled: either for all $m, l \in \mathbb{Z}$ the Hodge structures $H^m(M)/W_{D\,m+l-1}H^m(M)$ is $a_{\mathcal{I},l}$-effective, or $Gr_{m+1}^{W_D} H^m(M)$ is so (for all $m, l \in \mathbb{Z}$). Then the motive M satisfies the (equivalent) conditions of Theorem 3.3.3(2) (cf. Theorem 3.5.4(3)).

Proof. Since $\mathcal{I}$ is a staircase set, our two assumptions on M are easily seen to be equivalent (cf. the proof of Theorem 3.5.4(3)).

By the virtue of Theorem 3.3.3(2), it suffices to verify that M belongs to the extension-closure of $\cup_{i\in\mathbb{Z}}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)[-i]\langle a_{\mathcal{I},i}\rangle)$. So we fix certain $(i,j) \in \mathcal{I}$ and argue similarly to the proof of [Bon09, Proposition 7.4.2]. We choose the smallest $n \in \mathbb{Z}$ such that $l^j(M) \in DM_{gm}^j(k,\mathbb{Q})_{w^j_{\mathrm{Chow}} \geq -n}$. We should check that $n < i$.

Assume that the converse holds (i.e. $n \geq i$). Applying Proposition 1.3.2(2) we obtain that M is an extension of an element of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \geq -n}$ by that of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \leq -n-1}\langle j+1\rangle$. According to Proposition 1.4.2(4), this gives a choice of a weight complex $t(M) = (M^s)$ of M such that

$$M^s \in \operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle j+1\rangle$$

for $s > n$. Moreover, we can assume that $M^n = \mathcal{M}_{\mathbb{Q}}(P)$ for some $P \in \mathrm{SmPrVar}$ (since one can add a summand of the form $\dots \to 0 \to N \xrightarrow{id_N} N \to 0 \to \dots$ to $t(M)$, with N placed in degrees $n-1$ and n).

Next, recall that for any $q \in \mathbb{Z}$ we have

$$E_2^{-n,q}T(M) \cong \operatorname{Ker}(H^q_{\mathrm{sing}}(d_M^{n-1}))/\operatorname{Coker}(H^q_{\mathrm{sing}}(d_M^n));$$

here $d_M^* : M^* \to M^{*+1}$ are the boundaries of $t(M)$. Theorem 3.5.4(2) implies $E_2^{-n,q}T(M) \cong Gr_q^{W_D} H^{q-n}(M)$, so it is $j+1$-effective by our assumptions; recall that $j+1 \le a_{\mathcal{I},l} \le a_{\mathcal{I},n}$. Since the motive M^{n+1} is $j+1$-effective, we obtain that the Hodge structure $\operatorname{Ker}(H^q_{\mathrm{sing}}(d_M^{a-1}))$ is $j+1$-effective as well.

Now we need a more or less "standard" Hodge-theoretic argument to obtain a certain motivic splitting.

Our assumption A implies that the generalized Hodge conjecture (see Conjecture 7.5. of [PeS08]) is fulfilled for P (such that $M^a = \mathcal{M}_{\mathbb{Q}}(P)$); see Corollary 7.9 of [PeS08]. Hence there exists an open subvariety U of P such that the variety $Z = P \setminus U$ is of codimension more than j in P, and $\operatorname{Ker}(H^q_{\mathrm{sing}}(M^n) \to H^q_{\mathrm{sing}}(M^{n-1}))$ is supported on Z for all $q \ge 0$, that is, $\operatorname{Ker}(H^q_{\mathrm{sing}}(d_M^{n-1})) \subset \operatorname{Ker}(H^q_{\mathrm{sing}}(P) \to H^q_{\mathrm{sing}}(U))$. Now, the motive $C = \operatorname{Cone}(\mathcal{M}_{\mathbb{Q}}(U) \to \mathcal{M}_{\mathbb{Q}}(P))$ belongs to $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \le 0}\langle j+1\rangle$ according to Corollary 2.2.4(4). Next, there exists a choice of $C' = w_{\mathrm{Chow} \le 0} C$ that belongs to $\operatorname{Obj}\operatorname{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle j+1\rangle$ (see part (1) of the corollary). Since the morphism $\mathcal{M}_{\mathbb{Q}}(P) \to C$ factors through C' (see Proposition 1.2.4(9)), we obtain that $\operatorname{Ker}(H^q_{\mathrm{sing}}(d_M^{n-1})) \subset \operatorname{Im}(H^q_{\mathrm{sing}}(h))$ for some morphism $h \in \operatorname{Chow}^{\mathrm{eff}}(k,\mathbb{Q})(M^n, C')$ and all $q \ge 0$.

Next, recall that the category of polarizable pure Hodge structures is semi-simple (here one can either consider the direct sum of the corresponding categories for all weights $q \ge 0$ or treat the weights separately). Since the Hodge conjecture implies that any morphism between (the "total") H_{sing}-cohomology of Chow motives lifts to a morphism of these motives, we obtain the existence of a morphism $h' \in \operatorname{Chow}^{\mathrm{eff}}(k,\mathbb{Q})(M^n, M^{n-1} \bigoplus C')$ that fulfils the following conditions for all $q \ge 0$: the morphisms $H^q_{\mathrm{sing}}(h')$ are injective, and they induce injections of $\operatorname{Im}(H^q_{\mathrm{sing}}(d_M^{n-1}))$ into $H^q_{\mathrm{sing}}(M^{n-1})$ that split the surjections induced by $H^q_{\mathrm{sing}}(d_M^{n-1})$. Moreover, there also exists $h'' \in \operatorname{Chow}^{\mathrm{eff}}(k,\mathbb{Q})(C', M^a)$ such that $H^q_{\mathrm{sing}}(d_M^{n-1} \bigoplus h'')$ splits $H^q_{\mathrm{sing}}(h')$ for all $q \ge 0$. Thus the composition $(d_M^{n-1} \bigoplus h'') \circ h'$ is an automorphism of M^a according to our assumption B. Thus we can calculate a choice of a weight complex t_j of $l^j(M)$ as follows (according to Proposition 1.4.2(8)):

$$t_j \cong \cdots \to M_j^{n-1} \to M_j^n \to 0 \to \ldots \cong (M^{n-1} \bigoplus C')_j \xrightarrow{(d_M^{n-1} \bigoplus h'')_j} M_j^n \to 0 \to \ldots,$$

where the lower index j means that we apply the induced functor $\operatorname{Chow}^{\mathrm{eff}}(k,R) \to \underline{Hw}_{\mathrm{Chow}^j}$ (recall that $C' \in DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}}=0}\langle j+1\rangle$). Since the morphism $d_M^{n-1} \bigoplus h''$ splits, the same is true for its image $(d_M^{n-1} \bigoplus h'')_j$. Applying Proposition 1.4.2(5) we obtain that $l^j(M) \in DM^j_{gm}(k,\mathbb{Q})_{w^j_{\mathrm{Chow}} \ge 1-n}$, contrary to our assumption. $\square$

Remark 3.5.6.

(1) This proposition suggests that one can look for motives with "interesting" Chow-weight homology using singular and étale (co)homology.

(2) Clearly, our assumption B is a particular case of the well-known conservativity conjecture (that predicts the following: if $H^*(M) = 0$ for $H = H_{\mathrm{sing}}$ or $H = H_{et,\mathbb{Q}_\ell}$ and $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})$, then $M = 0$).

Moreover, assumption B is essentially equivalent to Theorem I of [Ayo18] (and formally a particular case of loc. cit.), whereas the full conservativity follows from Conjecture II of loc. cit.[4]

[4]Currently the proofs of the main results of ibid. contain a gap. Hopefully, it will be closed eventually.

(3) In this argument one can certainly replace singular cohomology by any other cohomology theory satisfying similar properties. A natural candidate here is the so-called mixed motivic (co)homology corresponding to the conjectural motivic t-structure on $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q}) \subset DM_{gm}(k,\mathbb{Q})$. One can easily see that the "standard" expectations on this functor (see §5.10A in [Bei87] and [Bon15, Definition 3.1.1(4) and Proposition 4.1.1]) imply that the conclusion of our proposition follows from them (over a perfect field k of arbitrary characteristic).

3.6. **Comparing integral and rational coefficients: Bounding torsion of homology.** Let r denote a fixed non-zero integer divisible by e. We deduce some consequences from our results by comparing $\mathbb{Z}[1/e]$-motives with $\mathbb{Q}$-linear and $\mathbb{Z}[1/r]$-linear ones.

Definition 3.6.1. We say that an object M of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])$ is torsion (resp. r-torsion) if there exists $E_M > 0$ (resp. $d > 0$) such that the morphism $E_M \operatorname{id}_M$ is zero (resp. $r^d \operatorname{id}_M = 0$).

Theorem 3.2.1 easily yields the following statement.

Proposition 3.6.2. *Set $R' = \mathbb{Q}$ (resp. $= \mathbb{Z}[1/r]$). Then the following statements are valid.*

I.(1) $DM_{gm}^{\mathrm{eff}}(k,R')$ is isomorphic to the Karoubi envelope of the localization of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])$ by its subcategory of torsion (resp., r-torsion) objects. We write $-\otimes R'$ for the connecting functor $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e]) \to DM_{gm}^{\mathrm{eff}}(k,R')$; then for any $X \in \mathrm{SmVar}$ we have $\mathcal{M}_{\mathbb{Z}[1/e]}(X) \otimes R' = \mathcal{M}_{R'}(X)$.

(2) The functor $-\otimes R'$ is weight-exact with respect to the Chow weight structures on $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])$ and $DM_{gm}^{\mathrm{eff}}(k,R')$, respectively.

II.(1) There exist natural isomorphisms

$$\mathrm{CWH}^i_j(-_K \otimes R', R') \cong \mathrm{CWH}^i_j(-_K, \mathbb{Z}[1/e]) \otimes_{\mathbb{Z}[1/e]} R'$$

(for all field extensions K/k, $i \in \mathbb{Z}$ and $j \geq 0$).

(2) Let $M \in \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])$, $(n,c) \in \mathbb{Z} \times [0,+\infty)$. Then the groups $\mathrm{CWH}^i_j(M_K)$ are torsion (resp. r-torsion) for all $i \geq n$, $0 \leq j < c$, and all function fields K/k, if and only if $l^{c-1}_{R'}(M \otimes R') \in DM^{c-1}_{gm}(k,R')_{w^{c-1}_{\mathrm{Chow}} \geq -n+1}$.[5]

Proof. I.(1) This result was proved in [Kel12] (see §A.2 of ibid.; cf. also the proof of Proposition 5.3.3 of [Kel17] and Proposition 1.3.3 of [BoK18]).

(2) The statement is immediate from the previous assertion by Proposition 1.2.4(5).

II.(1) The statement follows immediately from assertion I.2 (by the definition of Chow-weight homology).

(2) The statement is immediate from Theorem 3.2.1(2–3) (see also Theorem 3.3.3(2)) applied to $M \otimes R'$ (using the previous assertion). $\square$

[5] Recall that l^{c-1} for $c \in [0,+\infty]$ denotes the localization functor from $DM_{gm}^{\mathrm{eff}}(k,R)$ to $DM_{gm}^{\mathrm{eff}}(k,R)/DM_{gm}^{\mathrm{eff}}(k,R)\langle c\rangle$ for the corresponding R; consequently, it is the identity functor of $DM_{gm}^{\mathrm{eff}}(k,R)$ if $c = +\infty$.

Remark 3.6.3. The weight-exactness of $-\otimes R'$ yields that the Chow weight structure on $DM^{\text{eff}}_{gm}(k,R')$ is "determined" by the one on $DM^{\text{eff}}_{gm}(k,\mathbb{Z}[1/e])$. Thus it may be treated using the localization methods developed in [BoS18a] and [BoS19].

Now we proceed to prove a drastic improvement of Proposition 3.6.2(II.2).

Once again, one may consult §2.1, Proposition 2.2.1(1), and Definition 3.1.1 (along with Definition 2.2.2(5)) for other notation used in the following formulation.

Theorem 3.6.4. *Let $M \in \operatorname{Obj} DM^{\text{eff}}_{gm}(k,\mathbb{Z}[1/e])$, $\mathcal{I} \subset \mathbb{Z} \times [0,+\infty)$.*

I. *Then the group $\mathrm{CWH}^i_j(M_K)$ is torsion for any function field K/k and all $(i,j) \in \mathcal{I}$ if and only if $\mathrm{CWH}^i_j(M_{K_0})$ is torsion for all $(i,j) \in \mathcal{I}$, and a single universal domain K_0 containing k.*

II. *Assume in addition that $\mathcal{I}$ is a staircase set (in the sense of Definition* 3.3.1*) and r is a non-zero integer (that we assume to be divisible by p if $p > 0$).*

Then the following conditions are equivalent.

A. *The groups $\mathrm{CWH}^i_j(M_K)$ are torsion (resp. r-torsion) for all function fields K/k and all $(i,j) \in \mathcal{I}$.*

B. *$E_M \cdot \mathrm{CWH}^i_j(M_K) = \{0\}$, where E_M is a fixed non-zero integer (resp. a fixed power of r) for all field extensions K/k and all $(i,j) \in \mathcal{I}$.*

D. *For any integers n, n' there exists a distinguished triangle $T \to M \to N \to T[1]$ satisfying the following conditions: T is a torsion motive (resp. an r-torsion motive), and there exists a triangle $Q \to N \to N' \to Q[1]$ such that*

$$Q \in DM^{\text{eff}}_{gm}(k,\mathbb{Z}[1/e])_{w_{\text{Chow}} \geq -n'+1}$$

and such that for some choice of $w_{\text{Chow} \geq -n} N'$ (see Remark 1.2.3(2)*) we have $\mathrm{CWH}^i_j(w_{\text{Chow} \geq -n} N'_K) = \{0\}$ for all field extensions K/k and all $(i,j) \in \mathcal{I}$.*

E. *For any integers n, n' there exists a distinguished triangle $T \to M \to N \to T[1]$ along with a choice $t(N) = (N^i)$ of a weight complex of N such that N^i is $(j+1)$-effective whenever $(i,j) \in \mathcal{I} \cap ([n',n] \times [0,+\infty))$ and T is a torsion motive (resp. an r-torsion motive).*

E'. *For any integers n, n' there exists a distinguished triangle $T \to M \to N \to T[1]$ satisfying the following conditions: T is a torsion motive (resp. an r-torsion motive) and $\mathrm{CWH}^i_j(N_K) = \{0\}$ if $(i,j) \in \mathcal{I} \cap ([n',n] \times [0,+\infty))$.*

Proof. I. The statement is immediate from Proposition 2.3.4(II) applied to $M \otimes \mathbb{Q}$.

II. Clearly, Condition B implies Condition A.

Now assume D. We apply Proposition 4.2.1(2) of [BoS18c] for the following data: $\underline{C} = DM^{\text{eff}}_{gm}(k,\mathbb{Z}[1/e])$, K is the subcategory of torsion (resp. r-torsion) objects (it corresponds to $J = \mathbb{Z} \setminus \{0\}$ or to $J = \{r\}$ in the notation of loc. cit., respectively), $\underline{D}_i = DM^{\text{eff}}_{gm}(k,\mathbb{Z}[1/e])\langle i \rangle$, and $a_i = a_{\mathcal{I},i}$. Combining this proposition with Theorem 3.3.3(2) we obtain that for any integers n and n' there exists a distinguished triangle $T \to M \to N \to T[1]$ such that T is a torsion motive (resp. r-torsion motive) and N is an extension of an object of $DM^{\text{eff}}_{gm}(k,\mathbb{Z}[1/e])_{w_{\text{Chow} \geq n+1}}$, an object of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}\le n'-1}}$, and an element N' such that $l^{a_{\mathcal{I},i}-1}(N')\in DM_{gm}^{a_{\mathcal{I},i}-1}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}}^{a_{\mathcal{I},i}-1}\ge -i+1}$.[6] By the definition of $a_{\mathcal{I},i}$, we obtain $l^j(N')\in DM_{gm}^{j}(k,\mathbb{Z}[1/e])_{w^j_{\mathrm{Chow}}\ge 1-i}$ for any $(i,j)\in\mathcal{I}$. Clearly, a weight complex of any element of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}\ge n+1}}$ and $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}\le n'-1}}$ can be chosen so that all of its terms in the range $[n,n']$ are trivial (see Proposition 1.4.2(2)). Hence for any choice of a weight complex of N' we can choose a weight complex of N whose terms are the same as those of N' in the range $[n,n']$ (see part (4) of that proposition). By Theorem 3.3.3(2) there is a choice of a weight complex for N' such that its i-th term is $j+1$-effective whenever $(i,j)\in\mathcal{I}$. Thus we obtain E.

