QUALITATIVE CHANGES IN DYNAMICAL STATUS
THEORY AND OPEN PROBLEMS

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Preliminaries:

- What is a dynamical system?

  Any system whose status changes with time.

- Dynamical systems are specified by a number of variables, called states. Examples:

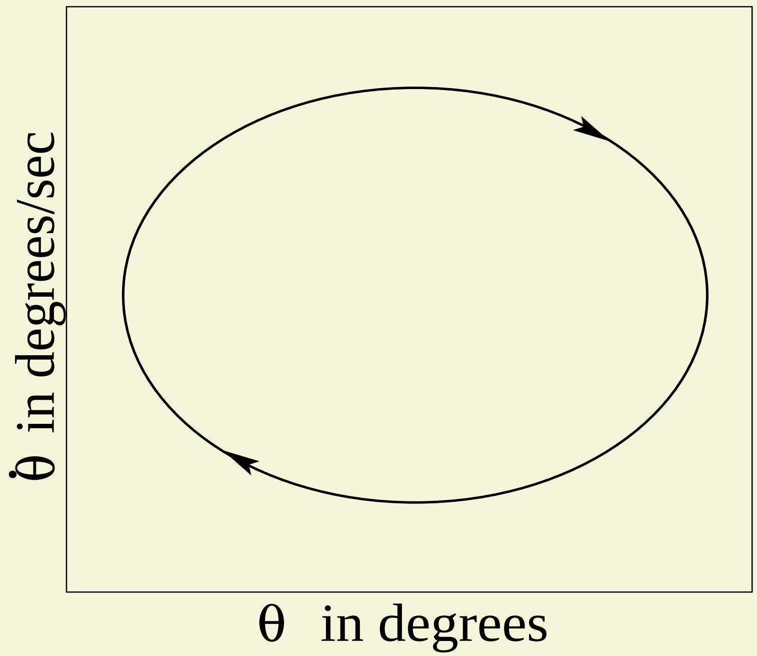
  - ➜ the position and momentum of a body,
  - ➜ the charge stored in a capacitor,
  - ➜ the current through an inductor.

To mathematically define the dynamics of a given system, we specify how the states change with time
$\Rightarrow$ differential equations.

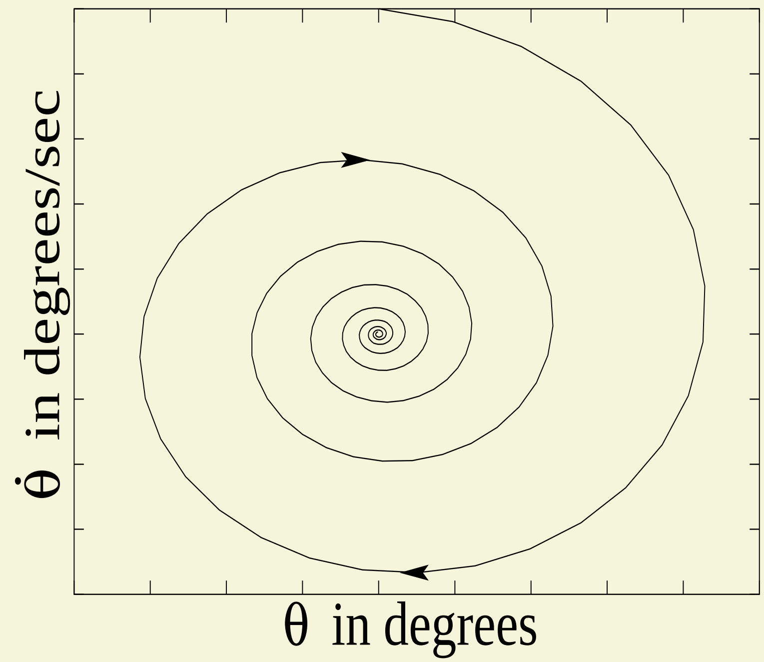
We then define a space with the states as the coordinates
$\Rightarrow$ State space or phase space.

Dynamics can be geometrically viewed as trajectories in the state space.

Trajectories in state space:



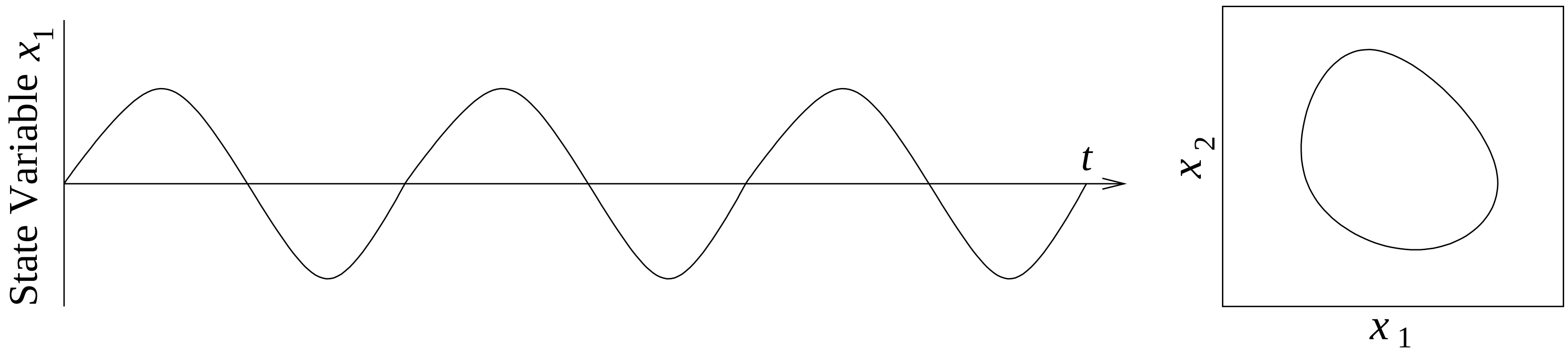
Simple pendulum

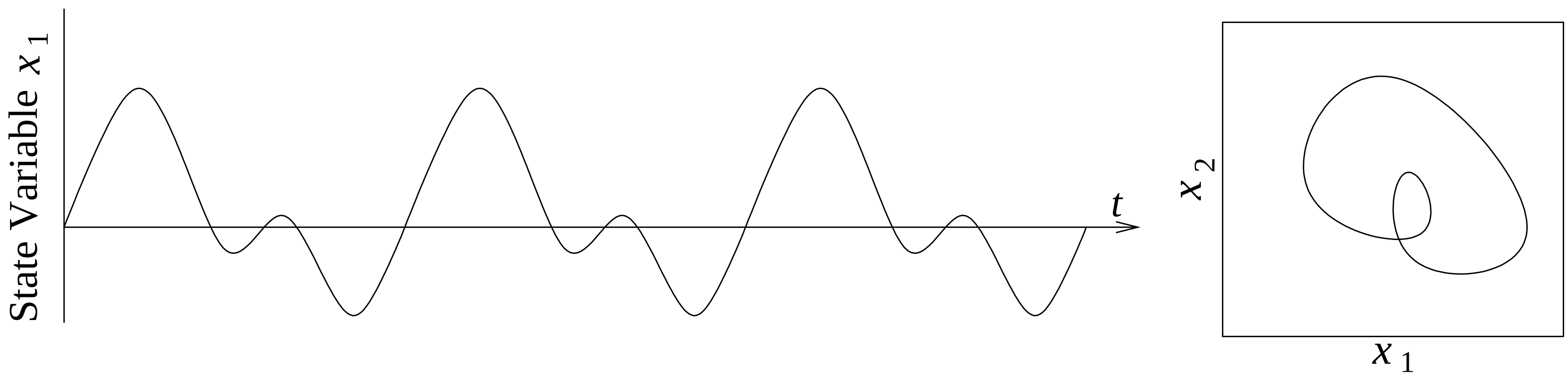


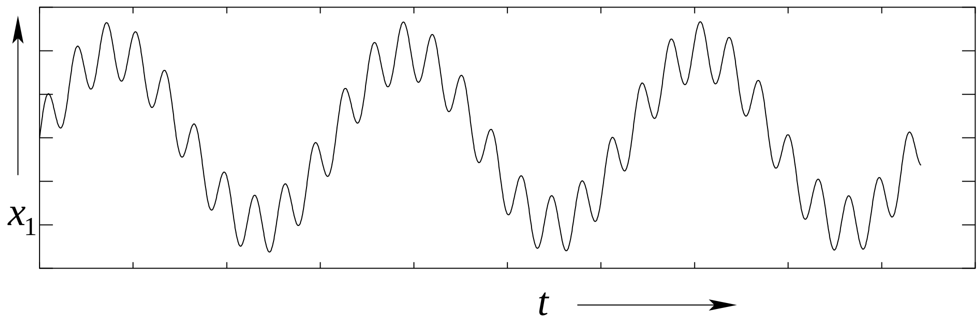
Pendulum with friction

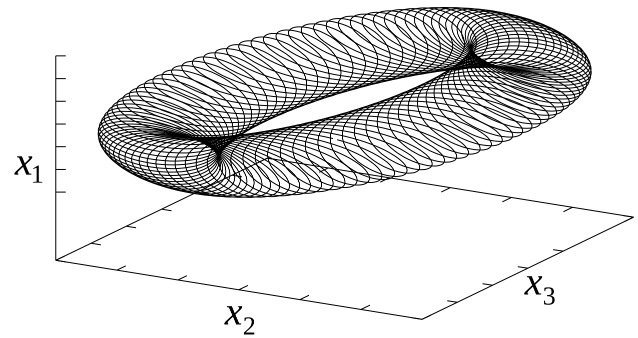
- Such simple trajectories are obtained by linear differential equations.
- Most systems found in nature or in engineering are nonlinear; linearity is a very special case.
- Nonlinear systems may exhibit many types of complex dynamical behaviours.

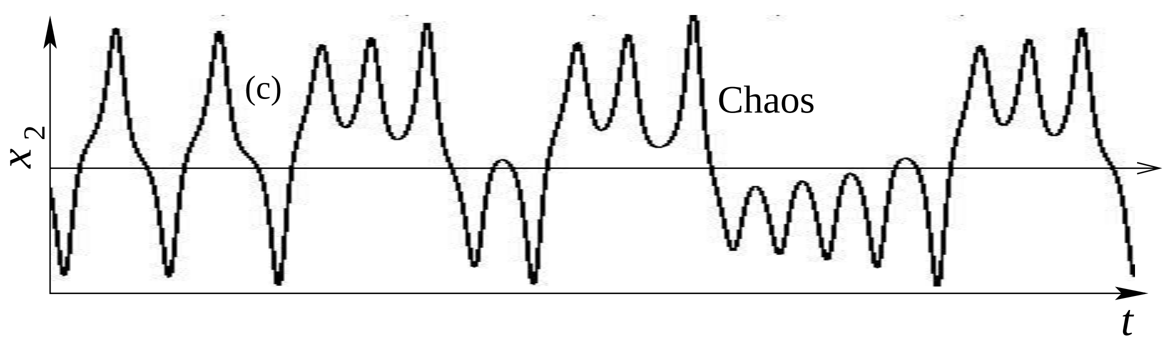
Examples of a few qualitatively different types of dynamical behaviour:

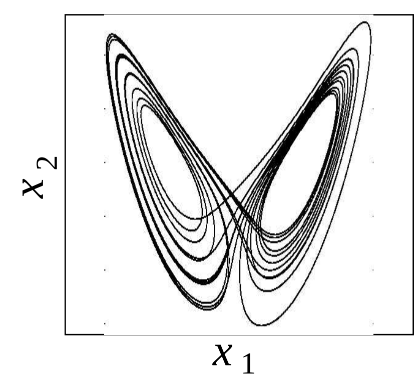





x_1
t


x_1
x_2
x_3


x_2
(c)
Chaos
t


x_2
x_1

CHAOS

- Aperiodic waveform
- Seemingly random, noise-like behavior
- Completely deterministic
- The orbit is sensitively dependent on the initial condition
- Statistical behaviour (average values of state variables, power spectrum etc.) completely predictable.
- Unstable at every equilibrium point, but globally stable. Waveform bounded.

Example: the Lorenz system

$$
\begin{aligned}
\dot{x} &= -\sigma(x - y) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz,
\end{aligned}
$$

Set $b = 8/3$ and $\sigma = 10$, and let r be a variable parameter.

What is a bifurcation?

In any system, as a parameter is varied, there is some change in the dynamical behavior. Most of the time these changes are only quantitative in nature. But there may also be situations where a small parameter change may result in a *qualitative change* in steady state behavior of a dynamical system. Such events are called bifurcations.

Naturally, bifurcations are very important dynamical events.

Bifurcation theory tries to answer the question:

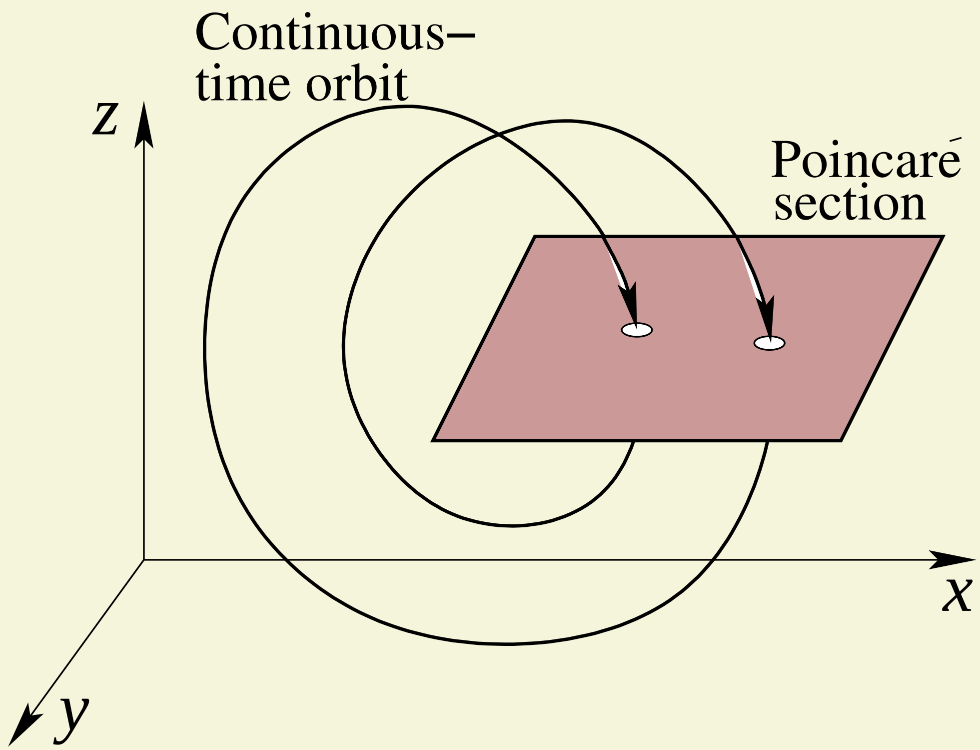
What is the underlying mathematical mechanism that may cause such qualitative change in the dynamical behaviour as a parameter is varied?

In dynamical systems we are interested in studying the asymptotically stable orbits, and how they change in response to changes in the parameters.

