## CHAPTER THREE

# CONCERNING THE RECTILINEAR MOTION OF A FREE POINT ACTED ON BY ABSOLUTE FORCES

p. 76]

## PROPOSITION 24.

### THEOREM.

**189.** *When the directions of the motion and of the force are along the same straight line, the motion will be rectilinear.*

### DEMONSTRATION.

Every body by its inertial force tries to continue its motion in the direction that it always performs, unless it is impeded (65). Truly there are two effects of a force on the motion of a body that we can show : the one by which the direction is unchanged, and the other by which the speed is unchanged. But the direction remains unchanged if the direction of the force lies in the direction that the point moves (128). Therefore in this case the point goes on traveling in a straight line. Q. E. D.

### Corollary 1.

**190.** We will only consider the case in this chapter in which the motion of the point and the direction of the force acting are placed on the same straight line.

### Corollary 2.

**191.** Moreover we see that this consistency can come about in two ways, as clearly either both [the inertial force and the applied force] are acting in the same direction, [p. 77] or in opposite directions. In the one case the speed of the point is increased, and in the other it is decreased 128).

### Scholium.

**192.** There are two things that have to be consider in this rectilinear motion, the first of which is the force upon which some point mass is acted, and the other is indeed the speed that the point has at any place. To these we may add also the time in which some interval of distance is traversed. Indeed these three variables are thus comparable, since with one given the remaining two can always be determined. In the first place therefore we will consider some force as given : then truly on that account we will find either the corresponding speed or time from the given force.

## PROPOSITION 25.

### PROBLEM.

**193.** *The point A is resting on the line* (Fig. 22) *AP, and is to be pulled forwards by a uniform force, or which acts with the same strength everywhere, and the speed of the point is to be determined at any position P.*

### SOLUTION.

The mass or the force of inertia of the point is set out by the letter *A*, and the [external] force by the letter *g*, which is constant since it is the same amount everywhere. Let the distance $AP = x$, and the required speed at *P* is put equal to *c*. An element of distance *Pp* is taken, which is equal to *dx*; and the increment of the speed *dc* is acquired by the point on completing the element *Pp,* acted on by the force *g* [p. 78]. With these put in place, it follows that $cdc = \frac{ngdx}{A}$ (157), since the force constantly pulls downwards, and on this account we put the motion to be accelerating.


Fig. 22.

[This integrated equation corresponds, from our point of view, to the usual kinematic equation $v^2 = u^2 + 2as$ where the initial speed *u* is zero, and the equation can be differentiated to give *vdv = ads*; which can of course be related to work done and the corresponding change in kinetic energy; in addition, the acceleration is given by Newton's second law of motion, *F/m* . Euler has however derived this equation for a general situation from first principles in Prop. 20, of which (157) is Cor. 3, in Ch. 2, in pre conservation of energy days, where *cdc* has been set up so that an integration can be performed, and this quantity is proportional to *g/A* (the force per unit mass), and the element of distance in the direction of the force *dx*, for which some constant *n* is inserted to relate the two sides of the equation, which otherwise are proportional quantities, according to Galileo's ideas. Thus, all the differential equations presented, that can be integrated, rely on proportions.]

From this equation, if it is integrated, there arises $cc = \frac{2ngx}{A} + \text{Const.}$, which constant must be determined from the condition that the speed of the point vanishes at A. Therefore from this it follows that $c = 0$, and $x = 0$, gives the constant as zero. On account of which we have $cc = \frac{2ngx}{A}$ or $c = \sqrt{\frac{2ngx}{A}}$ . Q. E. I.

## Corollary I.

**194.** Therefore the point A always falls along the straight line AP, and the speed at any point is as the square root of the distance now traversed.

## Corollary 2.

**195.** From these also, the descents of many points acted on by uniform or constant forces can be compared; for the speeds are in the ratio directly of the square root of the forces and the distances traversed, and inversely as the square root of the masses.

## Scholium 1.

**196.** This case agrees above all with the fall of bodies on the earth : for the force of gravity, which in turn takes the place of the force, is uniform for not too great distances from the surface of the earth. In so much as the weight of any body from the highest mountain to the deepest valley is found to be the same ; moreover from the weight the force of gravity is found. Therefore in the free fall of weights the speeds are as the square roots of the distances traversed. This is the proposal of Galileo himself, that he discovered first from experiment and then by reasoning [p. 79]. Moreover the descent should be made in a space from which the air has been evacuated, since the air resists the motion and this effect is avoided.

## Scholium 2.

**197.** In an empty space, which can be effected with the help of pneumatic pumps, it has been shown by many experiments that any bodies fall equally. From which it follows, if there should be no air, all bodies that fall from equal heights gain equal speeds. On account of this, if $g$ designates the force of gravity, by which any body A is moved, then $\frac{g}{A}$ is always the same constant. Hence the force of gravity is proportional to the quantity of matter in the body on which it acts. But that force is none other than the weight of the body ; whereby the weight in the *Princip. Phil.* confirms this too, and that besides is in agreement with pendulum experiments.

## Corollary 3.

**198.** Therefore any body on the surface of the earth fallen from a given height will acquire a step of the speed. Therefore with the height known, from which the body descended, [the speed] acquired from the descent will be known likewise.

## Scholium 3.

**199.** Therefore in order that we can measure these speeds, these heights are to put in place from which a weight falling to the surface of the earth acquires an equal speed [p. 80]. Indeed this height cannot be substituted in place of the speed itself, since the speeds are in the square root ratio to the height. Yet truly it will be possible for the height to denote the square of the speed.

## DEFINITION 15.

**200**. *Hereafter we will call the height corresponding to the speed that height, from which a weight falling to the surface of the earth, acquires that same speed.*

## Corollary 1.

**201.** This height must therefore be as the square of the speed, to which it refers. With the speed $c$ arising and with the due height $v$, $v$ shall be as $c^2$. [There is thus to be a proportionality between $c^2$ and $v$. People had a lot of trouble with non-standardised units in these days, and preferred to use proportionalities instead, whenever they could. Here of course an independent variable $v$ is introduced for the uniform height that can be integrated, without worrying about the more troublesome speed $c$. ]

## Scholium 1.

**202.** This far we have expressed the speed on a straight line, which can be traversed by the speed in a given time. But in the following it will be more convenient to introduce the corresponding height in place of this [Euler calls this the height *owing* or *due*, as one would do in a financial transaction; there is hence the understanding that something is conserved or changed from one form into another, in this analysis of a falling body; Euler uses the distance fallen as a means of representing the speed.] On account of this we put $v = cc$ and $c = \sqrt{v}$. We will therefore have in the preceding problem this equation : $v = \frac{2ngx}{A}$.

## Corollary 2.

**203.** Therefore in what follows, in place of the speed $c$ it will be permitted to put $\sqrt{v}$ or the square root of height that corresponds to the speed. [p. 81; Note the use of the symbols : $v$ here refers to verticalis for vertical (height), $c$ refers to celeritas for speed, s refers to spatium or interval of distance; some of these symbols have endured to the present time, as for example $c$ for the speed of light.]

## Corollary 3.

**204.** If the force $g$ denotes that of gravity itself, then $x$ will be the height corresponding to the speed $c$, and thus $v = x$. For indeed $v = \frac{2ngx}{A}$, from which it therefore follows that $n = \frac{A}{2g}$. From this we have gained a convenience, as we have determined the value of the letter $n$, which must maintain the same value in all cases (155).

## Scholium 2.

**205.** Since $g$ signifies the force of gravity [*i. e.* the weight of the body; do not confuse $g$ with our symbol for the acceleration of gravity, which it does *not* represent], then $\frac{g}{A}$ is a constant quantity (197). Therefore we can put this as 1, as that is allowed, since the force to the [mass] does not [yet] have a defined ratio. And hence it easily shows the ratio of $\frac{g}{A}$ in all cases, or the value of the applied force to the [mass of the] body. Certainly the ratio $\frac{g}{A}$ to 1 or g : A as the force *g*, acting on the body A, is to the weight [this should be mass], that the same body may have in our part of the world. Therefore the letter A will no longer denote the quantity of matter, but the weight *A* of the body itself, [since A.1 is the weight] if it should be placed on the surface of the earth. In this way we will compare all forces with weights, since that will add a great deal of light to the measurement of forces.

## Corollary 4.

**206.** Since $n = \frac{A}{2g}$, *g* denotes the force of gravity [i. e. the weight of the body and not the acceleration] and if $\frac{A}{g} = 1$, then $n = \frac{1}{2}$. That value will always be retained, if the speeds are to be expressed in terms of the appropriate square roots of the heights. And thus in our situation, this becomes $dv = \frac{gdx}{A}$ and $v = \frac{gx}{A}$. [p. 82. Thus, Euler chooses as his working dimensions not distance and time but acceleration and time. The time is measured in seconds, and with the acceleration of gravity $A/g = 1$, then $n = \frac{1}{2}$.]

## Corollary 5.

**207.** On this account in the general law $cdc = \frac{npds}{A}$ (157), if the height *v* of the corresponding speed is *c*, then $cdc = \frac{dv}{2}$, and thus on account of $n = \frac{1}{2}$ this law is obtained $dv = \frac{pds}{A}$ , where *p* is to *A* as the force *p* is to the weight of the body *A*. [For Euler has defined $c^2 = v$ and hence $dv = 2cdc = \frac{2npds}{A} = \frac{pds}{A}$ when $n = \frac{1}{2}$.]

## Corollary 6.

**208.** In a like manner, indeed the equations set out in (161) and (163) : $Acdc = npdy$ and $nprdx = Ac^2ds$, by substituting *v* in place of $c^2$ and $\frac{1}{2}$ in place of *n*, are transformed into $Adv = pdy$ and $prdx = 2Avds$, where *p* to *A* has the ratio in the manner given.

## Corollary 7.

**209.** And in (165) it is found that $r = \frac{2Av}{p}$ or $pr = 2Av$. Likewise in (165) there is obtained $Adv = pds$, and in the case of (167) there is obtained $Adv = -pds$. And in this manner we have reduced the previous variable quantities *n* and *c* to fixed values.

# PROPOSITION 26.

## THEOREM.

**210.** *The heights, by which equal small bodies fall to acquire equal speeds, vary inversely as the forces, under the hypothesis of different uniform* [gravitational] *forces.*

## DEMONSTRATION.

Let the mass or weight of some small body on the surface of the earth be *A*, the force [on some other celestial body] some constant value *g* and the corresponding height [p. 83] for this speed to be acquired be *v*. Truly the height shall be *x*, by which the small body *A* falling under the action of the force *g* shall acquire a speed [$c^2 =$ ] $v = \frac{2ngx}{A}$ (202). But $n = \frac{1}{2}$ (206), hence $v = \frac{gx}{A}$ or $Av = gx$. [On the Earth, $g/A = 1$ and $v = x$.] Whereby, since the speeds have been produced by different forces and the bodies have been made equal, then the quantity $Av$ is constant, and thus also $gx$. On account of this, $x$ will vary inversely as g, i. e. the height, by which the body *A* acquires the speed $\sqrt{v}$ by the action of the force *g*, varies inversely as the force *g*. Q. E. D.

[No doubt by now the reader has thrown up his or her hands in horror and said : *what about the conservation of mechanical energy* ? The factor $n = ½$ has arisen from the requirement that the acceleration of gravity is one. The problem for us lies in the lack of understanding that there was at the time about kinetic and potential energy and the conservation of the sum of these for a falling body. Euler has been very clever and set up his differential equations so that they can be scaled, and when it comes down to doing a numerical example, as with the flight of the cannonball in Ch. 4, he gets the correct answers. Thus, the answers right themselves when known experimental values are put in place for the time of fall of a body. It has been convenient to put the acceleration of gravity arbitrarily as 1, from which by (209) $n = ½$ (for Euler does his best to get rid of constants that always appear, a tradition that has been followed by theoreticians ever since!); but if the time is measured in seconds and the distance in scruples or thousandth parts of Rhenish feet, then the acceleration of gravity is not 1, but something around 32 ft/s$^2$ or 32000 scruples/s$^2$ . If fact, from the equation $V^2 = 2gH$ , for the speed *V* of a body dropped from rest from a height H is given by $V = \sqrt{2g}H \approx \sqrt{62500} \times H = 250H$ , the scaling factor used by Euler later. Hence when experimental results are imposed, the correct value for the acceleration of gravity results, and all is well.]

## Corollary 1.

**211.** Newton has shown that the impressed forces on the same body put in place on the surface, and acted upon towards the centre of, the Sun, Jupiter, Saturn, or the Earth, are as 10000, 835, 525, and 400. Therefore the heights from which the body acquires equal speeds in falling on the surface of the Sun, Jupiter, Saturn, and the Earth, are between themselves as $\frac{1}{10000}, \frac{1}{835}, \frac{1}{525}$ and $\frac{1}{410}$.

## Corollary 2.

**212.** Moreover Newton understood likewise that all bodies fall at equal rates on these surfaces, just as on the surface of the earth. Therefore there is no need to add the condition that the bodies are equal, for from the heights which are in the ratio to each other $\frac{1}{10000}, \frac{1}{835}, \frac{1}{525}$ and $\frac{1}{410}$, on the surfaces of the Sun, Jupiter, Saturn, and the Earth, any bodies dropped will acquire the same increase in their speed. [p. 84]

## Scholium 1.

**213.** It is understood from these that there is a two-fold effect on the body by any force : on the one hand, by which a certain force or effort is impressed on the body, and on the other, by how the body may be moved by that force. The one that is to be considered mainly in statics is the weight and how it should be measured, that the body has in an equal attempt to fall downwards, and it may be called the absolute strength of the force. In turn, truly the effect should be measured by the acceleration or the change in the speed, that is impressed on the body in a given time : this is proportional to that force divided by the mass of the body (154). This effect is called by Newton the accelerating force, and therefore the strength of the accelerating force is proportional to absolute force applied to the mass of the body, or the weight applied. On account of which, since $dv = \frac{pds}{A}$ (207) and $\frac{p}{A}$ denotes the strength of the acceleration, then $dv$ is equal to the product of the acceleration and the element of distance travelled. Thus the absolute force of gravity is proportional to the mass of the bodies on which it acts ; for the effect of these downwards is the cause or the weight that we have shown to be in proportion to the mass. Moreover the accelerating force of gravity is equal on all bodies, since they all fall equally and they gain equal speeds in equal intervals of time.