Proposition 3.1.2(2) easily yields that E implies E'.

Next, if T is a torsion (resp. an r-torsion) motive then there exists a non-zero integer (resp. a power of r) n_T such that $n_T\cdot id_T=0$. Hence all the Chow-weight homology groups of T are killed by (the multiplication by) n_T. Now assume that M belongs to $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{[-n+1,-n'-1]}$ and E' is fulfilled. Then the long exact sequences for $\mathrm{CWH}^i_j(-K)$ coming from the distinguished triangle $T\to M\to N\to T[1]$ (where $\mathrm{CWH}^i_j(N_K)=\{0\}$ for all $(i,j)\in\mathcal{I}\cap[n',n]\times[0,+\infty)$ and T is torsion) yield that $\mathrm{CWH}^i_j(M_K)$ is killed by the multiplication by n_T whenever $i\le n$ and $(i,j)\in\mathcal{I}$. Moreover, $\mathrm{CWH}^i_j(M_K)=\{0\}$ if $i\ge n+1$; hence it is also killed by the multiplication by n_T. Thus Condition E' implies B.

It remains to prove that Condition A implies D. Assume Condition A. According to Proposition 3.6.2 (combined with Theorem 3.3.3(2)), for any $i\in\mathbb{Z}$ we have $l_{R'}^{a_{\mathcal{I},i}-1}(M\otimes R')\in DM_{gm}^{a_{\mathcal{I},i}-1}(k,R')_{w_{\mathrm{Chow}}^{a_{\mathcal{I},i}-1}\le -i}$ (for $R'=\mathbb{Q}$ or $R'=\mathbb{Z}[1/r]$, respectively). Hence Proposition 4.2.1(2) of [BoS18c] yields that there exists a distinguished triangle $T\to M\to N\to T[1]$ satisfying the following conditions: T is a torsion motive (resp. an r-torsion motive), and there exists a triangle $Q\to N\to N'\to Q[1]$ such that $Q\in DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}}\ge -n'+1}$ and N' is an extension of an element $N''\in DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}\ge -n}}$ such that $l^{a_{\mathcal{I},i}-1}(N'')\in DM_{gm}^{a_{\mathcal{I},i}-1}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}}^{a_{\mathcal{I},i}-1}\ge -i+1}$ for any $(i,j)\in\mathcal{I}$, by an element of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}\le -n+1}}$. Since N' is an extension of N'' by an element $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])_{w_{\mathrm{Chow}\le -n+1}}$, N'' is a choice of $w_{\mathrm{Chow}\ge -n}N'$. By Theorem 3.3.3, $\mathrm{CWH}^j_{i,K}(w_{\mathrm{Chow}\ge -n}N')=\mathrm{CWH}^j_{i,K}(N'')=\{0\}$ for all field extensions K/k and $(i,j)\in\mathcal{I}$. Thus we obtain condition D. $\square$

Now we combine this theorem with the results of §3.4.

Corollary 3.6.5. *Let $M\in\operatorname{Obj}DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z}[1/e])$ and K_0 be a universal domain containing k.*

I. *Let $\mathcal{I}$ be a staircase set. Then the "main" versions of the (equivalent) Conditions A–E' of Theorem* 3.6.4(*II*) *(i.e., we ignore the versions in brackets that mention r) are also equivalent to each of the following assertions.*

[6]Note that $DM_{gm}^{-1}(k,\mathbb{Z}[1/e])$ is the zero category. Thus here and below the conditions on the images of motives with respect to l^{-1} are assumed to be vacuous; this corresponds to the case $a_{\mathcal{I},i}=0$.

(1) *For all rational extensions k'/k and all $(i,j) \in \mathcal{I}$ the group*

$$\mathrm{CWH}^i_{j-1}(M_{k'}, 1, \mathbb{Z}[1/e])$$

is torsion.

(2) *The group* $\mathrm{CWH}^i_j(M_{K_0}, \mathbb{Z}[1/e])$ *is torsion for all* $(i,j) \in \mathcal{I}$.

(3) *There exists an integer* $E_M > 0$ *such that* $E_M \,\mathrm{CWH}^i_{j-a}(M_{k'}, a, \mathbb{Z}[1/e]) = \{0\}$ *for all* $(i,j) \in \mathcal{I}$, $a \in \mathbb{Z}$, *and all field extensions* k'/k.

II. *The following conditions are equivalent.*

(1) $M \otimes \mathbb{Q} \in DM^{\mathrm{eff}}_{-}(k, \mathbb{Q})^{t^{\mathbb{Q}}_{hom} \leq 0}$.

(2) $h_{l,0}(M_{K_0}, \mathbb{Q}) = \{0\}$ *for all* $l < 0$.

(3) $\mathrm{CWH}^i_{j-a}(M_{k'}, a, \mathbb{Q}) = \{0\}$ *for all* $a \in \mathbb{Z}$, $i > j$, *and all field extensions* k'/k.

(4) *There exists an integer* $E_M > 0$ *such that* $E_M \,\mathrm{CWH}^i_{j-a}(M_{k'}, a, \mathbb{Z}[1/e]) = \{0\}$ *for all* $a \in \mathbb{Z}$, $i > j$, *and all field extensions* k'/k.

(5) *There exists an integer* $E'_M > 0$ *such that* $E'_M h_{l,0}(M_{k'}, \mathbb{Z}[1/e]) = \{0\}$ *for all* $l < 0$ *and all field extensions* k'/k.

III. *Assume that* M *belongs to* $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Z}[1/e])_{w_{\mathrm{Chow}} \geq 0}$. *Then for any* $c \geq 0$ *the following conditions are equivalent.*

(1) $M \otimes \mathbb{Q} \in DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})^{\langle c \rangle}_{\geq 0}$ *(see Definition* 3.3.6*).*

(2) $h_{2j,j}(M_{K_0}, \mathbb{Q}) = \{0\}$ *whenever* $0 \leq j < c$.

(3) *There exists* $E_M > 0$ *such that* $E_M h_{2j,j}(M_{k'}, \mathbb{Z}[1/e]) = \{0\}$ *for all* $0 \leq j < c$ *and all field extensions* k'/k.

Proof. I. Applying Proposition 3.4.1 to $M \otimes \mathbb{Q}$ we obtain that our conditions (I1)–(I2) are equivalent to Condition A of Theorem 3.6.4(II). It remains to note that Condition D of the theorem easily yields our condition (I3) (since the proof of the implication D $\Longrightarrow$ B in the theorem carries over to higher Chow-weight homology without any difficulty).

II. First we combine assertion I with Corollary 3.4.2 for the case $R = \mathbb{Q}$ (and with M replaced by $M \otimes \mathbb{Q}$). We obtain that our conditions (II1), (II3), and (II2) are equivalent. Moreover, the last of these conditions is clearly weaker than condition (II5).

Next, condition (II3) implies condition (II4) according to our assertion I (we take $\mathcal{I} = \{(i,j) :\ i > j\}$ in it). Thus it remains to verify that condition (II4) implies condition (II5).

Now we take the (Chow-) weight spectral sequence $T(M, k')$ converging to the (zero-dimensional) motivic homology of M over k':

$$E_1^{pq}(T(M,k')) = \mathrm{CH}_0(M^p_{k'}, -q, \mathbb{Z}[1/e]) \implies \mathrm{CH}_0(M_{k'}, -p-q, \mathbb{Z}[1/e])$$

(where $t_R(M) = (M^p)$). Clearly, $E_2^{pq}(T(M,k')) = \mathrm{CWH}^p_0(M_{k'}, -q, \mathbb{Z}[1/e])$. Since M is w_{Chow}-bounded, condition (II4) implies that a high enough power of E_M (that depends on M only) kills $h_{l,0}(M_{k'}, \mathbb{Z}[1/e])$ for all l and k'.

III. Applying Corollary 3.3.7(I) to the motive $M \otimes \mathbb{Q}$ we obtain the equivalence of conditions (III2) and (III3). It remains to combine Theorem 3.6.4(II) with Lemma 3.1.4(2) to obtain that these conditions are also equivalent to condition (III3). $\square$

Remark 3.6.6.

(1) It is quite remarkable that certain Chow-weight homology groups have finite exponents. Note that (in general) Chow-weight homology groups (as well as motivic homology ones) can certainly have really "weird" torsion.

In particular, our results can be applied to the case $M = \mathrm{Cone}(h)$, where h is a $\mathrm{Chow}^{\mathrm{eff}}(k, R)$-morphism (cf. Corollary 3.3.9); the resulting statement appears to be quite non-trivial and absolutely new.

(2) In the case where the set $\mathcal{I}$ satisfies some additional assumptions, there exist nicer re-formulations of the rather clumsy conditions II.D–E' in Theorem 3.6.4. They are given by Theorem 3.6.5(III–V) of [BoS14]; see also Condition II.C in loc. cit. and Condition II.2 in Corollary 3.6.6 of ibid. These statements follow from the results of [BoS18c, §4.2] easily (as well; cf. the proof of Theorem 3.6.4(II)).

4. Applications to motives and cohomology with compact support

In §4.1 we recall the theory of motives with compact support (of arbitrary varieties); in particular, their motivic homology gives Chow groups of varieties.

In §4.2 we use these results to obtain the main applications of our results to (motives and cohomology with compact support of) varieties. We relate the vanishing of lower (rational) Chow groups of varieties to the effectivity of the higher weight factors of their cohomology with compact support (see Theorems 4.2.1 and 4.2.3). We also obtain that the exponents of certain Chow groups (as well as of cokernels of certain homomorphisms between them) if these groups are torsion (cf. Theorem 3.6.4). Furthermore, in the case where k is finite we relate the effectivity conditions for motives (that can be checked using Chow-weight homology) to the number of points of varieties over k (modulo powers of $q = \#k$).

In §4.3 we study conditions ensuring that lower Chow groups of a smooth proper k-variety X are supported on its subvarieties of "small" dimension. In contrast to the case of a general X that was considered in §4.2, we are able to express these conditions in terms of certain decompositions of the diagonal of $X \times X$ (considered as an algebraic cycle). Consequently, we re-prove and extend the corresponding results of [Par94] and [Lat96]; this section also demonstrates the relation of our methods to earlier (and "more cycle-theoretic") ones.

In §4.4 we consider tensor products of motives. Combining the properties of motives with compact support with Corollary 3.3.7(II) we easily obtain that the vanishing ranges of lower Chow groups add when varieties multiply. Moreover, the conjectures mentioned in Proposition 3.5.5 imply several funny results essentially in the converse direction over characteristic 0 fields (both for varieties and motives). In particular, one may say that the "effectivity and connectivity" of the tensor product of (rational geometric) motives over a characteristic 0 field cannot exceed the sums of the effectivities and connectivities of the multipliers, respectively.

4.1. On motives with compact support and their relation to Chow groups.

Corollary 3.3.9 (along with Remark 3.3.10) can certainly be applied to morphisms of Chow motives that come from (closed) embeddings of smooth projective varieties. This gives conditions equivalent to the assumption that all algebraic cycles of dimension less than r_1 on a smooth projective variety X are "supported" on a smooth closed subvariety Z of X. However, we would like to demonstrate that our results can also be applied in the case where X or Z is singular.

For this purpose we need some basics on motives with compact support. We will start with the following definitions.

Definition 4.1.1.

(1) We will write SchPr for the wide subcategory of the category of k-varieties whose morphisms are the proper ones.
(2) If R is a unital $\mathbb{Z}[1/e]$-algebra then we will use the notation $\mathcal{M}_R^c$ (motive with compact support) for the composition of the functor $\mathcal{M}_{\mathbb{Z}[1/e]}^c$: $\mathrm{SchPr} \to DM_{-}^{\mathrm{eff}}(k, \mathbb{Z}[1/e])$ provided by Definition 5.3.1 of [Kel17] (cf. also §4.1 of [Voe00]) with the natural connecting functor $-\otimes R : DM^{\mathrm{eff}}(k, \mathbb{Z}[1/e]) \to DM^{\mathrm{eff}}(k, R)$ (see Proposition 1.3.3 of [BoK18]).[7]

Proposition 4.1.2. *Assume that k' is a perfect field extension of k. Then the functor $\mathcal{M}_R^c$ (motive with compact support) satisfies the following properties.*

(1) *$\mathcal{M}_R^c(P) = \mathcal{M}_R(P)$ whenever $P \in$ SmPrVar. Moreover, $\mathcal{M}_R^c(X)$ is an object of $DM_{gm}^{\mathrm{eff}}(k, R)$ for any $X \in$ Var.*
(2) *The k'-motive $\mathcal{M}_R^c(X)_{k'}$ is isomorphic to $\mathcal{M}_R^c(X_{k'})$.*
(3) *For any $j \geq 0$, $X \in$ Var, and any smooth quasi-projective k-variety U we have $\mathcal{M}_R(U)\langle j\rangle \perp \mathcal{M}_R^c(X)[i]$ for any $i > 0$.*

    Moreover, if $i \in \mathbb{Z}$ and U is of (constant) dimension d then the group $DM_{gm}^{\mathrm{eff}}(k', R)(\mathcal{M}_R(U_{k'})\langle j\rangle, \mathcal{M}_R^c(X)[i]_{k'})$ is isomorphic to the higher Chow group $\mathrm{CH}_{j+d}(U \times X_{k'}, -i, R)$ (cf. Theorem 5.3.14 of [Kel17] for the $\mathbb{Z}[1/e]$-version of this notation); in particular, if $i = 0$ then this group is isomorphic to the Chow group $\mathrm{CH}_{j+d}(U \times X_{k'}, R)$ of R-linear cycles of dimension $j+d$ on $U \times X_{k'}$ (cf. Remark 2.2.3).
(4) *If $i : Z \to X$ is a closed embedding of k-varieties and $U = X \setminus Z$ then there exists a distinguished triangle*

$$\mathcal{M}_R^c(Z) \xrightarrow{\mathcal{M}_R^c(i)} \mathcal{M}_R^c(X) \to \mathcal{M}_R^c(U) \to \mathcal{M}_R^c(Z)[1]. \tag{4.1}$$

(5) *If $X, Y \in$ Var then $\mathcal{M}_R^c(X \times Y) \cong \mathcal{M}_R^c(X) \otimes \mathcal{M}_R^c(Y)$.*
(6) *If Y is an affine bundle of dimension $r \geq 0$ over X then $\mathcal{M}_R^c(Y) \cong \mathcal{M}_R^c(X)\langle r\rangle$.*

Proof. In Definition 5.3.1, Lemma 5.3.6, Proposition 5.3.5, and Proposition 5.3.8 of [Kel17], respectively, the obvious $\mathbb{Z}[1/e]$-linear analogues of assertions (1), (4), and (5) were justified. Then the R-linear results in question follow immediately since the functor $-\otimes R : DM^{\mathrm{eff}}(k, \mathbb{Z}[1/e]) \to DM^{\mathrm{eff}}(k, R)$ in Definition 4.1.1(2) is an exact tensor functor that sends $\mathcal{M}_{\mathbb{Z}[1/e]}(Z)$ into $\mathcal{M}_R(Z)$ for any $X \in$ SmVar; see Proposition 1.3.3 of [BoK18].

Similarly, it suffices to prove the $\mathbb{Z}[1/e]$-linear version of assertion (6). In the case where $Y = \mathbb{A}^r \times X$ it is given by Corollary 5.3.9 of [Kel17]. To prove it in general we note that there exists a stratification of $X = \cup X_l$ such that for the preimages Y_l of X_l in Y we have $Y_l \cong \mathbb{A}^r \times X_l$. Hence one can apply the aforementioned Proposition 5.3.5 of ibid. (cf. (4.1)) along with the canonical comparison morphisms provided

[7] Below we will usually take R that is a localization of $\mathbb{Z}[1/e]$. In this case Proposition 3.6.2(I) is sufficient for our purposes; see Proposition 4.1.2(1).

by Proposition 5.3.12(ii) of ibid. to prove the statement by induction on the number of strata.[8]

Next, assertion (2) easily follows from description of motives with compact support provided by Proposition 8.10 of [CiD15]; see Proposition A.1(2) of [BoK20].

Lastly, combining Proposition 5.3.12(i) with Theorems 5.2.20, 5.2.21, and 5.3.14 of [Kel17] one easily obtains assertion (3) in the case $k' = k$. It remains to invoke assertion (2) to obtain the general case of the assertion. $\square$

Remark 4.1.3. Actually, $\mathcal{M}_R(X) = \mathcal{M}^c_R(X)$ whenever X is proper.

Now we relate motives with compact support to the weight structure w_{Chow}.