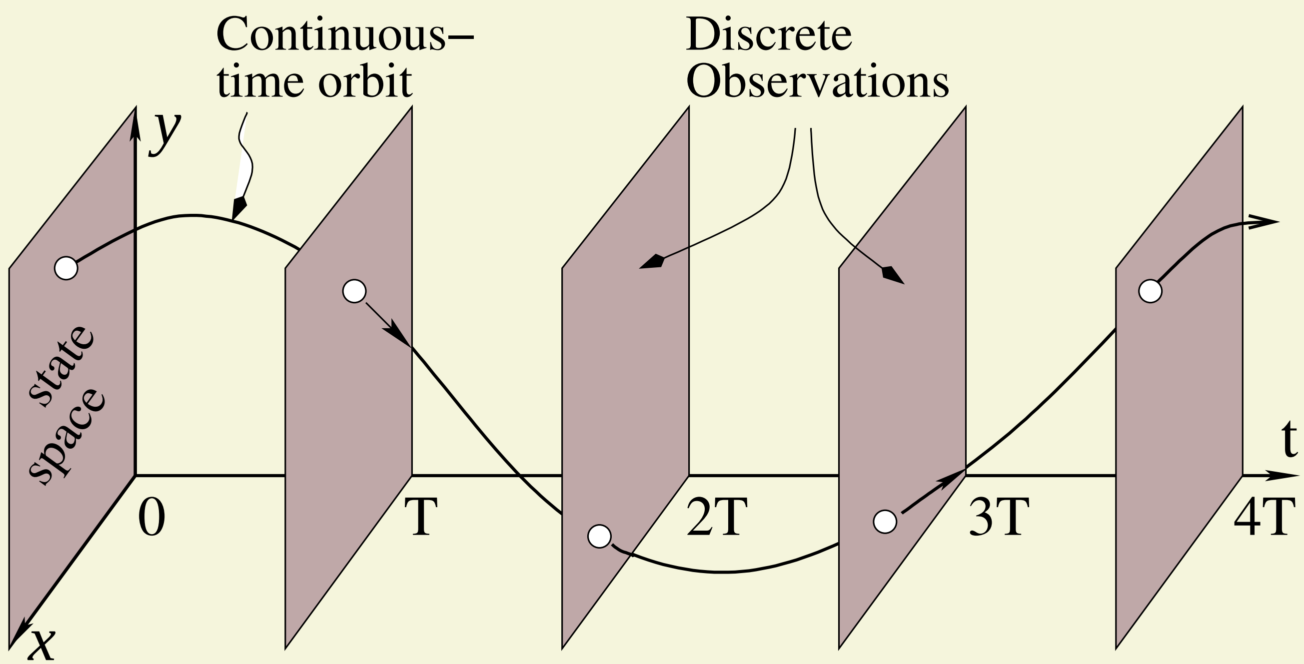
These are generally studied by obtaining discrete-time maps in the form

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n)$$

with the method of *Poincaré section*.



Obtaining Poincaré map from state-space trajectory for autonomous systems (where there is no external periodic input),



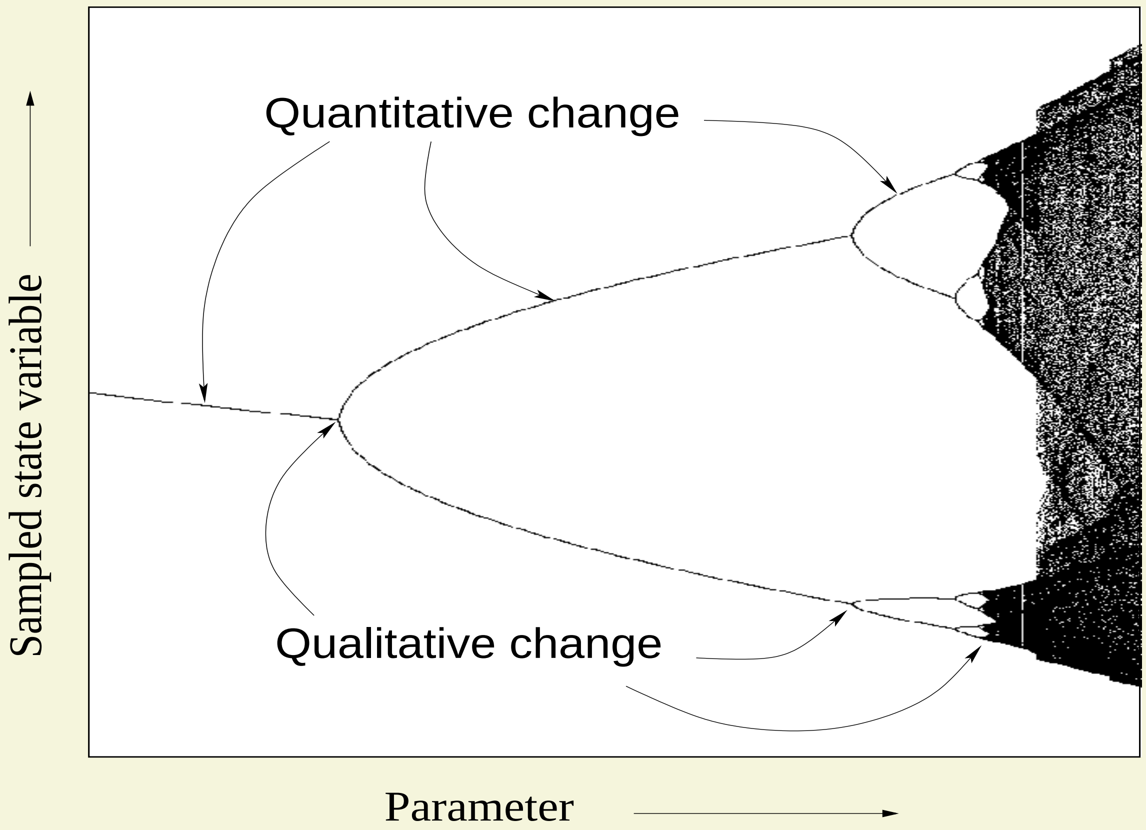
Obtaining Poincaré map from state-space trajectory for non-autonomous systems (where there is a periodic input).

Questions:

- How does the system behaviour change with the change in parameters?
- How can we explain the observed bifurcations?

Bifurcation diagrams

(panoramic view of stability status).



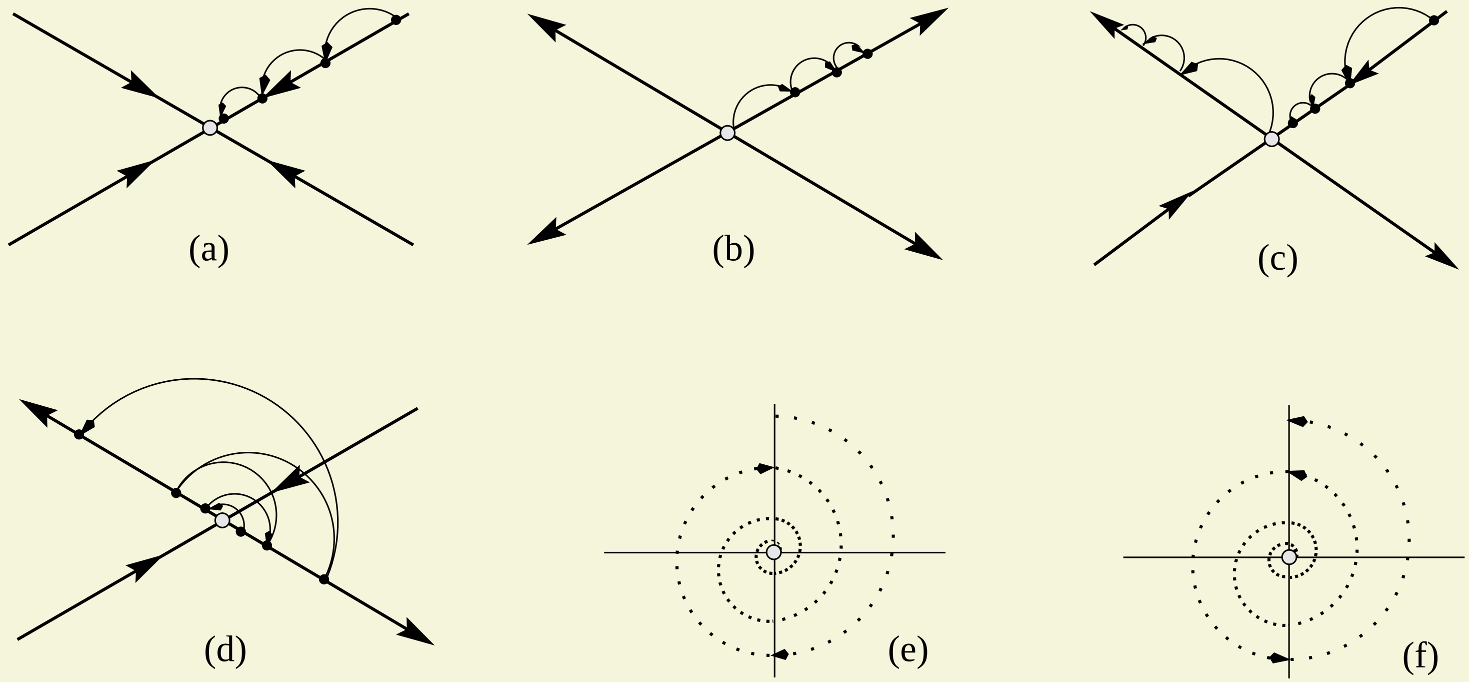
Standard method of studying bifurcations in the map