## Corollary 3.

**214.** Hence the sizes of the accelerations are to each other as the absolute forces, if the bodies are of equal mass. Whereby since the strength of the acceleration due to gravity is taken as 1, as we have put in place before (205), [p. 85] then the strength of the accelerating due to gravity on the surface of the sun is equal to 24.290; the strength of the acceleration on the surface of Jupiter is 2.036; the strength of the acceleration on the surface of Saturn is 1.280. And the strength of gravity has been taken on the moon by Newton as $\frac{1}{3}$.

## Corollary 4.

**215.** Whereby if from Proposition 25 the fall of the body to the surface of the Earth is to be accounted for, then $\frac{g}{A} = 1$, as we have done in (205). But truly the fall of bodies on the surface of the sun requires $\frac{g}{A} = 24.290$; or on the surface of Jupiter, $\frac{g}{A} = 2.036$ ; on the surface of Saturn $\frac{g}{A} = 1.280$; and for the fall of bodies on the surface of the moon, it will be $\frac{g}{A} = \frac{1}{3}$.

## Scholium 2.

**216.** Here we assume with Newton that all celestial bodies are similar to our Earth and bodies placed on the surface of these have a force pulling then to their centre, which is always similar to the force of terrestrial on bodies. Therefore, from Newton's exposition, it is apparent that a body, the weight of which here is 1 pound [lb.], will weigh on the surface of the Sun 24.290 lb ;on the surface of Jupiter, it will be 2.036 lb; on the surface of Saturn 1.280 lb ; and on the surface of the moon one third of a pound. [p. 86]

## Scholium 3.

**217.** Moreover in order that the nature of the gravitational forces can be more easily compared for celestial bodies, the individual equal elements of the bodies are understood to be equally affected by gravity. From which it follows, since now it agrees with experiment, that the forces of gravity which act on any bodies, are themselves proportional to the masses or quantities of matter. Truly it has been shown previously, that if the forces are in proportion to the masses of the bodies, then their effect on moving bodies is the equal (136). On which account it is shown from these that all bodies on the surface of the earth should descent equally, and likewise for all celestial bodies.

## PROPOSITION 27.

### PROBLEM.

**218.** *For the point A* (Fig. 22) *is to be moved forwards through the distance AP by a uniform force, and it is required to find the time in which the distance AP is completed.*

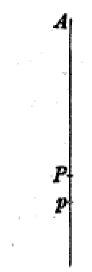
Fig. 22.

### SOLUTION.

As before, let the force acting be $g$, the distance $AP = x$ and the height corresponding to the speed at P should be v; on this account $n = \frac{1}{2}, v = \frac{gx}{A}$. Therefore the speed itself at $P$ is equal to $\sqrt{v} = \sqrt{\frac{gx}{A}}$. The time therefore be found, in which the element $Pp = dx$ is traversed, varies as $\frac{dx\sqrt{A}}{\sqrt{gx}}$. Let the time in which the distance $AP$ is completed be $t$ and put $dt = \frac{mdx\sqrt{A}}{\sqrt{gx}}$, it is necessary to perform a single experiment to determine the value of the letter [p. 87] $m$, so that the time in the given measurement can be found, reckoned in seconds. Indeed from that equation the time t will be produced by integration and we have $t = 2m\sqrt{\frac{Ax}{g}}$, to which there is no need to add on a constant quantity, since for the position $x = 0$ the time $t$ vanishes, as it should. Therefore on determining $m$ by experiment we have the time to fall given by $t = 2m\sqrt{\frac{Ax}{g}}$ sec. Moreover, so that the measurement of the time by this means shall be absolute, it is necessary that $x$ likewise is shown to be measured in agreement with this constant [i. e. in some standard unit of length]: therefore we will always determine the interval $x$ in scruples, i. e. thousandth parts of Rhenish feet ; for indeed the fraction $\frac{A}{g}$ is expressed in absolute numbers, in order that it will not be necessary to have a certain measurement for that. Therefore with the letter *m* defined, that we will make soon, the complete solution to the problem will be had. Q. E. I.

## Corollary 1.

**219.** If $g$ designates the force of gravity, then $\frac{A}{g} = 1$ (205) ; on account of this the time in which the body will fall to the earth from the height of $x$ scruples of Rhenish feet, will be $2m\sqrt{x}$ seconds.

## Corollary 2.

**220.** Moreover from experiments it has been ascertained that in a time of one second a body falls through a height of 15625 scruples of Rhenish feet [4.904 metres, which is in good agreement with the modern accepted value, although we do not know where the measurements were made for the full significant figures]. When on account of this, if the height is put as $x = 15625$, a time of one second will be produced : t = 1. But since $t = 2m\sqrt{x}$, then $1 = 2m\sqrt{15625}$, i. e. = 250*m*. Hence the letter *m* is found to be $m = \frac{1}{250}$.
[p. 88; when we compare this equation with the present value of the acceleration of gravity of around 9.8 m/s$^2$ in which case $x = \frac{1}{2}gt^2$, then in Euler's units of scruples, the acceleration of gravity *g* is given by 15625 × 2 = 31,250 scruples/s$^2$, corresponding to 9.81 m/s$^2$; we have discussed this in a slightly different way previously at the end of (210).]

## Corollary 3.

**221.** Since indeed the letter *m* retains the same value in all cases, in the case of the problem, the time will be $t = \frac{1}{125}\sqrt{\frac{Ax}{g}}$ sec. Therefore with the distance *x* expressed in scruples of Rhenish feet, the time as the number of seconds is given by $\frac{1}{125}\sqrt{\frac{Ax}{g}}$ for this space to be traversed [by falling from rest].

## Corollary 4.

**222.** And in all straight forward cases this value of *m* found can be applied. Indeed let the element of distance to be described be *ds*, with the height *v* corresponding to the speed in which this is traversed, the element of time is $dt = \frac{mds}{\sqrt{v}}$ and $t = m\int\frac{ds}{\sqrt{v}}$. From this equation, if *v* and *s* are expressed in scruples of Rhenish feet and we put $m = \frac{1}{250}$, the time will be produced in seconds : $t = \frac{1}{250}\int\frac{ds}{\sqrt{v}}$ sec.

## Scholium 1.

**223.** Therefore from this, since we specify the speeds by the square roots of the corresponding heights, which we take again for convenience in the following, as we always find the measure of the absolute time. For truly the experiment we have used from which the height is found that a weight falls in a time of one second, as Huygens found by experiments with pendulums to be 15 Paris feet, 1 digit, $2\frac{1}{18}$ lines, i. e. in decimal fractions 15.0796 Parisian feet. Moreover the ratio of Rhenish feet to Parisian feet we use is 1000 to 1035, from which the height to fall in one second comes to be 15.625 Rhenish feet or 15,625 [p. 89] scruples of the same feet; and this is the measure we prefer to have rather than the Parisian one, since this number is a square and in this way we avoid frequent root extractions. Besides, the number 250 that is found is easily remembered, by which $\int\frac{ds}{\sqrt{v}}$ (with $s$ and $v$ expressed in scruples of Rhenish feet) must be divided in order to find the time in seconds.

## Corollary 5.

**224.** Since $\frac{g}{A}$ denotes the strength of the acceleration of the force (213), the times will be, in which any distances are traversed under uniform forces, in the square root ratio composed from the distances and inversely as the strengths of the accelerations.
[i. e. $s \propto at^2$]

## Corollary 6.

**225.** With the speed $c$ put in place, that the point $A$ acquires with a force $g$ acting over the height $x$; $c$ varies as $\sqrt{\frac{gx}{A}}$ (193). Hence $ct$ will be as $x$, since $t$ is as $\sqrt{\frac{Ax}{g}}$ . Consequently the distances travelled through are in the ratio composed from the times in which they are described, and the speed that they gain in the descent, for whatever the forces acting in a uniform manner.

## Corollary 7.

**226.** And the distances that are traversed in equal intervals of time are as the strengths of the accelerations of the forces acting. [p. 90]

## Corollary 8.

**227.** Therefore the distances through which bodies fall in equal intervals of time on the surfaces of the Sun, Jupiter, Saturn, the Moon, and the Earth, are amongst themselves as 24390, 2036, 1280, 333, 1000. (214).

## Corollary 9.

**228.** For the hypothesis of the same acceleration under the same force, the times in which any distances are traversed , are as the speeds acquired, and so for the times as with the speeds, are in the ratio of the square roots of the distances described.

## Scholium 2.

**229.** Here we have always put the descending bodies to start the descent from rest or their initial speed to be zero. In the following indeed we will investigate these motions, which arise when the bodies are themselves in motion at the start and now they have some speed. Moreover with these both the times and the distances ought to be understood, which in the beginning have their own point where the speed vanishes, and all the equations found are thus comparable, as with the vanishing of $c$ or $v$ likewise $x$ and $t$ vanish.

## PROPOSITION 28.

### THEOREM.

**230.** *For a body falling through the distance AP* (Fig. 22)*, as we have put in place as hitherto, the* [final] *speed at P is of such a size, that if it progressed uniformly at this speed for the same time in which the body had fallen through AP, then it would be able to complete a distance twice as great as AP.*

[p. 91]

### DEMONSTRATION.

With everything kept, that we put in place in the preceding proposition, with the body $A$, the force $g$, with the distance described $x$, with the speed acquired at $P$ $\sqrt{v}$ and with the time of the descent $t$, then $t = \frac{1}{125}\sqrt{\frac{Ax}{g}}$ (221) and $v = \frac{gx}{A}$ (206). Then on account of this we have $\frac{g}{A} = \frac{v}{x}$, and thus $t = \frac{x}{125\sqrt{v}} = \frac{2x}{125\sqrt{v}}$. But this expression also gives the time, in which the distance $2x$ will be traversed with a uniform speed of $\sqrt{v}$, since $\frac{2x}{\sqrt{v}}$ is divided by 250, and then we find the number to be expressed in seconds (220). Consequently the distance $2x$ is travelled in the same time with the speed $\sqrt{v}$, in which the distance $x$ is fallen under a uniform acceleration. Q. E. D.

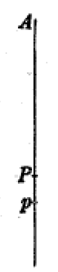

Fig. 22.

[Thus the final speed is twice the average speed for the motion of a body released from rest under gravity.]

## Corollary 1.

**231.** Therefore a body acted on by a uniform force falling in a time $t$ through a distance $x$ will acquire as great a speed, as that by which a body can progress uniformly through the same distance in half the time $t$.

## Corollary 2.

**232.** Since on the surface of the earth bodies are falling in a time of one second a distance of 15625 scruples of Rhenish feet, their final speed acquired in the fall will be as great as that with a uniform motion over a distance of 31250 in one second, or 15625 scruples traversed in a time of half a second. [p. 92]

## Corollary 3.

**233.** When the speeds are expressed, as we have established, from the square roots of the heights through which they have fallen, the speed will be as $\sqrt{15625}$ or as great as 125, in which time a body in one second can complete a distance of 31250 scruples.
[For which the constant of proportionality can now be evaluated : thus a distance of 15625 corresponds to a speed of 125 in the square root proportionality, for which the same distance of 15625 scruples corresponds to an actual speed of 31250 scruples/sec. = 250 × 125; or the true speed = 250 × $\sqrt{v}$ for the height $v$ fallen. Thus, the constant of proportionality is 250.]

## Corollary 4.

**234.** It is therefore easy to assign the distance that will be traversed in a time of one second with the speed expressed by $\sqrt{v}$. Indeed it happens that the distances described in the same time are in the same ratio as the speeds, thus 125 is to $\sqrt{v}$ thus as 31250 scruples is to $250\sqrt{v}$. For which by a factor $250\sqrt{v}$ expresses the distance in scruples traveled in one second, if indeed for the height $v$ it may be shown by such a proportionality that the distance can be completed with a speed $\sqrt{v}$ in a second, and which is clearly is equal to the motion.

## Example 1.

**235.** The fall of a body from a height of 1000 ped., will be $v = 1000000$ in scruples, whereby for this descent it will acquire as much speed as in one second it travels a distance 250000 scruples, i. e. it will be able to complete 250 feet in one second.

## Corollary 5.

**236.** And reciprocally, if the speed is expressed by the distance that is traversed in one second, as we did in the beginning, then this can hence be reduced to our method of taking square roots for the corresponding heights. Indeed if that distance is $a$ scrup., and the corresponding height for this speed is v scrup., then it follows that $250\sqrt{v} = a$ and $v = \frac{a^2}{62500}$ scrup. [p. 93]

## Example 2.

**237.** Let the body have such a speed that in a second it is able to traverse a distance of 1000 feet or 1000000 scrup., then the height that corresponds to this speed will be

$$=\frac{1000000000000}{62500}\text{ scrup.}$$

or 1600 feet.

## Scholium.

**238.** Therefore the way in which each speed is to be expressed in turn is clear, and how it may be required that the one can be reduced to the other. For initially we were expressing the speeds by the distances travelled in a second, or in some other interval of time. Afterwards, truly it was seen that the speeds were shown to be in agreement with the corresponding heights. Now truly we show how each way can be adapted to measure the speed.

## PROPOSITION 29.

### PROBLEM.

**239.** *The body now initially at B has a given speed along the line BP, and with a uniform force present acting along the line BP* (Fig. 23)*; the speed of this body is required at any point P of the line BP.*

### SOLUTION.

Fig. 23.

As before let the force of gravity be $g$ and the body $A$. Indeed the speed that it has in the beginning at $B$ is due to the height $c$. We call $BP = x$, and the speed at $P$, that we are looking for, shall correspond to the height $v$. As before (207) on account of the constant force $g$ acting on the element $Pp = dx$, it follows that $dv = \frac{gdx}{A}$.