Lemma 4.1.4. *Let* $X \in \mathrm{Var}$.

(1) *Then* $\mathcal{M}^c_R(X) \in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} \geq 0}$. *Moreover, if X is smooth and proper then* $\mathcal{M}^c_R(X) = \mathcal{M}_R(X) \in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} = 0}$.

(2) *For any $j \geq 0$ and any field extension k'/k the group* $\mathrm{CWH}^0_j(\mathcal{M}^c_R(X)_{k'})$ *is naturally isomorphic to* $\mathrm{CH}_j(X_{k'}, R)$.

(3) *If X is of dimension at most r (for some $r \geq 0$) then $\mathcal{M}^c_R(X)$ is an object of* $d_{\leq r} DM^{\mathrm{eff}}_{gm}(k,R)$.

(4) *For any* $Z \in \mathrm{Var}$ *there exists a smooth projective k-variety Y along with a morphism* $h : \mathcal{M}_R(Y) = \mathcal{M}^c_R(Y) \to \mathcal{M}^c_R(Z)$ *such that* $\dim Y = \dim Z$ *and h can be completed to a weight decomposition triangle for* $\mathcal{M}^c_R(Z)$.

(5) *Let* $M \in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} = 0}$ *and* $N \in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} \geq 0}$. *Then a morphism $h : M \to N$ gives a weight decomposition triangle for N if and only if the homomorphisms $h_{2j,j}(h_K, R)$ are surjective for all $j \geq 0$ and all function fields K/k.*

Proof. (1). The first part of the assertion is immediate from Proposition 4.1.2(3) (see Proposition 1.2.4(3), 2).

To get the "moreover" part it remains to recall Proposition 4.1.2(1) and Proposition 2.2.1(2).

(2). The statement is immediate from the previous assertion combined with Lemma 3.1.4(2); cf. Remark 2.2.3.

(3). Proposition 4.1.2(4) implies that it suffices to prove the statement under the assumption that X is smooth. Moreover, obvious induction allows us to assume that $\mathcal{M}^c_R(U) \in d_{\leq r-1} DM^{\mathrm{eff}}_{gm}(k,R)$ whenever U is of dimension at most $r-1$. Hence $\mathcal{M}^c_R(X') \in d_{\leq r} DM^{\mathrm{eff}}_{gm}(k,R)$ whenever X' is a smooth variety of dimension r that either possesses a smooth compactification (see Proposition 4.1.2(1)) or contains an open dense subvariety U' such that $\mathcal{M}^c_R(U') \in \mathrm{Obj}\, d_{\leq r} DM^{\mathrm{eff}}_{gm}(k,R)$.

Now, assume that $R = \mathbb{Z}_{(\ell)}$, where ℓ is an arbitrary prime distinct from p. Then Corollary 1.2.2(1) of [Bon11] implies that (for any smooth X of dimension r) there exists an open dense $U \subset X$ such that $\mathcal{M}_R(U)$ is a retract of $\mathcal{M}_R(U')$, where $\dim U' = r$ and U' possesses a smooth compactification. Next, the duality provided by Theorem 5.3.18 of [Kel17] immediately implies that $\mathcal{M}^c_R(U)$ is a retract of $\mathcal{M}^c_R(U')$ under these assumptions.

Thus we obtain our assertion in the case $R = \mathbb{Z}_{(\ell)}$. Applying this statement for all $\ell \in \mathbb{P} \setminus \{p\}$ along with Corollary 0.2 of [BoS15] and Proposition 3.6.2(I.1) (cf. also Appendix A.2 of [Kel12]) we obtain the result in question for $R = \mathbb{Z}[1/e]$ as well. Applying Proposition 1.3.3 of [BoK18] once again we conclude the proof.

[8]Note also that this assertion is mentioned in §5 of [Tot16].

(4). Immediate from assertions (1) and (3).

(5). Clearly, h yields a weight decomposition of N if and only if for $C = \mathrm{Cone}(h)$ we have $C \in DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}} \geq 1}$. Next, Theorem 3.2.1(3) says that the latter assumption is fulfilled if and only if $\mathrm{CWH}^i_j(C_K) = \{0\}$ for all $i,j \geq 0$ and all function fields K/k. Moreover, we have $\mathrm{CWH}^i_j(M_K) = \mathrm{CWH}^i_j(N_K) = \{0\}$ if $j \geq 0$ and $i \geq 1$, and $\mathrm{CWH}^i_j(M_K) = \{0\}$ also if $i < 0$ (and $j \geq 0$). Thus the long exact sequences relating Chow-weight homology of M, N, and C yields that h satisfies the condition in question if and only if the homomorphisms $\mathrm{CWH}^0_j(h_K)$ are surjective for all $j \geq 0$. Hence it remains to apply Lemma 4.1.4(2). □

Let us now concentrate on the case $R = \mathbb{Q}$ (yet cf. Remark 3.2.2(1)).

Lemma 4.1.5. *Let K_0 be a universal domain containing k, $X, Y, Z \in \mathrm{Var}$.*

(1) *Let $M \in DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}}=0}$ and $N \in DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \geq 0}$. Then a morphism $h : M \to N$ yields a weight decomposition of N if and only if the homomorphisms $h_{2j,j}(h_{K_0}, \mathbb{Q})$ are surjective for all $j \geq 0$.*

(2) *If $g : Y \to Z$ is a proper surjective morphism and $h = \mathcal{M}^c_{\mathbb{Q}}(g)$ then the homomorphisms $\mathrm{CH}_j(g_{K_0}, \mathbb{Q})$ and $\mathrm{CWH}^0_j(h_{K_0}, \mathbb{Q})$ are surjective.*

Moreover, if Y is smooth and proper then h gives a weight decomposition of $\mathcal{M}^c_{\mathbb{Q}}(Z)$.

(3) *Assume that X is proper. Then for any g as above, any closed embedding i of Z into X, and $U = X \setminus Z$ there exists a choice of $t(\mathcal{M}^c_{\mathbb{Q}}(U))$ of the form $\dots \mathcal{M}_{\mathbb{Q}}(Y) \xrightarrow{\mathcal{M}_{\mathbb{Q}}(i \circ g)} \mathcal{M}_{\mathbb{Q}}(X) \to 0 \to \dots$ (where $\mathcal{M}_{\mathbb{Q}}(X)$ is in degree 0).*

Proof. (1). This is an easy combination of Lemma 4.1.4(5) with Proposition 2.3.4(II); cf. Remark 3.2.2(1).

(2). According to Lemma 4.1.4(2), the surjectivity of $\mathrm{CWH}^0_j(h_{K_0}, \mathbb{Q})$ is equivalent to that of $\mathrm{CH}_j(g_{K_0}, \mathbb{Q})$. The latter surjectivity is rather obvious, since for any Zariski point z of Z_{K_0} one can choose a point y of Y_{K_0} that is of finite degree over z.

To obtain the "moreover" part of the assertions it remains to invoke assertion (1).

(3). Applying Proposition 4.1.2 (1, 3) along with Proposition 1.4.2(4) we obtain that it suffices to find a choice of $w_{\mathrm{Chow} \leq 0}\mathcal{M}^c_{\mathbb{Q}}(Z)$ and calculate the composed morphism $w_{\mathrm{Chow} \leq 0}\mathcal{M}^c_{\mathbb{Q}}(Z) \to \mathcal{M}^c_{\mathbb{Q}}(Z) \xrightarrow{\mathcal{M}^c_{\mathbb{Q}}(i)} \mathcal{M}^c_{\mathbb{Q}}(X)$. Hence it suffices to apply the functoriality of $\mathcal{M}^c_{\mathbb{Q}}$ along with assertion (2). □

Now we combine our lemmata with Corollary 3.3.7.

Proposition 4.1.6. *Assume that $r \in [0, +\infty]$, K_0 is a universal domain containing k, $g : Y \to X$ is a proper morphism of k-varieties, $Z = \mathrm{Im}\, g$, $U = X \setminus Z$.*

Denote $\mathcal{M}^c_{\mathbb{Q}}(g)$ by h, $M = \mathrm{Cone}(h)$, and $C = \mathcal{M}^c_{\mathbb{Q}}(U)$.

Then the following conditions are equivalent.

(1) *M belongs to $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r \rangle}_{\geq 0}$ (the notation comes from Definition 3.3.6; here we set $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle +\infty \rangle}_{\geq 0} = DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \geq 1}$).*

(2) *The homomorphisms $\mathrm{CH}_j(g_{K_0}, \mathbb{Q})$ are surjective for $0 \leq j < r$.*

(3) *$\mathrm{CH}_j(U_{K_0}, \mathbb{Q}) = \{0\}$ for $0 \leq j < r$.*

(4) *$C \in DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r \rangle}_{\geq 0}$.*

Proof. Let $j \geq 0$. Lemma 4.1.4(1), 2 implies that the motives $\mathcal{M}^c_{\mathbb{Q}}(Y)$, $\mathcal{M}^c_{\mathbb{Q}}(Z)$, $\mathcal{M}^c_{\mathbb{Q}}(X)$, M, and C belong to $DM^{\text{eff}}_{gm}(k,\mathbb{Q})_{w_{\text{Chow}}\geq 0}$. Moreover, $\text{CWH}^0_j(J_{K_0},\mathbb{Q}) \cong h_{2j,j}(J_{K_0},\mathbb{Q})$ if J equals either $\mathcal{M}^c_{\mathbb{Q}}(Y)$, $\mathcal{M}^c_{\mathbb{Q}}(Z)$, or $\mathcal{M}^c_{\mathbb{Q}}(X)$, and $\text{CWH}^i_j(J_{K_0},\mathbb{Q}) = \{0\}$ for all these motives and $i > 0$. Thus $\text{CWH}^i_j(M) = \text{CWH}^i_j(C) = \{0\}$ for all $i > 0$ and there is a long exact sequence

$$\begin{aligned} &\cdots \to \text{CWH}^{-1}_j(\mathcal{M}^c_{\mathbb{Q}}(X)) \to \text{CWH}^{-1}_j(M_{K_0},\mathbb{Q}) \to \text{CH}_j(Y_{K_0},\mathbb{Q}) \\ &\xrightarrow{\text{CH}_j(g_{K_0},\mathbb{Q})} \text{CH}_j(X_{K_0},\mathbb{Q}) \to \text{CWH}^0_j(M_{K_0},\mathbb{Q}) \to \{0\}. \end{aligned} \tag{4.2}$$

We combine it with Corollary 3.3.7(I) if $r < +\infty$, and with Theorem 3.2.1(3) if $r = +\infty$ (see also Remark 3.2.2)1)); this immediately gives the equivalence of our conditions (1) and (2). Similarly, these statements imply the equivalence of conditions (3) and (4).

Next, Proposition 4.1.2(4) implies that for the corresponding embedding $i : Z \to X$ we have $\text{Cone}(\mathcal{M}^c_{\mathbb{Q}}(i)) \cong C$. Thus we obtain a long exact sequence

$$\cdots \to \text{CH}_j(Z_{K_0},\mathbb{Q}) \to \text{CH}_j(X_{K_0},\mathbb{Q}) \to \text{CWH}^0_j(C_{K_0}) \to \{0\},$$

and arguing as above we obtain that our condition (4) is equivalent to the surjectivity of the homomorphism $\text{CH}_j(i,\mathbb{Q})$. Lastly, Lemma 4.1.5(2) implies that for the corresponding $g' : Y \to Z$ the homomorphism $\text{CH}_j(g',\mathbb{Q})$ is surjective. Hence the surjectivity of $\text{CH}_j(i,\mathbb{Q})$ is equivalent to condition (2). ☐

Remark 4.1.7.

(1) Note that the empty scheme is a variety by our convention. Its Chow groups are zero; thus if $Y = \emptyset$ in Proposition 4.1.6 then $U = X$ and we obtain that the motive $\mathcal{M}^c_{\mathbb{Q}}(U)$ belongs to $DM^{\text{eff}}_{gm}(k,\mathbb{Q})^{\langle r\rangle}_{\geq 0}$ if and only if $h_j(U_{K_0},\mathbb{Q}) = \{0\}$ for $0 \leq j < r$. We will often mention this case of the proposition below.

More generally, it is easily seen that for any coefficient ring R the motive $\mathcal{M}^c_R(U)$ belongs to $DM^{\text{eff}}_{gm}(k,R)^{\langle r\rangle}_{\geq 0}$ if and only if $h_j(U_K,R) = \{0\}$ for $0 \leq j < r$ and all function fields K/k.

(2) One can easily construct rich families of examples for part (1) of this remark. This clearly gives examples for Proposition 4.1.6 as well, and one can take Y and Z to be non-empty in them.

Let $T \in \text{Var}$, $r > 0$, and U is an affine bundle of dimension r over T (say, $U = T \times \mathbb{A}^r$). Then combining Proposition 4.1.2 (6, 4) with Lemma 4.1.4(1) and Corollary 2.2.4(1) we obtain $\mathcal{M}^c_R(U) \in DM^{\text{eff}}_{gm}(k,R)_{w_{\text{Chow}}\geq 0}\langle r\rangle \subset DM^{\text{eff}}_{gm}(k,R)^{\langle r\rangle}_{\geq 0}$ for any R. Moreover, the aforementioned statements easily imply that for any open dense embedding $U' \to U$ the motive $\mathcal{M}^c_R(U')$ belongs to $DM^{\text{eff}}_{gm}(k,R)^{\langle r\rangle}_{\geq 0}$ whenever $\mathcal{M}^c_R(U)$ does.

Moreover, Remark 4.1.5(3) of [BoS14] describes certain $X \in \text{Var}$ such that for $M = \mathcal{M}^c_R(X)$ we have $\text{CWH}^i_j(M_K,R) = \{0\}$ for all function fields K/k and all (i,j) that belong to a given staircase set $\mathcal{I}$.[9]

(3) In the case $Y = Z = \emptyset$ (see part (1) of this remark) the equivalent conditions of Proposition 4.1.6 can also be re-formulated as follows: there exists a smooth projective k-variety P of constant dimension $s \geq 0$ and a $\mathbb{Q}$-linear

[9]Actually, in loc. cit. the case $R = \mathbb{Q}$ is considered; yet this assumption is not necessary.

algebraic cycle η of dimension $s+r$ in $P \times X$ that (if considered as a correspondence via Proposition 4.1.2(3)) induces a surjection $\mathrm{CH}_{j-r}(P_{K_0}, \mathbb{Q}) \to \mathrm{CH}_j(X_{K_0}, \mathbb{Q})$ for all $j \geq 0$; here we set $\mathrm{CH}_{j-r}(P_{K_0}, \mathbb{Q}) = \{0\}$ if $j < r$. Indeed, the "if" implication is obvious (see condition (3) of Proposition 4.1.6) and it suffices to combine Corollary 3.3.7(I) (see condition (I3) in it) with the obvious "correspondence version" of Lemma 4.1.5(1) to obtain the converse implication.

We will give a "decomposition of the diagonal" re-formulation of this condition in the case where X is smooth (and possesses a smooth compactification) in §4.3 below.

(4) It is easily seen not to be sufficient to assume that $g : Y \to Z$ is (proper and) surjective to claim that $h = \mathcal{M}_R^c(g)$ gives a weight decomposition of $\mathcal{M}_R^c(Z)$ (see Lemmata 4.1.4(4) and 4.1.5(2)) in the case of a general coefficient ring R.

Hence one needs some more restrictive assumptions on the morphism g to ensure that all the R-linear versions of the conditions in Proposition 4.1.6 are equivalent (i.e., to ensure that condition (3) implies condition (2)).

We need some more preparation for the next subsection. To relate our results to "the usual" cohomology with compact support we need the following statement.

Proposition 4.1.8.

(1) *For the cohomological functor $H = H_{et,\mathbb{Q}_\ell}$ mentioned in Definition* 3.5.3(4), *any $X \in$* Var, *$i \in \mathbb{Z}$, and $M = \mathcal{M}_{\mathbb{Q}}^c(X)$ (see Definition* 4.1.1(2)*) the $\mathbb{Q}_\ell[G]$-module $H^i(M) = H(M[-i])$ is canonically isomorphic to the module $H^i_{c,et}(X_{k^{alg}})$ of i-th étale cohomology of $X_{k^{alg}}$ with compact support. Moreover, these isomorphisms are* SchPr*-natural.*

(2) *Assume that k is a subfield of $\mathbb{C}$. Then for any $X \in$* Var *the factors of the Deligne weight filtration on the MHS-valued singular cohomology of $X_{\mathbb{C}}$ with compact support are* SchPr*-naturally isomorphic to the weight factors of $H^*_{\mathrm{sing}}(\mathcal{M}_{\mathbb{Q}}^c(X))$ (see Definition* 3.5.3(3)*).*

Proof. (1) Recall that $H_{et,\mathbb{Q}_\ell}$ is the restriction to $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$ of the cohomological functor $H^0_{et}(-_{k^{alg}}, \mathbb{Q}_\ell)$ from $DM_{gm}(k, \mathbb{Q})$ into $\mathbb{Q}_\ell[G] - \mathrm{Mod}$ coming from Proposition 7.2.21 and Theorem 7.2.24 of [CiD16]. Now, $H_{et}(-_{k^{alg}}, \mathbb{Q}_\ell)$ possesses the corresponding "compact support" property by loc. cit.; see here Proposition 8.10 of [CiD15] for the "six functor" description of motives with compact support.

