

Chapter 1

Radiation Exchange Between Surfaces

1.1 Motivation and Objectives

Thermal radiation, as you know, constitutes one of the three basic modes (or mechanisms) of heat transfer, i.e., conduction, convection, and radiation. Actually, on a physical basis, there are only two mechanisms of heat transfer; diffusion (the transfer of heat via molecular interactions) and radiation (the transfer of heat via photons/electromagnetic waves). Convection, being the bulk transport of a fluid, is not precisely a *heat* transfer mechanism.

The physics of radiation transport are distinctly different than diffusion transport. The latter is a *local* phenomena, meaning that the rate of diffusion heat transfer, at a point in space, precisely depends only on the local nature about the point, i.e., the temperature gradient and thermal conductivity at the point. Of course, the temperature field will depend on the boundary and initial conditions imposed on the system. However, the diffusion heat flux at, say, one point in the system does not *directly* effect the diffusion flux at some distant point. Radiation, on the other hand, is not local; the flux of radiation at a point will, in general, be directly and instantaneously dependent on the radiation flux at all points in a system. Unlike diffusion, radiation can act over a distance. Accordingly, the mathematical description of radiation transport will employ an *integral* equation for the radiation field, as opposed to the *differential* equation for heat diffusion.

Our objectives in studying radiation in the short amount of time left in the course will be to

1. Develop a basic physical understanding of electromagnetic radiation, with emphasis on the properties of radiation that are relevant to heat transfer.
2. Describe radiation exchange among surfaces, in which the surfaces can be perfect absorbers of radiation (black) or diffusely absorbing (gray).
3. Introduce the topic of radiation transfer in a participating medium that absorbs, emits, and scatters radiation, and describe the formulation and application of the radiative transport equation.

1.2 Basic Radiation Properties

For our purposes, it is useful to view radiation as the transport of energy in electromagnetic waves. Radiation can also be viewed as the transport of energy by discrete photons, and the basic relationship between the energy of a photon, ϵ , and the frequency ν or wavelength $\lambda = c/\nu$ of the wave is given by Planck's relation;

$$\epsilon = h\nu = \frac{hc}{\lambda} \quad (1.1)$$

where h is Planck's constant and c is the speed of light in a vacuum.

Equation (1.1) indicates that the energy of the radiation is inversely proportional to the radiation wavelength. *Thermal* radiation refers to the spectrum of radiation in the visible ($\lambda = 0.4 - 0.7 \mu\text{m}$) to infrared (IR, $\lambda = 0.7 \sim 10 - 100 \mu\text{m}$) wavelengths. Radiation at these wavelengths can excite the rotational and vibrational energy levels of molecules and thus transfer energy to molecules in the form of *heat*. That is, the absorption of thermal radiation by molecules will act to raise the temperature of the system. Radiation having shorter wavelengths (UV, x-rays, γ -rays) can excite the electronic energy levels of molecules and atoms and/or ionize or break molecular bonds. This spectra of radiation is often referred to as *ionizing* radiation. On the other hand, longer wavelengths (microwaves, radio waves) will not, in general, couple with the energy storage levels in molecules; although an obvious exception are the microwave wavelengths used in the common microwave oven.

In practically all engineering-relevant applications, the source of thermal radiation is *thermal emission*. All bodies at any finite temperature will emit radiation. It was theoretically established by Boltzmann, and experimentally confirmed by Stephan, in the 19th century that the maximum possible emission rate from a surface is given by

$$e_b = \sigma T^4 \quad (1.2)$$

where $\sigma = 5.67 \times 10^8 \text{ W/m}^2 \text{ K}^4$. The above formula is known as the *Stephan-Boltzmann* law, and e_b , having units of W/m^2 , is the *blackbody emissive power*, which depends solely on the absolute temperature T of the surface. All real surfaces will emit radiation at a rate smaller than e_b ; an 'ideal' surface which attained an emission of e_b would be referred to as a *blackbody*.

It is possible to construct devices which come very close to meeting the ideal emission of a blackbody. Typically, these devices are formed from an isothermal cavity (i.e., a hollow sphere), with a small opening from which the radiation escapes. The radiation emitted by a blackbody at a specified T will be distributed over wavelength λ , the spectrum of which will also depend solely on T . Although the nature of the blackbody spectrum was experimentally known in the late 19th century, theoretical prediction of the spectrum defied the 'classical' physical understanding of the day. Planck, at the beginning of the 20th century, used the concept of discrete wavelength energy levels to develop a theoretical prediction of the blackbody spectrum which was consistent with experiments. His formula for the spectral blackbody emissive power is

$$e_{b\lambda} = \frac{C_1}{\lambda^5 (\exp[C_2/(\lambda T)] - 1)} \quad (1.3)$$

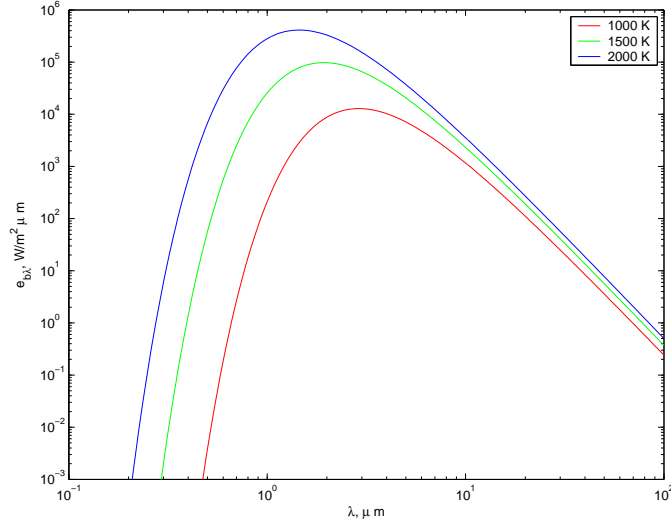


Figure 1.1: Blackbody spectral power distribution

in which $C_1 = 2\pi hc^2 = 3.7413 \times 10^8 \text{ W } (\mu\text{m})^4/\text{m}^2$ and $C_2 = hc/k_B = 14,388 \mu\text{m K}$. The quantity $e_{b\lambda}d\lambda$ denotes the energy emitted from the black surface, per unit area, within a wavelength interval $d\lambda$ about wavelength λ . The energy over all wavelengths is obtained from

$$e_b = \int_0^\infty e_{b\lambda} d\lambda = \sigma T^4 \quad (1.4)$$

i.e., the total emissive power is given by the Stephan–Boltzmann law, as it must.