[For in (207) Euler has defined $c^2 = v$ and hence $dv = 2cdc = \frac{2npds}{A} = \frac{pds}{A}$.]

By integrating [p. 94] we therefore have $v = \frac{gx}{A} + Const.$, which constant quantity is to be determined from this, since by taking $x = 0$ then $v = c$ (by hyp.); and therefore the constant is equal to $c$. Consequently we have :

$v = c + \frac{gx}{A}$ and speed itself : $\sqrt{v} = \sqrt{c + \frac{gx}{A}}$. Q. E. I.

## Corollary 1.

**240.** If we put $\frac{g}{A} = 1$, as in the case of ordinary gravity, then $v = c + x$. Therefore the height corresponding to the speed at $P$ is the sum of the initial height at $B$ and the distance traversed.

## Scholium.

**241.** The solution of the problem comes to mind in another way; for we put the motion along *BP* with the initial speed $\sqrt{c}$ in B to be considered as part of the motion from rest along the line AP, as we have put the term, in which the body as it came from A to B has the proposed speed $\sqrt{c}$. Therefore if this distance *AB* = *k*, then $c = \frac{gk}{A}$ (206) and $v = \frac{g(k+x)}{A} = c + \frac{gx}{A}$, as now it has been found. Moreover the distance AB is $\frac{Ac}{g}$.

## PROPOSITION 30.

### PROBLEM.

**242.** *With everything put in place as in the preceding proposition, to determine the time in which the distance BP* (Fig. 23) *is run through.*

### SOLUTION.

Let the time of for the distance *BP* = *t*, then $dt = \frac{mdx}{\sqrt{(c+\frac{gx}{A})}}$ (218) ; $\sqrt{(c+\frac{gx}{A})}$ is the speed that the body has in traversing the element *Pp* , as we have found in the preceding proposition. On integrating, this therefore becomes $t = \frac{2mA}{g}\sqrt{(c+\frac{gx}{A})} + \text{Const.}$ This constant amount is to be added from this can be defined, since for the position $x = 0$ , it must make $t = 0$. Therefore the constant produced will be : $\text{Const.} = \frac{-2mA}{g}\sqrt{c}$.

Consequently we have $t = \frac{2mA}{g}\sqrt{(c+\frac{gx}{A})} - \frac{2mA}{g}\sqrt{c}$. With *c* and *x* expressed in scruples of Rhenish feet put in place, $m = \frac{1}{250}$ (220) , and there comes about the time expressed in seconds. Q. E. I.

## Corollary.

**243.** Since in our terrestrial regions $\frac{g}{A} = 1$, the time in which the distance BP is completed with the speed given at the start at B by $\sqrt{c}$ , is equal to

$$\frac{1}{250}\sqrt{(v+c)} - \frac{1}{125}\sqrt{c} \text{ seconds,}$$

if *c* and *x* are expressed in scruples of Rhenish feet.

## Scholium.

**244.** In a similar way we can affirm another solution for this problem, which is added to the previous as a scholium. For on putting the line *AB* = *k*, from which the body falling from *A* to *B* gains the speed corresponding to the height *c*, the time will be, in which this distance is completed, equal to $\frac{1}{125}\sqrt{\frac{Ak}{g}}$ and the time, in which the distance AP is run through, will be $\frac{1}{125}\sqrt{\frac{A(k+x)}{g}}-\frac{1}{125}\sqrt{\frac{Ak}{g}}$. Therefore $k=\frac{Ac}{g}$ (241). Consequently this time sought for the distance BP becomes $\frac{A}{125g}\sqrt{(c+\frac{gx}{A})}-\frac{1}{125}\sqrt{c}$, as we found before, if $\frac{1}{250}$ is put in place of *m*. And these are the solutions of the problems concerned with the descent bodies according to the hypothesis of uniform forces that had to be set out. Therefore I can go on to rectilinear ascent, in which the direction of the speed is in the opposite direction to the force, that also is now uniform or that I make constant.

## PROPOSITION 31.

### PROBLEM.

**245.** *The body has a given speed in the upwards direction and with a uniform force pulling in downwards ; the speed of the body is required at any point in the interval BA that it has travelled through while rising.*

Fig. 24.

### SOLUTION.

It is evident that in this case that the progression of the motion along the straight line to be one of retardation (191), since the direction of its motion is acting against the direction of the force. And thus the speed of the body at *B* is due to the altitude *c*, and the body is put in place arriving at *P*. The altitude to which the speed at this place is owed, is called *v*, and the distance *BP* now traversed is called *x*. Take $Pp = dx$, and the altitude to which the speed is owed [i. e. corresponding] at *p* will be $v + dv$. Moreover since the force, that I put equal to *g*, is contrary in direction to the motion, the total speed is diminished. On account of this , it is necessary to put *dv* equal to $\frac{-gdx}{A}$, with A denoting the mass of the body. And thus $-dv=\frac{gdx}{A}$, and on integration, it becomes $C-v=\frac{gx}{A}$. In order that the constant C can be found, put $x = 0$, in which case *v* is changed into *c*; and hence it becomes $C = c$. From which the equation itself is produced : $c-v=\frac{gx}{A}$ or $v=c-\frac{gx}{A}$, which determines the speed of the body to be described in any point of the ascent. Q. E. I.

## Corollary 1.

**246.** The speed therefore vanishes when $c=\frac{gx}{A}$, i. e. when it arrives at the height $x=\frac{Ac}{g}$. Let BA be equal to that height $\frac{gc}{A}$, hence $c=\frac{BA.g}{A}$, from which it is understood that *BA* is the height itself, from which the body A acted on by the force *g* by falling acquired the speed *c* (206). Therefore the body with that speed, that it gained by falling through the given height, progresses upwards again to reach the same height that it had at the start of its motion when it was sent off.

## Corollary 2.

**247.** Besides the body ascending through the distance BA has the same speeds at the individual points, that it has at the same points if it is descending from A. For with AP = y the speed at P descending, coming from *A*, is equal to $\sqrt{\frac{gy}{A}}$ ; but the speed at the same place P in the ascent from B leaves $\sqrt{(c-\frac{gx}{A})}$ . Moreover, since $x+y=BA=\frac{Ac}{g}$, it is apparent that these expressions of the speed are equal , surely $\frac{gy}{A}=c-\frac{gx}{A}$.

## Corollary 3.

**248.** Therefore the motion of the ascending body agrees with the motion of the descending body, and the speeds of each at the same points are equal, *i. e.* which are placed at the same distance from the upper point at which the speed is zero.

## Corollary 4.

**249.** From these it is likewise evident that the time of ascent through the distance *BA* is equal to the time of descent through the same distance. Whereby by calling *BA = a,* the time of descent is equal to $\frac{1}{125}\sqrt{\frac{Aa}{g}}$ sec. (221), and that must be equal to the time of ascent through *BA*; or to the place with the value $\frac{Ac}{g}$, the time of the whole ascent will be $\frac{A}{125g}\sqrt{c}$ .

## Corollary 5.

**250.** In a similar manner the time of the ascent through any part BP can be defined; for keeping $AP=y$ the time for either the ascent or the descent through $AP$ $=\frac{1}{125}\sqrt{\frac{Ay}{g}}$ , that taken from the time of the whole ascent , which is equal to $\frac{1}{125}\sqrt{c}$ , there is left the time to pass through the part BP. This distance is given by $y=\frac{Ac}{g}-x,$ whereby the time to ascend through BP is given by $\frac{A}{125g}\sqrt{c}-\frac{A}{125g}\sqrt{(c-\frac{gx}{A})}$.

## Scholium 1.

**251.** Evidently in the first place the equation of the ascent has been shown from the action of the force. Since indeed in the ascent the force takes away as much from the speed, as it adds on during the descent, it is evident that there is a complete equality between each motion, and one can be turned into the other without any distinction, unless the succession of the time, by knowing which the one case is only turned into the other.

Indeed the reasoning is also the same of all the motion produced by the absolute forces: for the body can be turned around in its path with the same speeds, if indeed in the return motion the impressed forces are the same, as those in the departing motion, but to be clearly in the opposite direction. Thus the planets would go round the sun in contrary orbits, which now move in ellipses, if in the first place they had been sent off in a motion contrary to these. For through the same element of distance, the effect of the force in changing the speed is always the same, and for that reason it shall be negative, when the body returns. Moreover the effect, which is devoted to changing the direction of motion of the body, in each case remains the same, which is, that the body in coming and going retraces the same path. But this will become more apparent below, where motion of this kind will be established. But truly, if it is decreased by resistance, this similitude between the ascending and the descending motion vanishes : for in each case the resistance to the motion is made less, and neither is the effect of this on either of the other in the opposite direction, as usually comes about for absolute forces acting.

## Scholium 2.

**252.** Enough therefore with rectilinear motion, which arises from uniform forces, and it is time to move on to different kinds of forces, which in different places exert other forces on bodies, and that needs to be examined, for whatever the motions of the bodies, in as much as they shall be on a straight line, they will vary from the above. Indeed all the forces that we observe in the world suffer from this kind of difference, that no force can be assigned to a body put in any place that will equally affect the body. Thus the planets, where they are closer to the sun, [p. 100] are attracted to the sun by a stronger force; and also where a body more removed from the surface of the earth, the weight or the exertion downwards shall be less on that body. It almost comes about in the same manner, where we observe the magnet to attract an iron filing more at shorter distances, and the attraction to be weaker at greater distances. Therefore we will elicit the laws, for whatever ratio of the forces with distance they hold, according to which the motion of the body acted upon can be changed. And in the first place we are indeed to contemplate forces that vary according to the distance of the body from a fixed point raised to some power.

## Definition 16.

**253.** *That fixed point is called the centre of attraction, to which bodies are attracted by a force, which depends on the distance from this point, or which is as some fraction of this distance.*

## Corollary 1.

**254.** Therefore given the distance from this centre of force in which the body placed is drawn to the centre by as large a force, as if it should be the force of gravity acting on this, and placed on the surface of the earth.

## Corollary 2.

**255.** Therefore with this distance and the law of the attraction known, clearly given by a function of the distance, to which the attraction is proportional, the ratio of this force is known, for any position of the body trying to fall towards the centre of force, to the force of gravity on the same body acted on if it should be on the surface of the earth. [p. 101]

## Corollary 3.

**256.** Thus in this manner, any variable forces are allowed to be compared with the force of gravity, since the effect of this force on the body is known, and thus also the effect of any force on the body can be determined.

## Scholium 1.

**257.** I put this attraction of the centres of the forces similar to the force of gravity, thus in order that likewise the forces of different bodies placed in the same position are as the masses themselves, and thus the magnitudes of the accelerations are all the same (212). Therefore in handling these problems, it is not necessary to call upon the mass of the body in the computation of the motion, but only the magnitude of the acceleration, to which the force of attraction to the centre divided by the mass is in proportion. Moreover it can be compared with the acceleration of the force of gravity, that we put equal to one, and we will compare all the accelerations due to the magnitudes of these forces with this acceleration of unity, clearly homogeneous quantities.

## Corollary 4.

**258.** Thus when we talk about the forces as being as the distances from the centre of force or proportional to a certain function of these, it is not only the forces that the bodies have to the centre, but also the accelerations associated with these force, i. e. it is the ratio of the force to the mass, that should be understood.

## Corollary 5.

**259.** Therefore since the direction of the force which presses upon the body, always pulls towards the centre of force, [p. 102] it is evident, if the body is either at rest or it has motion, the direction of which passes through the centre of motion, then the body must be moving on this straight line perpetually crossing through the centre of force (189).

## Definition 17.

**260.** *The force, which presses upon bodes to the centre of this kind of force, is called the centripetal force. And that, if it is negative, in order that the body is repelled from the centre, is called the centrifugal force.*

## Corollary 1.

**261.** Since this will be the question about the motion, the centripetal force will be for us the magnitude of the acceleration, or the force pulling the body towards the centre divided by the mass of the body.

## Corollary 2.

**262.** Therefore the effort or the striving that the body has towards the centre of force, is expressed by the strength of the centripetal force [i. e. the acceleration] multiplied by the mass of the body. On account of which it will be to the weight of the same body, if it were put in place on the surface of the earth, as the strength of the centripetal force or the strength of the acceleration to unity. (257).

## Scholion.

**263.** Newton, who mainly talks about the centripetal force [in the Definitions in the *Principia*], has paid attention to three ways in which the effects of the same can be measured. In the first case, it provides a measure of the absolute quantity of the effectiveness of the centre of the force, without regard to the [mass of the] body being attracted ; thus he asserts, in the case of the larger loadstone the greater the absolute quantity of the centripetal force present, and in the case of the lesser lodestone, the corresponding centripetal force is smaller too [p. 103; See Def. VI. *Principia* ; though the mass of the orbiting body would not cancel, if such a motion were possible, as in *e/m* experiments with a uniform magnetic field.] And in a similar manner, following this theorem, the absolute quantity is greater on the sun than it is on the earth. Moreover this comparison is to be understood from the similarity of the centres of force, i. e. according to the same function of the distances, and with attractions; indeed the comparison does not have differences of this kind at the same place. Hence this absolute quantity of force is to be measured from a known effort, which the body has exerted on it at a given distance from the given centre of force. Moreover in place of this consideration, I put in place the distance into which the body can be put with a force equal to its weight pushing towards the centre (254). In the second place the strength of the centripetal force has the magnitude of an acceleration, which is perceived by the senses at the object itself, where the centripetal force itself is acting (261); indeed it is measured by the ratio of the effort applied to the mass. In the third case, the strength of the centripetal force leads us to the magnitude of the motive force, which is specified by the force the bodies experience on approaching the centre of force; the motive force is the quantity of motion, and that is usually measured by the product of the speed by the mass, and which is produced in a given time proportional to this effort itself. And for this force $p$, and the mass $A$, the increment in the speed in a given element of time varies as $\frac{p}{A}$ (154), that multiplied by the mass $A$ gives the increment in the quantity of motion, that is proportional to the force $p$.