(2) Theorem 3 of [GiS96] says that the quotients of the weight filtration on $H^i_{c,sing}(X_{\mathbb{C}})$ are functorially isomorphic (as pure Hodge structures) to the corresponding E_2-terms of the weight spectral sequences (similarly to Theorem 3.5.4(2)). Now, these E_2-terms in loc. cit. are expressed (cf. Proposition 1.4.5(2)) in terms of their weight complex $W(X)$ of X as provided by Theorem 2 of ibid. (cf. Remark 1.4.3(2)). Thus it remains to apply Theorem 3.1 of [KeS17] (or recall that the composition $t \circ \mathcal{M}_{\mathbb{Q}}^c$ is essentially isomorphic to the weight complex functor of ibid. according to Proposition 6.6.2 of [Bon09]; cf. Remark 1.4.3(2)). □

Remark 4.1.9. The authors do not know whether the known properties of singular cohomology of motives are sufficient to verify that the singular cohomology of $\mathcal{M}_{\mathbb{Q}}^c(X)$ is isomorphic to the corresponding cohomology of X with compact support as mixed Hodge structures. Yet this statement is most probably true.

4.2. **On cohomology with compact support and the number of points of varieties.** Let us apply results of previous sections to motives with compact support of varieties.

Theorem 4.2.1. *Let* $U \in \mathrm{Var}$, $r \geq 0$, K_0 *is a universal domain containing* k, *and assume that* $\mathrm{CH}_j(U_{K_0}, \mathbb{Q}) = \{0\}$ *for* $0 \leq j < r$.

(1) *Then there exists* $E > 0$ *such that* $E\,\mathrm{CH}_j(U_{k'}, \mathbb{Z}[1/e]) = \{0\}$ *for all* $0 \leq j < r$ *and all field extensions* k'/k.

(2) *If* k *is a subfield of* $\mathbb{C}$ *then the* q*-th (Deligne) weight factor of* $H^q_c(U_{\mathbb{C}})$ *of the (*$\mathbb{Q}$*-linear) singular cohomology of* U *with compact support is* r*-effective as a pure Hodge structure (see Definition* 3.5.3(1)*).*

Moreover, the same property of Deligne weight factors of $\mathbb{Q}_\ell$*-étale cohomology* $H^q_c(U_{k^{alg}})$ *is fulfilled (in the sense of Definition* 3.5.3(2)*) if* k *is an essentially finitely generated field (see Definition* 2.3.1(1)*) and* $\ell \neq p$.

In particular, these factors are zero if $q < 2r$.

(3) *Assume that* $U = X \setminus Z$, *where* Z *is the image of a proper morphism* $g : Y \to X$ *of* k*-varieties. Then there exists* $E > 0$ *such that the cokernel of the homomorphism* $\mathrm{CH}_j(g_{k'}, \mathbb{Z}[1/e])$ *is annihilated by* E *whenever* $0 \leq j < r$ *and* k'/k *is a field extension. Moreover, if* k *and* H *are as in assertion* (2) *then the object* $\mathrm{Ker}(W_{D_q} H^q_c(X) \to W_{D_q} H^q_c(Y))$ *is* r*-effective (in the sense of Definition* 3.5.3*).*

(4) *The motive* $\mathcal{M}^c_{\mathbb{Q}}(U)$ *(see Definition* 4.1.1(2)*) is an extension of an element of* $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})_{w_{\mathrm{Chow}} \geq 1}$ *(see Proposition* 2.2.1(1)*) by an object of* $\mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Q})\langle r \rangle$ *(see* §2.1*).*

Proof. All of these statements are rather easy implications of earlier results.

We take $M = \mathcal{M}^c_{\mathbb{Z}[1/e]}(U)$ (this corresponds to $R = \mathbb{Z}[1/e]$ in Definition 4.1.1(2)). Then $M \in DM^{\mathrm{eff}}_{gm}(k, \mathbb{Z}[1/e])_{w_{\mathrm{Chow}} \geq 0}$ by Lemma 4.1.4(1). Moreover, $\mathcal{M}^c_{\mathbb{Z}[1/e]}(T)$ belongs to $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Z}[1/e])_{w_{\mathrm{Chow}} \geq 0}$ whenever T is equal either to X, Y, or Z in assertion (3). Furthermore, Proposition 4.1.6 implies that $M \otimes \mathbb{Q} = \mathcal{M}^c_{\mathbb{Q}}(U) \in DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})^{\langle r \rangle}_{\geq 0}$. Hence assertion (1) follows from Corollary 3.6.5(III) (see condition III3 in it); see also Proposition 4.1.2(3) and Remark 2.2.3.

Given assertion (1), assertion (2) easily follows from Theorem 3.5.4(3) combined with (the corresponding parts of) Proposition 4.1.8; note also that r-effective pure Hodge structures and Galois representations are of weight at least $2r$.

Next, assertion (4) follows from Corollary 3.3.7(I).

To prove assertion (3) we argue similarly to the proof of Proposition 4.1.6. Firstly we complete the morphism $\mathcal{M}^c_{\mathbb{Z}[1/e]}(Y) \to \mathcal{M}^c_{\mathbb{Z}[1/e]}(Z)$ to a distinguished triangle

$$\mathcal{M}^c_{\mathbb{Z}[1/e]}(Y) \to \mathcal{M}^c_{\mathbb{Z}[1/e]}(Z) \to J \to \mathcal{M}^c_{\mathbb{Z}[1/e]}(Y)[1]. \tag{4.3}$$

Then for any $j \geq 0$ and k'/k we have a long exact sequence

$$\cdots \to \mathrm{CH}_j(Y_{k'}, \mathbb{Z}[1/e]) \to \mathrm{CH}_j(Z_{k'}, \mathbb{Z}[1/e]) \to h_{2j,j}(J_{k'}, \mathbb{Z}[1/e]) \to \{0\}.$$

Next, $J \otimes \mathbb{Q} \in DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})_{w_{\mathrm{Chow}} \geq 1}$ according to Lemma 4.1.5(2) (combined with Proposition 4.1.6; one should take $r = +\infty$ in it). Applying Theorem 3.6.4(II) we obtain that the groups

$$h_{2j,j}(J_{k'}, \mathbb{Z}[1/e]) \cong \mathrm{Coker}(\mathrm{CH}_j(Y_{k'}, \mathbb{Z}[1/e]) \to \mathrm{CH}_j(Z_{k'}, \mathbb{Z}[1/e]))$$

are annihilated by some constant $E' > 0$ (and E' does not depend on j and k'). Similarly, the functor $M \mapsto Gr_q^{W_D} H^q(M)$ is cohomological (for H that equals either H_{sing} or $H_{et,\mathbb{Q}_\ell}$ and $q \geq 0$); since $W_{D_q} H^q(\mathcal{M}^c_{\mathbb{Q}}(Y)[1]) = 0$ (apply Theorem 3.5.4(3) once again), we obtain that $W_{D_q} H^q(\mathcal{M}^c_{\mathbb{Q}}(Y))$ surjects onto $W_{D_q} H^q(\mathcal{M}^c_{\mathbb{Q}}(Z))$. Thus it suffices to verify that the cokernels of homomorphisms $\mathrm{CH}_j(Z_{k'}, \mathbb{Z}[1/e]) \to \mathrm{CH}_j(X_{k'}, \mathbb{Z}[1/e])$ are annihilated by some constant E'' (for all field extensions k'/k), and that the object $\mathrm{Ker}(W_{D_q} H^q_c(X) \to W_{D_q} H^q_c(Z))$ is r-effective for H that is either étale or singular cohomology (here we invoke Proposition 4.1.6 once again). Hence considering the long exact sequences

$$\cdots \to \mathrm{CH}_j(Z_{k'}, \mathbb{Z}[1/e]) \to \mathrm{CH}_j(X_{k'}, \mathbb{Z}[1/e]) \to \mathrm{CH}_j(U_{k'}, \mathbb{Z}[1/e]) \to \{0\}$$

and

$$0 \to W_{D_q} H^q_c(U) \to W_{D_q} H^q_c(X) \to W_{D_q} H^q_c(Z) \to \ldots$$

we reduce assertion (3) to assertion (2). □

Remark 4.2.2.

(1) We did not formulate all possible statements of this sort above. In particular, we could have considered Chow-weight homology for various staircase sets $\mathcal{I}$; cf. Theorems 3.3.3 and 4.2.3.

Moreover, in §5.1 below we study the (more general) case where certain $\mathbb{Q}$-linear Chow and Chow-weight homology groups are finite dimensional. In particular, Corollary 5.1.6(2) below generalizes parts (2) and (4) of our theorem.

(2) Recall also that the assumption of the r-effectivity of the q-th (Deligne) weight factor of $H^q_c(U_{\mathbb{C}})$ of the singular cohomology of U with compact support is conjecturally equivalent to the vanishing of $\mathrm{CH}_j(U, \mathbb{Q})$ for $0 \leq j < r$; one should just combine our theorem with Proposition 3.5.5.

(3) Recall that a large family of examples to our theorem can be constructed by means of Remark 4.1.7(2); however, these examples may also be treated "directly".

So it may be more interesting to apply our theorem to the case where g is (proper and) surjective (and for any $r > 0$; see Lemma 4.1.5(2)); the resulting statement appear to be new.

Applying part II of Corollary 3.6.5 instead of its part III (that was used in the proof of Theorem 4.2.1) we easily obtain the following statement (in which the vanishing of lower Chow groups condition is replaced by the vanishing of higher Chow groups of 0-cycles).

Theorem 4.2.3. *Let U, r, K_0 be as in Theorem 4.2.1, and assume $\mathrm{CH}_0(U_{K_0}, j, \mathbb{Q}) = \{0\}$ (cf. Theorem 5.3.14 of [Kel17]) for $0 \leq j < r$.*

(1) *Then there exists $E > 0$ such that $E\,\mathrm{CH}_0(U_{k'}, j, \mathbb{Z}[1/e]) = \{0\}$ for all $0 \leq j < r$ and all field extensions k'/k.*

(2) *If k is a subfield of $\mathbb{C}$ then for any $q, s \geq 0$ the $q-s$-th (Deligne) weight factor of $H^q_c(U)$ of the singular cohomology of U with compact support and is r-effective as a pure Hodge structure. Furthermore, the same property of Deligne weight factors of $H^q_c(U)$ is fulfilled for the $\mathbb{Q}_\ell$-étale cohomology of $U_{k^{alg}}$ with compact support if k is an essentially finitely generated field (see Definition 2.3.1(1)) and $\ell \neq p$.*

Proof. The proof is quite similar to that of Theorem 4.2.1(1)–(2); one should only recall that $\mathrm{CH}_0(U_{k'}, j, \mathbb{Q}) \cong h_{j,0}(\mathcal{M}^c_{\mathbb{Z}[1/e]}(U)_{k'}, \mathbb{Z}[1/e]) = \{0\}$ if $j < 0$, and apply Corollary 3.6.5(II) to the motive $\mathcal{M}^c_{\mathbb{Z}[1/e]}(U)[-r]$. $\square$

Now we discuss the relation of our results to the number of points of varieties over finite fields. Proposition 4.2.4 is essentially a combination of Theorem 3.2.1 with the consequences of the Grothendieck-Lefschetz trace formula that are probably well-known to experts in the field.

Proposition 4.2.4.

(1) *Assume that k is a subfield of the finite field $\mathbb{F}_q$. Then there exists a function* Card_q *from* $\mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$ *into the ring A of integral algebraic numbers such that for any distinguished triangle $M \to N \to O \to M[1]$ in $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$ we have*

$$\mathrm{Card}_q(N) = \mathrm{Card}_q(M) + \mathrm{Card}_q(O) \tag{4.4}$$

and for any $X \in \mathrm{Var}$ and $M = \mathcal{M}^c_{\mathbb{Q}}(X)$ we have $\mathrm{Card}_q(M) = \#X(\mathbb{F}_q)$ *(the number of $\mathbb{F}_q$-points of X).*

Moreover, for any M in $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})\langle 1\rangle$ the number $\mathrm{Card}_q(M)$ *is divisible by q in A.*

(2) *Assume that X is a proper k-variety; take the morphism $h : M = \mathcal{M}_{\mathbb{Q}}(X) = \mathcal{M}^c_{\mathbb{Q}}(X) \to \mathbb{Q} = \mathcal{M}^c_{\mathbb{Q}}(\mathrm{pt})$ corresponding to the projection $X \to \mathrm{Spec}\, k$ (see Definition 4.1.1(2)) and set $\tilde{M} = \mathrm{Cone}(h)$. Then $\mathrm{Card}_q(X) \equiv 1 \mod q$ whenever either of the following equivalent conditions is fulfilled:*
   (i) $\tilde{M} \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})\langle 1\rangle$*;*
   (ii) $\mathrm{CWH}^i_0(\tilde{M}_{K_0}, \mathbb{Q}) = \{0\}$ *(see Definition 3.1.1) for all $i \in \mathbb{Z}$ and a universal domain K_0 containing k;*
   (iii) $\mathrm{CWH}^0_0(M_{K_0}, \mathbb{Q}) = \mathbb{Q}$ *and* $\mathrm{CWH}^i_0(M_{K_0}, \mathbb{Q}) = \{0\}$ *for all $i \neq 0$.*

Proof. (1) We use the étale cohomology functor $H_{et,\mathbb{Q}_\ell} = H_{et,\mathbb{Q}_\ell}(-_{\mathbb{F}})$ mentioned in Definition 3.5.3(4), where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$. Let us recall that for any $X \in \mathrm{Var}$ and $i \in \mathbb{Z}$ the $\mathbb{Q}_\ell$-vector spaces $H^i_{et,\mathbb{Q}_\ell}(X_{\mathbb{F}})$ are well-known to be finite-dimensional and almost all of them (when i varies) are zero; hence the same is true for the corresponding cohomology of Chow motives. Since the subcategory $\mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Q})$ densely generates $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$, we obtain that these finiteness properties extend to $\{H^i_{et,\mathbb{Q}_\ell}(M_{\mathbb{F}}),\ i \in \mathbb{Z}\}$ for any $M \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$ as well.

We will write $Frob_q : x \mapsto x^q$ for the (arithmetic) Frobenius automorphism of $\mathbb{F}$. Our candidate for $\mathrm{Card}_q(M)$ will be the trace of the action of the geometric Frobenius automorphism $g = Frob_q^{-1} \in G$ on the (finite dimensional $\mathbb{Q}_\ell$-vector space) $\bigoplus_{i\in\mathbb{Z}} H^i_{et,\mathbb{Q}_\ell}(M_{\mathbb{F}})$; a priori we have $\mathrm{Card}_q(M) \in \mathbb{Q}_\ell$. Since H is a cohomological functor, it converts distinguished triangles into long exact sequences; this obviously implies the property (4.4).

Now we study the values of Card_q. Theorem 5.2.2 of [DeK73] says that the eigenvalues of the action of g on $H^i_{c,et}(X_{\mathbb{F}})$ are integral algebraic numbers (i.e., belong to A) for any $X \in \mathrm{Var}$ and $i \in \mathbb{Z}$. Hence these properties are also fulfilled for $H^i_{et}(M_{\mathbb{F}})$ for any $M \in \mathrm{Obj}\, \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Q})$; thus they are valid for any $M \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$ as well. To conclude the proof it obviously suffices to note that for any object M of $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$ we have $\mathrm{Card}_q(M\langle 1\rangle) = q\, \mathrm{Card}_q(M)$ (once again, it suffices to verify this equality for $M \in \mathrm{Obj}\, \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Q})$ only).

(2) The previous assertion implies that $1 - \#X(\mathbb{F}_q) = \mathrm{Card}_q(\tilde{M})$. Moreover, if condition (i) is fulfilled then this (integral!) number is divisible by q. Next, conditions (ii) and (iii) are obviously equivalent. It remains to note that condition (i) is equivalent to condition (ii) according to Theorem 3.2.1(1). □

Remark 4.2.5.