A plot of $e_{b\lambda}$ vs. wavelength λ is given in Fig. 1.1. As T increases two things happen: 1) the spectral emissive power at all wavelengths increase, and 2) the wavelength corresponding to the maximum power shifts towards the shorter wavelengths. The value of the wavelength at which the maximum occurs is predicted by the Wien displacement law,

$$\lambda_{max} = \frac{2898 \mu\text{m K}}{T} \quad (1.5)$$

Equation (1.4) can be rearranged to identify a *fractional distribution* function. Assume that T is constant, then we can write

$$\frac{1}{\sigma T^5} \int_0^\infty e_{b\lambda} d(\lambda T) = 1 \quad (1.6)$$

Now combine with Eq. (1.3), and observe that λT becomes the sole variable of the distribution. We can therefore define

$$f'(\lambda T) = \frac{C_1/\sigma}{(\lambda T)^5 (\exp[C_2/(\lambda T)] - 1)} \quad (1.7)$$

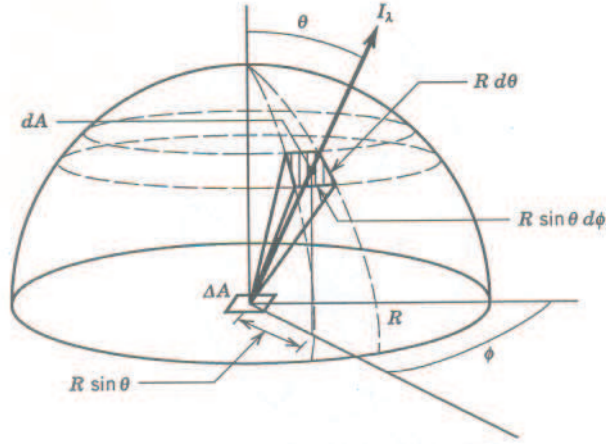


Figure 1.2: Intensity definition

so that

$$f(\lambda T) = \int_0^{\lambda T} f'(x) dx \quad (1.8)$$

represents the fraction of emitted energy between 0 and λT , relative to the total emitted energy.

Intensity

The previous section identified the blackbody emissive power from a surface, which is a quantity that has units of W/m^2 , that is, the same units of a heat flux. The emissive power, however, is not a heat flux – the latter is precisely a vector quantity which can be resolved into directional components. The emissive power, on the other hand, describes the energy leaving a surface per unit area of the surface; it does not describe the directional characteristics of the radiation as it leaves the surface.

To describe the directional properties of radiation as it propagates through space, we need to introduce a quantity referred to as the radiation *intensity*. The intensity, denoted I , is defined by use of Fig. 1.2. Radiation is emitted from a small surface element ΔA and travels in all directions. A portion of the radiation lands on the small area element on a hemisphere enclosing ΔA , denoted as dA . With regard to geometrical considerations, it should be easy to see that the net amount of energy falling on dA , which is denoted as dP , will be proportional to

1. the *projected area* of the emitting surface, which would be $\Delta A \cos \theta$,
2. the area of the ‘target’, dA , and

3. the inverse square of the distance between the source and the target, $1/R^2$.

The last proportionality comes from the fact that the area of the hemisphere scales as R^2 , and the total power falling on the hemisphere – which would equal the total power emitted by the source – would be constant in the absence of a participating medium. So the power per unit area on the hemisphere would go as $1/R^2$.

The target area dA can be related to the distance R by the polar coordinates;

$$dA = R^2 \sin \theta d\theta d\phi \quad (1.9)$$

Note that

$$\int dA = R^2 \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi = 2\pi R^2 \quad (1.10)$$

as expected. Now dP is proportional to dA/R^2 , and

$$\frac{dA}{R^2} = \sin \theta d\theta d\phi \equiv d\Omega \quad (1.11)$$

defines a differential *solid angle* $d\Omega$. The units of solid angle are the *steradian* (abbreviated str), and 4π steradians encompass all directions about the origin of a spherical coordinate system.

With our solid angle definition, we now see that the power falling on the target will be proportional to the projected area of the source and the solid angle subtended by the target with respect to the source. The power will also be proportional to the magnitude and directionality of the radiation leaving the source and propagating through space. Since dP will be in watts, and dP will be proportional to $\Delta A \cos \theta d\Omega$, it follows that radiation propagating through space must have units of $\text{W}/\text{m}^2 \text{ str}$. This quantity is referred to as the *intensity*, and is defined by

$$I = \frac{dP}{\Delta A \cos \theta d\Omega} \quad (1.12)$$

taken in the limit of $\Delta A \rightarrow 0$ and $d\Omega \rightarrow 0$.

The intensity is a difficult quantity to grasp. Although it represents the directional distribution of propagating radiation, the intensity *is not a vector* in the sense that it cannot be resolved into vector components. Rather, the intensity is a scalar that is dependent on the directional coordinates θ and ϕ as well as the usual spacial coordinates and time. In a sense, I is a scalar which is relevant to the 5-D ‘space’ defined by x, y, z and θ, ϕ .

It was implicitly assumed that the intensity, defined in the Eq. (1.12), represents a wavelength-integrated (or *total*) quantity. The *spectral intensity* I_λ is defined by a relation similar to Eq. (1.12), except now on a per-unit-wavelength basis;

$$I_\lambda = \frac{dP_\lambda}{\Delta A \cos \theta d\Omega d\lambda} \quad (1.13)$$

where the limit is again taken, and dP_λ is the power, in W, falling on the surface dA within the wavelength interval $d\lambda$ about λ . It follows that

$$I = \int_0^\infty I_\lambda d\lambda \quad (1.14)$$

Blackbody Intensity

If the surface ΔA is a black surface – meaning that it emits as a perfect blackbody – the intensity distribution leaving the surface will be independent of direction, i.e., the blackbody intensity I_b landing on the hemisphere in Fig. 1.2 would not be a function of θ . It is important to note that the power per unit area landing on the hemisphere, dP/dA , would be a function of θ ; this quantity would be proportional to the projected area of the source, $\Delta A \cos \theta$. The intensity, however, represents the power per unit projected area, and so this quantity would remain constant.