[According to Cohen, on p. 406 of his translation of the *Principia*, which is very good on this point, Newton's summary of the three effects of centripetal force are : *absolute, accelerative, and motive*. One may take the first to mean that the centripetal force lies in the category of absolute forces; the second that it involves an acceleration; and the third that the momentum of the body is related to it from the impulse divided by time relation, as Euler sketches above. One may raise the odd eyebrow at Def. 8, which tells us that it is the opposing centrifugal force which prevents the body from falling...... Well!! It may also be appropriate to note what the modern usage is by contrast, at least according to this translator's understanding : Newton's Second Law, which he never actually enunciated himself in the form '$F = ma$ ' is a cause – effect relation. The cause can be some phenomenologically determined relation such as the law of universal gravitation, Coulomb's law of electrostatics, etc: this takes care of the first property. The effect is what the mass does in response. If the force is always applied at right angles to the direction of motion, then we get the centripetal acceleration, which is a purely kinematic quantity: which is the second property. The third property is an amalgamation of these two, whereby the rate of change of momentum can be used as a measure of the central force, as for example for a moon rotating around a planet, etc. This is all viewed in the realm of classical physics. Newton was to correct his unfortunate assertion made above, when he discussed cannon balls falling towards the centre of the earth when in orbit.]

## PROPOSITION 32.

### PROBLEM.

**264.** *Let C be the centre of the forces* (Fig. 25) *that attract bodies in some ratio of the distance, and by this force a body at rest at A is drawn forwards; the speed of this body is then sought at any point in the interval AC.* [p. 104]

### SOLUTION.

Fig. 25.

Let $AC = a$, $AP = x$; and the speed that the body has at $P$ is that corresponding to having fallen from the height $v$. The attraction shall be given as the ratio of the distance raised to some power $n$, and $f$ is taken as the distance from C, at which the force on the body towards C is equal to the weight of the body, if it should be placed on the surface of the earth. Therefore the strength of the acceleration, by which the body at P is pulled towards C, will be as the strength of gravity, that I put equal to 1, as $CP^n$, *i. e.* as $(a-x)^n$ to $f^n$; on account of which the acceleration is expressed by $\frac{(a-x)^n}{f^n}$. Therefore by taking $Pp = dx$ then $dv = \frac{(a-x)^n dx}{f^n}$. For $dv$ is equal to $dx$ multiplied by the strength of the acceleration (213). This integrated equation produces $v = C - \frac{(a-x)^{n+1}}{(n+1)f^n}$. For the constant C to be defines put $x = 0$, in which case by hypothesis it must become $v = 0$; therefore $C = \frac{a^{n+1}}{(n+1)f^n}$. It is therefore found that $v = \frac{a^{n+1}-(a-x)^{n+1}}{(n+1)f^n}$. Or by putting $a - x = CP = y$ it becomes $v = \frac{a^{n+1}-y^{n+1}}{(n+1)f^n}$. From which equation the speed of the body at any point of the interval $AC$ is known. Q. E. I.

[Thus, the force on the body at position $a - x$ is given by some function $k(a-x)^n$, and the acceleration is $k(a-x)^n/m$, where $m$ is the mass of the accelerated body and $k$ is a constant of proportionality. However, when the body is at $f$, it is considered to have the force equal to its weight acting on it, which is just $m$, as $g$ is taken as equal to 1; hence

$kf^n = m.1$, or $k = m / f^n$, giving the force as $m(a-x)^n / f^n$, and the acceleration as $(a-x)^n / f^n$. In addition, for motion under gravity $2vg = V^2$, where *V* is the final speed at the point and *g* is taken as 1, then *VdV* = *gdv* and also *VdV* = *adx* for the motion under the new force; hence *adx* = *dv,* where *a* is the acceleration under the new force, and the point masses under gravity and under the new force have the same speed and increment in the speed; as these are not in proportion in general, each point must correspond to a different release point. This gives the ratio of the accelerations under gravity and the force, which is a function of the distance. From which it follows that $dv = \frac{(a-x)^n dx}{f^n}$, which can then be integrated as above. We would now proceed in a slightly different manner , and set the acceleration $dV / dt = VdV / dx$; in which case $VdV / dx = (a-x)^n / f^n$ and $\frac{1}{2}V^2 = v = C - \frac{(a-x)^{n+1}}{(n+1) f^n}$, etc. We may note also that this first integral is in fact just the conservation of energy, as the sum of the kinetic and potential energy is related to the constant C. This relation obviously breaks down when infinite quantities are involved.]

## Corollary 1.

**265.** If $n + 1$ is a positive number, $y^{n+1}$ vanishes when $y = 0$. Therefore in this case [p. 105] the altitude corresponding to the speed, that the body has on arriving at C, will correspond to $\frac{a^{n+1}}{(n+1) f^n}$. But if $n + 1$ is a negative number, $y^{n+1}$ will become infinitely large when *y* is made zero : hence in this case the body on arriving at C will have an infinitely large speed.

## Corollary 2.

**266.** But if $n + 1 = 0$ or n = -1, the value found from the equation itself may not be known on account of the numerator and the denominator vanishing. Because of this, it will be necessary to repeat the differential equation. Moreover, it follows that $dv = \frac{fdx}{a-x}$, the integral of which is $v = C - fl(a-x)$ . And it must be that $C = fla$, on account of which $v = fl\frac{a}{a-x} = fl\frac{a}{y}$ [$= f\log(\frac{a}{y})$]. Which is the true value of *v*, when *n* has the value -1, *i. e.* when the centripetal force varies inversely with the distance from the centre of the force.

## Corollary 3.

**267.** Therefore in this case, $n = -1$, when the body arrives at the centre *C*, its speed is infinitely great; for it shall be that $v = fl\,\infty$. This infinite step is to be deplored, and if a nearby value is taken, it is finite ; however if $n + 1$ should exceed zero a little, then the speed at *C* suddenly becomes finite.

## Corollary 4.

**268.** Moreover since $n + 1$ should be a positive number, since then the height corresponding to the speed at C is $\frac{a^{n+1}}{(n+1)f^n}$, then the speeds of many bodies falling [p. 106] towards the centre C, and which they have at C, are as $a^{\frac{n+1}{2}}$, *i. e.* as the $\frac{n+1}{2}$ power of the distances from which they have began the fall.

## Scholium 1.

**269.** Moreover, after the body arrives at *C* from *A* in *C*, where then it shall keep on moving forwards, it is not so easy to be defined. Indeed it is observed, that if *y* is made negative in the expression found, the height corresponding to the speed at *Q* should be emerging ; which if it is positive, then the body again returns to *Q* ; but truly if it is negative, from the evidence, this body never reaches beyond *C* into the region *CQ*. In truth this way of continuing the motion is not always possible to be adhered to; often indeed the hypothesis itself, by which the attractive force is placed before and beyond *C* towards the centre is opposite. In as much as the body proving to be at *P*, since it is being pulled down, when it arrives at *Q*, it is pushed up by an equal force, if $CQ = CP$. This force, on account of the nature of the force which is acting on the body at *Q*, is negative with the former ratio and thus is to be expressed by a negative quantity. Therefore the force at *P* expressed by $\frac{(a-x)^n}{f^n}$ or $\frac{y^n}{f^n}$ must be the negative of itself, when $-y$ is put in place of *y*, and that never happens, unless *n* is either an odd number or a fraction, of which the numerator and denominator are uneven. Therefore for these cases the value of *v* is produces, when the body arrives at *Q* ; always in the remaining cases, since in calculating the force acting on the body at *Q* when indeed with the value not in agreement, the quantity elicited for the letter *v* is not the true value [p. 107]. If indeed *n* is an even number, the attracting force at *Q* by making *y* negative is equal to the force at *P* and clearly falls in the same place. From which it shall be, that as the body crosses the centre *C* on the line *CQ* must continue to fall to infinity, that calculation also makes clear. Because when it disagrees with the hypothesis, it is seen that in these cases the motion of the body, after it has arrived at C, cannot be defined by the formula defined. Moreover it is seen to be more absurd, when $n = \frac{1}{2}$ or another fraction of this kind, which changes $y^n$ into an imaginary quantity with $-y$ put in place of *y*; because that may indicate that that not only is the body not attracted to *C*, but the force of attraction also becomes imaginary, which is indeed not possible to understand.

## Corollary 5.

**270.** Therefore if *n* is an odd number, the value of *v* itself, which is $\frac{a^{n+1}-y^{n+1}}{(n+1)f^n}$, does not change with $-y$ put in place of $+y$, since the even number $n + 1$ of *y* avoids the [sign change in the] exponent. From which it is apparent that the speed of the body at *Q* is equal to that which it had before at *P*, if indeed *CQ* = *CP*. Therefore the motion is equal in the manner in which the body recedes in the direction *CQ*, by which before it approached along *AC*; and it shall reach as far as *B*, thus in order that *CB* = *AC*, where it loses all its speed. And thus it reverts again in the same way to *C*, and then it arrives at *A* again. Which reciprocal motion, unless decreased by friction, will be carried out indefinitely.[p. 108]

## Corollary 6.

**271.** Nevertheless the case when $n = -1$, since $-1$ is an odd number is to be undertaken. For with *y* made negative $v = fl\frac{-a}{y}$, which is an imaginary amount. From which it is seen that the body never goes beyond *C*. Hence another judgement is seen to be brought down, when *n* is a negative number, even if it is odd. For a similar example of this kind occurs beyond, if $n = -3$ (355).

## Scholium 2.

**272.** Indeed this is seen to be less in agreement with the truth; for the reason is hardly apparent why the body with its infinite speed that it acquires at *C* should be about to progress into CB rather than another region, especially when the direction of this infinite speed should follow into this region. But whatever it shall be, here the calculation rather than our judgement being trusted and established, the jump if it is made from the infinite to the finite, is not thoroughly understood. Moreover, this opinion is further confirmed by a similar example for which a full explanation is given below, (665), if $n = -2$; for in this case the speed of the body arriving at *C* is also infinite and directed along *CB* ; by no less truth, in this case the body does not progress beyond *C*, but suddenly reverts from *C* equally and approaches towards A. From which it is understood, that as often as an infinite speed should arise at C, judgement about the further motion of the body should be suspended. So for the time being only this shall be done, until we come to considering motion along curves [p. 109] ; and with these indeed which are rectilinear, and [the resolution of this problem is] clearly connected to these(762). For neither then is the calculation which is put in place subject to this inconvenience, as it is in disagreement with the hypothesis ; but whatever is put equal to the centripetal force is not in opposition to the calculation.

## Scholion 3.

**273.** But always, when the speed at *C* is not infinitely great, because that happens, when the size of the number $n + 1$ is positive, the whole motion of the body is known by our judgement, even if the calculation is insufficient. For if the speed at C is finite and has the direction along *CB*, that by necessity is should have, then it may not be possible to happen, as no motion can be continued along *CB*. But in a like manner the motion may be continuing to recede from *C*, when before it was approaching along *AC*, and at some point Q it has the same speed that before it had at the point P placed at an equal distance from C, thus as can be understood from § 251. Therefore the motion occurs perpetually between A to B and back again, and in returning the body completes the motion.

# PROPOSITION 33.

## PROBLEM.

**274.** *With the attraction from the centre C* (Fig. 26) *to be in some ratio of a multiple of the distances, the body at D now has a given speed ; the point A is required on the line CD produced, from which the descent of the body towards C begins, so that it has acquired this speed when it arrives at D.* [p. 110]

## SOLUTION.

With $n$ denoting as above the exponent of the ratio of multiplication, in what shall be the centripetal force, and $f$ the distance at which the centripetal force is equal to the force of gravity; let $CD = b$, the speed at $D$ corresponding to the height $h$, and the distance $CA$ sought which is put equal to $q$. Since here therefore $q$ denotes that same distance as $a$ [*i. e.* CA] in the above proposition, and $b$ likewise that of $y$ [*i. e.* CP], and $h$ likewise here represents $v$, this equation is formed: $h = \frac{q^{n+1} - b^{n+1}}{(n+1)f^n}$.

[*c. f.* $v = \frac{a^{n+1} - y^{n+1}}{(n+1)f^n}$ above.] From which

$q^{n+1} = b^{n+1} + (n+1)hf^n$ and $q = (b^{n+1} + (n+1)hf^n)^{\frac{1}{n+1}}$. Moreover in the particular case, when $n = -1$, there is obtained

$h = fl\frac{q}{b}$ and hence $q = e^{\frac{h}{f}}b$, where $e$ is the number, the logarithm of which is unity. Q. E. I.


Fig. 26.

## Corollary 1.

**275.** If the centripetal force varies directly as the distance, then $n = 1$ and $q = \sqrt{(b^2 + 2fh)}$. Which is always a finite quantity, but only if $b$, $f$ and $h$ are such. Likewise it arises, provided $n + 1$ is a positive number. And also in the case $n = -1$ the distance $q$ is never infinite.

## Corollary 2.

**276.** But if $n + 1$ is a negative number, for example $-m$, since $n = -m - 1$, then $q = b.\sqrt[m]{\frac{f^{m+1}}{f^{m+1} - mb^m h}}$, for which the height is infinite when $h$ is given by $h = \frac{f^{m+1}}{mb^m}$, and if $h$ is a quantity greater than this, [p. 111] $q$ is negative, or rather infinitely greater, or even imaginary. From which it is understood from these cases that only by falling from infinity is the body able to acquire as much speed at $D$.

## Corollary 3.

**277.** By keeping $n + 1$ equal to the negative number $-m$ and the distance $h$ for the point *A* at infinity will be $h = \frac{f^{m+1}}{mb^m}$. And the distance from the centre C at which the body, falling from the infinite distance, will have the speed $\sqrt{h}$, which is equal to $f\sqrt[m]{\frac{f}{mh}}$.

## Corollary 4.

**278.** If the centripetal force is inversely proportional to the square of the distance, then $m = 1$. On account of which $q = \frac{bf^2}{f^2 - bh}$. When, therefore $h = \frac{f^2}{b}$, the distance AC, *i. e. q*, shall be infinitely great.