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n) :$$

1. Locate the fixed point of the map

$$\mathbf{x}_{n+1} = \mathbf{x}_n = \mathbf{x}^*$$

2. Locally linearize the discrete system in the neighborhood of a fixed point by obtaining the Jacobian matrix.
3. Obtain the eigenvalues of the Jacobian matrix. The eigenvalues indicate the type of the fixed point.



(a) An attractor: eigenvalues real, $0 < \lambda_1, \lambda_2 < 1$.

(b) A repeller: eigenvalues real, $\lambda_1, \lambda_2 > 1$.

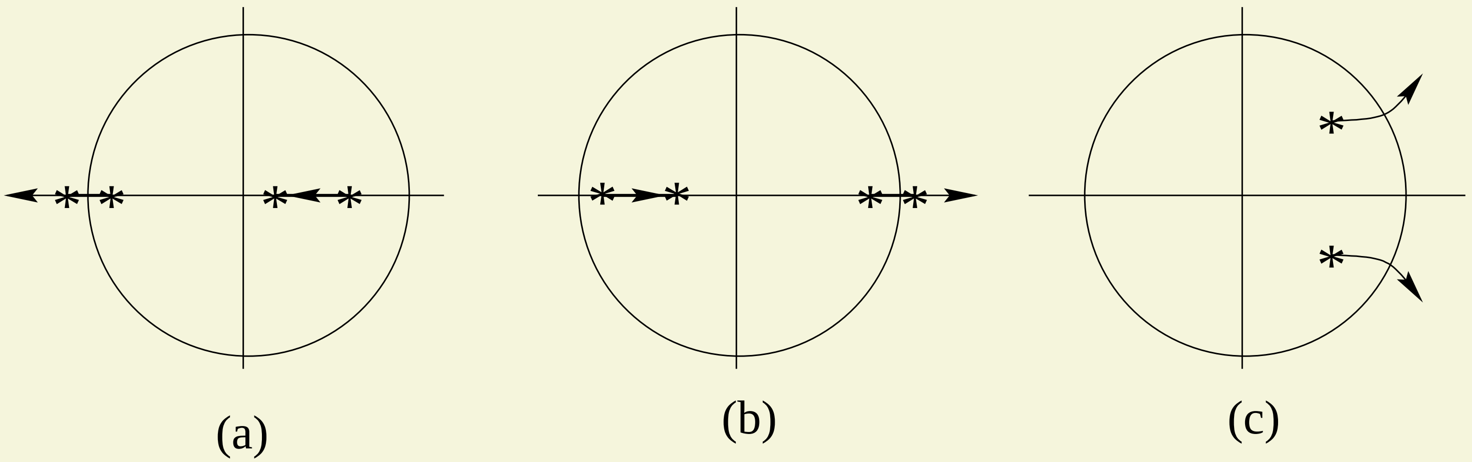
(c) A regular saddle: eigenvalues real, $0 < \lambda_1 < 1$, $\lambda_2 > 1$.

(d) A flip saddle: eigenvalues real, $0 < \lambda_1 < 1$, $\lambda_2 < -1$.

(e) A spiral attractor: eigenvalues complex, $|\lambda_1|, |\lambda_2| < 1$.

(f) A spiral repeller: eigenvalues complex, $|\lambda_1|, |\lambda_2| > 1$.

- Bifurcation occurs when a fixed point loses stability.
- Condition of stability of a fixed point: $|\lambda| < 1$, i.e., Eigenvalues should remain inside the unit circle.
- The classification of bifurcations depends on where an eigenvalue crosses the unit circle.
- Smooth systems can lose stability in three possible ways.

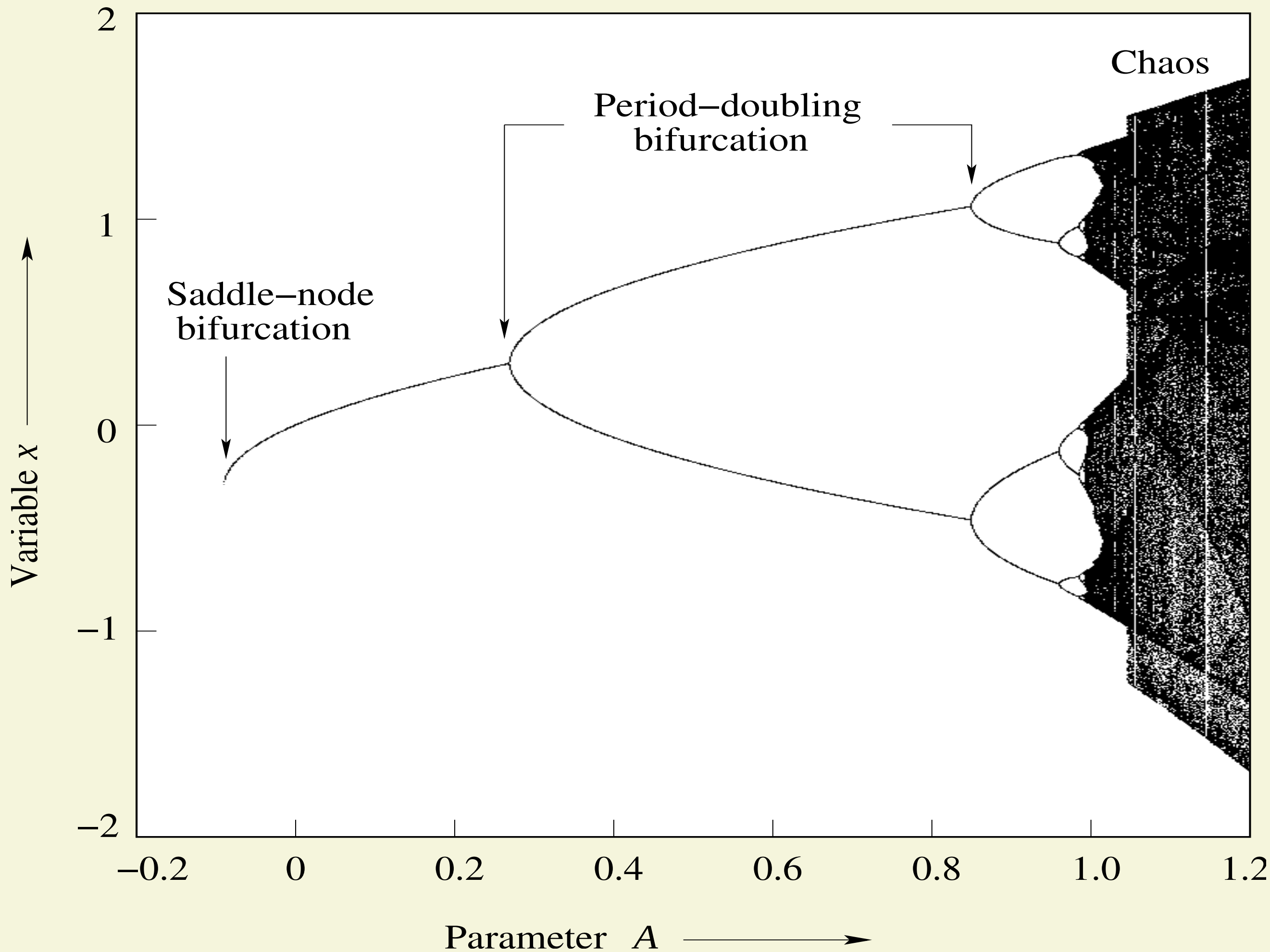


(a) A period doubling bifurcation: eigenvalue crosses the unit circle on the negative real line,