The total power leaving the black source would be $P = e_b \Delta A$, and in the absence of an intervening medium the total power leaving the source would be the total power landing on the hemisphere. If we use Eq. (1.12), and note that the intensity $I = I_b$ is constant, then

$$\begin{aligned} P &= e_b \Delta A \\ &= \int dP = I_b \Delta A \int_{\Omega=4\pi} \cos \theta d\Omega \\ &= I_b \Delta A \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos \theta \sin \theta d\theta \\ &= \pi I_b \Delta A \end{aligned} \quad (1.15)$$

or

$$I_b = \frac{e_b}{\pi} \quad (1.16)$$

Radiative Heat Flux

As mentioned previously, the intensity, which is a scalar, describes the directional *distribution* of radiation energy transfer. The radiative heat flux, \bar{q}_R , is a vector which describes the *net* direction and magnitude of radiant energy transfer in space.

Say we take a small surface element in space, dA , oriented so that its normal points in the positive z direction. Radiation, arriving from all directions with an intensity distribution $I(\theta, \phi)$, passes through the element. The net rate at which energy is transported across the surface, denoted as dP , will be

$$dP = dA \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I(\theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

Note that the $\cos \theta$ term appears because $dA \cos \theta$ is the projected area of the element with respect to the θ direction. The quantity dP will be either positive, negative, or zero; if it is positive, it

means that the net transfer of energy across dA is in the positive z direction, negative means the net energy transfer is in the negative z direction, and zero means that there is no net transfer. The last condition does not necessarily imply that $I = 0$; rather, it implies that the positive and negative contributions cancel out. Such would be the case if the intensity distribution were *isotropic*, i.e., independent of direction.

The radiative heat flux, in the z direction, would be dP/dA . The three components of radiative flux with respect to a cartesian coordinate system are given by

$$q_{R,x} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I(\theta, \phi) \sin^2 \theta \cos \phi \, d\theta \, d\phi \quad (1.17)$$

$$q_{R,y} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I(\theta, \phi) \sin^2 \theta \sin \phi \, d\theta \, d\phi \quad (1.18)$$

$$q_{R,z} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \quad (1.19)$$

1.2.1 Surface Properties

Surfaces can emit, absorb, and reflect radiation. You are certainly familiar with the basic concepts of surface absorptivity and reflectivity, i.e., the absorptivity α represents the fraction of incident radiation on a surface that is absorbed by the surface. We will need to develop more precise definitions than this to account for the directional and spectral dependencies of the radiation falling on or emitted by a surface.

Emissivity

As mentioned in the previous section, the intensity emitted by a blackbody is independent of direction and given by $I_b = e_b/\pi$. Likewise, the spectral (or wavelength-dependent) blackbody intensity is independent of direction and given by $I_{b\lambda} = e_{b\lambda}/\pi$. The emitted spectral intensity from a real surface, into some direction θ , ϕ and within a small wavelength interval about λ , will always be less than the spectral blackbody intensity for a surface at the same temperature. We can therefore define the *spectral directional emissivity* ϵ'_λ as

$$\epsilon'_\lambda = \frac{I_{e,\lambda}(\theta, \phi)}{I_{b\lambda}} \quad (1.20)$$

in which $I_{e,\lambda}$ is the *emitted* intensity from the surface; it does not include components due to other sources such as reflected intensity. The spectral directional emissivity represents the most fundamental, or irreducible, information on the emission properties of a surface. In general, it would be a function of direction, wavelength, temperature, and the physical and chemical properties of the surface.

We can obtain averaged emissivity properties by integration of ϵ'_λ . The *spectral hemispherical emissivity* represents the directional average of ϵ'_λ , and is defined by

$$\begin{aligned}\epsilon_\lambda &= \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{e,\lambda}(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{b,\lambda} \cos \theta \sin \theta \, d\theta \, d\phi} \\ &= \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \epsilon'_\lambda(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{\pi} \\ &= \frac{e_\lambda}{e_{b,\lambda}}\end{aligned}\tag{1.21}$$

That is, ϵ_λ is the ratio of the emitted spectral power from the surface, e_λ to the blackbody power $e_{b,\lambda}$.

Likewise, the *total directional emissivity* ϵ' is obtained from an appropriate wavelength average of ϵ'_λ ;

$$\begin{aligned}\epsilon' &= \frac{\int_0^\infty I_{e,\lambda} \, d\lambda}{\int_0^\infty I_{b\lambda} \, d\lambda} \\ &= \frac{\int_0^\infty \epsilon'_\lambda I_{b,\lambda} \, d\lambda}{I_b}\end{aligned}\tag{1.22}$$

Finally, the *total hemispherical emissivity* ϵ is obtained from either a directional average of ϵ' or a wavelength average of ϵ_λ ; either would yield the same result, which is

$$\begin{aligned}\epsilon &= \frac{\int_{\lambda=0}^\infty \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{e,\lambda}(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \, d\lambda}{\int_{\lambda=0}^\infty \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{b\lambda} \cos \theta \sin \theta \, d\theta \, d\phi \, d\lambda} \\ &= \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \epsilon'(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{\pi} \\ &= \frac{\int_{\lambda=0}^\infty \epsilon_\lambda I_{b\lambda} \, d\lambda}{I_{b\lambda}} \\ &= \frac{e}{e_b}\end{aligned}\tag{1.23}$$

Absorptivity

Consider now a surface that is exposed to an incident source of spectral intensity, denoted as $I_{\lambda}^{-}(\theta, \phi)$. The $-$ superscript indicates that the radiation is moving downwards onto the surface. When the radiation strikes the surface a fraction of it will be absorbed by the surface, the remainder will be reflected¹. Denote the absorbed intensity as $I_{a,\lambda}$. The *spectral directional absorptivity* α'_{λ} is defined by

$$\alpha'_{\lambda} = \frac{I_{a,\lambda}}{I_{\lambda}^{-}} \quad (1.24)$$

That is, it is the fraction of incident spectral intensity that was absorbed by the surface.

Similar to the spectral directional emissivity ϵ'_{λ} , the spectral directional absorptivity describes the fundamental absorption properties of the surface. And as was done with the emissivity, we can define averages, with respect to direction, wavelength, or both, by appropriate integrations.