(1) Recall that in (Theorem 1.1 of) [Esn03] essentially a particular case of Proposition 4.2.4(2) was established (actually, K_0 equal to the algebraic closure of $k(X)$ instead of being a universal domain was considered; yet one can easily look at our proofs and note that this is a minor distinction that does not affect any applications; cf. Proposition 5.2.3(1)). X was assumed to be smooth projective; hence $\mathrm{CWH}^i_j(M_{K_0}, \mathbb{Q}) = \{0\}$ for $i \neq 0$ and $\mathrm{CWH}^0_0(M_{K_0}, \mathbb{Q}) \cong h_{0,0}(M_{K_0}, \mathbb{Q}) \cong \mathrm{CH}_0(X_{K_0}, \mathbb{Q})$. Next, the corresponding statement was applied to smooth *rationally chain connected varieties*, that is, one assumes that (for K_0 as above) any two closed points of X_{K_0} can be linked by a connected chain of rational projective curves (cf. Definition IV.3.2.1, Exercise IV.3.2.5, Corollary IV.3.5.1, and Proposition IV.3.6.2 of [Kol96]); recall that this condition is fulfilled for Fano varieties.

Certainly, our proposition (and actually the whole paper) says nothing new on this number on points matter when restricted to the case where X is (proper and) smooth.

However (as demonstrated by J. Kollár's example in [BlE08, §3.3]) the situation becomes more complicated if X is allowed to be singular. Consequently, we suggest to look at the negative degree Chow-weight homology of M (or $\tilde{M}$) in the case where X is a non-smooth rationally chain connected variety.

(2) More generally, if k is an extension of $\mathbb{F}_q$ and $g : X \to Y$ is a proper morphism then for $\tilde{M}' = \mathrm{Cone}(\mathcal{M}^c_{\mathbb{Q}}(g))$ we clearly have the following: if $\tilde{M}' \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, R)\langle r\rangle$ for some $r > 0$ then $\#X(\mathbb{F}_q) \equiv \#Y(\mathbb{F}_q) \mod q^r$. Thus it does make sense to consider (also, higher-dimensional) Chow-weight homology of motives $\tilde{M}'$ of this sort.

Recall also that in the case where g is a dominant morphism of smooth proper varieties (consequently, Chow-weight homology of $\mathcal{M}^c_{\mathbb{Q}}(X)$ and $\mathcal{M}^c_{\mathbb{Q}}(Y)$ vanishes in non-zero degrees once again) and $r = 1$ this statement essentially coincides with Corollary 1.3 of [FaR05]. However, one can clearly "multiply" any example of this sort by an arbitrary k-variety V. Then clearly $\tilde{M}' \times \mathcal{M}^c_{\mathbb{Q}}(V) \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, R)\langle 1\rangle$ and $\#X \times V(\mathbb{F}_q) \equiv \#Y \times V(\mathbb{F}_q) \mod q$; yet one cannot deduce these facts from the properties of Chow groups of $X \times V$ and $Y \times V$ directly (unless V is smooth and proper).

(3) We could have based our proof on Theorem 8.1 of [Kah09] (cf. also Theorem 9.1 of ibid.); then we would obtain that all the values of our function Card_q are actually integral.

4.3. **On the support of Chow groups of proper smooth varieties.** Now we study in detail the case where X is proper and smooth in the setting of Proposition 4.1.6. The point is that in this case the endomorphisms of $\mathcal{M}^c_R(X)$ can be expressed in terms of algebraic cycles on $X \times X$; consequently, we are able to prove certain (partially new) statements that are formulated in this language.

Proposition 4.3.1. *Let $r > 0$; assume that K_0 is a universal domain containing k.*

Let $g : Y \to X$ be a morphism of smooth proper k-varieties, $Z = \operatorname{Im} g$, $U = X \setminus Z$ (cf. Proposition 4.1.6), and denote $\mathcal{M}^c_{\mathbb{Q}}(g)$ by h.

Then the following conditions are equivalent.

(1) $\mathrm{CH}_j(U_{K_0}, \mathbb{Q}) = \{0\}$ *for* $0 \le j < r$.
(2) *The equivalent conditions of Corollary 3.3.9 are fulfilled for the morphism* $\mathcal{M}_{\mathbb{Q}}(Y) \xrightarrow{h} \mathcal{M}_{\mathbb{Q}}(X)$ *of Chow motives,* $c_1 = 0$, *and* $c_2 = r$.
(3) *The diagonal of $X \times X$ (considered as a cycle on it) is rationally equivalent to the sum of a cycle supported on $Z \times X$ and a cycle supported on $X \times X'$, where $X' \subset X$ is a closed subvariety of codimension r.*

Proof. According to Proposition 4.1.6, condition (1) is equivalent to the surjectivity of the homomorphisms $\mathrm{CH}_j(g_{K_0}, \mathbb{Q})$ for $0 \le j < r$, i.e., to condition (1) of Corollary 3.3.9; thus conditions (1) and (2) are equivalent.

Next, the easy arguments described in Remark 3.3.10(1) immediately yield that condition (2) is equivalent to (3). □

Remark 4.3.2.

(1) Recall that for any closed subvariety Z of X there exists some proper $g : Y \to X$ such that Y is smooth and $\operatorname{Im} g = Z$ according to the seminal result of de Jong (cf. the stronger Gabber's Corollary 2.1.15 of [Kel17]). Note also that here we can choose Y whose dimension equals that of Z.
(2) Now we demonstrate that our proposition implies Proposition 6.1 of [Par94].

   So, for a smooth projective k-variety X, closed subvarieties V_j of X for $0 \le j < r$, and K_0 as above we assume that $\mathrm{CH}_j((X \setminus V_j)_{K_0}, \mathbb{Q}) = \{0\}$ for $0 \le j < r$. Then we can take $Z = \cup_{0 \le j < r} V_j$ and apply Proposition 4.3.1; hence condition (3) says that the diagonal in $X \times X$ is rationally equivalent to the sum of a cycle supported on $Z \times X$ and a cycle supported on $X \times X'$, where X' is of codimension r in X. Decomposing the first of these cycles into the sum of cycles supported on $V_j \times X$ (for $0 \le j < r$) we obtain loc. cit.
(3) Certainly, the authors would like to suggest the readers to study the negative degree Chow-weight homology of $C = \mathcal{M}^c_{\mathbb{Q}}(U)$ as well (note that computations of this sort are closely related to cohomology; cf. Theorem 3.5.4 and Proposition 3.5.5 and Theorem 4.2.1). Obviously, one can argue similarly to Corollary 3.3.9 and Remark 3.3.10(1) to obtain certain equivalent conditions in terms of algebraic cycles provided that the weight complex $t = t(C)$ or (equivalently) $t' = t(\mathcal{M}^c_{\mathbb{Q}}(Z))$ is known.

Thus it makes sense to recall that t can be expressed in the (more or less) obvious way in terms of an arbitrary smooth proper hypercover of Z (here one can apply the h-topological $\mathbb{Q}$-linear version of [Kel17, Theorem 4.0.7] noting that the arguments in the proof of loc. cit. give this modification without any difficulty); cf. also Remark 1.4.3(2).

In particular, if $\{Z_i\}$ are irreducible components of Z and (all Z_i and) the intersections of all subsets of $\{Z_i\}$ are smooth then one can take the $-n$-th term of t to be equal to $\bigoplus_{J \subset I,\ \#J=n} \mathcal{M}_{\mathbb{Q}}(\cap_{i \in J} Z_i)$ and the boundary morphisms to be the obvious ones; cf. Proposition 6.5.1 of [Bon09].

Recall also that any smooth U can be presented in this form (i.e., as $X' \setminus (\cup Z'_i)$ for some smooth proper X' and a normal crossing divisor $\cup Z'_i$) if $p = 0$.

One can also say something about $t_R(C)$ in the case $R \neq \mathbb{Q}$ (even if $p < 0$); see Remark 4.3.2(4) of [BoS14].

Now we want to discuss certain conditions that are equivalent to (combinations of) collections of support assumptions (motivated by Theorem 1.7 of [Lat96]). Our methods allow us to study the case of a general R here (in contrast to ibid.); however, in this case we need the following substitute of Proposition 4.3.1.

Lemma 4.3.3. *Assume that X is smooth and proper, and for a closed subvariety Z of X and $U = X \setminus Z$ the groups $\mathrm{CH}_j(U_K, R)$ vanish for $0 \leq j < r$ (for some $r > 0$) and all function fields K/k.*

Then $\mathcal{M}_R(X)$ is a retract of $\mathcal{M}_R(Y) \bigoplus \mathcal{M}_R(Q)\langle r\rangle$ for some $Y, Q \in \mathrm{SmPrVar}$ with $\dim Y = \dim Z$.

Proof. According to Lemma 4.1.4(4), there exists a smooth projective k-variety Y with $\dim Y = \dim Z$ along with a morphism $h : \mathcal{M}^c_R(Y) \to \mathcal{M}^c_R(Z)$ such that $\dim Y = \dim Z$ and h gives a weight decomposition of $\mathcal{M}^c_R(Z)$; hence the homomorphisms $\mathrm{CH}_j(h_K)$ are surjective for all function fields K/k and $j \geq 0$ (see Lemma 4.1.4(5)). Next, the long exact sequence (4.1) yields that $\mathrm{CH}_j(Z_K)$ surjects onto $\mathrm{CH}_j(X_K)$ for all function fields K/k and $0 \leq j < r$. Thus the composed morphism $h' : \mathcal{M}^c_R(Y) \to \mathcal{M}^c_R(X)$ gives a surjection of the corresponding Chow groups as well. Applying Corollary 3.3.9 for $c_1 = 0$ and $c_2 = r$ we conclude that the morphism id_h factors through $\mathcal{M}_R(Y) \bigoplus \mathcal{M}_R(Q)\langle r\rangle$ for some $Q \in \mathrm{SmPrVar}$ (cf. Remark 0.5). $\square$

Proposition 4.3.4. *Let X be a smooth proper variety, $r \geq 0$, and $c > 0$.*

Then the following conditions are equivalent.

1. *The motive $M = \mathcal{M}_R(X)$ is a retract of a Chow motive of the form $\bigoplus_{0 \leq j \leq c} \mathcal{M}_R(P_j)\langle j\rangle$, where $P_j \in \mathrm{SmPrVar}$ for all j and $\dim P_j \leq r$ for $j < c$.*
2. *There exist closed subvarieties $V_j \subset X$ for $0 \leq j < c$ such that for all j we have $\dim V_j \leq j + r$ and $\mathrm{CH}_j((X \setminus V_j)_K, R) = \{0\}$ (i.e., the group $\mathrm{CH}_j(X_K, R)$ is "supported on" $V_{j,K}$) for all field extensions K/k.*
3. *The diagonal Δ of $X \times X$ (considered as an algebraic cycle on it) is rationally equivalent to the sum $\sum_{j=0}^{c} \Delta_j$, where the cycle Δ_j is supported on $W_j \times V_j$ for $j < c$ and on $W_c \times X$ for $j = c$ and V_j (for $0 \leq j < c$) are closed subvarieties of X of dimension at most $j + r$ and W_j (for $0 \leq j \leq c$) are closed subvarieties of X of codimension at least j.*

Moreover, if $R = \mathbb{Q}$ then one can take a single universal domain K_0 containing k for K in condition (2).

Proof. Once again, Proposition 2.3.4(II) implies that in the case $R = \mathbb{Q}$ condition (2) is equivalent to its K_0-version.

Thus it suffices to prove the main part of the statement. We fix some X, r, and c as above, and recall that $M = \mathcal{M}_R(X)$ is a Chow motive itself according to Lemma 4.1.4(1).

First we prove that condition (1) implies (2). Assume that condition (1) is fulfilled; we will check the support condition for certain $j = j_0$, $0 \leq j_0 < c$. Denote by p the corresponding split surjective morphism $p : \bigoplus_{0 \leq j \leq c} \mathcal{M}_R(P_j)\langle j\rangle \to M$; p_K

clearly gives a surjection of the h_{2j_0,j_0}-groups. Moreover, $h_{2j_0,j_0}(\mathcal{M}_R(P_{jK})\langle j\rangle, \mathbb{Q}) = \{0\}$ whenever $j > j_0$; hence for $N_{j_0} = \bigoplus_{0\le j\le j_0} \mathcal{M}_R(P_j)\langle j\rangle$ the corresponding retract p_{j_0} of p is converted by the functor $h_{2j_0,j_0}(-_K, R)$ into a surjection as well.

Now we choose a presentation of p_{j_0} as an algebraic cycle on $Q_{j_0} = (\sqcup_{0\le j\le j_0} P_j)\times X$; this cycle is supported on a subvariety R_{j_0} of Q_{j_0} of dimension at most $r + j_0$. Then the definition of the action of correspondences on cycles implies that $\mathrm{CH}_{j_0}(X_K)$ is supported on the image of $R_{j_0,K}$ in X_K (with respect to the projection $Q_{j_0,K} \to X_K$). Since the latter has dimension not greater than that of R_{j_0} (and comes by base change from the corresponding k-variety), we obtain the implication in question.

Next we prove that condition (3) implies condition (2) by an argument rather similar to the one that we have just used. We fix j_0, $0 \le j_0 < c$, and find a support k-variety for $\mathrm{CH}_{j_0}(X_K)$ (for all K). Arguing similarly to the proof of Proposition 2.2.6(3) we easily obtain that for any $j > j_0$ the endomorphism h_j of M corresponding to the cycle Δ_j factors through $\mathrm{Chow}^{\mathrm{eff}}(k, R)\langle j\rangle$; hence its action on the group $\mathrm{CH}_{j_0}(X_K)$ is zero. Therefore it suffices to note that for $0 \le j \le j_0$ the elements of $h_{j*}(\mathrm{CH}_{j_0}(X_K))$ are supported on $V_{j,K}$ (by the classical theory of correspondences), and the dimensions of these V_j are at most $j_0 + r$.

Now we prove that condition (2) implies condition (1). Assume that condition (2) is fulfilled (for our X, r, and c). Then Lemma 4.3.3 implies that for each j, $0 \le j < c$, the morphism id_M may be factored through $\mathcal{M}_R(Y_j) \bigoplus \mathcal{M}_R(Q_j)\langle j+1\rangle$ for some $Y_j, Q_j \in \mathrm{SmPrVar}$ such that $\dim Y_j \le j + r$ (for all j). We "compose these factorizations" starting from the last one, i.e., we factor id_M through the chain of objects $M \to \mathcal{M}_R(Y_{c-1}) \bigoplus \mathcal{M}_R(Q_{c-1})\langle c\rangle \to \mathcal{M}_R(Y_{c-2}) \bigoplus \mathcal{M}_R(Q_{c-2})\langle c-1\rangle \to \dots \mathcal{M}_R(Y_0) \bigoplus \mathcal{M}_R(Q_0)\langle 1\rangle \to M$. This gives a decomposition of id_M into 2^c summands e_l such that each of these endomorphisms factors either through $\mathcal{M}_R(Y_{c-i}) \bigoplus \mathcal{M}_R(Q_{c-i})\langle c-i+1\rangle$ at the "ith step". It obviously suffices to verify that each of e_l factors through certain $\mathcal{M}_R(P)\langle j\rangle$ such that $P \in \mathrm{SmPrVar}$ and either $j = c$ or $0 \le j < c$ and $\dim P_j \le r$. Now we choose one of these e_l and consider the smallest i such that e_l factors through $\mathcal{M}_R(Q_{c-i})\langle c-i+1\rangle$. If there is no such i then e_l factors through $\mathcal{M}_R(Y_0)$; thus we can take $j = 0$ and $P = Y_0$. If this minimal i equals 1 then we can take $j = c$ and $P = Q_c$. In other cases the morphism e_l factors firstly through $\mathcal{M}_R(Y_{c-i+1})$ and through $\mathcal{M}_R(Q_{c-i})\langle c-i+1\rangle$ after that; thus Proposition 2.2.6(3) implies that e_l factors through $\mathcal{M}_R(P)\langle c-i+1\rangle$ for some P of dimension at most $\dim Y_{c-i+1} - (c-i+1) \le r$.