## Corollary 5.

**279.** If this problem is joined with the preceding one, the motion of the body can be easily determined, since it begins to descent to *C* from *B* with the speed $\sqrt{h}$. From the preceding indeed the descent of the body is observed to be made from A, since it begins to descend in the part from D with the speed $\sqrt{h}$, *CP* is called equal to *y*, and the speed, that the body has at *P*, corresponding to the height *v*, is $v = \frac{q^{n+1} - y^{n+1}}{(n+1)f^n}$ (264). [p. 112]

But $q^{n+1} = b^{n+1} + (n+1)hf^n$. Hence it becomes

$$v = \frac{b^{n+1} + (n+1)hf^n - y^{n+1}}{(n+1)f^n} = \frac{b^{n+1} - y^{n+1}}{(n+1)f^n} + h.$$

## Corollary 6.

**280.** From this expression for *v*, when the descent begins from *D* with the speed corresponding to the height *h*, it does not differ from that which is produced, if the descent were made from rest, except that this is quantity is always a distance greater than *h*.

## Scholium.

**281.** Since the time delay, in which the distance AC or any part of this is completed for any hypothetical centripetal force (Fig. 25), that is easily known from the known speeds. Generally the time for any letter *n* cannot be shown in a finite number of terms, clearly the time to traverse *AP* is found to equal

$$\int \frac{dx\sqrt{(n+1)f^n}}{\sqrt{(a^{n+1} - (a-x)^{n+1})}},$$

which quantity generally neither can be integrated nor reduced to the quadrature of any known curve. But yet in various cases of *n* itself, it can be expressed neatly enough , on account of which from the general case set out, particular special cases will be examined in the following propositions.

## PROPOSITION 34.

### PROBLEM.

**282.** *If the centripetal force is in proportion to the distance from the centre C* (Fig. 27) *and the body falls from A as far as C,* [p. 113] *it is required to determine the time in which the body completes any part of this distance.*

### SOLUTION.

With $AC = a$, and the distance from the centre $C$, in which the centripetal force is equal to the force of gravity , equal to $f$, some part of the distance $CP = y$ and the speed at $P$ corresponding to the height $v$. Therefore the time, in which the distance $CP$ is completed, is equal to $\int \frac{dy}{\sqrt{v}}$; with the fraction $\frac{1}{250}$ ignored, since this can be used for the known time in seconds and can be added as desired. Truly from Prop. 32 (264) by making $n = 1$ $v = \frac{a^2-y^2}{2f}$, hence $\sqrt{v} = \frac{\sqrt{(a^2-y^2)}}{\sqrt{2f}}$.

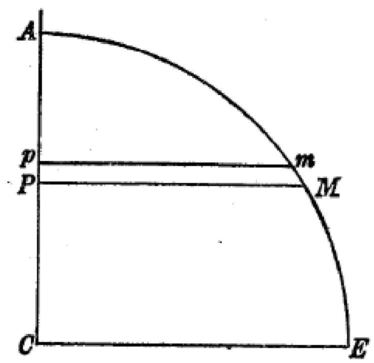

Fig. 27.

From which the time to travel through $PC$

$$= \int \frac{dy\sqrt{2f}}{\sqrt{(a^2-y^2)}} = \frac{\sqrt{2f}}{a}\int \frac{ady}{\sqrt{(a^2-y^2)}}.$$

Upon AC the quadrant of a circle is constructed AME, and to this the lines CE and PM as axis. From which is made, as agreed, the arc $EM = \int \frac{ady}{\sqrt{(a^2-y^2)}}$. On account of which the time to traverse PC becomes $= \frac{EM.\sqrt{2f}}{a}$. The time therefore of the total descent through AC will be $\frac{AME.\sqrt{2f}}{a}$. Hence the time of descent through $AP = \frac{AM.\sqrt{2f}}{a}$. From these therefore the time of descent through any distance travelled through can become known, and that in seconds, if these expressions are divided by 250 and the length $f$ is shown in thousandth parts of Rhenish feet. Q. E. I.

## Corollary 1.

**283.** Let $1 : \pi$ denote the ratio of the diameter to the circumference , then it becomes $2AME : a = \pi : 1$ and $\frac{AME}{a} = \frac{\pi}{2}$. Hence on account of this, the time of descent through AC is equal to $\frac{\pi}{2}\sqrt{2f}$. [p. 114] Since that does not depend on the height dropped or travelled through $a$, but whatever amount this shall be, it keeps the same value. Therefore all bodies, which are released towards this centre, reach that in equal amounts of time.

## Scholium.

**284.** This equality of the time follows from the expression for the time $\frac{\sqrt{(a^2-y^2)}}{\sqrt{2f}}$, in which $a$ and $y$ are required to have one dimension. Indeed the amount comes about, the times, in which any distances $a$ are travelled through, must be equal to each other. (46).

## Corollary 2.

**285.** If besides there should be another of centre of force of this kind, but with a different effectiveness provided, thus in order that the distance at which the centripetal force is equal to the force of gravity is $F$, the times of the descents shall be to each other as $\sqrt{f}$ ad $\sqrt{F}$. But the effectiveness of each are themselves in this case in the inverse ratio of the distances $f : F;$ indeed these are as the forces, which these forces exercise at equal distances. Wherefore the times of descent to the different centres of force are in the inverse ratio of the square roots of their effectiveness. Which ratio indeed holds in all similar centres of force in place, if the distances traversed are equal to each other, as will be taught in what follows.

## CAPUT TERTIUM

## DE MOTU RECTILINEO PUNCTI LIBERI A POTENTIIS ABSOLUTIS SOLLICITATI

[p. 76]

## PROPOSITIO 24.

### THEOREMA.

**189.** *Quando potentiae et motus directiones in eadem sitae sunt recta, motus erit rectilineus.*

### DEMONSTRATIO.

Omne corpus vi insita conatur motum suum in directum continuare, id quod semper praestat, nisi impediatur (65). Potentiae vero in corpus motum duplicem esse ostendimus effectum, alterum, quo eius directio immutatur, alterum, quo celeritas eius. At directio manet immutata, si potentiae directio cum ea in directum iacet (128). Hoc igitur casu punctum in linea recta progredi perget. Q. E. D.

### Corollarium 1.

**190.** In hoc igitur capite alios non considerabimus casus, nisi in quibus motus et potentiae directiones in eadem recta sunt positae.

### Corollarium 2.

**191.** Videmus autem duobus modis hanc congruentiam evenire posset, prout scilicet ambae [p. 77] hae directiones vel in eandem plagam vel in oppositas tendunt. In quorum illo puncti celeritas augetur, in hoc vero diminuitur (128).

### Scholion.

**192.** In motu hoc rectilineo duo sunt consideranda, quarum primum est potentia, a qua punctum ubivis sollicitatur, alterum vero celeritas, quam habet in quolibet spatii loco. His praeterae adiungimus tertium, quod est tempus, quo quaevis spatii portio percurritur. Tria vero ista ita sunt comparata, ut dato uno reliqua duo semper possint determinari. Primo igitur potentiam tanquam datam considerabimus : deinde vero eam ex data vel celeritatum vel temporum ratione investigabimus.

# PROPOSITIO 25.

## PROBLEMA.

**193.** *Protrahatur punctum in A* (Fig. 22) *quiescens in recta AP, a potentia uniformi seu quae ubique punctum eadem vi sollicitat, determinare celeritatem puncti in quovis loco P.*

## SOLUTIO.

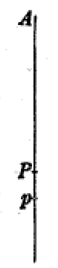
Fig. 22.

Exponatur massa seu vis inertiae puncti litera *A* et potentia litera *g*, quae erit constans seu ubique eiusdem quantitatis. Sit spatium $AP = x$, et celeritas in *P*, quae quaeritur, ponatur = *c*. Sumatur elementum spatii *Pp*, quod erit = *dx*; atque incrementum celeritatis, quod punctum, dum elementuun *Pp* absolvitur, a potentia *g* accepit, erit *dc*. [p. 78] His positis erit $cdc = \frac{ngdx}{A}$ (157), quia potentiam perpetuo deorsum trahere, et propterea motum accelerare ponimus. Ex hac aequatione, si integratur, oritur $cc = \frac{2ngx}{A} + \text{Const.}$, quae constans ex eo debet determinari, quod celeritas in A evanescat. Factis igitur $c = 0$, et $x = 0$, prodibit Const. = 0, quanobrem habebitur $cc = \frac{2ngx}{A}$ seu $c = \sqrt{\frac{2ngx}{A}}$. Q. E. I.

## Corollarium I.

**194.** Punctum A igitur perpetuo in recta AP descendet, et celeritas in quovis loco erit ut radix quadrata ex spatio iam percurso.

## Corollarium 2.

**195.** Ex his etiam plurium punctorum a potentiis uniformibus seu constantibus descensus poterunt comparari; erunt enim celeritatis in ratione subduplicata composita ex directis potentiarum et percursorum et inversa massarum.

## Scholion 1.

**196.** Casus hic apprime convenit cum lapsu corporum super terra: gravitas enim, quae potentiae vices sustinet, est uniformis in non nimis magnis a terrae superficie distantiis. Namque idem pondus cuiusvis corporis reperitur in altissimis montibus et profundissimis vallibus; ex pondere autem gravitas innotescit. In descensu igitur gravium libero celeritates sunt ut radices quadratae ex altitudinibus percuris. Haecque est ipsa GALILAEI proposito, quam primus tum ex experimentis tum ex ratione detexit [p. 79]. Descensus autem in spatio ab aere vacuo fieri debet, quia aer motui resistit hancque regulam evertit.

## Scholion 2.

**197.** In spatio ab aere vacuo, quod ope antliae pneumaticae efficitur, plurimus experimentis est demonstratum corpora quaecunque aequaliter descendere. Ex quo consequitur, si nullus esset aer, omnia corpora ex aequalibus altitudinibus delapsa aequales adipisci celeritates. Hanc ob rem si $g$ designet vim gravitatis, qua quodvis corpus A cietur, erit $\frac{g}{A}$ quantitas semper constans. Vis igitur gravitatis proportionalis est quantitati materiae corporis, in quod agit. Illa autem vis nil aliud est nisi pondus corporis; quare pondera in *Princ. Phil.* quoque affirmat eamque praeterea ex experimentis pendulorum probat.

## Corollarium 3.

**198.** Corpus igitur quodcunque in superficie terrae ex data altitudine delapsum datum acquiret celeritatis gradum. Cognita ergo altitudine, ex qua corpus descendit, innotescet simul eius descensu acquisita.

## Scholion 3.

**199.** Ad celeritates igitur mensurandas poterimus has altitudines adhibere, ex quibus grave [p. 80] in terrae superficie descendens aequalem acquirit celeritatem. Haec quidem altitudo non potest loco ipsius celeritatis substitui, quia celeritates sunt in altitudinum ratione subduplicata. Verum tamen altitudine commode quadratum celeritatis denotari poterit.

## DEFINITIO 15.

**200**. *Altitudinem celeritati cuidam debitam vocabimus posthac eam altitudinem, ex qua grave in superficia terrae descendens eandem illam acquirit celeritatem.*

## Corollarium 1.

**201.** Haec igitur altitudo debita est ut quadratum celeritatis, ad quam refertur. Celeritate ergo existente $c$ et ipsi debita altitudine $v$, erit $v$ ut $c^2$.

## Scholion 1.

**202.** Hactenus celeritatem expressimus linea recta, quae dato tempore ex celeritate percurri potest. In posterum autem commodius erit altitudinem debetam eius loco introducere. Hanc ob rem ponemus $v = cc$ et $c = \sqrt{v}$ . Habebimus ergo in problemate praecedents hanc aequationem $v = \frac{2ngx}{A}$ .

## Corollarium 2.

**203.** In posterum igitur semper loco celeritatis c ponere licebit $\sqrt{v}$ seu radicem quadratam ex altitudine celerati debita. [p. 81]

## Corollarium 3.

**204.** Si potentia $g$ denotet ipsam vim gravitatis, erit $x$ ipsa altitudo celeritati $c$ debita, adeoque $v = x$. Est vero $v = \frac{2ngx}{A}$, ex quo igitur erit $n = \frac{A}{2g}$. Hoc igitur assecuti sumus commodum, ut literam $n$ determinaverimus, quae in omnibus casibus tenet eundem valorem (155).

## Scholion 2.

**205.** Quia hic $g$ vim gravitatis significat, erit $\frac{g}{A}$ quantitas constans (197). Hanc ergo ponemus 1, id quod licebit, cum potentiae ad corpora definitam rationem habere nequeant. Atque hinc facile erit in aliis casibus valorem ipsius $\frac{g}{A}$ seu potentiae applicatae ad corpus exhibere. Erit nempe $\frac{g}{A}$ ad 1 seu g : A ut vis g, qua corpus A sollicatur, ad pondus, quod idem corpus haberet in nostris regionibus. Litera igitur A non amplius materiae quantitatem denotabit, sed ipsum corporis A pondus, si super terra esset positum. Hoc igitur modo omnes potentias cum ponderibus comparabimus, id quod in potentiis mensurandis ingentem lucem foenerabitur.

## Corollarium 4.

**206.** Cum in $n = \frac{A}{2g}$ g denotet vim gravitatis positumque sit $\frac{A}{g} = 1$, erit $n = \frac{1}{2}$. Quem valorem semper retinebit, si modo celeritates per radices quadratas altitudinum ipsis debitarum exprimantur. Ideoque erit in nostro casu $dv = \frac{gdx}{A}$ et $v = \frac{gx}{A}$. [p. 82]

## Corollarium 5.

**207.** Propterea in hac lege generali $cdc = \frac{npds}{A}$ (157), si sit altitudo celebritati $c$ debita $v$, erit $cdc = \frac{dv}{2}$, adeoque ob $n = \frac{1}{2}$ habebitur haec lex $dv = \frac{pds}{A}$, ubi $p$ est ad $A$ ut vis $p$ ad pondus corporis $A$.

