their conciseness certain questions of emphasis. But as for the subject matter, there is no feeling of compression; on the contrary the author frequently pauses to suggest problems, explain difficulties and guiding principles, and emphasize values. For some theorems there are developed several distinct proofs, each bringing to light new aspects of the theory and new points of view. The essentials of the point set theory of separable spaces are developed at an early stage so that one may read the entire volume without outside reference. The material is well arranged and the printing is excellent, with exceptionally few errors. The book constitutes, in short, a notable presentation of an important chapter in modern mathematics.

P. A. Smith

TONELLI ON TRIGONOMETRIC SERIES

Serie Trigonometriche. By Leonida Tonelli. Bologna, Nicola Zanichelli, 1928. viii+523 pp.

The extent of the existing literature on the theory of trigonometric series is tremendous and keeps increasing very rapidly. During the quarter of a century after 1900 the theory has made remarkable progress and it would not be an exaggeration to say that now it is of equal and fundamental importance for all branches of the modern mathematics including the theory of numbers on the one hand, and mathematical physics on the other. In spite of, or perhaps because of this, there is practically no place in the literature where an adequate account of the theory is given, except for the second volume of Hobson's *Theory of Functions* and two excellent but short reports by M. Plancherel (L'Enseignement Mathématique, vol. 24 (1925), and by E. Hilb and M. Riesz (Encyklopädie der mathematischen Wissenschaften, vol. II, 3_2, 1924).

Under such circumstances the publication, by a mathematician of Tonelli's rank, of a large volume devoted exclusively to the theory of trigonometric series must be considered as a significant event, even if it does not represent a step toward the solution of the difficult problem of creating an all-inclusive treatise on trigonometric series.

The work under review originated as a course of lectures on trigonometric series delivered at the University of Bologna in 1924–1925 with a view "to expounding in a systematic manner the classical results, together with more recent investigations on these series." The author found it more convenient to abandon the usual order of treatment of trigonometric series. The book begins with a study of general trigonometric series, in order "to reveal at once the properties which are common to all such series," and subsequently passes to the discussion of the special properties of Fourier series. An advantage of such a treatment lies, according to the author's opinion, in the fact that "the theory of the general trigonometric series can be presented in a form essentially elementary in character, while the theory of Fourier series, for its complete development, requires speculations of a more advanced nature." No space is given to the "beautiful investigations of Denjoy concerning the computation of the coefficients of an everywhere convergent trigonometric series, nor to the important theory of quasi-periodic functions" of H. Bohr. The author expects to devote a separate volume to the applications of the theory to geometry and mathematical physics.

The topics which are most completely treated in the monograph under review are: the absolute convergence of the general trigonometric series, Riemann's method of summation, the representation of functions by trigonometric polynomials and, above all, double Fourier series.

A short historical sketch is followed by Chapter I (General trigonometric series; §1, necessary conditions of convergence; §2, sufficient conditions; §3, necessary and sufficient conditions). §1 contains a detailed analysis of the Cantor theorem and of the problem of absolute convergence of trigonometric series. The reader will find here some important theorems (Steinhaus, Denjoy-Lusin) which are not so well known as they should be. §2 treats primarily of the uniform convergence and of the convergence almost everywhere on $(0, 2\pi)$. The principal tool in treating the former of these two problems is of course the classical Abel transformation which figures here under the name of Brunacci-Abel transformation, with a reference to the *Corso di Matematica Sublime* by V. Brunacci (vol. 1, Firenze, 1804). As to the latter problem, we find here an important theorem due to Kolmogoroff-Seliverstoff and to Plessner: If $\sum(a_n^2+b_n^2)\log n$ converges, then the corresponding trigonometric series converges almost everywhere. It should be observed that the method of proof used is hardly of an "elementary" nature; on the one hand there is used an unproved lemma (p. 65) in the general theory of infinite series, which, so far as the reviewer is aware, is not stated in the standard treatises, like those of Bromwich or Knopp; on the other hand a reference is made to the Riesz-Fischer theorem, which is proved considerably later. The last §3 of Chapter I is devoted to the fundamental theorem of Riemann.

Chapter II (Representation of functions by means of trigonometric series) contains three parts: §1, conditions for the representation; §2, uniqueness of the expansion in a trigonometric series; §3, Fourier series. §§1 and 2 contain an interesting treatment of Riemann's method of summation and Cantor's uniqueness theorem, while in §3 the definition of Fourier series, together with some elementary examples, is given and the theorem of Du Bois-Reymond-Lebesgue is proved.

Chapter III (Approximate representation by means of trigonometric polynomials) contains four parts: §1, trigonometric interpolation; §2, method of least squares; §3, trigonometric polynomials of Fejér; §4, method of approximation of Tchebychev.