Lastly we prove that condition (1) implies condition (3). It clearly suffices to verify for $0 \le j \le c$ that an endomorphism h_j of M that factors through $\mathcal{M}_R(P_j)\langle j\rangle$, where $P_j \in \mathrm{SmPrVar}$ and $\dim P_j \le r$ if $j < c$, can be presented by a cycle Δ_j that satisfies the support assumptions of condition (3). Consequently, we present h_j as a composition $M \xrightarrow{a} \mathcal{M}_R(P_j)\langle j\rangle \xrightarrow{b} M$. Now, Proposition 2.2.6(3) gives the existence of an open embedding $w : W' \to P$ such that $W_j = P \setminus W'$ is of codimension j in P and $a \circ \mathcal{M}_R(w) = 0$. Hence we can choose a presentation of a as an algebraic cycle supported on W_j. Next (similarly to the proof (1) $\Longrightarrow$ (2)), we consider the support variety R_j for some cycle in $P_j \times P$ that represents b, and take V_j to be the image of R_j in P. Obviously, V_j is of dimension at most $j + r$ if $j < c$. It remains to note that the composition $b \circ a = h_j$ is clearly supported on $W_j \times V_j$ as an algebraic cycle. $\square$

Remark 4.3.5.

(1) In the case $K = K_0$ and $R = \mathbb{Q}$ our conditions (3) and (2) are precisely conditions (i) and (ii) of [Lat96, Theorem 1.7].

(2) Now let us discuss possible variations of the argument that we used to deduce condition (1) from condition (2).

One can certainly re-formulate it inductively to obtain the following: condition (1) is fulfilled if and only if M is a retract both of a motive of the form $\bigoplus_{0\le j\le c-1}\mathcal{M}_R(P'_j)\langle j\rangle$, where $P'_j \in \mathrm{SmPrVar}$ for all j and $\dim P'_j \le r$ for $j < c-1$, and also of $\mathcal{M}_R(Y_{c-1})\bigoplus\mathcal{M}_R(Q_{c-1})\langle c\rangle$ for some $Y_{c-1}, Q_{c-1} \in \mathrm{SmPrVar}$ such that $\dim Y_{c-1} \le c + r - 1$ (see Lemma 4.3.3).

Now we pass to a "triangulated" version of the equivalence of these conditions. The proof of this result is also somewhat similar to the aforementioned part of the proof of Proposition 4.3.4.

Proposition 4.3.6. *Let $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)$, $r \ge 0$, and $c > 0$.*
Then the following conditions are equivalent.

(1) *M is an object of the subcategory $\underline{D}_{r,c}$ of $DM^{\mathrm{eff}}_{gm}(k,R)$ densely generated by $\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle c\rangle \cup (\cup_{0\le j<c}\operatorname{Obj}(d_{\le r}\mathrm{Chow}^{\mathrm{eff}}(k,R))\langle j\rangle)$.*

(2) *M is an object both of $\underline{D}_{r,c-1}$ and of the category*

$$\underline{E}_{r,c} = \langle \operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle c\rangle \cup \operatorname{Obj}(d_{\le r+c-1}\mathrm{Chow}^{\mathrm{eff}}(k,R))\rangle.$$

(3) *M is an object of $\underline{E}_{r,j}$ for all $0 < j \le c$.*

Proof. Obviously, condition (1) implies condition (2), and the latter implies condition (3). Moreover, obvious induction (cf. Remark 4.3.5(2)) implies that it suffices to verify that condition (2) implies condition (1) for all $c > 0$ (whereas we can assume r to be fixed).

So we assume that condition (2) is fulfilled. Similarly to Corollary 2.2.4(1,3), Proposition 1.2.4(8) implies that the Chow weight structure on $DM^{\mathrm{eff}}_{gm}(k,R)$ restricts to $\underline{D}_{r,j}$ and $\underline{E}_{r,j}$ for any $j \ge 0$, and the corresponding hearts $\underline{HD}_{r,j}$ and $\underline{HE}_{r,j}$ are the Karoubi-closures in $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ of the sets

$$\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j\rangle \bigoplus \Big(\bigoplus_{0\le l<j}\operatorname{Obj}(d_{\le r}\mathrm{Chow}^{\mathrm{eff}}(k,R))\langle l\rangle\Big)$$

and of $\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,R)\langle j\rangle \bigoplus \operatorname{Obj}(d_{\le r+j-1}\mathrm{Chow}^{\mathrm{eff}}(k,R))$, respectively.

Now, Proposition 2.2.6(3) easily implies that any morphism from $\underline{HE}_{r,c}$ into $\underline{HD}_{r,c-1}$ factors through $\underline{HD}_{r,c}$ (cf. the proof that condition (2) implies (1) in Proposition 4.3.4). Thus applying Proposition 1.9 of [Bon18a] (cf. also Remark 2.3(2) of ibid.) we obtain the result in question. □

Remark 4.3.7.

(1) The authors do not know of any "nice" if and only if criteria for $M \in \operatorname{Obj} DM^{\mathrm{eff}}_{gm}(k,R)$ to be an object of the subcategory $\underline{E}_{r,j} \subset DM^{\mathrm{eff}}_{gm}(k,R)$ (see the previous proposition). However, M is clearly an object of $\underline{E}_{r,j}$ whenever it is an extension of an object of M_1 of $d_{\le r+j-1}DM^{\mathrm{eff}}_{gm}(k,R)$ by an object M_2 of $DM^{\mathrm{eff}}_{gm}(k,R)\langle j\rangle$. Moreover, we can check whether M_2 is an object of $DM^{\mathrm{eff}}_{gm}(k,R)\langle j\rangle$ by looking at its Chow-weight homology; see Theorem 3.2.1(1).

(2) Furthermore, Proposition 4.1.2(4) says that the motive $M = \mathcal{M}^c_R(X)$ for $X \in \mathrm{Var}$ is an extension of $M_2 = \mathcal{M}^c_R(X \setminus Z)$ by $M_1 = \mathcal{M}^c_R(Z)$ whenever Z is a closed subvariety of X. Now, M_1 is an object of $d_{\le r+j-1} DM^{\mathrm{eff}}_{gm}(k, R)$ if Z is of dimension at most $r+j-1$ by Lemma 4.1.4(3); thus to prove that M is an object of the subcategory $\underline{E}_{r,j}$ it suffices to suppose in addition that $\mathrm{CWH}^i_r(M_{2,K}) = \{0\}$ for all $i \in \mathbb{Z}$, $0 \le r < j$, and all function fields K/k.

Note also that one can check if a motive M_1 belongs to

$$d_{\le r+j-1} DM^{\mathrm{eff}}_{gm}(k, R)$$

by looking at its *Chow-weight cohomology*; see Proposition 5.2.1 below.

(3) Clearly, all the "motivic" conditions of this subsection (see condition (2) in Proposition 4.3.1, condition (1) in Proposition 4.3.4, and Proposition 4.3.6(1)) easily imply certain properties for (co)homology of M; cf. Proposition 3.5.1 and Theorem 3.5.4.

4.4. **On (tensor) products.** First we deduce a simple corollary from Corollary 3.3.7(II).

Corollary 4.4.1. *Assume that $U = U_1 \times U_2$, where $U_1, U_2 \in \mathrm{Var}$, and for some $r_1, r_2 \ge 0$ and a universal domain $K_0 \supset k$ we have $h_j(U_{iK_0}, \mathbb{Q}) = \{0\}$ for $0 \le j < r_i$ and $i = 1, 2$.*

Then $h_j(U_{K_0}, \mathbb{Q}) = \{0\}$ for $0 \le j < r_1 + r_2$.

Proof. According to Proposition 4.1.6 (see Remark 4.1.7(1)), our vanishing assumptions imply that $\mathcal{M}^c_{\mathbb{Q}}(U_i) \in DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})^{\langle r_i \rangle}_{\ge 0}$ for $i = 1, 2$. Hence Corollary 3.3.7(II) along with Proposition 4.1.2(5) imply that $\mathcal{M}^c_{\mathbb{Q}}(U_1 \times U_2)$ belongs to $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})^{\langle r_1 + r_2 \rangle}_{\ge 0}$. It remains to apply the converse implication in Remark 4.1.7(1). □

Remark 4.4.2.

(1) Corollaries 4.4.1 and 4.4.5 are quite non-trivial since there certainly cannot exist any Künneth-type formulae for Chow groups (or Chow-weight homology) of general varieties and motives.

(2) One can easily prove the following R-linear version of Corollary 4.4.1 (for any $\mathbb{Z}[1/e]$-algebra R): if $h_j(U_{iK}, R) = \{0\}$ for all $0 \le j < r_i$, all function fields K/k, and $i = 1, 2$, then $h_j(U_K, R) = \{0\}$ for $0 \le j < r_1 + r_2$ and all K of this sort. These vanishing assumptions are equivalent to $\mathcal{M}^c_R(U_i) \in DM^{\mathrm{eff}}_{gm}(k, R)^{\langle r_i \rangle}_{\ge 0}$; see Remark 4.1.7(4).

Now we will try to deduce some curious statements on motives and varieties from simple properties of Hodge structures. Unfortunately, this requires assumptions A and B of Proposition 3.5.5 (and confines us to $\mathbb{Q}$-linear motives).

Proposition 4.4.3. *Assume that $p = 0$, assumptions* A *and* B *of Proposition 3.5.5 are fulfilled, $M_1, M_2 \in \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})$, and $r_1, r_2 > 0$. Then the following statements are fulfilled.*

(1) *If $M_1 \notin \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})\langle r_1 \rangle$ and $M_2 \notin \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})\langle r_2 \rangle$ then $M_1 \otimes M_2 \notin \mathrm{Obj}\, DM^{\mathrm{eff}}_{gm}(k, \mathbb{Q})\langle r_1 + r_2 - 1 \rangle$.*

(2) *If* $M_1 \notin \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})^{t_{hom}^{\mathbb{Q}}\le r_1}$ *and* $M_2 \notin \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})^{t_{hom}^{\mathbb{Q}}\le r_2}$ *then* $M_1\otimes M_2 \notin DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})^{t_{hom}^{\mathbb{Q}}\le r_1+r_2+1}$.

(3) *If* M_1 *belongs to* $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{w_{\mathrm{Chow}}\ge 0} \setminus DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{\ge 0}^{\langle r_1\rangle}$ *and* $M_2 \in DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{w_{\mathrm{Chow}}\ge 0}\setminus DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{\ge 0}^{\langle r_2\rangle}$ *then* $M_1\otimes M_2\notin DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{\ge 0}^{\langle r_1+r_2-1\rangle}$.

Proof. Firstly we note that both M_1 and M_2 are defined over some countable subfield k' of k (see Proposition 5.2.3(1)). Next, all the conditions on motives in this proposition can be "detected" by (the vanishing of the corresponding) Chow-weight homology of C_{K_0}, where C either M_1, M_2, or $M_1\otimes M_2$, and K_0 is a universal domain that contains k; see Corollaries 3.3.7(I) and 3.4.2 along with Proposition 3.4.1(3). Thus we can assume $k=k'$; hence there exists an embedding of k into $\mathbb{C}$.

Recall now that the "total Hodge" version of singular cohomology as provided by Theorem 2.3.3 of [Hub00] is a tensor exact functor. Consequently, for any $m\in\mathbb{Z}$ we have a (Künneth) filtration on $H^m_{\mathrm{sing}}(M_1\otimes M_2)$ whose factors are $H^l_{\mathrm{sing}}(M_1)\otimes H^{m-l}_{\mathrm{sing}}(M_2)$ for l running through integers, where H_{sing} is the "mixed Hodge" version of singular cohomology (see Definition 3.5.3(4)). Next, if $F^{c_1}V^1_{\mathbb{C}}\neq V^1_{\mathbb{C}}$ and $F^{c_2}V^2_{\mathbb{C}}\neq V^2_{\mathbb{C}}$ for (effective) mixed Hodge structures V^1 and V^2 and $c_1,c_2>0$ then $F^{c_1+c_2-1}(V^1\otimes V^2)_{\mathbb{C}}\neq (V^1\otimes V^2)_{\mathbb{C}}$; see Examples 3.2(2) of [PeS08].

Now we apply Proposition 3.5.5. The assumptions of assertion (1) imply (see Theorem 3.2.1(1)) that there exist $q_1,q_2\in\mathbb{Z}$ such that $F^{r_1}H^{q_1}_{\mathrm{sing}}(M_1)\neq H^{q_1}_{\mathrm{sing}}(M_1)$ and $F^{r_2}H^{q_2}_{\mathrm{sing}}(M_2)\neq H^{q_1}_{\mathrm{sing}}(M_2)$. Hence $F^{r_1+r_2-1}H^{q_1+q_2}_{\mathrm{sing}}(M_1\otimes M_2)\neq H^{q_1+q_2}_{\mathrm{sing}}(M_1\otimes M_2)$. Thus $M_1\otimes M_2\notin \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})\langle r_1+r_2-1\rangle$ indeed by Theorem 3.5.4(3); this gives assertion (1).

The proofs of assertions (2) and (3) are similar.

Recall that for any $r\in\mathbb{Z}$ if $N\in \operatorname{Obj} DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})^{t_{hom}^{\mathbb{Q}}\le r}$ then N fulfils all the conditions of Theorem 3.3.3(2) for $\mathcal{I}$ being the staircase set $\mathcal{I}[-r]=\{(i,j): i-r>j\ge 0\}$; see Corollary 3.4.2. Thus Proposition 3.5.5 implies that there exist $q_1,q_2,w_1,w_2>0$ such that

$$F^{w_i}(H^{q_i-r_i}_{\mathrm{sing}}(M_i)/W_{D\,q_i+w_i-1}H^{q_i-r_i}_{\mathrm{sing}}(M_i))\neq H^{q_i-r_i}_{\mathrm{sing}}(M_i)/W_{D\,q_i+w_i-1}H^{q_i-r_i}_{\mathrm{sing}}(M_i)$$

for $i=1,2$. Hence the mixed Hodge structure $V=(H^{q_1-r_1}_{\mathrm{sing}}/W_{D\,q_1+w_1-1})(M_1)\otimes (H^{q_2-r_2}_{\mathrm{sing}}/W_{D\,q_2+w_2-2})(M_1)$ is not w_1+w_2-1-effective. Now, the definition of the tensor product of mixed Hodge structures (see loc. cit.) easily implies that V is a quotient of $V'=H^{q_1-r_1}_{\mathrm{sing}}(M_1)\otimes H^{q_2-r_2}_{\mathrm{sing}}(M_2)/W_{D\,q_1+w_1+q_2+w_2-1}(H^{q_1-r_1}_{\mathrm{sing}}(M_1)\otimes H^{q_2-r_2}_{\mathrm{sing}}(M_2))$. Looking at the Künneth filtration of $H^{q_1-r_1+q_2-r_2}_{\mathrm{sing}}(M_1\otimes M_2)$ and applying Theorem 3.5.4(3) once again we conclude the proof of assertion (2).

Lastly, the assumptions of assertion (3) imply (cf. Theorem 3.5.4(3)) that there exist $q_1,q_2\in\mathbb{Z}$ such that both $H^{q_i}(M_i)/W_{D\,q_i-1}H^m(M_i)$ are not r_i-effective. It follows that $H^{q_1+q_2}(M_1\otimes M_2)/W_{D\,q_1+q_2-1}H^m(M_1\otimes M_2)$ is not r_1+r_2-1-effective. Thus $M_1\otimes M_2\notin DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{\ge 0}^{\langle r_1+r_2-1\rangle}$ according to Theorem 3.5.4(3). $\square$

Remark 4.4.4.

(1) Clearly, no analogue of this proposition holds for motives with integral coefficients, since $\mathbb{Z}/l\mathbb{Z}\otimes\mathbb{Z}/m\mathbb{Z}=0$ in $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Z})$ for any mutually prime integers m and l.

(2) Surprisingly, it appears that Proposition 4.4.3 does not extend to the case $p>0$. Indeed, to demonstrate this it suffices to find objects of $\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})$

that are not 1-effective whereas their product is. Now, R. van Dobben de Bruyn's answer [RvD20] hints that tensor powers of (retracts of) motives of abelian varieties over finite fields can give an example of this sort.

(3) One can easily prove some more statements similar to the parts of our proposition. In particular, one can prove the following strengthening of Proposition 4.4.3(3): if $M_i \notin DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r_i\rangle}_{\geq 0}$ for $i=1,2$ then $M_1\otimes M_2 \notin DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r_1+r_2-1\rangle}_{\geq 0}$. We leave this claim as an exercise to the reader since we will not apply it below.

Now we can establish a certain "converse" to Corollary 4.4.1.

Corollary 4.4.5. *Assume that $U=U_1\times U_2$, where $U_1,U_2\in \mathrm{Var}$, $r_1,r_2\geq 0$, $K_0\supset k$ is a universal domain, and assumptions* A *and* B *of Proposition* 3.5.5 *are fulfilled. Suppose $h_j(U_{iK_0},\mathbb{Q})=\{0\}$ for $0\leq j<r_i$ and $i=1,2$, and $h_{r_i}(U_{iK_0},\mathbb{Q})\neq\{0\}$.*

Then $h_{r_1+r_2}(U_{K_0},\mathbb{Q})\neq\{0\}$ as well.