The *spectral hemispherical absorptivity* α_{λ} is

$$\begin{aligned} \alpha_{\lambda} &= \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{a,\lambda}(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{\lambda}^{-}(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi} \\ &= \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \alpha'_{\lambda}(\theta, \phi) I_{\lambda}^{-}(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{q_{\lambda}^{-}} \\ &= \frac{q_{a,\lambda}}{q_{\lambda}^{-}} \end{aligned} \quad (1.25)$$

where q_{λ}^{-} and $q_{a,\lambda}$ are the downward spectral flux on the surface and the spectral absorbed flux.

The *total directional absorptivity* α' is

$$\begin{aligned} \alpha' &= \frac{\int_0^{\infty} I_{a,\lambda} \, d\lambda}{\int_0^{\infty} I_{\lambda}^{-} \, d\lambda} \\ &= \frac{\int_0^{\infty} \alpha'_{\lambda} I_{\lambda}^{-} \, d\lambda}{I^{-}} \end{aligned} \quad (1.26)$$

¹Some of the radiation might also be transmitted through the surface, but at this point we will not make this distinction; if the radiation is not reflected, then it went into the surface material and was absorbed by it

and the *total hemispherical absorptivity* α is

$$\begin{aligned}
\alpha &= \frac{\int_{\lambda=0}^{\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{a,\lambda}(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \, d\lambda}{\int_{\lambda=0}^{\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{\lambda}^{-} \cos \theta \sin \theta \, d\theta \, d\phi \, d\lambda} \\
&= \frac{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \alpha'(\theta, \phi) I^{-} \cos \theta \sin \theta \, d\theta \, d\phi}{q^{-}} \\
&= \frac{\int_{\lambda=0}^{\infty} \alpha_{\lambda} q_{\lambda}^{-} \, d\lambda}{q^{-}} \\
&= \frac{q_a}{q^{-}} \tag{1.27}
\end{aligned}$$

All these definitions look analogous to those for the emissivity. An important distinction, though, is in regard to the ‘weighting function’ used to obtain the averages. The blackbody intensity I_b is *not* a function of direction, so it could be removed from the integrals over direction in Eq. (1.21) and (1.23) for the hemispherical emissivities. On the other hand, the incident intensity I^{-} (either spectral or total) is, in general, a function of direction, and it cannot be removed from the corresponding formulas for hemispherical absorptivity in Eqs. (1.25) and (1.27). This points out an important fact: the hemispherical absorptivity will be a function of the directional intensity distribution falling on a surface. For example, a surface illuminated by the sun from the normal direction would, in general, have a different hemispherical absorptivity than the same surface illuminated by the sun at an oblique angle, or by the sun on a cloudy day (diffuse illumination). Likewise, the total absorptivity (either directional or spectral) will be a function of the spectral distribution of the incident radiation. A surface illuminated by visible light would likely have a different total absorptivity than the same surface illuminated by IR radiation.

Reflectivity

Reflectivity is one step more complicated than absorptivity. On the most basic level, the reflectivity will depend on the angle of the incident radiation as well as the angle of the reflected radiation. As before, denote as $I_{\lambda}^{-}(\theta, \phi)$ the incident intensity, and let $I_{\lambda}^{+}(\theta', \phi')$ denote the reflected intensity from the surface in the direction θ', ϕ' . The *spectral bidirectional reflectivity* ρ_{λ}'' is defined by

$$\rho_{\lambda}''(\Omega, \Omega') = \frac{I_{\lambda}^{+}(\Omega')}{\pi I_{\lambda}^{-}(\Omega)} \tag{1.28}$$

The reason for the π in the denominator will become obvious shortly. The *spectral directional-hemispherical reflectivity* ρ'_λ is obtained by integration of the reflected intensity over the hemisphere;

$$\begin{aligned}\rho'_\lambda &= \frac{\int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi/2} I_\lambda^+(\theta', \phi') \cos \theta' \sin \theta' d\theta' d\phi'}{\pi I_\lambda^-(\theta, \phi)} \\ &= \frac{1}{\pi} \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi/2} \rho''_\lambda(\theta, \phi, \theta', \phi') \cos \theta' \sin \theta' d\theta' d\phi'\end{aligned}\quad (1.29)$$

All of the radiation incident on the surface is either absorbed by the surface (again we assume transmission through the material counts as absorption) or reflected. Consequently,

$$\alpha'_\lambda + \rho'_\lambda = 1 \quad (1.30)$$

Formulas for the hemispherical-hemispherical reflectance are analogous to used for the hemispherical absorptivity.

Kirchoff's Law

Only when one goes to the spectral directional level does the absorptivity become independent of the properties (spectral, directional) of the intensity falling on the surface. For a given surface, α'_λ , at a particular direction and wavelength, would not depend upon whether the incident intensity at this direction and wavelength was produced from, say, a laser or an incandescent source.

The spectral directional absorptivity α'_λ is therefore a function solely of the surface material properties – as is the case with ϵ'_λ . Indeed, it can be shown from thermodynamic principles that the emissivity and absorptivity are equal on the spectral directional level,

$$\alpha'_\lambda = \epsilon'_\lambda \quad (1.31)$$

This equality is commonly referred to as Kirchoff's law.

It is easy to show, using Kirchoff's law and Eqs. (1.25–1.27), that if the incident intensity arriving at a surface originates from a blackbody that a) completely surrounds the surface, so that the incident intensity is independent of direction, and b) is at the same temperature of the surface, so that the incoming intensity has the same spectra as a blackbody emission from the surface, then emissivity and absorptivity will be equal on the hemispherical and total levels. This condition, i.e., equal temperatures of source and target, would correspond to thermal equilibrium for which there could be no net heat transfer between the source and target. In most engineering applications of relevance, the incident intensity on a surface will be directionally depend, and will originate from a source that is not at the surface temperature. And for such cases Kirchoff's law, in general, will not hold at either the directional or the hemispherical levels.

We can, however, apply approximations to extend the application of Kirchoff's law. Firstly, a surface with emission and absorption properties that are independent of wavelength will have equal emissivity and absorptivity on the total level. From Eqs. (1.22) and (1.26), it can be seen that if $\epsilon'_\lambda = \epsilon' \neq \text{func}(\lambda)$, then $\epsilon' = \alpha'$. Surfaces with wavelength-independent properties are referred to as *gray* surfaces.