(b) A saddle-node or fold bifurcation: an eigenvalue touches the unit circle on the positive real line,

(c) A Hopf or Naimark bifurcation: a complex conjugate pair of eigenvalues cross the unit circle.

- In a period doubling bifurcation, a fixed point becomes unstable and another stable double-periodic orbit emerges.
- In a saddle-node bifurcation, a pair of new fixed points are created – one stable and the other unstable; responsible for periodic windows.
- In a Naimark bifurcation, a periodic orbit changes to a quasiperiodic orbit (summation of two incommensurate frequencies).



Hénon map:

$$x_{n+1} = A - x_n^2 + 0.4\, y_n,$$
$$y_{n+1} = x_n.$$

Hybrid systems are dynamical systems with continuous-time evolution punctuated by discrete events.

Examples:

- Power electronic circuits
- Systems involving relays
- Impacting mechanical systems
- Systems involving dry friction (stick-slip motion)
- Nonlinear circuits like the Colpitt's oscillator, Chua's circuit etc.
- Walking robots
- Hydraulic systems with on-off valves, the human heart
- Continuous systems controlled by discrete logic.

In hybrid dynamical systems, certain discrete events occur when certain conditions on the state variables are satisfied. The discrete events signify some change in the continuous-time state variable equations.

Mathematically, these systems can be described equations of the form

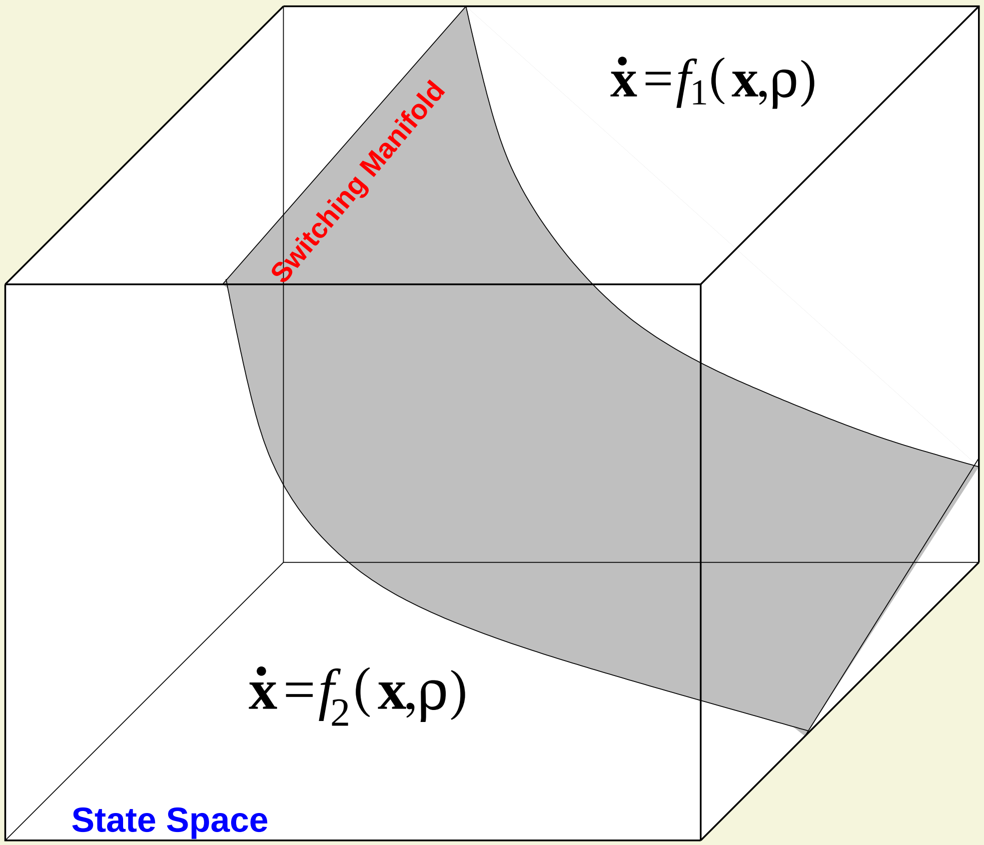
$$\dot{\mathbf{x}} = f(\mathbf{x}, \rho) = \begin{cases} f_1(\mathbf{x}, \rho) & \text{for} \quad \mathbf{x} \in R_1 \\ f_2(\mathbf{x}, \rho) & \text{for} \quad \mathbf{x} \in R_2 \\ \vdots & \\ f_n(\mathbf{x}, \rho) & \text{for} \quad \mathbf{x} \in R_n \end{cases}$$

where R_1, R_2 etc. are different regions of the state space, and ρ is a system parameter.

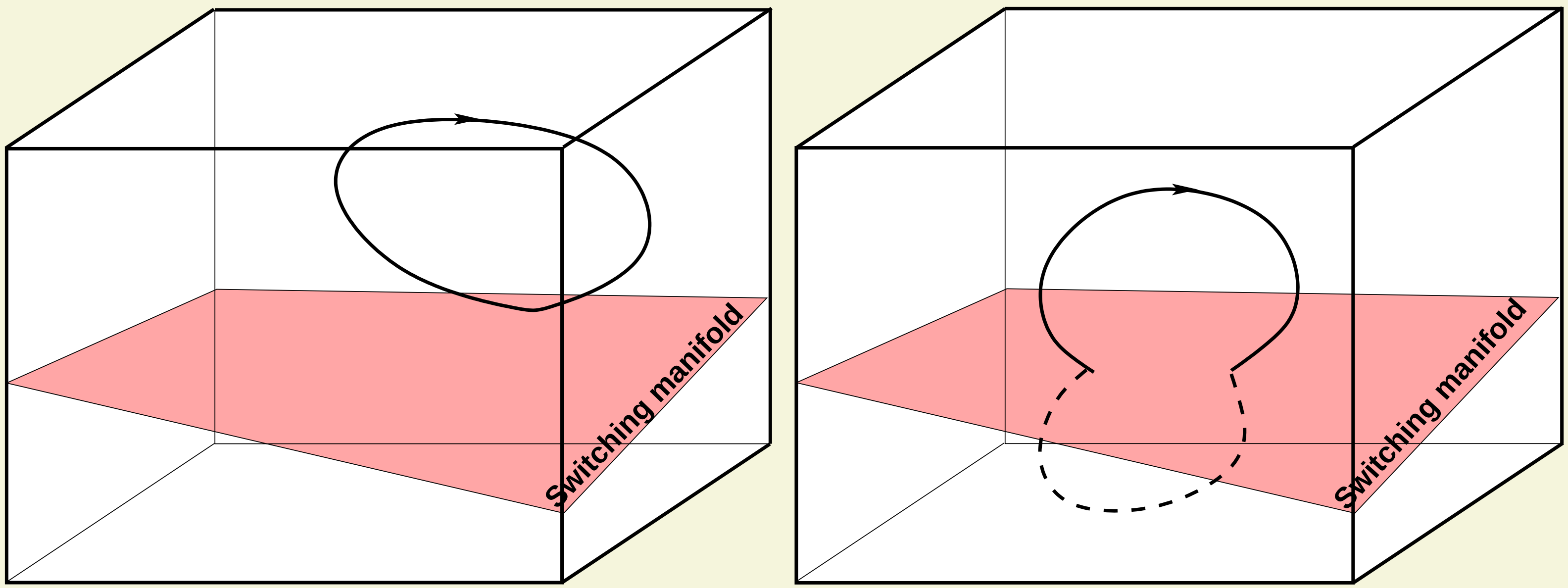
The regions are divided by the discrete event conditions. In the state space these are $(n-1)$ dimensional surfaces given by algebraic equations of the form

$$\Gamma_n(\mathbf{x}) = 0.$$

These are the "switching manifolds."