Proof. By Corollary 4.4.1, $h_i(U_{K_0},\mathbb{Q})=\{0\}$ for $i<r_1+r_2$.

Next take $M_i=\mathcal{M}^c_{\mathbb{Q}}(U_i)$ for $i=1,2$. Then we have:

$$M_i\in DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r_i\rangle}_{\geq 0}\setminus DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r_i+1\rangle}_{\geq 0}$$

by Proposition 4.1.6. Thus $M_1\otimes M_2\notin DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})^{\langle r_1+r_2+1\rangle}_{\geq 0}$ according to Proposition 4.4.3(3). Lastly, applying Proposition 4.1.6 once again (see also Remark 4.1.7(1)) we conclude that $h_{r_1+r_2}(U_{K_0},\mathbb{Q})\neq\{0\}$ indeed. $\square$

5. Supplements: small Chow-weight homology, Chow-weight cohomology, and remarks

In this section we deduce some more implications from the previous results.

In §5.1 we consider motives in $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})$ whose Chow-weight homology groups in a "staircase range" $\mathcal{I}$ are finite dimensional (over $\mathbb{Q}$); thus we extend Theorems 3.3.3(2) in the case $R=\mathbb{Q}$. We also prove a motivic criterion for the lower $\mathbb{Q}$-linear Chow groups of a variety X (over a universal domain) to be finite dimensional; it follows that the corresponding weight factors of the singular or étale cohomology of X with compact support are Artin-Tate ones (cf. Theorem 4.2.1).

In §5.2 we dualize Theorem 3.2.1; this allows to bound the dimensions of motives and also their weights (from above) via calculating their *Chow-weight cohomology*. We also note that to verify the vanishing of Chow-weight homology of M (in higher degrees) over arbitrary extensions of k it suffices to compute these groups over (rational) extensions of k of bounded transcendence degrees.

In §5.3 we make some more remarks on our main results. In particular, we propose (briefly) a "sheaf-theoretic" approach to our results, and discuss their possible extensions to motives over a base.

5.1. On motives with "small" Chow-weight homology.

Definition 5.1.1. We write either AT^{eff} or AT^{eff}_k for the class $\{\mathcal{M}_{\mathbb{Q}}(k')\langle j\rangle\}$, where $j\geq 0$ and k' runs through finite extensions of k, and $EAT^{\mathrm{eff}}=\cup_{i\in\mathbb{Z}}AT^{\mathrm{eff}}[i]$.

Theorem 5.1.2. *Assume that K_0 is a universal domain containing k, $\mathcal{I}$ is a staircase set (see Definition* 3.3.1*), and $M\in DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}}\geq i_0}$ for some $i_0\in\mathbb{Z}$.*

Then the groups $\mathrm{CWH}^i_j(M_{K_0},\mathbb{Q})$ *are finite-dimensional* $\mathbb{Q}$*-vector spaces for all* $(i,j)\in\mathcal{I}$ *if and only if* M *belongs to the envelope (see* §1.1*) of the set*

$$\cup_{i\le -i_0}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle a_{\mathcal{I},i}\rangle\cup AT^{\mathrm{eff}})[-i].$$

Proof. Clearly, for any object N of $\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})[-i]\langle a_{\mathcal{I},i}\rangle$ we have:

$$\mathrm{CWH}^{i'}_{j'}(N_{K_0},\mathbb{Q})=\{0\}\text{ for all }(i',j')\in\mathcal{I}.$$

Moreover, the only non-zero Chow-weight homology group of the motive $T=\mathcal{M}_{\mathbb{Q}}(\operatorname{Spec}k')\langle j\rangle[-i]$ (over K_0; here k' is a finite extension of k) is $\mathrm{CWH}^i_j(T_{K_0},\mathbb{Q})=\mathbb{Q}^{[k':k]}$. Since Chow-weight homology functors are homological, we obtain that any element of the envelope in question does have finite-dimensional CWH^i_j-homology over K_0 for $(i,j)\in\mathcal{I}$.

Now we verify the converse implication. Clearly, the number of non-zero Chow-weight homology groups of M is finite.

Assume that $k=K_0$; then any element of $\mathrm{CWH}^i_j(M_{K_0})$ gives a morphism $\mathbb{Q}\langle j\rangle[-i]\to t(M)$. Thus in this case there exists a $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q}))$-morphism $\bigoplus_{l>0}\mathbb{Q}\langle j_l\rangle[-i_l]\to t(M)$ for some $i_l\le -i_0$ and $j_l\ge 0$, such that for its cone C we have $\mathrm{CWH}^i_j(C)=\{0\}$ for all $(i,j)\in\mathcal{I}\cup[1-i_0,+\infty)\times[0,+\infty)$. Applying the $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q}))$-version of Theorem 3.3.3(2) (see Remark 3.3.5) we obtain that C belongs to the $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q}))$-extension-closure of

$$\cup_{i\le -i_0}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})[-i]\langle a_{\mathcal{I},i}\rangle).$$

Hence there exists a choice of $t(M)=(M^s)$ such that $M^s=0$ if $s>i_0$ and $M^s=E^s\langle a_{\mathcal{I},-s}\rangle\bigoplus(\bigoplus_l\mathbb{Q}\langle j_l^s\rangle)$ for some Chow motives E^s and $j_l^s\ge 0$; see Proposition 1.4.2(4). It remains to apply Proposition 1.4.2(5) to conclude the proof in this case.

Now we prove our assertion in the general case step-by-step. Firstly, the "if" implication that we have just proved implies that it suffices to verify the "only if" implication in the case where K_0 is of infinite transcendence degree over k. Hence Lemma 5.1.3 below (this is a rather easy Suslin rigidity-type statement) implies that the aforementioned morphism $\bigoplus_{l>0}\mathbb{Q}\langle j_l\rangle[-i_l]\to t(M)$ is defined over the algebraic closure of k. Arguing as above we obtain that $M_{k^{alg}}$ belongs to the envelope (and actually, also the extension-closure) of the set $\cup_{i\le -i_0}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k^{alg},\mathbb{Q})\langle a_{\mathcal{I},i}\rangle\cup AT^{\mathrm{eff}}_{k^{alg}})[-i]$.

Next, the "continuity" of motivic categories discussed in Remark 1.3.3(4) of [Bon20a] easily yields the existence of a finite extension K/k such that M_K belongs to the envelope of $\cup_{i\le -i_0}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(K,\mathbb{Q})\langle a_{\mathcal{I},i}\rangle\cup AT^{\mathrm{eff}}_K)[-i]$ (actually, Artin-Tate motives can be replaced by Tate motives here).

It remains to apply a rather standard descent argument. Denote the corresponding morphism $\operatorname{Spec}K\to\operatorname{Spec}k$ by f. Then Remark 1.2 and Corollary 3.2(2) of [CiD15] (cf. also Appendix A of [BoK20] or sections A.5 and C in the introduction to [CiD19]) give the existence of (the "effective geometric" version of) the functor f_* that is right adjoint to the functor $-_K$ in Definition 2.1.2(3). Moreover, the composition $f_*\circ -_K$ is isomorphic to the functor $\mathcal{M}_{\mathbb{Q}}(\operatorname{Spec}K)\otimes -$, and for a variety X over K we have $f_*\mathcal{M}_{\mathbb{Q}K}(X)\cong\mathcal{M}_{\mathbb{Q}}(X)$ (we consider X as a $\operatorname{Spec}k$-scheme in the right hand side). Thus applying the functor f_* to the fact that M_K belongs to the envelope of $\cup_{i\le -i_0}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(K,\mathbb{Q})\langle a_{\mathcal{I},i}\rangle\cup AT^{\mathrm{eff}}_K)[-i]$ we obtain that $f_*(M_K)\cong M\otimes\mathcal{M}_{\mathbb{Q}}(\operatorname{Spec}K)$ belongs to the envelope of $\cup_{i\le -i_0}(\operatorname{Obj}\operatorname{Chow}^{\text{eff}}(k,\mathbb{Q})\langle a_{\mathcal{I},i}\rangle \cup AT^{\text{eff}})[-i]$. It remains to note that $\mathbb{Q}$ is a retract of $\mathcal{M}_{\mathbb{Q}}(\operatorname{Spec} K)$ (this is an easy property of Artin motives); hence M is a retract of $M\otimes \mathcal{M}_{\mathbb{Q}}(\operatorname{Spec} K)$ and we obtain the result. □

To conclude the proof we need the following statement.

Lemma 5.1.3. *Let k be an algebraically closed field.*

(1) *Denote the category of smooth connected affine schemes by* SmAffVar, *and suppose that F is a functor* $\text{SmAffVar}^{op}\to \mathbb{Q}-\text{Mod}$ *that satisfies the following condition (*): $F(f)$ is injective whenever f is an* SmAffVar-*morphism that is either an open embedding or finite and flat.*

Assume in addition that there exists an algebraically closed field extension K_0/k of infinite transcendence degree over k such that $\tilde{F}(\operatorname{Spec} K_0)$ is finite dimensional over $\mathbb{Q}$; here $\tilde{F}$ is the natural extension of F to pro-smooth affine connected k-schemes (that is, we set $\tilde{F}(\varprojlim X_i)=\varinjlim F(X_i)$, where X_i is any projective system of smooth connected affine varieties; cf. §1.4 of [Deg11]*).*

Then F is a constant functor; thus for the morphism $p_0:\operatorname{Spec} K_0\to \mathrm{pt}$ the homomorphism $\tilde{F}(p_0)$ is bijective.

(2) *Condition (*) of assertion* (1) *is fulfilled whenever F equals the Nisnevich sheafification of the presheaf $G: X\mapsto H_0((h_{2j+l,j}(M^i\otimes \mathcal{M}_{\mathbb{Q}}(X),\mathbb{Q})))$ (the zeroth homology of this complex) for any fixed $j,l\in\mathbb{Z}$ and any complex M^i as in Proposition* 2.3.4. *Moreover, $\tilde{F}(\operatorname{Spec} K)$ is isomorphic to $H_0((h_{2j+l,j}(M^i_K,\mathbb{Q})))$ whenever K is an extension of k.*

Proof. (1) Assume that $\tilde{F}(\operatorname{Spec} K_0)\cong \mathbb{Q}^d$ (for some $d\ge 0$). Arguing as in the proof of Proposition 2.3.4(I) we easily obtain that $\tilde{F}(\operatorname{Spec}(i))$ is injective whenever i is a k-linear embedding of algebraically closed extensions of k, and there exists an (algebraically closed) extension K_1/k of finite transcendence degree such that $\tilde{F}(\operatorname{Spec} K_1)\cong \mathbb{Q}^d$. Moreover, applying (*) we obtain that $\mathbb{Q}$-dimension of $\tilde{F}(Y)$ is at most d if Y is either a smooth affine variety or (the spectrum) of its generic point; moreover, there exists $Y_0\in$ SmAffVar such that $F(Y_0)\cong\mathbb{Q}^d$. Being more precise, the homomorphisms $F(Y)\to F(\operatorname{Spec} k(Y))\to F(\operatorname{Spec} K_0)$ are injective for any $Y\in$ SmAffVar and any k-linear field embedding $k(Y)\to K_0$.

Since any $C\in$ SmAffVar has a k-rational point, the homomorphism $F(c\times \mathrm{id}_{Y_0})$ is bijective for the structure morphism c of any $C\in$ SmAffVar. It obviously follows that for any two SmAffVar-morphisms $f_1,f_2: C_1\to C_2$ we have $F(f_1\times\mathrm{id}_{Y_0})=F(f_2\times \mathrm{id}_{Y_0})$. Since Y_0 has a k-point, it follows that $F(f_1)=F(f_2)$. In particular, for (C,c) as above and a morphism $i:\operatorname{Spec} k\to C$ (coming from any point of C) both $F(i\circ c)$ and $F(c\circ i)$ are identical.

Therefore F is constant indeed. It obviously follows that $\tilde{F}(p_0)$ is bijective.

(2) This is a rather simple motivic exercise. Let us consider the functors F and G, as well as the functors G^i below, as presheaves on the Nisnevich site $\mathrm{SmVar}_{\mathrm{Nis}}$ of all smooth k-varieties; note that this extension of the domain is compatible with the Nisnevich sheafification.

Since for any $i\in\mathbb{Z}$ the functor $G^i: X\mapsto h_{2j+l,j}(M^i\otimes\mathcal{M}_{\mathbb{Q}}(X),\mathbb{Q})$ is additive and factors through $\mathcal{M}_{\mathbb{Q}}$, it yields a *homotopy invariant presheaf with transfers*; see Definitions 2.4 and 2.15, and Theorem 14.11 of [MVW06]. Since the category of homotopy invariant presheaves with transfers is abelian, and the forgetful functor from it into presheaves on $\mathrm{SmVar}_{\mathrm{Nis}}$ is well-known to be exact, G yields a homotopy invariant presheaf with transfers as well. By Theorems 13.1 and 13.8 of ibid., it follows that F comes from a homotopy invariant (Nisnevich) sheaf with transfers.

Now let us recall that any homotopy invariant sheaf H with transfers satisfies condition (*). Firstly, if f is flat then Lemma 2.3.5 of [SuV00] immediately implies that $H(f)$ is injective.[10] Moreover, Lemma 22.8 of [MVW06] implies that $F(f)$ is injective whenever f is a (dense) open embedding of smooth k-varieties.

Lastly, the zeroth homology of the complex $(h_{2j+l,j}(M_K^i,\mathbb{Q}))$ is clearly isomorphic to $\tilde{G}(\operatorname{Spec} K)$. It remains to recall that spectra of fields give points in the Nisnevich topology; thus $\tilde{G}(\operatorname{Spec} K)\cong \tilde{F}(\operatorname{Spec} K)$. □

Remark 5.1.4.

(1) Our arguments in the proof of Theorem 5.1.2 also yield that a motive $M\in DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{w_{\mathrm{Chow}}\geq i_0}$ satisfies its assumptions if and only if M is a retract of M' that belongs to the extension-closure of the set $\cup_{i\leq -i_0}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle a_{\mathcal{I},i}\rangle\cup AT^{\mathrm{eff}})[-i]$.

(2) Since geometric motives are w_{Chow}-bounded below, we obtain that for any object M of $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})$ the groups $\mathrm{CWH}_j^i(M_{K_0},\mathbb{Q})$ are finite-dimensional $\mathbb{Q}$-vector spaces for all $(i,j)\in\mathcal{I}$ if and only if M belongs to the envelope of the set $(\cup_{i\in\mathbb{Z}}(\operatorname{Obj}\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle a_{\mathcal{I},i}\rangle)[-i])\cup EAT^{\mathrm{eff}}$.

    On the other hand, one can easily generalize our theorem and establish for any staircase sets $\mathcal{I}$ and $\mathcal{I}'$ a similar envelope criterion for the groups $\mathrm{CWH}_j^i(M_{K_0},\mathbb{Q})$ to vanish if $(i,j)\in\mathcal{I}'$ and to be $\mathbb{Q}$-finite dimensional if $(i,j)\in\mathcal{I}$. The formulation of Theorem 5.1.2 corresponds to the case $\mathcal{I}'=[1-i_0,+\infty)\times[0,+\infty)$.

(3) One can define another notion of "smallness" of Chow-weight homology using certain "Chow-weight" cycle classes into singular and étale homology (and asking whether they are injective); this matter is discussed in Remark 5.1.3 and Proposition 5.1.4 of [BoS14]. Loc. cit. gives a certain generalization of [Voi14, Theorem 3.18].

(4) The proof of Lemma 5.1.3(1) was inspired by [Sus83]; this is a certain "rigidity" statement.

Let us now relate Theorem 5.1.2 to étale and singular cohomology; cf. Theorem 3.5.4.

Definition 5.1.5.

(1) We say that a (pure) object of weight m of the category MHS_{eff} is an Artin-Tate one if it is a direct sum of copies of the pure Hodge structures $\mathbb{Q}((-m)/2)$.[11]

(2) Let k be an essentially finitely generated field; G is the absolute Galois group of k. Then we will say that a pure object V of weight m (see Definition 3.5.3(2)) in the category $\mathbb{Q}_\ell[G]-\mathrm{Mod}$ is an Artin-Tate one if there exists a finite extension K/k such that V becomes a direct sum of copies of the representation $\mathbb{Q}_l((-m)/2)$ as a $\mathbb{Q}_\ell[\mathrm{Gal}(K)]$-module.

[10] Actually, this statement holds for any presheaf with transfers. The authors are deeply grateful to Prof. D.-Ch. Cisinski for this argument.