Likewise, if the spectral directional emissivity is constant for all directions, i.e., $\epsilon'_\lambda = \epsilon_\lambda \neq \text{func}(\theta, \phi)$, then $\epsilon_\lambda = \alpha_\lambda$. Surfaces with directionally-independent properties are referred to as *diffuse*.

Only for surfaces that are both gray and diffuse can Kirchoff's law be applied at the total hemispherical level, i.e.,

$$\epsilon = \alpha, \quad \text{gray, diffuse surfaces} \quad (1.32)$$

The diffuse approximation is relatively accurate for surfaces that have a roughness on the scale of the radiation wavelength or greater, such as oxidized metals, wood, paper etc. Diffuse absorbers/emitters will also be diffuse *reflectors*, meaning that the reflection of intensity from a surface is isotropic (independent of direction), regardless of incident direction of the intensity.

The gray approximation is more of a stretch of reality. It is fairly accurate when the source of incident radiation is a blackbody at a temperature close to the temperature of the surface onto which it is falling; for such cases the spectra of the incident and emitted intensities will be about equal.

The gray assumption can fail miserably, however, when the incident radiation has a significantly different spectra than the emitted radiation. A common example is sunlight falling on a solar collector. The radiation spectrum of sunlight is similar to that of a blackbody at $T_s \approx 5800$ K, and is concentrated mainly in the visible wavelengths. Solar collectors, on the other hand, will typically operate at a temperature of around $T_c \approx 350$ K, and emission at this temperature will be concentrated in the mid IR wavelengths. For such conditions, a collector with a high spectral emissivity in the visible yet a small spectral emissivity in the IR (which is a desirable quality for collectors) would have $\alpha \gg \epsilon$.

1.3 Radiosity and irradiance

The basic idea of this section is as follows: given N surfaces, which can exchange radiation heat transfer among each other, calculate the net rate of heat transfer to each surface.

We will assume that the surface exchanging radiation have *diffuse* surface properties, in that emissivity and absorptivity are not a function of direction. However, the properties are initially taken to be wavelength-dependent, which implies that, in general, $\alpha \neq \epsilon$. Once the formulations are complete, we will examine the simplified situation of gray surface properties.

We begin by defining some basic quantities for use in radiation exchange. Consider a surface, denoted 'surface 1' and having area A_1 . The properties of the surface, including the temperature

T_1 , total (hemispherical) emissivity ϵ_1 and absorptivity α_1 , are assumed to be uniformly constant over the surface.

The *irradiance*, H , is the flux of radiant energy falling on a surface, averaged out over the surface area of the surface. That is, the total radiant energy falling on the surface is

$$H_1 A_1 = \int_{A_1} \int_{\Omega} I_1^- \cos \theta_1 d\Omega dA_1 \quad (1.33)$$

in which I_1^- denotes the intensity falling on the surface, which is (implicitly) a function of incident direction θ , ϕ . Brewster uses the symbol q^- for the irradiance. The *radiosity*, J , is the flux of radiant energy leaving the surface, averaged over the surface area. The total rate of radiant energy leaving the surface is

$$J_1 A_1 = \int_{A_1} \int_{\Omega} I_1^+ \cos \theta_1 d\Omega dA_1 \quad (1.34)$$

The radiosity can be related to the irradiance by

$$J_1 = \epsilon_1 e_{b1} + (1 - \alpha_1) H_1 \quad (1.35)$$

i.e., radiosity will consist of emission from the surface (in which $e_{b1} = \sigma T_1^4$) plus the reflected part of the irradiance (with $\rho_1 = 1 - \alpha_1$). On the other hand, irradiance H_1 will depend explicitly on the incoming radiation field at the surface, which, in turn, will depend on the outgoing radiation fields from all the other surface which can ‘view’ surface 1. We’ll encounter the explicit formulas shortly.

The *net* average radiative flux from the surface, denoted q , will simply be the difference between the flux leaving the surface and the flux arriving at the surface, i.e.,

$$q_1 = J_1 - H_1 \quad (1.36)$$

Note that this formula is not, explicitly, a function of the surface properties ϵ or α . However, an alternative formula for q can be stated, in which q is the difference between the emissive flux from the surface and the absorbed incident flux,

$$q_1 = \epsilon_1 e_{b,1} - \alpha_1 H_1 \quad (1.37)$$

If the surfaces are in steady state (which we assume to be the case), then the net radiative heat transfer rate *from* the surface, $q_1 A_1$, will equal the net rate of heat transfer *to* the surface by other means such as conduction or convection. That is, $q_1 A_1$ is the rate of ‘external’ heat transfer required to keep the surface at a constant temperature. If the surface is adiabatic, then $q_1 A_1 = 0$.

By eliminating H_1 among the previous two equation, we get

$$q_1 = \frac{1}{1 - \alpha_1} (\epsilon_1 e_{b1} - \alpha_1 J_1) \quad (1.38)$$

This equation is not too useful for a black surface (i.e., $\alpha = \epsilon = 1$). For this special case Eq. (1.35) shows that $J = e_b$, but we will need to use either Eqs. (1.36) or (1.37) to get the heat transfer q_1 .

1.4 The Configuration Factor

Say our ‘system’ contains N surfaces, on each of which the temperature T_i is specified. We want to calculate the net heat transfer rate to each surface per Eq. (1.38). To do so, we need to determine the radiosity J_i at each surface (assume, for the moment, that the surface are not black). To get the radiosity, however, we’ll need to know the irradiance at the surface, per Eq. (1.35). And to get the irradiance, we need to know how radiation is exchange among the various surfaces. The relevant formula to evaluate is Eq. (1.33), repeated here as

$$H_1 A_1 = \int_{A_1} \int_{\Omega} I_1^- \cos \theta_1 d\Omega dA_1 \quad (1.39)$$