## Corollarium 6.

**208.** Simili modo, quae in (161) et (163) tradita sunt, nempe aequationes $Acdc = npdy$ et $nprdx = Ac^2ds$, substituendo $v$ loco $c^2$ et $\frac{1}{2}$ loco $n$, transmutantur in has $Adv = pdy$ et $prdx = 2Avds$, ubi $p$ ad $A$ habet rationem modo dictam.

## Corollarium 7.

**209.** Atque in (165) havebitur $r = \frac{2Av}{p}$ seu $pr = 2Av$. Item in (165) havebitur $Adv = pds$, et in casu (167) habebitur $Adv = -pds$. Hocque modo ante usitatas quantitas vagas $n$ et $c$ ad determinatos valores reduximus.

## PROPOSITIO 26.

### THEOREMA.

**210.** *In diversis potentiarum uniformium hypothesibus altitudines, ex quibus aequalia corpuscula descendentia aequales acquirunt celeritates, sunt reciproce ut potentiae.*

### DEMONSTRATIO.

Sit uniuscuiusque corpusculi massa seu pondus in superficie terrae *A*, potentia quaevis uniformis *g* et altitudo [p. 83] celeritati acquisitae debita *v*. Altitudo vero, ex qua corpusculum *A* a potentia *g* sollicitatum aequalem acquirit descendendo celeritatem, sit *x*, erit $v=\frac{2ngx}{A}$ (202). At est $n=\frac{1}{2}$ (206). Ergo sit $v=\frac{gx}{A}$ seu $Av = gx$. Quare cum celeritates a diversis potentiis productae et corpuscula ponantur aequalia, erit *Av* quantitas constans ideoque etiam *gx*. Propterea erit *x* reciproce ut g, i. e. altitudo, ex qua corpusculum *A* a potentia *g* sollicitatum acquirit celeritatem $\sqrt{v}$, erit reciproce ut potentia *g*. Q. E. D.

## Corollarium 1.

**211.** Ostendit Neutonus eiusdem corporis in superficiebus Solis, Iovis, Saturni et Terrae positi nisum seu potentiam,qua ad eorum centra sollicitatur, esse ut 10000, 835, 525, et 400. Altitudines igitur, ex quibus corpus in superficiebus Solis, Iovis, Saturni et Terrae descendens aequalea acquirit celeritates, sunt inter se ut $\frac{1}{10000},\frac{1}{835},\frac{1}{525}$ et $\frac{1}{410}$.

## Corollarium 2.

**212.** Statuit autem idem NEUTONUS omnia corpora in his superficiebus aequaliter descendere, pariter ut in superficie Terrae. Non igitur opus est hanc adiicere conditionem, quod corpora sint aequalia, sed ex altitudinibus, quae sunt ut $\frac{1}{10000},\frac{1}{835},\frac{1}{525}$ et $\frac{1}{410}$, in superficiebus Solis, Iovis, Saturni et Terrae quaecunque corpora delabentia eundem acquirunt celeritatis gradum. [p. 84]

## Scholion 1.

**213.** Intelligitur ex his duplicem cuiusvis potentiae esse effectum in corpora, alterum, quo certum nisum seu conatum corporibus imprimit, alterum, quo ea reipsa movet. Ille in Statica potissimum consideratur et mensurandus est pondere, quod aequalem habet conatum deorsum, poteritque vocari vis potentiae absoluta. Posterior vero effectus mensurari debet acceleratione seu celeritatis incremento, quod corpori dato tempore imprimit : proportionalis igitur est illi conatui divisio per corporis massam (154). Vocatur hic effectus a NEUTONO vis accelerans, et propterea vis potentiae accelerans proportionalis est vi eius absolutae ad massam corporis seu pondus applicatae. Quapropter cum sit $dv=\frac{pds}{A}$ (207) et $\frac{p}{A}$ denotet vim accelerantem, erit *dv* aequale facto ex vi accelerante in elementum spatii percursi. Ita vis gravitatis absoluta est massae corporum, in quae agit, proportionalis; nisum enim eorum deorsum causatur seu pondus, quod massae proportionale esse ostendimus. Vis autem accelerans gravitatis in omnibus corporibus est aequalis, cum omnia aequaliter descendant aequalibusque temporibus aequales adipiscantur celeritates.

## Corollarium 3.

**214.** Vires ergo potentiarum acceleratrices sunt inter se ut vires absolutae, si corpora sint aequalia. Quare cum vis acceleratrix gravitatis sit 1, ut ante posuimus (205), [p. 85] erit vis acceleratrix gravitatis solaris = 24.290; vis acceleratrix gravitatis in superficie Iovis = 2.036; vis acceleratrix gravitatis, quae est in superficie Saturni, = 1.280. Atque gravitatis vim Lunae statuit NEUTONUS = $\frac{1}{3}$.

## Corollarium 4.

**215.** Quare si Propositio 25 ad lapsum corporum in superficie Terrae accommodari debeat, erit $\frac{g}{A} = 1$, quemadmodum fecimus (205). Sin vero ad lapsum corporum in superficie Solis, erit $\frac{g}{A} = 24.290$; sin ad lapsum corpurum in superficie Iovis, erit $\frac{g}{A} = 2.036$; sin ad lapsum corpurum in superficie Saturni, erit $\frac{g}{A} = 1.280$; sin ad lapsum corpurum in superficie Lunae, erit $\frac{g}{A} = \frac{1}{3}$.

## Scholion 2.

**216.** Assumimus hic cum NEUTONO omnia coelestia Terrae nostrae esse similia atque corpora in eorum superficiebus posita vim habere ad eorum centra tendentem, quae similis sit gravitati corporum terrestrium. Ex tradititis igitur NEUTONIANIS apparet corpus, cuius hic pondus sit 1 librae, in superficie Solis positum ponderare 24.290; in superficie Iovis, erit $2.036$ libras; in superficie Saturni $1.280$; et in superficie Lunae tertiam librae partem. [p. 86]

## Scholion 3.

**217.** Quo autem facilius gravitatis similiumque potentiarum in corporibus coelestibus natura perspiciatur, singula corporum elementa aequalia aequaliter a gravitate affici concipienda sunt. Ex quo sequitur, quod iam experienta constat, vires gravitatis, quibus quaeque corpora sollicitantur, esse ipsorum massis seu quantitatibus materiae proportionales. Ante vero iam est demonstratum, si potentiae sint massis corporum, quae sollicitant, proportionales, effectus earum in corporibus movendis esse aequales (136). Quamobrem ex his manifestum est omnia corpora in superficie Terrae aequaliter descendere debere atque etiam pariter in omnibus corporibus coelestibus.

## PROPOSITIO 27.

### PROBLEMA.

**218.** *Puncto A* (Fig. 22) *a potentia uniformi per spatium AP promoto, definire tempus, quo spatium AP absolvitur.*

### SOLUTIO.

Sit ut ante potentia sollicitans $g$, spatium $AP = x$ et altitudo celeritati, quam in P habet, debita v; erit ob $n = \frac{1}{2}, v = \frac{gx}{A}$. Ipsa igitur celeritas in $P$ erit $= \sqrt{v} = \sqrt{\frac{gx}{A}}$. Habebitur ergo tempus, quo elementum $Pp = dx$ percurritur, ut $\frac{dx\sqrt{A}}{\sqrt{gx}}$. Sit tempus, quo spatium $AP$ absolvitur, $= t$ ponaturque $dt = \frac{mdx\sqrt{A}}{\sqrt{gx}}$, oportebit ex unico experimento determinare [p. 87] literam m, quo tempus in data mensura, puta in minutis secundis, reperiatur. Ex illa vero aequatione prodit integrando $t = 2m\sqrt{\frac{Ax}{g}}$, ad quod constantem quantitatem adiicere non est opus, quia posito $x = 0$ etiam $t$ evanescit, prout debet. Determinato igitur $m$ ex experimento habebitur $t = 2m\sqrt{\frac{Ax}{g}}$ minut. sec. Quo autem huiusmodi mensura temporis absoluta resultet, oportet, ut $x$ quoque secundum constantem mensuram exhibeatur : determinabimus igitur semper spatium $x$ in scrupulis, i. e. partibus millesimis pedis Rhenani; fractio enim $\frac{A}{g}$ in numeris absolutis exprimetur, ita ut non opus sit ad eam certam mensuram adhibere. Definita ergo litera $m$, id quod mox faciemus, habibitur plena problematis solutio. Q. E. I.

## Corollarium 1.

**219.** Si $g$ designet gravitatem, erit $\frac{A}{g} = 1$ (205) ; hanc ob rem tempus, quo corpus terrestre ex altitudine $x$ scrup. pedis Rhenani delabitur, erit $2m\sqrt{x}$ minutorum secundorum.

## Corollarium 2.

**220.** Experimentis autem compertum est corpus minuto secundo altitudinem 15525 scrup. pedis Rhenani descendendo absolvere [4.904 metres]. Quam ob rem, si ponatur $x =$ 15625, debet prodire t = 1. Cum autem sit $t = 2m\sqrt{x}$, erit $1 = 2m\sqrt{15625}$, i. e. $= 250m$. Reperitur ergo literae $m = \frac{1}{250}$. [p. 88]

## Corollarium 3.

**221.** Quoniam vero litera *m* in omnbs casibus eundem retinet valorem, erit in casu problematis $t = \frac{1}{125}\sqrt{\frac{Ax}{g}}$ minut. sec. Expresso igitur spatio percurso x in scruplis pedis Rhenani dabit $\frac{1}{125}\sqrt{\frac{Ax}{g}}$ numerum minutorum secundorum, quibus hoc spatium percurritur.

## Corollarium 4.

**222.** Atque ad omnes prorsus casus hic valor ipsius m inventus accommodari potest. Sit enim elementum spatii descripti $ds$, altitudo celeritati, qua hoc percurritur, debita $v$, erit temporis elementum $dt = \frac{mds}{\sqrt{v}}$ et $t = m\int \frac{ds}{\sqrt{v}}$. Ex qua aequatione, si $v$ et $s$ in scrup. pedis Rhenani exprimantur et ponatur $m = \frac{1}{250}$, prodibit tempus in minutis secundis,

$t = \frac{1}{250}\int \frac{ds}{\sqrt{v}}$ min. sec.

## Scholion 1.

**223.** Ex hoc igitur, quod celeritates per radices quadratas altitudinum debitarum denotamus, istud porro assecuti sumus commodum, quod temporum absolutam mensuram semper inveniamus. Usi vero sumus experimento, quo definitur altitudo, ex qua grave minuto secundo delabitur, quam HUGENIUS per experimenta pendulorum invenit 15 ped. Paris. 1 dig. $2\frac{1}{18}$ lineas, i. e. in fractionibus decimalibus 15.0796 ped. Parisinos. Rationem autem pedis Rhenani ad Parisinum adhibemus 1000 ad 1035, ex qua altitudo minuto secundo cadendo percursa provenit 15.625 ped. Rhenanos seu 15625 [p. 89] scrupula eiusdem pedis. Hancque mensuram malumus adhibere quam Parisinam, quia hic numerus est quadratus eoque evitamus frequentes radicis extractiones. Numerus praeterea, per quem $\int \frac{ds}{\sqrt{v}}$ ($s$ et $v$ in scrupulis pedis Rhenani expressis) dividi debet, ut tempus in minutis sec. reperiatur, est 250, qui facillime memoria teneri potest.

## Corollarium 5.

**224.** Cum $\frac{g}{A}$ denotet potentiae vim accelerantem (213), erunt tempora, quibus spatia quaecumque a potentiis uniformibus percurruntur, in ratione subduplicata composita ex directa spatiorum et reciproca virium accelerantium.

## Corollarium 6.

**225.** Posita celeritate, quam punctam $A$ ex altitudine $x$ a potentia $g$ sollicitatum acquirit, $c$, est $c$ ut $\sqrt{\frac{gx}{A}}$ (193). Ergo $ct$ erit ut $x$, quia $t$ est ut $\sqrt{\frac{Ax}{g}}$. Consequenter spatia percursa sunt in ratione composita temporum, quibus describuntur, et celeritatum, quas descensu adipiscuntur, quaecunque sint potentiae sollicitantes, modo sint uniformes.

## Corollarium 7.

**226.** Atque spatia, quae aequalibus temporibus percurruntur, sunt ut vires potentiarum sollicitantium accelerantes. [p. 90]

## Corollarium 8.

**227.** Spatia igitur, per quae corpora aequalibus temporibus in superficiebus Solis, Iovis, Saturni, Luni et Terrae delabuntur, sunt inter se ut 24390, 2036, 1280, 333, 1000. (214).

## Corollarium 9.

**228.** In eadem vis accelerantis hypothesi tempora, quibus spatia quaecunque percurrentur, sunt ut celeritates acquisitatae, atque tam tempora quam celeritates sunt in ratione subduplicata spatiorum descriptorum.

## Scholion 2.

**229.** Hic semper ponimus corpora descendentia descensum a quiete inchoare seu eorum celeritatem in initio descensus esse nullam. In sequentibus vero investigabimus eos motus, qui oriuntur, quando corpora in ipso motus initio iam habent quandam celeritatem. In his autem tempora et spatia ea debent intelligi, quae initium suum habent in ipso celeritatis evanescentis puncto, et aequationes inventae omnes ita sunt comparatae, ut evanescente $c$ vel $v$ simul $x$ et $t$ evanescant.