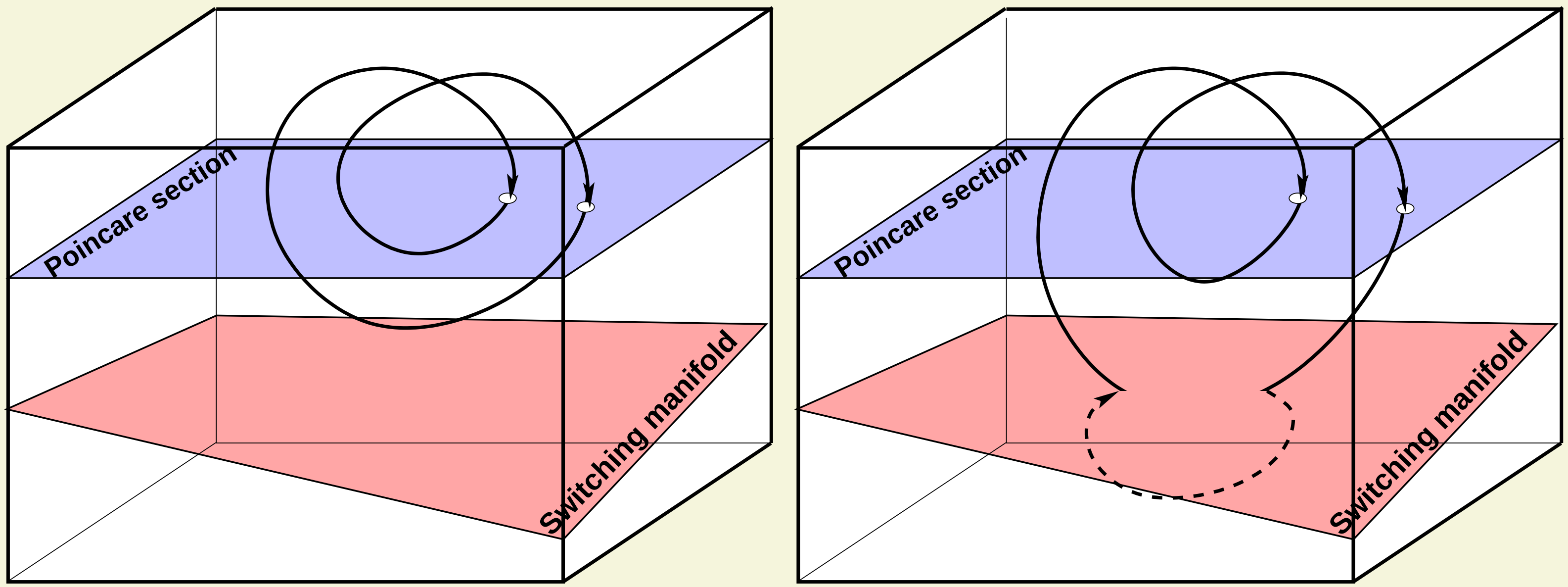


Schematic diagram showing the structure of the state space of a hybrid system.

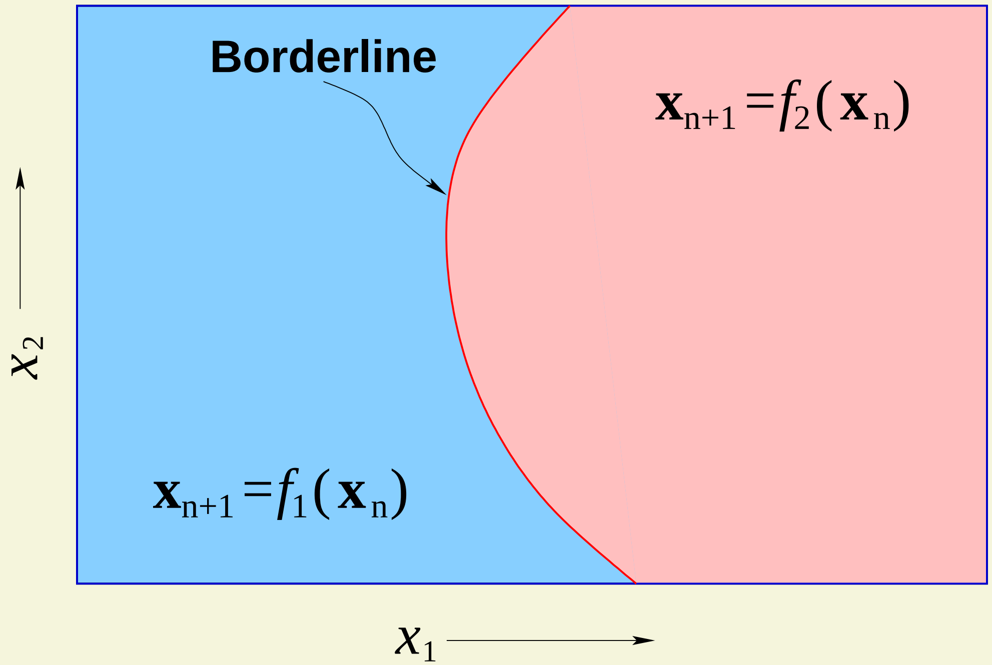


In case of hybrid systems there can be two (or more) different types of orbits depending on which regions in the state space are visited.

Therefore the Poincaré section must yield different functional forms of the map depending on the number of crossing of the switching manifold.



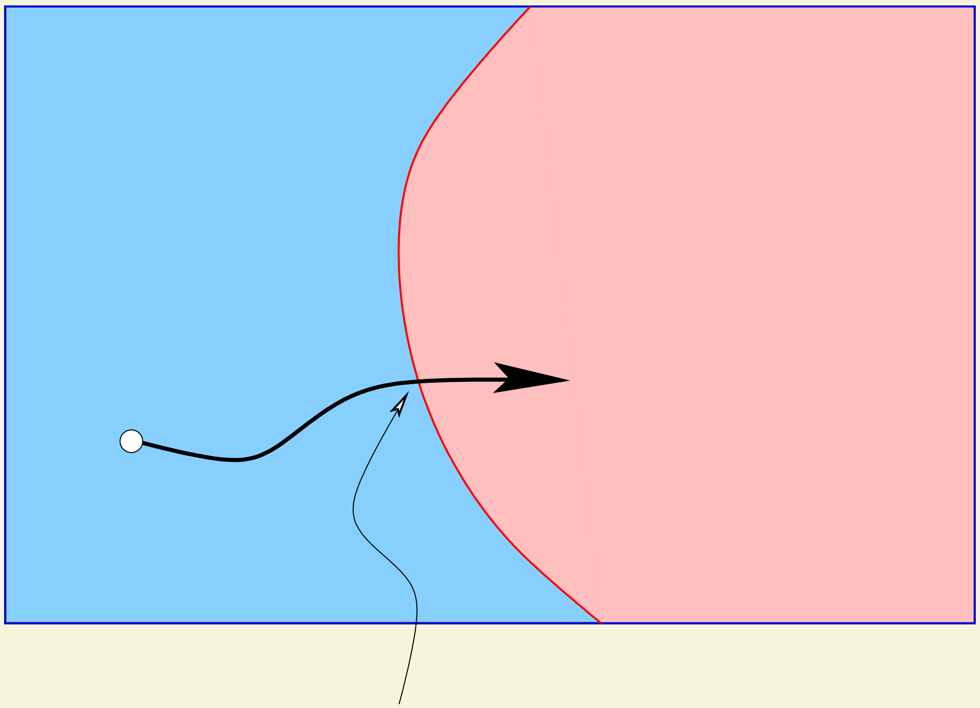
This implies that the structure of the discrete state space for a hybrid system must be piecewise smooth (PWS).