To simplify the evaluation of this (without a tremendous loss in generality), take the system to consist of a pair of surfaces, 1 and 2. The ‘background’ (i.e., what surrounds 1 and 2) is taken to be black at zero K, so that no radiation originates from the background. In this case all of the radiation arriving at 1 originates (either through emission or reflection) at 2, and the integral over solid angle in Eq. (1.39) will include only those directions which point towards surface 2. Say a point on 2 is located a distance R from a point on 1. The differential solid angle $d\Omega$ in Eq. (1.39) will be, by definition,

$$d\Omega = \frac{\cos \theta_2 dA_2}{R^2} \quad (1.40)$$

in which θ_2 is the angle between the normal on 2 and the direction vector from 1 to 2. Alternatively, $\cos \theta_2 dA_2$ is the projected area of dA_2 as seen from the point on 1. The medium between 1 and 2 is non-participating; it does not absorb or emit radiation along the path. Consequently, for a specified path between 1 and 2, $I_1^- = I_2^+$. That is, the intensity arriving at 1 along the path is the same as the intensity leaving 2 along the same path. Finally, the surfaces are assumed to be diffuse, so

$$I_2^+ = \frac{J_2}{\pi} \quad (1.41)$$

Now replace the two previous equations into Eq. (1.39). The radiosity J_2 is not a function of position on surface 2 (recall that it is averaged over the surface area), so it can be taken out of the integrals. We get

$$H_1 A_1 = J_2 \int_{A_1} \int_{A_2} \frac{\cos \theta_1 \cos \theta_2}{\pi R^2} dA_2 dA_1 \quad (1.42)$$

The cluster of integrals depends only on the geometrical configuration of surfaces 1 and 2, and defines a *configuration factor* F_{2-1}

$$F_{2-1} A_2 \equiv \int_{A_1} \int_{A_2} \frac{\cos \theta_1 \cos \theta_2}{\pi R^2} dA_2 dA_1 \quad (1.43)$$

so that

$$H_1 A_1 = J_2 F_{2-1} A_2 \quad (1.44)$$

for our 2-surface system.

The configuration factor F_{i-j} represents the fraction of radiant energy leaving i that arrives at j . Basic properties of the configuration factor are *reciprocity*,

$$F_{i-j}A_i = F_{j-i}A_j \quad (1.45)$$

which follows directly from Eq. (1.43) by exchanging the subscripts 1 and 2, and *summation*,

$$\sum_{j=1}^N F_{ij} = 1 \quad (1.46)$$

in which N is the total number of surface that can view surface i . This latter property simply states that all the radiation leaving i must end up somewhere. Note also that $j = i$ must also be included in the summation, as F_{i-i} is not necessarily zero.

1.5 Exchange equations

Equation (1.44) can be generalized to an N -surface system,

$$H_i A_i = \sum_{j=1}^N J_j F_{j-i} A_j \quad (1.47)$$

We can now use reciprocity, i.e., $F_{j-i}A_j = F_{i-j}A_i$, in the above and cancel out the area A_i ;

$$H_i = \sum_{j=1}^N J_j F_{i-j} \quad (1.48)$$

Replacing this into Eq. (1.35) gives a system of equations for the radiosities,

$$J_i = \epsilon_i e_{bi} + (1 - \alpha_i) \sum_{j=1}^N J_j F_{i-j} \quad (1.49)$$

with $i = 1, 2, \dots, N$. If we know the temperature of each surface (from which $e_{bi} = \sigma T_i^4$) and we also know the configuration factors, then the system of equations can be solved for the radiosity at each surface. And once we know the radiosity, we can get the heat transfer per Eqs. (1.36–1.38).

Frequently, the heat transfer to a surface is known, and the temperature of the surface becomes an unknown. A typical example is the insulated surface, for which $q_i = 0$. For such cases Eq. (1.49)

will not be useful, since e_{bi} will be unknown. To remedy the problem we use Eq. (1.36) to formula the radiosity equations, to get

$$q_i = J_i + \sum_{j=1}^N J_j F_{i-j} \quad (1.50)$$

Once we solve for the radiosities, the temperature of the surface can be obtained from Eq. (1.37). This equation has a network interpretation: if we multiply the J_i term by $\sum_j F_{i-j} = 1$ (recall the summation property), then

$$q_i A_i = \sum_{j=1}^N (J_i - J_j) F_{i-j} A_i \quad (1.51)$$

The heat transfer from i can therefore be interpreted as a sum of currents flowing from i to all other surfaces, with $J_i - J_j$ being the potential (or voltage) difference and $1/F_{i-j} A_i$ the resistance between i and j .

The general procedure is as follows: say our system has $M < N$ surfaces on which the temperature is prescribed, and $N - M$ surfaces on which q is prescribed. We apply Eq. (1.49) to the M surfaces with specified temperature, and Eq. (1.50) to the $N - M$ surfaces with specified q . Altogether we obtain N linear equations for the radiosities J_1, J_2, \dots, J_N . And once we have solved for these quantities, we can calculate either the heat flux or the temperature of the surface.

1.5.1 Spectral considerations

The previous formulas can be applied on a spectral level by simply appending the λ subscript to all relevant quantities. In doing so, however, the spectral heat flux $q_{\lambda,i}$ to surface i can no longer be viewed as the heat transfer rate to the surface by external means. Rather, the total flux, i.e.,

$$q_{\lambda,i} = \int_0^\infty q_{\lambda,i} d\lambda \quad (1.52)$$

is the heat transfer by external means. This means that, in general, it is difficult to explicitly state specified heat flux boundary conditions in the spectral exchange equations, because we don't (beforehand) know the heat flux on a spectral level. In particular, an adiabatic surface has a total heat transfer rate of zero, yet the spectral flux to the surface could be positive or negative at various wavelengths in such a way that the total, when integrated out per the previous formula, is zero.

It is also difficult to accurately apply the exchange equations on a total (wavelength integrated) basis to surfaces that strong wavelength variations in emissivity/absorptivity (i.e., non-gray surfaces), especially in conditions in which significant temperature differences exist among the surfaces. The problem here is obtaining an estimate of the total absorptivity on each surface *prior* to solving the equations. The total absorptivity of a surface depends on the spectrum of the incident radiation, yet the incident radiation (the irradiance) will depend on the emission and reflection of

radiation throughout the entire system. Indeed, the exchange formulation presented above, with the corresponding configuration factors, does not offer a clean way of predicting the radiation spectrum that falls on a given surface. To see this, note that the absorbed flux (on a total basis) at surface i will be

$$q_{abs,i} = \alpha_i H_i = \sum_{j=1}^N J_j F_{i-j} = \sum_{j=1}^N (\epsilon_j e_{b,j} + (1 - \alpha_j) H_j) F_{i-j}$$

Now α_i depends on the spectrum of H_i , yet H_i is seen to depend on emission *and* reflection from all other surfaces. We can predict the emission spectrum, but we cannot predict the reflection spectrum without considering the incident radiation on surface j . And so on....