# PROPOSITIO 28.

### THEOREMA.

**230.** *Corporis per AP* (Fig. 22) *descendentis, ut hactenus posuimus, celeritas in P tanta erit, ut ea aequaliter progrediens eodem tempore, quo per AP est delapsum, spatium duplo maius quam AP absolvere possit.*

[p. 91]

### DEMONSTRATIO.

Manentibus, quae in praecedentibus posuimus, corpore $A$, potentia $g$, spatio descripto $x$, celeritate in $P$ acquisita $\sqrt{v}$ et tempore descensus t, erit $t=\frac{1}{125}\sqrt{\frac{Ax}{g}}$ (221) et $v=\frac{gx}{A}$ (206). Hanc ob rem habetur $\frac{g}{A}=\frac{v}{x}$, ideoque $t=\frac{x}{125\sqrt{v}}=\frac{2x}{125\sqrt{v}}$. At haec expressio dat tempus quoque, quo spatium $2x$ celeritate uniformi $\sqrt{v}$ percurritur, quia $\frac{2x}{\sqrt{v}}$ est divisum per 250, quem numerum invenimus ad tempus in minutis sec. experimendum (220). Consequenter spatium $2x$ eodem tempore celeritate $\sqrt{v}$ percurritur, quo spatium $x$ descensu uniformiter accelerato. Q. E. D.

## Corollarium 1.

**231.** Corpus igitur a potentia uniformi sollicitatum descendens tempore $t$ per spatium $x$ tantam acquirit celeritatem, qua aequbiliter progrediens idem spatium $x$ dimidio tempore $t$ percurrere poterit.

## Corollarium 2.

**232.** Quia in superficie terrae corpora tempore minuti secundi per spatium 15625 scrup. pedis Rhenani delabuntur, tanta erit eorum celeritas hoc lapsu acquisita, qua uniformi motu spatium 31250 scrup. minuto secundo seu 15625 scrup. semi-minuto secundo percurrent. [p. 92]

## Corollarium 3.

**233.** Cum celeritates per radices quadratas ex altitudinibus, ex quibus lapsu acquiruntur, exprimere instituerimus, erit celeritas $\sqrt{15625}$ seu 125 tanta, qua minuto secundo spatium 31250 scrup. absolvi potest.

## Corollarium 4.

**234.** Facile igitur erit spatium assignare, quod celeritate hoc modo expressa $\sqrt{v}$ minuto sec. percurritur. Fiat enim, quia spatia eodem tempore descripta sunt ut celeritates, ut 125 ad $\sqrt{v}$ ita 31250 scrup. ad $250\sqrt{v}$. Quo facto erit $250\sqrt{v}$ spatium in scrupulis expressum, siquidem altitudo v in talibus exhibeatur, quod celeritate $\sqrt{v}$ minuto secundo absolvi potest, motu scilicet aequabili.

## Exemplum 1.

**235.** Delapsum sit corpus ex altitudine 1000 ped., erit in scrupulis $v$ – 1000000, quare ex hoc descensu tantam acquiret celeritatem, qua minuto secundo spatium 250000 scrup., i. e. 250 pedes, absolvere posset.

## Corollarium 5.

**236.** Et reciproce si celeritas per spatium, quod ea minuto secundo percurritur, exprimatur, ut initio fecimus, reduci hinc ea poterit ad receptum nostrum modum per radices ex debitis altitudinibus. Sit enim spatium illud a scrup. et altitudo huic celeritati debita v scrup., erit $250\sqrt{v} = a$ atque $v = \frac{a^2}{62500}$ scrup.

[p. 93]

## Exemplum 2.

**237.** Habeat corpus tantam celeritatem, qua minuto secundo spatium 1000 ped. seu 1000000 scrup. percurrere potest. erit altitudo huic celeritati debita

$$= \frac{1000000000000}{62500} \text{ scrup.}$$

seu 1600 pedes.

## Scholion.

**238.** Perspicitur igitur, quomodo utrumque celeritates exprimendi modum invicem comparari alterumque ad alterum reduci oporteat. Initio enim celeritates per spatia exprimebamus, quae minuto secundo seu alio dato termpore percurruntur. Postmodum vero celeritates per altitudines debitas exhibere magis congruum visum erat. Nunc vero monstratum, quomodo uterque exprimendi modus ad celeritates mensurandas accommodantus sit.

## PROPOSITIO 29.

### PROBLEMA.

**239.** *Potentia existente uniformi secundum rectum BP* (Fig. 23) *trahente habeat corpus in initio B iam celeritatem datam secundum eundem directionem BP; requiritur eius celeritas in quovis puncto P rectae BP.*

### SOLUTIO.


Fig. 23.

Sit ut ante potentia g et corpus *A*. Celeritas vero, quam habet in initio *B*, ponatur debita altitudina *c*. Vocetur *BP* = *x*, et celeritas in *P*, quam quaerimus, debita sit altitudini *v*. Erit ut ante (207) ob potentiam constantem *g* per elementum *Pp* = *dx* sollicitantem $dv=\frac{gdx}{A}$. Integrando [p. 94] igitur sit $v=\frac{gx}{A}+Const.$, quae quantitas constans ex eo est determinanda, quod facto $x = 0$ fiat $v = c$ (per hyp.); erit ergo Const. = c. Consequenter habebimus $v=c+\frac{gx}{A}$ et ipsam celeritatem $\sqrt{v}=\sqrt{c+\frac{gx}{A}}$. Q. E. I.

## Corollarium 1.

**240.** Ponatur, ut fit in case gravitatis ordinariae, $\frac{g}{A}=1$, erit $v = c + x$. Altitudo igitur celeritati in *P* debita est aggregatum altitudinis celeritati initiali in *B* et in spatii percursi.

## Scholion.

**241.** Alia hic occurrit solutio problematis; possumus enim motum per BP cum celeritate initiali $\sqrt{c}$ in B considerare ut partem motus per lineam AP ex quiete, ut ante posuimus factum, in quo corpus, cum ex A in B pervenerit, habeat propositam $\sqrt{c}$. Sit igitur hoc spatium *AB* = *k*, erit $c=\frac{gk}{A}$ (206) et $v=\frac{g(k+x)}{A}=c+\frac{gx}{A}$, ut iam est inventum. Spatium autem AB erit $\frac{Ac}{g}$.

## PROPOSITIO 30.

### PROBLEMA.

**242.** *Iisdem quibus in praecedente propositione positis, determinare tempus, quo spatium BP* (Fig. 23) *percurritur.*

### SOLUTIO.

Sit tempus per spatium *BP* = *t*, erit $dt = \frac{mdx}{\sqrt{(c+\frac{gx}{A})}}$ (218) ; est enim celeritas, qua [p. 95] elementum *Pp* percurritur, $\sqrt{(c+\frac{gx}{A})}$ , ut in praecendent propositione repperimus. Integrando igitur fiet $t = \frac{2mA}{g}\sqrt{(c+\frac{gx}{A})} + \text{Const.}$ Constans haec vero quantitas addenda ex hoc definietur, quod posito *x = 0* fieri debeat *t = 0*. Prodibit igitur $\text{Const.} = \frac{-2mA}{g}\sqrt{c}$. Consequenter habetur $t = \frac{2mA}{g}\sqrt{(c+\frac{gx}{A})} - \frac{2mA}{g}\sqrt{c}$. Expressis c et x in scrupulis ped. Rhen. positoque $m = \frac{1}{250}$ (220) proveniet tempus in minutis secundis expressum. Q. E. I.

## Corollarium.

**243.** Quia in nostris terrestribus regionibus est $\frac{g}{A} = 1$, erit tempus, quo spatium BP cum celeritate in B initiali $\sqrt{c}$ absolvitur,

$$= \frac{1}{250}\sqrt{(v+c)} - \frac{1}{125}\sqrt{c} \text{ minut.secund.},$$

siquidem *c* et *x* in scrupulis pedis Rhenani exprimantur.

## Scholion.

**244.** Simili modo aliam huius problematis affermimus solutionem, quo praecedentis in scholio annexo. Posita enim recta *AB = k*, ex qua corpus *A* delabens in *B* adipiscitur celeritatem altitudini c debitam, erit tempus, quo hoc spatium *AB* absolvitur, $= \frac{1}{125}\sqrt{\frac{Ak}{g}}$ et tempus, quo spatium AP percurritur, erit $\frac{1}{125}\sqrt{\frac{A(k+x)}{g}} - \frac{1}{125}\sqrt{\frac{Ak}{g}}$ . Est vero $k = \frac{Ac}{g}$ (241). Consequenter hoc tempus quaesitum per BP fiet $\frac{A}{125g}\sqrt{(c+\frac{gx}{A})} - \frac{1}{125}\sqrt{c}$ , ut ante invenimus, si quidem ibi loco *m* ponatur $\frac{1}{250}$ . Atque haec sunt, quae de punctorum descensu rectilineo [p. 96] in hypothesi potentiae uniformis exponenda erant. Pergo igitur ad ascensus rectilineos, in quibus celeritatis directo est directe contraria directioni potentiae, quam etiam nunc uniformem seu contantem ponam.

## PROPOSITIO 31.

### PROBLEMA.

**245.** *Potentia uniformi tendente deorsum habeat corpus in B* (Fig. 24) *celeritatem datam sursus directam; requiritur eius celeritas in quovis puncto spatii BA, quod ascensu percurrit.*


Fig. 24.

### SOLUTIO.

Perspicuum est hoc casu corpus in linea recta esse progressurum motu retardato (191), quia eius directio motus directe est contraria potentiae sollicitantis directioni. Sit itaque celeritas in *B* debita altitudini *c*, et ponatur corpus iam pervenisse in *P*. Vocetur altitudo, cui celeritas hoc loco debetur, *v* spatiumque ipsum iam percursum *BP x*. Capiatur *Pp* = *dx*, erit in *p* altitudo celeritati debita *v* + *dv*. Quia autem potentia, quam pono = *g*, motui est contraria, tota in diminuendo motu consumitur. Quam ob rem *dv* aequae poni oportet ipsi $\frac{-gdx}{A}$ denotante A corporis massam. Cum itaque sit $-dv=\frac{gdx}{A}$, erit integrando $C-v=\frac{gx}{A}$. Ad constantem C definiendam ponatur x = 0, quo casu v in c transmutari debebit; eritque ideo C = c. Ex quo prodibit ista aequatio $c-v=\frac{gx}{A}$ seu $v=c-\frac{gx}{A}$, [p. 97] quae determinat celeritatem corporis in quovis puncto spatii ascensu descripti. Q. E. I.

## Corollarium 1.

**246.** Celeritas igitur corporis evanescit, quando sit $c=\frac{gx}{A}$, i. e. quando pervenit ad altitudinem $x=\frac{Ac}{g}$. Sit BA ista altitudo ideoque aequalis $\frac{gc}{A}$, inde $c=\frac{BA.g}{A}$, ex quo intelligitur *BA* eam ipsam esse altitudinem, ex qua corpus A a potentia *g* sollicitatum descendendo acquirit celeritatem *c* (206). Corpus igitur ea celeritate, quam ex data altitudine delapsum est adeptu, sursum progrediens ead eam ipsam pertinget altitudinem, antiquam motum suum amittit.

## Corollarium 2.

**247.** Praeterea corpus ascendus per spatium BA in singulis punctis eas ipsas habet celeritates, quas ibidem haberet, si ex A descendisset. Posita enim AP = y erit celeritas in P descensu ex A nata $=\sqrt{\frac{gy}{A}}$; at est celeritas in eodem loco P ascensu ex B relicta $\sqrt{(c-\frac{gx}{A})}$. Quia autem est $x+y=BA=\frac{Ac}{g}$, patet has expressiones celeritatem esse aequales, nempe $\frac{gy}{A}=c-\frac{gx}{A}$.

## Corollarium 3.

**248.** Congruit igitur motus corporis ascendentis cum motu descendentis, atque utriusque celeritates in iisdem locis, i. e. in iisdem a puncto supremo, quo celeritas evanescit, distantiis, erunt aequales. [p. 98]

## Corollarium 4.

**249.** Ex his perspicitur tempus quoque ascensus per spatium *BA* aequale esse tempori descensus per idem spatium. Quare cum dicto $BA = a$ tempus descensus sit $= \frac{1}{125}\sqrt{\frac{Aa}{g}}$ min. sec. (221), eisem valori aequale esse debebit tempus ascensus per *BA*; seu posito loco a valore $\frac{Ac}{g}$ erit tempus integri ascensus $= \frac{A}{125g}\sqrt{c}$.

## Corollarium 5.

**250.** Simili modo tempus ascensus per quamvis portion BP definietur; manente enim *AP* $= y$ erit tempus sive ascensus sive descensus per $AP = \frac{1}{125}\sqrt{\frac{Ay}{g}}$, quod ablatum ab integro tempore ascensus $= \frac{1}{125}\sqrt{c}$ relinquit tempus per portionem BP. Est vero $y = \frac{Ac}{g} - x$, quare tempus ascensus per BP habebitur $= \frac{A}{125g}\sqrt{c} - \frac{A}{125g}\sqrt{(c - \frac{gx}{A})}$.

## Scholion 1.

**251.** Evidenter vero haec aequalitas ascensuum a priori ostendi potest ex ipsa potentiarum actione. Cum enim in ascensu potentia tantum de celeritate auferat, quantum in descensu ad eam addit, perspicuum est perfectam aequalitatem inter utrumque motum versari debere neque aliud esse discrimen, nisi temporis ordinem, qui cogitatione duntaxat inversus alterum casum in alterum transformat. Similis vero est ratio etiam omnium motuum a potentiis absolutis productorum : iisdem enim celeritatibus per eandem viam [p. 99] reverti poterit corpus, si quidem in reditu impressiones easdem, quas in itu, sed contrarias patitur. Sic planetae in plagas contrarias circa solem eodem modo, quo nunc, in ellipsibus moverentur, si initio motus ipsis fuissent his contrarii. Nam per idem spatii elementum effectus potentiae ad celeritatem immutandam idem est semper atque sit ratione corporis negativus, quando revertitur. Effectus autem, qui ad directionem corporis immutandam impeneditur, utroque casu manet, quo sit, ut corpus in itu et reditu eandem percurrat semitam. Sed haec infra clarius patebunt, ubi de huius modi motibus ex instituto agetur. At vero, si adest resistentia, haec inter ascensus et descensus similitudo evanescit : nam in utroque casu resistentia motum corporis minuit, neque effectus eius in altero alterius est oppositus, quemadmodum usu venit, si potentia sollicitans est absoluta.