The borderline in discrete domain corresponds to the condition where the orbit grazes the switching manifold in the continuous-time system.

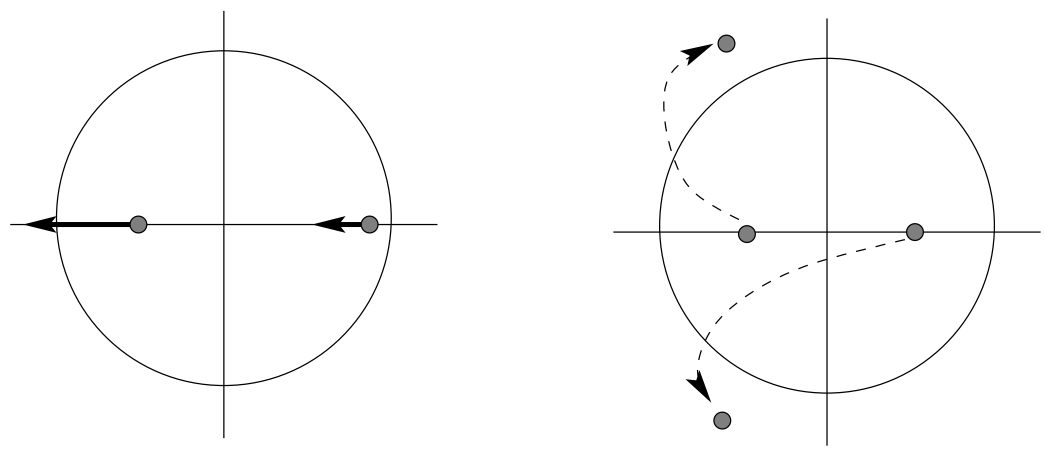
Dynamics of Piecewise Smooth Maps

- If a fixed point loses stability while in either side, the resulting bifurcations can be categorized under the generic classes for smooth bifurcations.
- But what if a fixed point crosses the borderline as some parameter is varied?



The Jacobian elements discretely change at this point

- The eigenvalues may jump from any value to any other value across the unit circle.
- The resulting bifurcations are called *Border Collision Bifurcations*.



Continuous movement of eigenvalues in a smooth bifurcation

Discontinuous jump of eigenvalues in a border collision bifurcation

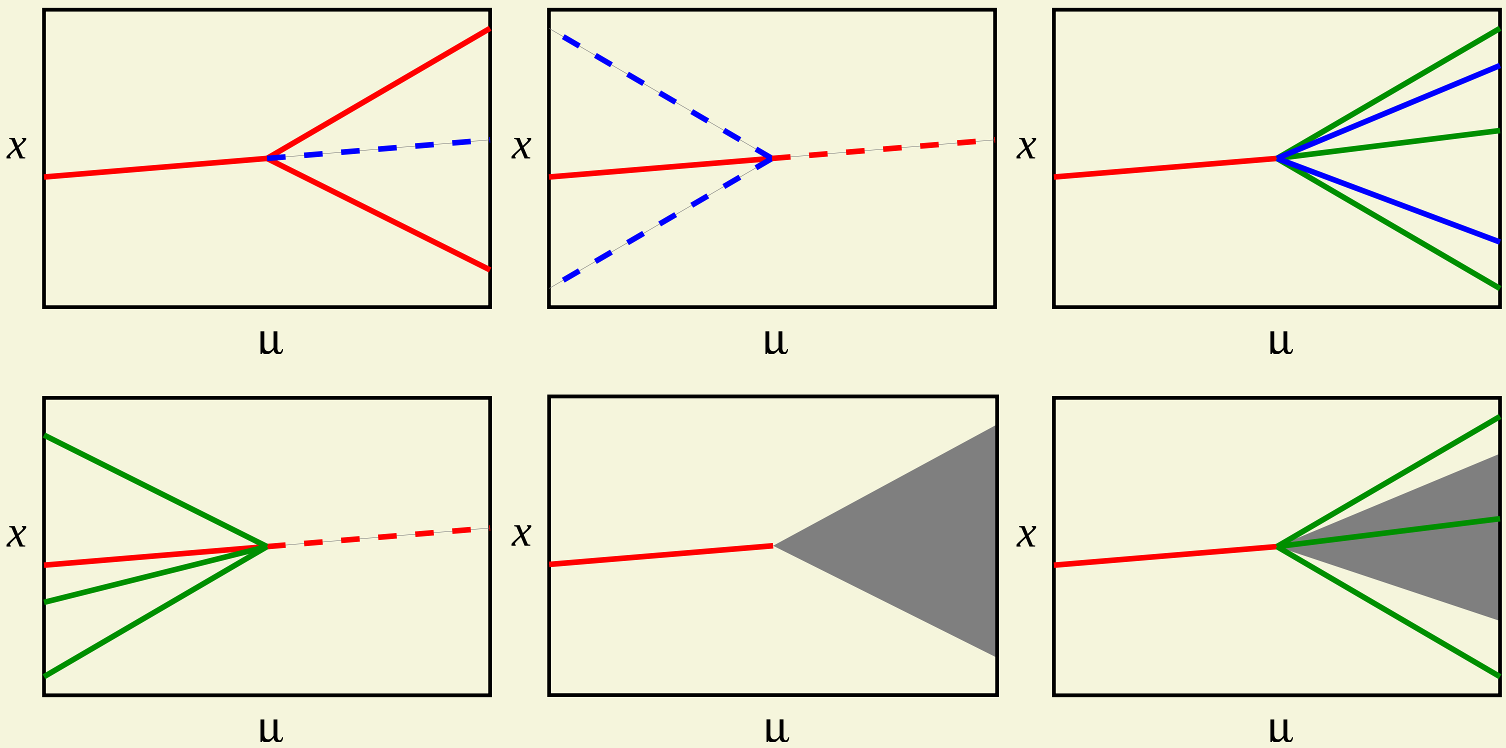
My contribution has mainly been to develop the mathematical theory of border collision bifurcations:

1. Soumitro Banerjee, Celso Grebogi, "Border Collision Bifurcations in Two-Dimensional Piecewise Smooth Maps", *Physical Review E*, Vol.59, No.4, 1 April, 1999, pp.4052-4061. Times cited: 57
2. Soumitro Banerjee, James A. Yorke and Celso Grebogi, "Robust Chaos", *Physical Review Letters*, Vol.80, No.14, 6 April 1998, pp. 3049-3052. Times cited: 43
3. Parag Jain and Soumitro Banerjee, "Border Collision Bifurcations in One-Dimensional Discontinuous Maps," *International Journal on Bifurcation and Chaos*, Vol. 13, No. 11, November 2003, pp.3341-3352. Times cited: 8
4. Anindita Ganguli and Soumitro Banerjee, "Dangerous bifurcation at border collision — when does it occur?" *Physical Review E*, Vol.71, No.5, May 2005.

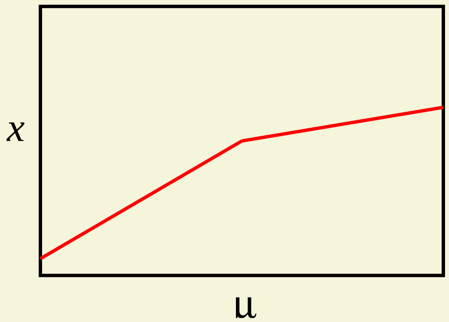
And to apply it in various fields of science and engineering.