Chapter IV (Fourier successions) is subdivided into three parts: §1, Fourier sequences and series; §2, necessary conditions for Fourier successions; §3, sufficient conditions for Fourier successions. In §1 a short discussion of the uniqueness of determination of a function by means of the sequence of its Fourier coefficients is given. §2 contains the Riemann-Lebesgue theorem, an estimate of the order of magnitude of the Fourier coefficients of a function under various conditions, the Parseval identity, the Young-Hausdorff theorem, the approximation theorem of Bernstein, the theorem of Szász and, finally, a discussion of convergence of the series $\sum a_n/n, \sum b_n/n, \sum a_n, \sum b_n$. The proof of the Young-Hausdorff theorem proceeds along the lines of the proof of F. Riesz with one deviation, however, which makes the proof more difficult; in establishing the existence of a minimum of a certain integral the reader is referred, not to the classical paper by F. Riesz in Mathematische Annalen, vol. 69, but to a general theorem of the calculus of variations as given in the second volume of the author's *Fondamenti di Calcolo delle Variazioni*. §3 is devoted to the proof of the Riesz-Fischer theorem and its corollaries.

Chapter V (Convergence of Fourier series) consists of six parts: §1, absolute convergence; §2, simple convergence; §3, uniform convergence; §4, partial convergence; §5, conjugate series; §6, degree of approximation. In §1 the theorem of Bernstein and some of the author's results are proved. §2 contains a discussion of the Dirichlet integral and of the classical criteria of convergence. One of these criteria (that of Young) is not mentioned and a new special criterion (due to the author) is added, namely, a "criterion of convergence at singular isolated points." The usefulness of this important paragraph would have been considerably increased if the author had given space to a comparative analysis of various criteria and showed examples of their applications, his own criterion included. In §4 we find an interesting theorem of Kolmogoroff: If $S_n(x)$ is the nth partial summation of the Fourier series of a function $f(x)$, then the sequence $\{S_{n_k}(x)\}$ converges to $f(x)$ almost everywhere, provided there is a constant $\lambda > 1$, such that $n_{k+1}/n_k > \lambda$. The treatment of the conjugate series as given in §5 is rather short and hardly gives a right idea of the present situation of this question whose importance becomes more and more obvious.

Chapter VI (Operations on Fourier series) consists of three parts: §1, addition and multiplication; §2, integration; §3, differentiation. A remarkable theorem of Lusin, given in §2, deserves to be mentioned separately: If a trigonometric (not necessarily a Fourier) series converges at all points of (a, b) to a sum $S(x)$ which is integrable on (a, b), then the series can be integrated term by term over any subinterval which is entirely interior to (a, b).

Chapter VII (Singularities of the Fourier series: §1, singularities of the Fourier series of continuous functions; §2, singularities of the Fourier series of discontinuous functions) gives a discussion of singularities of Du Bois-Reymond and Lebesgue, and an account of the Gibbs phenomenon.

Chapter VIII (Classical integrals: §1, Poisson integral; §2, Fourier integral). §1 contains an exposition of classical results concerning the Poisson integral, including some of the results of Fatou. The treatment of the Fourier integral in §2 proceeds along the lines of Jordan-Pringsheim and of the Acta Mathematica paper by Hahn. The end of this paragraph is devoted to a discussion of the Sommerfeld integral.

Chapter IX (Double Fourier series: §1, preliminaries; §2, convergence of the double Fourier series; §3, summation by rows and columns; §4, Fejér trigonometric polynomials in two variables; §5, supplements). This concluding chapter is perhaps one of the most interesting in the volume. It is based to great extent upon the author's own investigations and contains many results which reach farther than those known before. Considerable progress and simplification were made possible by introducing a new notion of a function of bounded variation in two variables, which contains most of those given before (Hardy, Vitali-Lebesgue-Fréchet, Arzelà): A function $f(x, y)[a \leqq x \leqq b,\ a \leqq y \leqq b]$ is of bounded variation in (x, y) if (i) $f(x, y)$ is of bounded variation in y for almost all values of x and of bounded variation in x for almost all values of y; (ii) the corresponding total variations are integrable as functions of x or of y respectively.

The selection of material in a course of lectures which is not to be considered as a treatise is to great extent a matter of taste. It is to be regretted, however, that such important topics as the theory of Fourier transforms, the general Cesàro summability of Fourier series (only C_1 summability is given consideration in the book), the relationship between the Fourier and power series, are not discussed in the volume under review. It is hardly possible to obtain a correct idea of the recent development of the theory of trigonometric series and integrals without touching these topics. But even if we concentrate our attention on the topics treated in the book, we might find several facts and names which perhaps deserve somewhat more than to be merely mentioned in a few footnotes. Such are, in the reviewer's opinion, important investigations of Hardy-Littlewood on the theory of Fourier constants (=Fourier successions), on the so-called strong summability of the Fourier series by the same, the theory of convergence factors by various writers, some investigations by Zygmund, Titchmarsh, Wiener, etc. There are also some names which should be referred to, at least in the footnotes, for instance, the name of Carleman, in connection with the discussion of the exponent of convergence of a Fourier succession of a continuous function, of the strong summability of Fourier series, etc.

Despite these criticisms we are convinced that *Serie Trigonometriche* by Tonelli will be of great value to anyone seriously engaged in studying the immense field of trigonometric series, and we are looking forward to the pleasure of reading the promised volume on the applications of trigonometric series.

J. D. TAMARKIN