[11] It would be certainly more natural to call these Hodge structures Tate ones. Our reason to call them Artin-Tate ones is just to make the formulation of Corollary 5.1.6 shorter.

Corollary 5.1.6. *Assume that either k is a subfield of $\mathbb{C}$ and $H = H_{\mathrm{sing}}$ or that k is an essentially finitely generated field and $H = H_{et,\mathbb{Q}_\ell}$ (for $\ell \neq p$; see Definition* 3.5.3(4)*).*

(1) *Suppose that $\mathcal{I}$ is a staircase set and an object M of $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})$ satisfies the equivalent conditions of Theorem* 5.1.2.

Then for all $m, l \in \mathbb{Z}$ the object $Gr^{W_D}_{m+l} H^m(M)$ vanishes (resp. an Artin-Tate one, see Definition 5.1.5*) if $m + l$ is less than $2a_{ii,l}$ and is odd (resp. even), and it is $a_{ii,l}$-effective if $m + l \geq 2a_{ii,l}$.*

(2) *Assume that $X \in \mathrm{Var}$, K_0 is a universal domain containing k, and $r > 0$.*

Then the vector spaces $\mathrm{CH}_j(X_{K_0},\mathbb{Q})$ are finite dimensional if $j < r$ if and only if the motive $\mathcal{M}^c_{\mathbb{Q}}(X)$ belongs to the envelope of

$$(\cup_{i>0} \mathrm{Obj}\, \mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})[i]) \cup \mathrm{Obj}\, \mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q})\langle r\rangle \cup AT^{\mathrm{eff}}$$

(see Definition 5.1.1*).*

Furthermore, if these conditions hold then for any $0 \leq m \leq r$ the object $Gr^{W_D}_{m+l} H^{2m-1}_c(X)$ vanishes and $Gr^{W_D}_{2m} H^{2m}_c(X)$ (this is the corresponding Deligne weight factor of the cohomology of X with compact support; see Definition 3.5.3(3) *and Theorem* 4.2.1(2)*) is an Artin-Tate one.*

Proof. (1) This is an easy combination of Proposition 3.5.1(1) with Theorem 3.5.4(2) (that relates Deligne's weight filtration to the Chow-weight one); one should only note that $(m + l)/2$-effective (pure) Hodge structures and pure representations of weight q are zero if $q < m + l$ (cf. Theorem 4.2.1(2)).

(2) Recall that the motive $M = \mathcal{M}^c_{\mathbb{Q}}(X)$ belongs to $DM^{\mathrm{eff}}_{gm}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \geq 0}$ and $\mathrm{CH}_j(X_{K_0},\mathbb{Q}) \cong \mathrm{CWH}^0_j(M_{K_0},\mathbb{Q})$ for all $j \geq 0$; see Lemma 4.1.4 (1, 2). Thus our finite dimensionality assumption is fulfilled if and only if $\mathrm{CWH}^i_j(M_{K_0},\mathbb{Q})$ is finite dimensional over $\mathbb{Q}$ for $(i,j) \in \mathcal{I}^{\langle r\rangle}_0$ (see Definition 3.3.6). Applying Theorem 5.1.2 in the case $i_0 = 0$ we obtain the equivalence part of this assertion.

Combining it with assertion (1) of this proposition we obtain the cohomological part of the assertion as well. □

5.2. **Chow-weight cohomology and the dimension of motives.** Now we dualize (parts (1) and (3) of) Theorem 3.2.1 along with some other properties of Chow-weight homology.

To this end we note that Proposition 2.2.1(1) yields the following: the Poincaré duality for $DM_{gm}(k,R)$ "respects" w_{Chow}, i.e., the image under the duality functor of $DM_{gm}(k,R)_{w_{\mathrm{Chow}} \leq 0}$ is $DM_{gm}(k,R)_{w_{\mathrm{Chow}} \geq 0}$ (and also vice versa). Moreover, the categorical duality (cf. Proposition 1.2.4) essentially respects weight complexes (at least, for motives; see Remark 1.5.9(1) of [Bon10a] and Corollary 3.5 of [Sos19] along with its proof which is essentially self-dual). Thus one easily obtains the following results.

Proposition 5.2.1. *For an object M of $DM_{gm}(k,R)$, $j, l, i \in \mathbb{Z}$, (M^*) that is a choice of a weight complex for M, and a field extension K/k let us define $\mathrm{CWC}^{j,i}(M_K,R)$ as the ith homology of the complex $DM_{gm}(K^{perf},R)(M^{-*},R\langle j\rangle)$.*

I. *The following properties of these cohomology theories are valid.*

(1) $\mathrm{CWC}^{j,i}(-_K,R)$ *yields a cohomological functor on $DM_{gm}(k,R)$.*

(2) $\mathrm{CWC}^{j,i}(-_K)$ *vanishes on $d_{\leq n} DM^{\mathrm{eff}}_{gm}(k,R) \subset DM_{gm}(k,R)$ if $j - i > n$.*

II. *Assume that M is an object of $d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)$ for some $n \ge 0$. Then M belongs to $d_{\le n-s}DM^{\mathrm{eff}}_{gm}(k,R)$ (for some $s \in [1,n]$) if and only if $\mathrm{CWC}^{j,i}(M_K,R)=\{0\}$ for all $i\in\mathbb{Z}$, $j\in[n-s+1,n]$, and all function fields K/k.*

III. *For M as above and $q\in\mathbb{Z}$ M belongs to $DM^{\mathrm{eff}}_{gm}(k,R)_{w_{\mathrm{Chow}}\le q}$ if and only if $\mathrm{CWC}^{j,i}(M_K)=\{0\}$ for all $i>q$, $j\in[1,n]$, and all function fields K/k.*

IV. *Now let $R=\mathbb{Q}$. Then it suffices to verify any of the assertions in parts II and III of the proposition for a single universal domain K containing k.*

Proof. Recall that the Poincaré dual of $d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)$ is $d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)\langle -n\rangle$, and that the dual to $\operatorname{Obj} d_{\le n-s}DM^{\mathrm{eff}}_{gm}(k,R)$ can (also) be described as

$$\operatorname{Obj} d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)\langle s-n\rangle \cap \operatorname{Obj} d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)\langle -n\rangle$$

(see Proposition 2.2.6(4)). Along with the observations made prior to this proposition, this easily reduces our assertions I–III to their duals given by Proposition 3.1.2(1,2) and Theorem 3.2.1(1,3), respectively.

Lastly, assertion IV easily follows from Proposition 2.3.4(II); cf. Proposition 3.4.1(3). □

Remark 5.2.2.

(1) Certainly, one can dualize Theorems 3.3.3 and 3.5.4, Propositions 3.5.1 and 3.5.5, and the results of §3.4 in a similar way as well.

Moreover, one may consider higher Chow-weight cohomology groups of motives; see Proposition 5.2.1 of [BoS14] (and Proposition 3.4.1).

(2) Since Chow-weight cohomology yields a mighty tool for computing the dimension of an (effective) motive, it makes all the more sense to make the main “arithmetical” observation of this subsection (that appears to be more interesting either if $R\ne\mathbb{Q}$ or if we study motives over essentially finitely generated fields).

(3) One can define dimensions of not necessarily effective motives as follows: for $m\in\mathbb{Z}$ and $M\in\operatorname{Obj} DM_{gm}(k,R)$ we say that M is of dimension at most m if M belongs to $\langle \mathcal{M}_R(P)\langle c\rangle,\ P\in\mathrm{SmPrVar},\ c\in\mathbb{Z}, \dim P\le m-c\rangle$. This definition is easily seen to be coherent with the formulations of this section.

Now let M be an object of $d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)$ (for some $n\ge 0$). We recall that in the proof of Theorem 3.2.1(2) we have studied the question whether $g: w^{c-1}_{\mathrm{Chow}\le t}l^{c-1}(M)\to l^{c-1}(M)$ is zero. By our assumption on M, we can choose $w^{c-1}_{\mathrm{Chow}\le t}l^{c-1}(M)$ to be of dimension at most d (in $DM^{c-1}_{gm}(k,R)$). Hence the corresponding application of Proposition 3.1.2(5) reduces the verification of $g=0$ to the vanishing of the corresponding $\mathrm{CWH}^i_j(M_{k(P)})$, whereas the dimension of P_j not greater than $n-j$.

Thus we obtain the following statement; we will call the transcendence degrees of function fields over k their *dimensions* in it.

Proposition 5.2.3. *Let M be an object of $d_{\le n}DM^{\mathrm{eff}}_{gm}(k,R)$ (for some $n\in\mathbb{Z}$). Then the following statements are valid.*

(1) *To verify any of the conditions in Theorem* 3.2.1 *(resp. condition* (4) *in the setting of Proposition* 3.4.1(2)*, resp. condition (2) of Corollary* 3.4.2*) it suffices to compute the corresponding* $\mathrm{CWH}^i_j(M_K)$ *(resp. motivic homology*

groups over K^{perf}) for K running through function fields of dimension at most $d-j$ (resp. for K/k of dimension at most d) only.

(2) *In Proposition* 3.4.1(2) *it suffices to verify condition* (3) *for rational extensions K/k of transcendence degree at most $d-j+1$.*

(3) *For $R=\mathbb{Q}$, in the assertions mentioned in part (1) of this proposition it suffices to take K to be the algebraic closure of $k(t_1,\dots,t_{d-j})$ (resp. of $k(t_1,\dots,t_d)$) instead.*

Remark 5.2.4.

(1) Thus, if M does **not** satisfy the (motivic) equivalence conditions of the statements mentioned in the previous proposition, there necessarily exists a function field K/k of "small dimension" such that (at least) one of the corresponding Chow-weight homology (resp. motivic homology) groups does not vanish over K.

Note also that it is actually suffices to consider dimensions of fields over a field of definition for M (that certainly may be smaller than k).

(2) The question whether these dimension restrictions are the best possible ones seems to be quite difficult in general (especially if we consider geometric motives only). Note however that in the case $d=1$, $R=\mathbb{Q}$, and a finite k it is clearly not sufficient to compute Chow-weight homology over algebraic extensions of k only.

5.3. **Some more remarks; possible development.** We make some more remarks on our main results; some of them concern torsion phenomena. Possibly the matters mentioned below will be studied in consequent papers.

Remark 5.3.1.

(1) It would certainly be interesting to relate the results of this paper to earlier statements on effectivity of cohomology (of singular varieties); cf. Theorem 1.2 of [BEL05].

(2) The results of the current paper can be easily combined with the main statements in [Bac18] and [Bon20a] to obtain certain Chow-weight homology criteria for the effectivity and connectivity of motivic spectra (that is, objects of the R-linear version $SH_R(k)$ of the stable homotopy category $SH(k)$, where R is a localization of $\mathbb{Z}[1/e]$); see §5.3 of [BoS14].

(3) The main formulations of this paper are easier to apply when $R=\mathbb{Q}$ (or R is a $\mathbb{Q}$-algebra). Now we describe some ideas related to motives and homology with integral and torsion coefficients.

Firstly we note that a bound on the dimension of a motive clearly yields some information on its (co)homology. In particular, the $\mathbb{Z}_\ell$-étale homology H of an object M of $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ of dimension at most d is concentrated in degrees $[-2d,0]$ (here we take a prime $\ell\neq p$, a coefficient ring not containing $1/\ell$, and consider the étale homology over an algebraically closed field of definition; we apply our convention for enumerating homology). Moreover, considering the relation between $\mathbb{Z}_\ell$-homology and $\mathbb{Z}/\ell\mathbb{Z}$-one one obtains that $H_{-2d}(M)$ is torsion-free.

One can use these simple remarks for studying the E_2-terms of Chow-weight spectral sequences for H; cf. Theorem 3.5.4. In particular, the latter of them can be applied for "comparing M with $M\otimes\mathbb{Q}$"; cf. [Voi14, Remark 3.11]. Note however that the groups $E_2^{**}T(H,M)$ cannot be recovered from the weight filtration on $H^*(M)$ in general; see [GiS96, §3.1.3] (cf. the proof of Proposition 4.1.8(2)).

(4) In the current paper we treat Chow-weight homology (of a fixed object M of $DM_{gm}^{\mathrm{eff}}(k,R)$) as functors that associate to field extensions of k certain R-modules. Yet one can apply a "more structured" approach instead; it seems to be especially actual for $R \neq \mathbb{Q}$.

For any $U \in \mathrm{SmVar}$ and $t_R(M) = (M^*)$, $j, l \in \mathbb{Z}$, one can consider the homology of the complex $DM_{gm}(k,R)(\mathcal{M}_R(U)\langle j\rangle[l], M^*)$. Next the functors obtained can be sheafified with respect to U; this yields a collection of certain *Chow-weight homology sheaves* (for any (j,l)). Moreover, if $j \geq 0$ then the sheafifications of $U \mapsto DM_{gm}(k,R)(\mathcal{M}_R(U)\langle j\rangle[l], M^i)$ (that were called the *Chow sheaves* of M^i in [KaL10]) are birational (in U, i.e., they convert open dense embeddings of smooth varieties into isomorphisms; see Remark 2.3 of [HuK06]). Hence the corresponding Chow-weight homology sheaves are birational as well.

Moreover, these observations can probably be extended to the setting of motives (with rational coefficients) over any "reasonable" base scheme S; one should study the corresponding *dimensional homotopy invariant* Chow sheaves for S-motives (recall that those are conjecturally Rost's cycle modules over S) and apply the results of [BoD17].

(5) $\mathrm{Chow}^{\mathrm{eff}}(k,R)$-complexes of length 1 yield a simple counterexample to the natural analogue of Theorem 3.2.1(3) for motives whose Chow-weight homology vanishes in degrees **less** than n (along with the corresponding analogues of Theorem 3.2.1(2) and Theorem 3.3.3(2)). Assume $R = \mathbb{Q}$, $k \subset K = \mathbb{C}$ (actually, any K that is not an algebraic extension of a finite field is fine for our purposes); take a smooth projective P/k (say, an elliptic curve) that possesses a 0-cycle c_0 of degree 0 that is rationally non-torsion. We also use the notation c_0 for the corresponding morphism $\mathbb{Q} = M_{gm}^{\mathbb{Q}}(\mathrm{pt}) \to M_{gm}^{\mathbb{Q}}(P)$; let M be the cone of c_0 (i.e., $M = \dots 0 \to \mathbb{Q} \xrightarrow{c_0} \mathcal{M}^{\mathbb{Q}}(P) \to 0 \to \dots$; $\mathcal{M}^{\mathbb{Q}}(P)$ is in degree 0).

Since c_0 is rationally non-trivial (as a cycle with $\mathbb{Q}$-coefficients), the map $h_{00}(c_0,\mathbb{Q})$ is an injection (and $h_{2j,j}(c_{0,K},\mathbb{Q})$ is injective for any $j \geq 0$ and K/k as well). Hence $\mathrm{CWH}^i_j(M_K,\mathbb{Q}) = \{0\}$ whenever $i \neq 0$ (and any field extension K/k). On the other hand, c_0 does not split since it is numerically trivial as a cycle. Thus M does not belong to $K^b(\mathrm{Chow}^{\mathrm{eff}}(k,\mathbb{Q}))_{w_{\mathrm{Chow}} \leq 0}$ (or to $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})_{w_{\mathrm{Chow}} \leq 0}$ if we "put it into" $DM_{gm}^{\mathrm{eff}}(k,\mathbb{Q})$). Hence the vanishing of the Chow-weight homology in negative degrees does not imply that the weights of a motive M are non-negative.

Moreover, tensor products of examples of this type behave "even worth" from this point of view; see Remark 5.4.1(6) of [BoS14] for more detail. Thus Chow-weight homology cannot be used for bounding weights from above. On the other hand, the argument used in the proof of Proposition 3.5.5 can easily be modified to prove that the weight filtration on singular homology does yield bounds of this sort (if one assumes conjectures A and B in the proposition); the corresponding version of Theorem 3.5.4 is valid as well.

(6) In the current paper we mostly study geometric Voevodsky motives; these are certainly the most important ones. Yet in [BoK20] some of our main results are extended to w_{Chow}-bounded below objects of $DM_{-}^{\mathrm{eff}}(k, R)$. These generalizations allow treating *slices* of motives (in §2.3 of ibid.); note that slices of geometric motives do not have to be geometric. As an application, a generalization of Corollary 3.4.2 (that is new for geometric motives as well) was obtained. Possibly, some more properties of slices can be obtained using our methods. The authors are deeply grateful to Prof. M. Levine for the suggestion to study this problem (and for mentioning Theorem 7.4.2 of [KaL10] as an interesting example of the calculation of slices).

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