Introduction to Double Categories, Part 2

Susan Niefield

Union College
Schenectady, NY

February 11, 2021

Last Time:

A double category $\mathbb{D}$ is a pseudo internal category in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \quad (\mathrm{id}^\bullet : \mathbb{D}_0 \to \mathbb{D}_1)$$

- horizontal (tight) morphisms, like f_0
- vertical (loose) morphisms, like v
- cells, like φ

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle v}\downarrow\!\bullet & \varphi & \bullet\!\downarrow{\scriptstyle w} \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array}$$

$\mathbb{D}$ is fibrant if every $f : X \longrightarrow Y$ has a companion f_* and conjoint f^*

$$X \underset{f^*}{\overset{f_*}{\rightleftarrows}} Y$$

Examples: $\mathbb{T}\mathrm{op}$, $\underset{\text{Groth. toposes}}{\mathcal{S}\text{-}\mathbb{T}\mathrm{opos}}$, $\mathbb{L}\mathrm{oc}$, $\mathbb{Q}\mathrm{uant}^{op}$, $\mathcal{V}\text{-}\mathbb{C}\mathrm{at}$, $\mathbb{R}\mathrm{ing}$, $\mathbb{Q}\mathrm{uant}$

Double Categories and Glueing

Paré: "Read Shulman" Saw cells in exponential constructions.

For spaces (1978), locales/toposes (1980), categories (2000), and posets (2001), proved:

Theorem $Y \subseteq B$ exponentiable $\iff$ locally closed

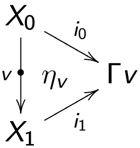
Proof of $(\Leftarrow)$ for locales/toposes used Artin-Wraith glueing.

Idea: Prove $(\Leftarrow)$ via "glueing" for all 5 using double categories.

Note: These $\mathbb{D}$ have a "Sierpinski" object 2 and $\mathbb{D}_1 \simeq D_0/2$.

Cotabulators

A **cotabulator** of $v\colon X_0 \to\!\!\!\!\!\bullet\; X_1$ is a universal cell $\eta_v\colon v \to \mathrm{id}^\bullet_{\Gamma v}$



i.e., $\forall \varphi\colon v \to \mathrm{id}^\bullet_Y$, $\exists!\ f\colon \Gamma v \to Y$ such that $\eta_v \cdot f = \varphi$.

Exercise: $\mathbb{D}$ has cotabulators $\iff$ $\mathbb{D}_0 \xrightarrow{\mathrm{id}^\bullet} \mathbb{D}_1$ has a left adjoint

Note: If $\mathbb{D}$ has a horizontal terminal object 1, i.e., $\mathrm{id}^\bullet_1$ is a terminal object in $\mathbb{D}_1$, then Γ induces a functor

$$\Gamma_2\colon \mathbb{D}_1 \to \mathbb{D}_0/2$$

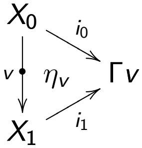
where $2 = \Gamma(\mathrm{id}^\bullet_1)$.

Cotabulator Examples

$\mathcal{V}$-$\mathbb{C}$at: Γv is the collage $X_0 \underset{v}{\sqcup} X_1$

$\Gamma v(x_0, x_1) = v(x_0, x_1)$

2 is the category $0 \to 1$



$\mathbb{T}$op: $\Gamma v = X_0 \underset{v}{\sqcup} X_1$ with $U = U_0 \sqcup U_1$ open when $\underbrace{U_1 \subseteq v(U_0)}_{\text{gives } \eta_v}$

2 is the Sierpinski space $\{0, 1\}$

$\mathbb{L}$oc: $\Gamma v = \{\, (x_0, x_1) \mid x_1 \leq v(x_0) \,\}$ Artin-Wraith glueing

$i_{0*}(x_0) = (x_0, v(x_0)), \quad i_{1*}(x_1) = (\top, x_1)$

2 is the Sierpinski locale $\mathcal{O}(2)$

$\mathbb{T}$opos: Artin-Wraith glueing

2 is the Sierpinski topos $\mathcal{S}^2$

Glueing Categories

A **glueing category** is a fibrant double $\mathbb{D}$ such that

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0^v} & \\ v \downarrow\!\bullet & \eta_v & \Gamma v \\ X_1 & \xrightarrow{i_1^v} & \end{array}$$

G1: $\mathbb{D}$ has cotabulators

G2: $\mathbb{D}$ has a horizontal initial object 0 and terminal object 1

G3: $\Gamma_{\mathbb{2}} \colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0/\mathbb{2}$ is an equivalence, where $\mathbb{2} = \Gamma(\mathrm{id}_1^\bullet)$, and the following are pullbacks in $\mathbb{D}_0$

$$\begin{array}{ccc} X_0 & \xrightarrow{i_0^v} & \Gamma v \\ \downarrow & & \downarrow \Gamma_{\mathbb{2}} v \\ 1 & \xrightarrow[i_0]{} & \mathbb{2} \end{array} \quad \text{and} \quad \begin{array}{ccc} X_1 & \xrightarrow{i_1^v} & \Gamma v \\ \downarrow & & \downarrow \Gamma_{\mathbb{2}} v \\ 1 & \xrightarrow[i_1]{} & \mathbb{2} \end{array}$$

Remark: Companions and conjoints are used for $\Gamma_{\mathbb{2}}^{-1}$.