In view of these problems, the common practice is to either apply exchange equations on a gray basis ($\alpha = \epsilon$), or to use a spectral formulation and solve on a wavelength-by-wavelength basis. Alternatively, the Monte Carlo procedure (to be discussed in the near future) offers a way of modeling a non-gray yet wavelength-averaged system.

1.6 Gray Approximation

1.6.1 Two surface systems

Often we deal with simple problems in which radiation is exchanged between a pair of surface, such as two parallel plates or a pair of coaxial cylinders. In this case the exchange equations reduce to a pair of equations for J_1 and J_2 . Because the overall system is in steady state, the heat transfer rates must be balanced by

$$q_1 A_1 + q_2 A_2 = 0$$

It is easy enough to solve two linear equations for two unknowns, yet for the general non-gray case ($\alpha \neq \epsilon$) the resulting equation for the heat transfer rate $q_1 A_1$ is algebraically complex and need not be presented here.

A considerable simplification will occur if we examine the gray simplification, for which $\alpha_i = \epsilon_i$. For this case, the heat flux, from Eq. (1.38), becomes

$$q_1 A_1 = \frac{\epsilon_1 A_1}{1 - \epsilon_1} (e_{b1} - J_1), \quad \text{gray approx.} \quad (1.53)$$

which has the same network form (current=voltage drop/resistance) as Eq. (1.51). The resistance in the above, i.e., $(1 - \epsilon)/\epsilon A$, can be viewed as a surface resistance, whereas the resistance in Eq. (1.51), $1/F_{1-2} A_1$, is a geometrical (or space) resistance. In the two-surface problem the network is equivalent to a series circuit with three resistances; two surface resistances and a single space resistance. And by adding up the resistances, we get

$$q_1 A_1 = \frac{e_{b1} - e_{b2}}{\frac{1 - \epsilon_1}{\epsilon_1 A_1} + \frac{1}{F_{1-2} A_1} + \frac{1 - \epsilon_2}{\epsilon_2 A_2}} \quad (1.54)$$

The multiple reflection model

An alternative way of modelling radiation exchange in a simple, two-surface system is to view the exchange process as a series of reflections. For simplicity, consider a system of parallel flat plates, for which $F_{1-2} = F_{2-1} = 1$. Assume also that surface 1 is at a finite temperature yet 2 is at zero K, so that emission occurs only from surface 1. This does not limit the generality of the approach, for we can model heat transfer between two surfaces at finite temperature as a superposition of two heat transfers, with each of the two heat transfers corresponding to one surface at zero K and the other at the finite temperature.

Surface 1 emits heat at a rate $\epsilon_1 e_{b1}$. This emission travels to surface two, and a fraction ρ_2 is reflected. The reflected fraction travels back down to 1, and a fraction ρ_1 of this is reflected back towards 2. And so on. With this picture, the net rate at which radiation leaves 1, i.e., the radiosity at 1, is

$$J_1 = \epsilon_1 e_{b1} \left(1 + \rho_2 \rho_1 + (\rho_2 \rho_1)^2 + \dots \right) = \epsilon_1 e_{b1} \frac{1}{1 - \rho_1 \rho_2}$$

The second line comes from the power series expansion of $1/(1-x)$ for $x < 1$. Likewise, the irradiance on surface 1 will consist of the first reflection of the emission, $\rho_2 \epsilon_1 e_{b1}$, plus all multiple reflections,

$$H_1 = \rho_2 \epsilon_1 e_{b1} \left(1 + \rho_2 \rho_1 + (\rho_2 \rho_1)^2 + \dots \right) = \rho_2 \epsilon_1 e_{b1} \frac{1}{1 - \rho_1 \rho_2}$$

The heat transfer is $q_1 = J_1 - H_1$, and using $\rho = 1 - \alpha$ we get

$$q_1 = \epsilon_1 e_{b1} \frac{1 - \rho_2}{1 - \rho_1 \rho_2} = e_{b1} \frac{\epsilon_1 \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1 \alpha_2}$$

If you now set $\alpha = \epsilon$ (the gray approximation) and perform a little extra algebra, the above result will be equivalent to Eq. (1.54) with $e_{b2} = 0$.

1.6.2 More than two surfaces

For gray, nonblack surfaces the exchange equations become

$$\epsilon_i e_{bi} = J_i - (1 - \epsilon_i) \sum_{j=1}^N J_j F_{i-j}, \quad \text{specified } T_i \quad (1.55)$$

$$q_i = J_i + \sum_{j=1}^N J_j F_{i-j}, \quad \text{specified } q_i \quad (1.56)$$

These can be solved for J_i , $i = 1, 2, \dots, N$ by standard methods for linear equations (matrix inversion, iteration). Once the radiosities are obtained, the heat transfer fluxes at surfaces with specified temperature are obtained from

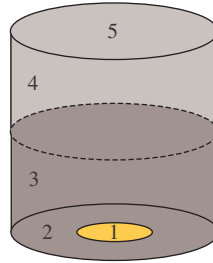
$$q_i = \frac{\epsilon_i}{1 - \epsilon_i} (e_{bi} - J_i) \quad (1.57)$$

and on surfaces with specified q_i , the emissive power (and, from which, the temperature) would be obtained from

$$e_{bi} = \frac{1 - \epsilon_i}{\epsilon_i} q_i + J_i \quad (1.58)$$

1.6.3 Gray/diffuse exchange example

Consider the system illustrated below. The system consists of a cylindrical cavity, of outer radius $R = 3$ m and height $L = 6$ m. A circular surface on the lower end (surface 1) has a radius of $R_1 = 1$ m and properties of $\epsilon_1 = 0.8$, $T_1 = 1000$ K. Surfaces 2 and 3 are adiabatic and $L_3 = L_4 = 3$ m. Surfaces 4 and 5 are at 400 K and have $\epsilon = 0.2$.



We want to compute the net heat transfer rates from surfaces 1, 4, and 5 and the temperatures of surfaces 2 and 3.