1. Guohui Yuan, Soumitro Banerjee, Edward Ott and James A. Yorke, “Border-collision Bifurcations in the Buck Converter”, *IEEE Transactions on Circuits & Systems – I*, Vol.45, No.7, July 1998, pp.707-716. Times cited: 78
2. S. Banerjee, P. Ranjan and C. Grebogi, “Bifurcations in two-dimensional piecewise smooth maps — theory and applications in switching circuits”, *IEEE Transactions on Circuits & Systems – I*, Vol. 47, No. 5, May 2000, pp. 633-643. Times cited: 58
3. Krishnendu Chakrabarty, Goutam Poddar and Soumitro Banerjee, “Bifurcation Behaviour of the Buck Converter”, *IEEE Transactions on Power Electronics*, Vol.11, No.3, May 1996, pp.439-447. Times cited: 51
4. Soumitro Banerjee and Krishnendu Chakrabarty, “Nonlinear Modeling and Bifurcations in the Boost Converter”, *IEEE Transactions on Power Electronics*, Vol.13, No.2, March 1998, pp.252-260. Times cited: 47

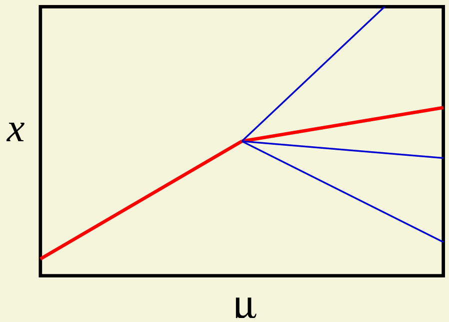
Scenario 1: A fixed point loses stability as it moves across the border.



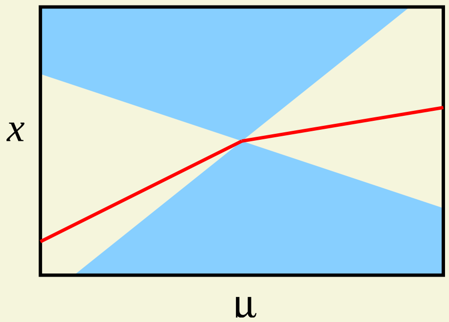
Scenario 2: A fixed point remains stable. But ...



The “normal” case.

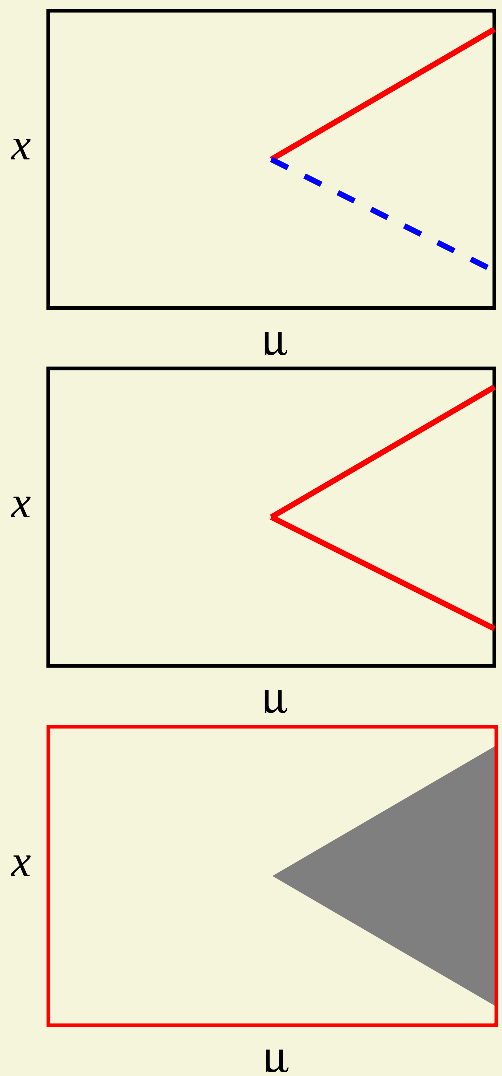


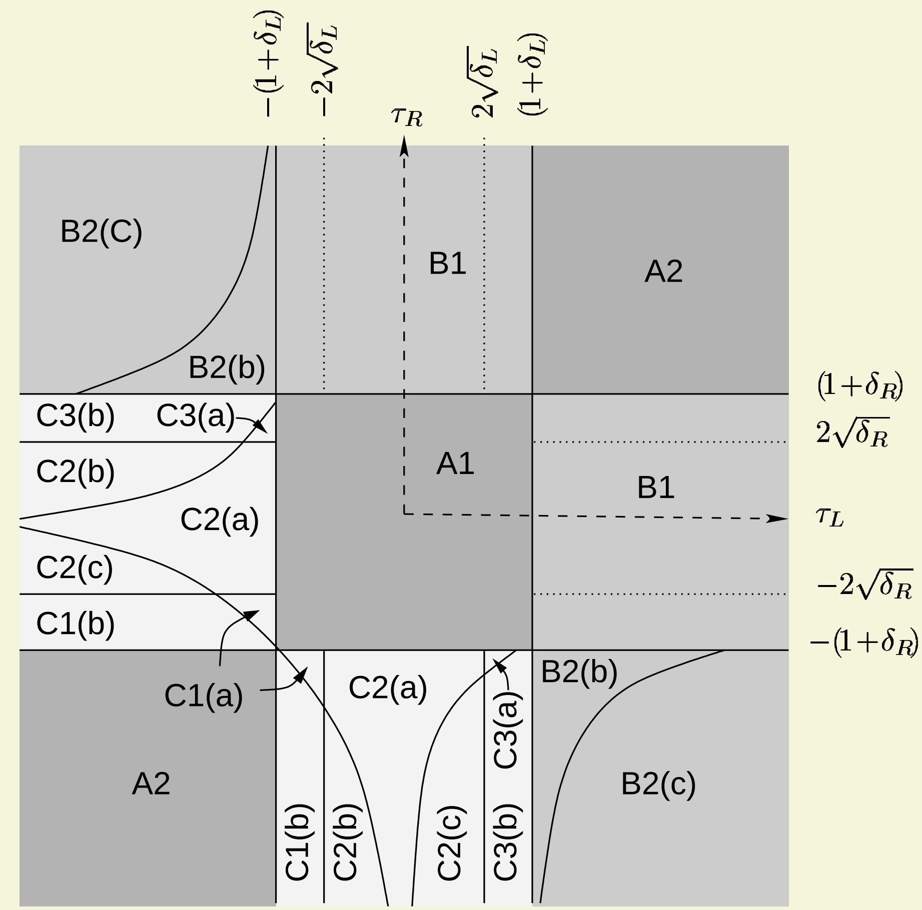
M. Dutta, H. E. Nusse, E. Ott, J. A. Yorke and G-H. Yuan, PRL, **83**, 1999.



Anindita Ganguli and Soumitro Banerjee, “Dangerous bifurcation at border collision — when does it occur?” *PRE*, Vol.71, No.5, 2005.

Scenario 3: A pair of fixed points are born. But ...





The conditions for the occurrence of such bifurcations are now available in terms of the trace and the determinant of the Jacobian matrices at the two sides of the borderline.

In practical systems, if such phenomena are observed,

- obtain the eigenvalues before and after a border collision,
- obtain the trace and the determinant, and
- match with the available theory.

➜ Prediction of bifurcation

➜ Control of bifurcation.

The theory has been used in understanding bifurcation phenomena in

- power electronic circuits
- impacting mechanical systems
- stick-slip oscillations
- internet packet transfer
- walking robots
- cardiac alternans
- neuronal dynamics

Thank You