Examples: $\mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}$, $\mathbb{P}\mathrm{os}$, and $\mathcal{V}$-$\mathbb{C}\mathrm{at}$.

Locally Closed Inclusions

Suppose $\mathbb{D}$ is a glueing category. Then $X_0 \rightarrowtail X$ is **open** (resp., $X_1 \rightarrowtail X$ is **closed**) if there is a morphism $p\colon X \longrightarrow \mathbb{2}$ such that the diagram on the left (resp., right) is a pullback in $\mathbb{D}_0$

$$
\begin{array}{ccc}
X_0 & \rightarrowtail & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
1 & \underset{i_0}{\rightarrowtail} & \mathbb{2}
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_1 & \rightarrowtail & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
1 & \underset{i_1}{\rightarrowtail} & \mathbb{2}
\end{array}
$$

A pullback of an open and a closed is called **locally closed**.

Examples: This agrees with the usual definitions for spaces, locales, and toposes, and the ones given for categories (2000), where the inclusion of a full subcategory is called locally closed if it is a UFL.

Exponentiable Morphisms

Recall: A morphism $X \xrightarrow{p} B$ is called **exponentiable** in a category $\mathcal{D}$, if the functor $\mathcal{D}/B \xrightarrow[-\times p]{-\times_B X} \mathcal{D}/B$ has a right adjoint.

Lemma If $\mathbb{D}$ is a glueing category and $B = \Gamma(b)$, then Γ_2 induces an equivalence of categories $\mathbb{D}_1/b \simeq D_0/B$.

Proof. $\mathbb{D}_1/b \simeq (\mathbb{D}_0/2)/p \cong \mathbb{D}_0/B$, where $p = \Gamma_2(b \to \mathrm{id}_1^\bullet)$.

□

Exercise: For $n = 0, 1$, the functor $\mathbb{D}_1/b \xrightarrow[-\times \beta_n]{-\times_b b_n} \mathbb{D}_1/b$ corresponds to $\mathbb{D}_0/B \xrightarrow[-\times i_n]{-\times_B B_n} \mathbb{D}_0/B$ via Γ_b, where $i_n \colon B_n \to B$ and

$$
\begin{array}{ccc}
B_0 & \xrightarrow{\mathrm{id}_{B_0}} & B_0 \\
{\scriptstyle b_0}\downarrow & \beta_0 & \downarrow{\scriptstyle b} \\
0 & \longrightarrow & B_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
0 & \longrightarrow & B_0 \\
{\scriptstyle b_1}\downarrow & \beta_1 & \downarrow{\scriptstyle b} \\
B_1 & \xrightarrow[\mathrm{id}_{B_1}]{} & B_1
\end{array}
$$

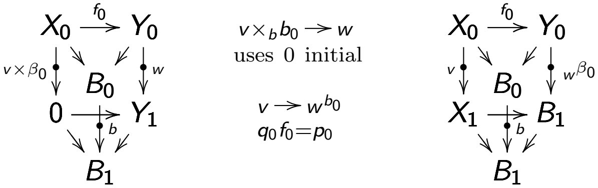
Exponentiability of Locally Closed Morphisms

Theorem Suppose $\mathbb{D}$ is a glueing category. Then open, closed, and hence, locally closed morphisms are exponentiable in $\mathbb{D}_0$.

Proof. To show $b_0 \xrightarrow{\beta_0} b$ is exponentiable, where $B_0 \xrightarrow{b_0} 0$, take

$$
\begin{array}{ccc}
X_0 & \xrightarrow{p_0} & B_0 \\
{\scriptstyle v}\downarrow & \pi & \downarrow{\scriptstyle b} \\
X_1 & \xrightarrow[p_1]{} & B_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y_0 & \xrightarrow{q_0} & B_0 \\
{\scriptstyle w}\downarrow & \theta & \downarrow{\scriptstyle b} \\
Y_1 & \xrightarrow[q_1]{} & B_1
\end{array}
$$

Then $v \times \beta_0 \colon X_0 \to 0$, and define $w^{\beta_0} \colon Y_0 \xrightarrow{q_{0*}} B_0 \xrightarrow{b} B_1$



The result follows, since both are determined by $q_0 f_0 = p_0$.

Similarly, for $0 \xrightarrow{\beta_1} B_1$, define $w^{\beta_1} \colon B_0 \xrightarrow{b} B_1 \xrightarrow{q_1^*} Y_1$. □

Examples

Corollary Locally closed inclusions are exponentiable in $\mathbb{T}\mathrm{op}_0$, $\mathbb{L}\mathrm{oc}_0$, $\mathbb{T}\mathrm{opos}_0$, $\mathbb{P}\mathrm{os}_0$, and $\mathcal{V}$-$\mathbb{C}\mathrm{at}_0$, for symmetric monoidal $\mathcal{V}$.

Remark: The converse holds in each case, but not via double categories.

References

- M. Bunge and S. Niefield, Exponentiability and single universes, J. Pure Appl. Algebra, 148, 2000, 217?-250.

- M. Grandis and R. Paré, Limits in double categories, Cahiers 40 (1999),162–220.

- S. B. Niefield, The glueing construction and double categories, J. Pure Appl. Algebra 216 (2012), 1827–1836.

- R. Paré, Yoneda theory for double categories, TAC 25 (2011), 436–489.

- M. Shulman, Framed bicategories and monoidal fibrations, TAC 20 (2008), 650–738.