Configuration factor calculation

The formula for parallel concentric circles on P. 500 of the text can be used to obtain all of the needed configuration factors. For a pair of circles (1 and 2, say) separated by a distance L , the formula is

$$X = \frac{R_1}{L}, \quad Y = \frac{R_2}{L}, \quad Z = 1 + \frac{1 + Y^2}{X^2}$$

$$F_{1-2} = \frac{1}{2} \left[Z - \left(Z^2 - 4 \left(\frac{Y}{X} \right)^2 \right)^{1/2} \right]$$

Applying this formula gives

$$F_{1-5} = 0.1965, \quad F_{(1+2)-5} = 0.1716$$

$$F_{5-1} = \frac{R_1^2}{R^2} F_{1-5} = 0.0218, \quad F_{5-(1+2)} = F_{(1+2)-5} = 0.1716$$

$$F_{5-2} = F_{5-(1+2)} - F_{5-1} = 0.1497, \quad F_{2-5} = \frac{R^2}{R^2 - R_1^2} F_{5-2} = 0.1685$$

Let surface 6 be the imaginary circular surface formed by the dotted line.

$$F_{5-6} = 0.3820, \quad F_{1-6} = 0.4861$$

Then

$$F_{5-4} = 1 - F_{5-6} = 0.6180, \quad F_{1-3} = 1 - F_{1-6} = 0.5139$$

$$F_{4-5} = \frac{R^2}{2RL_4} = 0.3090, \quad F_{3-1} = \frac{R_1^2}{2RL_3} F_{1-3} = 0.2855$$

By symmetry and summation,

$$F_{3-(1+2)} = F_{4-5} = F_{3-1} + F_{3-2} : \quad F_{3-2} = F_{4-5} - F_{3-1} = 0.2805$$

$$F_{2-3} = \frac{2RL_3}{R^2 - R_1^2} = 0.6311$$

Now use summation:

$$F_{1-4} = 1 - F_{1-5} - F_{1-3} = 0.2896, \quad F_{5-3} = 1 - F_{5-4} - F_{5-2} - F_{5-1} = 0.2104$$

$$F_{2-4} = 1 - F_{2-5} - F_{2-3} = 0.2005$$

$$F_{4-1} = \frac{R_1^2}{2RL_4} F_{1-4} = 0.0161, \quad F_{3-5} = \frac{R^2}{2RL_3} F_{5-3} = 0.1052$$

$$F_{4-2} = \frac{R^2 - R_1^2}{2RL_4} F_{2-4} = 0.0891$$

By the symmetry of the problem and summation,

$$F_{3-(1+2)} = F_{4-5} = F_{3-1} + F_{3-2} : \quad F_{3-2} = 0.2805$$

$$F_{2-3} = \frac{A_3}{A_2} F_{3-2} = 0.6311$$

What leaves 3 and lands on 6 (the imaginary surface) must land on either 4 or 5, so

$$F_{3-6} = F_{3-4} + F_{3-5}$$

But $F_{3-6} = F_{4-5}$ by symmetry, so

$$F_{3-4} = F_{4-5} - F_{3-5} = 0.2038 = F_{4-3}$$

Only surfaces 3 and 4 can see themselves. Use summation:

$$F_{3-3} = F_{4-4} = 1 - F_{3-1} - F_{3-2} - F_{3-4} - F_{3-5} = 0.3820$$

Exchange equations

For surfaces 1, 4, and 5, upon which the temperature is specified, the exchange equations are

$$J_i - (1 - \epsilon_i) \sum_{j=1}^N J_j F_{i-j} = \epsilon_i e_{bi}, \quad i = 1, 4, 5 \quad (1.59)$$

and on the adiabatic surfaces, 2 and 3, the exchange equations are

$$J_i - \sum_{j=1}^N J_j F_{i-j} = 0, \quad i = 2, 3 \quad (1.60)$$

The following is a listing of the Mathematica code used to solve the equations. I had previously calculated the configuration factors and stored them in the arrays $f[i, j]$. I use the symbol $jf[i]$ for J_i in the code.

```
In[29]:=sigma=5.67*^-8;eps[1]=0.8;eps[4]=0.2;eps[5]=0.2;
t[1]=1000;t[4]=400;t[5]=400;
eb[1]=sigma t[1]^4;eb[4]=sigma t[4]^4;eb[5]=sigma t[5]^4;
```

```
In[37]:=vars=Table[jf[i],{i,1,5}]
Out[37]={jf[1],jf[2],jf[3],jf[4],jf[5]}
```

```
In[40]:=eqns=Table[
  If[i>1&& i<4,
    jf[i]-Sum[jf[j] f[i,j],{j,1,5}]==0,
    jf[i]-(1-eps[i])Sum[jf[j] f[i,j],{j,1,5}]==eps[i] eb[i]
  ],
  {i,1,5}]
```

```
Out[40]=
{jf[1]-0.2 (0.513878 jf[3]+0.28963 jf[4]+0.196491 jf[5])==45360.,
 jf[2]-0.631053 jf[3]-0.200488 jf[4]-0.168458 jf[5]==0,
 -0.0285488 jf[1]-0.280468 jf[2]+0.618034 jf[3]-0.20382 jf[4]
 -0.105197 jf[5]==0,
 jf[4]-0.8 (0.0160906 jf[1]+0.089106 jf[2]+0.20382 jf[3]
 +0.381966 jf[4]+0.309017 jf[5])==290.304,
 -0.8 (0.0218324 jf[1]+0.14974 jf[2]+0.210393 jf[3]
 +0.618034 jf[4])+jf[5]==290.304}
```

```
In[42] := soln = Solve[eqns, vars] [[1]]
```

```
Out[42] = {jf[1] -> 47060.5, jf[2] -> 8808.58, jf[3] -> 9753.96,  
jf[4] -> 7088.69, jf[5] -> 7314.03}
```

```
In[44] := t[2] = (jf[2]/sigma)^(1/4)/.soln  
t[3] = (jf[3]/sigma)^(1/4)/.soln
```

```
Out[44] = 627.814
```

```
Out[45] = 644.02
```

```
In[48] :=
```

```
q[1] = eps[1] a[1] / (1 - eps[1]) (eb[1] - jf[1]) / .soln
```

```
q[4] = eps[4] a[4] / (1 - eps[4]) (eb[4] - jf[4]) / .soln
```

```
q[5] = eps[5] a[5] / (1 - eps[5]) (eb[5] - jf[5]) / .soln
```

```
Out[48] = 121133.
```

```
Out[49] = -79693.6
```

```
Out[50] = -41439.7
```

```
In[51] := q[4] + q[5]
```

```
Out[51] = -121133.
```