## Scholion 2.

**252.** Satis igitur exposito motu rectilineo, qui a potentiis uniformibus oritur, pergendum est ad potentias difformes, quae aliis in locis alias exercent in corpora vires, atque expondendum, quomodo motus corporum, quatenus fiunt in linea recta, ab iis varientur.

Huiusmodi enim difformitati omnes potentiae, quas in mundo observamus, sunt obnoxiae, neque ulla potentia potest assignari, quae corpus, in quocunque loco sit positum, aequaliter afficiat. Sic planetae, quo soli sunt propiores, [p. 100] vehementius ad solum attrahuntur, et quo magis etiam corpus a superficie terrae removetur, minor in eo sit gravitas seu nissus deorsum. Simili haec fere accidunt modo, quo magnetem observamus idem ferri frustum in minore distantia fortius, in maiore vero debilius attrahere. Quamcunque igitur potentiae distantiarum seu positionum corporis teneant rationem, leges eruemus, secundum quas motus corporis sollicitati immutatur. Et primo quidem potentias potestati cuidam distantiarum corporis a puncto fixo proportionales contemplabimur.

## Definitio 16.

**253.** *Centrum virium vocatur punctum illus fixum, ad quod corpora attrahuntur vi, quae pendet a distania ab hoc puncto, seu quae est ut fractio distantiae quaecunque.*

## Corollarium 1.

**254.** Datur igitur distantia ab hoc centro virium, in qua corpus positum tanta vi ad centrum trahitur, quanta foret vis eius gravitatis, si in superficie terrae versaretur.

## Corollarium 2.

**255.** Cognita ergo hac distantia et lege attractionis, scilicet functione distantiae, cui attractio est proportionalis, innotescet ratio, quam habet corporis ubicunque positi conatus accedenti ad centrum virium, ad eiusdem corporis vim gravitatis, si esset in terrae superficie. [p. 101]

## Corollarium 3.

**256.** Cum itaque hoc modo vires utcunque variabiles cum vi gravitatis comparare liceat, quia huius in corpora effectus est cognitus, cuiuscunque etiam vis in corpora effectus poterit determinari.

## Scholion 1.

**257.** Pono hic attractionem centrorum virium similem vi gravitatis, ita ut quoque diversorum corporum in eadem distantia positorum conatus ad centrum sint ut massae ipsorum, ideoque vires accelerationes omnium aequales (212). In his igitur pertractandis massam corporis moti non opus est in computum vocare, sed vim accelerationem duntaxat, quae conatui accedendi ad centrum diviso per massam est proportionalis. Comparabitur ea autem cum vi gravitatis acceleratrice, quam ponimus = 1, atque ad hanc unitatem revocabimus omnes vires potentiarum acceleratrices, quippe quantitates homogeneas.

## Corollarium 4.

**258.** Quando itaque dicemus vires esse distantiis a centro virium seu cuidam functioni earum proportionales, id non de solis nisibus, quos corpora habent ad centrum, sed de viribus acceleratricibus, i. e. de nisibus ad massas corporum applicatis, intelligi debebit.

## Corollarium 5.

**259.** Quoniam igitur potentiae directio, qua corpus urtetur, semper tendit ad centrum virium, [p. 102] perspicuum est, si corpus vel quietscit vel motum habet, cuius directio per centrum virium transit, tum corpus in hac linea recta per centrum virium transeunte perpetuo moveri debere (189).

## Definitio 17.

**260.** *Potentia, quae corpora ad huiusmodi virium centrum urget, vocatur vis centripeta. Haecque, si sit negativa, ut corpora ab hoc centro repellat, vocatur vis centrifuga.*

## Corollarium 1.

**261.** Cum hic de motu sit quaestio, vis centripeta nobis erit vis acceleratrix seu conatus corporis ad centrum tendentis divisus per corporis massam.

## Corollarium 2.

**262.** Conatus igitur seu nisus, quem habet corpus ad centrum virium, exprimitur vi centripeta in corporis massam ducta. Quamobrem erit ad pondus eiusdem corporis, si in superficie terrae esset positum, ut vis centripeta seu vis acceleratrix ad unitatem (257).

## Scholion.

**263.** NEUTONUS, qui voce vis centripetae potissimum utitur, triplici modo eandem mensurari posse animadvertit. Primo quantitate eius absoluta, qua efficaciam ipsius centri virium metitur, sine respectu habito ad corpora attracta ; sic dicit in maiore magnete maiorem inesse vis centripetae quantitatem [p. 103] absolutam, in minore minorem. Et simili modo secundum eius theoriam in sole quantitas absolutas maior est quam in terra. Haec comparatio autem intelligenda est de centris virium similiter, i. e. secundum eandem distantiae functionem, attrahentibus; in dissimilibus enim huiusmodi comparatio locum non habet. Haec ergo quantitas absolutas mensuranda est ex conatu, quem datum corpus in data distantia habet versus virium centrum. Loco huius autem considerationis hic adhibeo distantiam, in quam corpus positum vi aequali ponderi eius nititur ad centrum (254). Secundo habet vis centripetae quantitatem accelerantem, quae apud ipsum eodem accipitur sensu, quo hic ipsa vis centripeta (261); mensuratur enim conatu ad massam applicato. Tertio inducit vis centripetae quantitatem motricem, qua nihil alius denotat nisi ipsum conatum, quem corpora habent ad centrum virium accedendi; quantitas motus enim, quam mensurare soliti sunt celeritate ducta in massam, quaeque dato tempore generatur, proportionalis est ipsi conatui. Posito namque conatu hoc $p$, massa $A$, est celeritas incrementum dato tempusculo ut $\frac{p}{A}$ (154), quod ductum in massam $A$ dat incrementum quantitas motus, quod itaque ipsi $p$ erit proportionale.

## PROPOSITIO 32.

### PROBLEMA.

**264.** *Sit centrum virium C* (Fig. 25), *quod attrahat ad se in ratione quacunque multiplicata distantiarum, ad hocque trahatur corpus in A quiescens; quaeritur eius celeritas in quovis puncto spatii AC.* [p. 104]

### SOLUTIO.

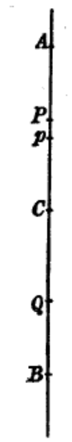
Fig. 25.

Sit $AC = a$, $AP = x$; et celeritas, quam in $P$ habebit, sit debita altitudini $v$. Fiat attractio in ratione $n$-plicata distantiarum, et designet $f$ eam distantium a C, in qua corporis conatus ad C aequalis est ponderi corporis, si esset in terrae superficie positum. Vis igitur acceleratrix, qua corpus in P urgetur ad C, erit ad vim gravitatis, quam pono = 1, ut $CP^n$, i. e. ut $(a-x)^n$ ad $f^n$; quamobrem ea exprimetur per $\frac{(a-x)^n}{f^n}$. Sumpto ergo $Pp = dx$ erit $dv = \frac{(a-x)^n dx}{f^n}$. Est enim dv aequale dx multiplicato per vim acceleratricem (213). Integrata hac aequatione prodibit $v = C - \frac{(a-x)^{n+1}}{(n+1)f^n}$. Ad constantem C definendam ponatur $x = 0$, quo casu fieri debet per hypothesin $v = 0$; fiet ergo $C = \frac{a^{n+1}}{(n+1)f^n}$. Habebitur ergo $v = \frac{a^{n+1}-(a-x)^{n+1}}{(n+1)f^n}$. Seu posito $a - x = CP = y$ erit $v = \frac{a^{n+1}-y^{n+1}}{(n+1)f^n}$. Ex qua aequatione celeritas corporis in quovis loco spatii $AC$ cognoscitur. Q. E. I.

## Corollarium 1.

**265.** Si $n + 1$ est numerus positivus, evanescet $y^{n+1}$ facto $y = 0$. Hoc igitur casu [p. 105] altitudo celeritati, quam corpus in C perveniens habebit, debita erit $= \frac{a^{n+1}}{(n+1)f^n}$. At si $n + 1$ est numerus negativus, fiet $y^{n+1}$ facto $y = 0$, infinite magnum : hoc ergo casu corporis in C pervenientis celeritas erit infinita magna.

## Corollarium 2.

**266.** Sed si $n + 1 = 0$ seu n = - 1, ex inventa aequatione valor ipsius non cognoscitur ob numeratorem et denominatorem evanescentes. Quamobrem hunc casum ex ipsa aequatione differentiali oportebit repetere. Erit autem $dv = \frac{fdx}{a-x}$ ,cuius integralis est $v = C - fl(a-x)$. Debebit autem esse $C = fla$, quocirca $v = fl\frac{a}{a-x} = fl\frac{a}{y}$. Qui est verus valor ipsius $v$, quando habetur $n$ = - 1, i. e. quando vis centripeta est reciproca ut distantia a centro virium.

## Corollarium 3.

**267.** Hoc igitur casu, $n = -1$, cum corpus pervenerit in centrum $C$, celeritas eius erit infinite magna; sit enim $v = fl\ \infty$. Qui infinitorum gradus est infimus et quasi proximus finito ; quantumvis enim parum $n + 1$ cyphram excedat, subito celeritas in $C$ sit finita.

## Corollarium 4.

**268.** Cum autem fuerit $n + 1$ numerus affirmativus, quia tum altitudo celeritati in C debita est $\frac{a^{n+1}}{(n+1)f^n}$, erunt ipsae celeritates plurium corporum [p. 106] ad centrum C delabentium, quas in C habebunt, ut $a^{\frac{n+1}{2}}$, i. e. in ratione $\frac{n+1}{2}$ - plicata distantiarum, ex quibus lapsum inchoaverunt.

## Scholion 1.

**269.** Postquam autem corpus ex $A$ in $C$ pervenit, quo tum promoveatur, non tam facile potest definiri. Videtur quidem, si in expressione inventa $y$ ponatur negativum, proditura esse altitudo celeritati in $Q$ debita; quae si est affirmativa, corpus revera in $Q$ perveniet; sin vero sit negativa, indicio hoc est corpus nunquam ultra $C$ in plagam $CQ$ esse perventurum. Verum hic motum prosequendi modus non semper adhiberi potest; saepe enim ipsi hypothesi, qua ponitur vis attractiva cis et ultra $C$ versus centrum eadem, est contrarius. Namque corpus in $P$ existens, quia trahitur deorsum, cum in $Q$ pervenerit, sursum urgebitur aequali vi, si est $CQ = CP$. Hanc ob rem vis, qua corpus in $Q$ sollicitatur, sit negativa ratione prioris atque idcirco quantitate negativa est exprimenda. Vis igitur in $P$ per $\frac{(a-x)^n}{f^n}$ seu $\frac{y^n}{f^n}$ expressa sui fieri debet negativa, cum $-y$ ponitur loco $y$, id quod nunquam evenit, nisi sit $n$ vel numerus impar vel fractio, cuius numerator et denominator sunt numeri impares. His igitur in casibus prodibit verus valor ipsius $v$, cum corpus in $Q$ pervenerit; in reliquis semper, quia in calculo vis sollicitans corpus in $Q$ cum vero eius valore non congruit, veritati non consentanea [p. 107] litterae $v$ quantitas elicitur. Si enim est $n$ numerus par, vis attrahens in $Q$ facto $y$ negativo aequalis erit vi in $P$ et tendet in eandem plagam deorsum scilicet. Ex quo sit, ut corpus transgressum centrum $C$ in recta $CQ$ in infinitum deberet descendere, id quod etiam calculus declarat. Quod cum pugnet cum hypothesi, perspicuum est his casibus motum corporis, postquam in C pervenit, ex inventa formula definiri non posse. Absurdum autem magis elucet, quando est $n = \frac{1}{2}$ vel alia huiusmodi fractio, quae $y^n$ transmutat in quantitatem imaginariuam posito $-y$ loco $y$; id quod indicaret corpus ultra $C$ progressum non solum non ad $C$ attrahi, sed vim attrahentem etiam fieri imaginariam, quod quid sit nequidem intelligi potest.

## Corollarium 5.

**270.** Si igitur *n* est numerus impar, valor ipsius *v*, qui est $\frac{a^{n+1}-y^{n+1}}{(n+1)f^n}$, non mutator positio $-y$ loco $+y$, quia $n+1$ exponens ipsius *y* evadit numerus par. Ex quo apparet celeritatem corporis in *Q* aequalem fore ei, quam habet in *P*, si quidem est $CQ = CP$. Pari ergo modo corpus in directione *CQ* recedit, qua ante per *AC* accesserat; pertingetque in *B* usque, ita ut sit $CB = AC$, ubi celeritatem suam perdet omnem. Revertetur itaque simili modo ad *C*, et tum rursus in *A* perveniet. Quos motus reciprocos, nisi adest resistentia, perpetuo perficiet. [p. 108]

## Corollarium 6.

**271.** Excipiendus est tamen casus, quo $n = -1$, quanquam $-1$ est numerus impar. Facto enim *y* negativo sit $v = fl\frac{-a}{y}$, quae est quantitas imaginaria. Ex quo perspicitur corpus nunquam ultra *C* esse transgressurum. Aliud ergo iudicium ferendum esse videtur, quando *n* est numerus negativus, etiamsi impar. Huiusmodi enim simile exemplum infra occurret, si $n = -3$ (355).

## Scholion 2.

**272.** Hoc quidem veritati minus videtur consentaneum; vix enim apparet ratio, cur corpus celeritate sua infinite magna, quam in *C* acquisivit, in aliam potius plagam quam in *CB* sit progressurum, praesertim cum huius celeritatis infinitae directio sit secundum hanc plagam. Quicquid autem sit, hic calculo potius quam nostro indicio est fidendum atque statuendum, nos saltum, si fit ex infinitio in finitum, penitus non comprehendere. Eo autem magis in hac sententia confirmamur simili exemplo, quod infra plene explanatum occurret (665), si est $n = -2$; hoc enim casu corporis in *C* pervenientis celeritas quoque est infinita et secundum *CB* directa; nihilo vero minus corpus non ultra C progreditur, sed subito ex C versus A revertitur pariter ac accesserat. Ex quo perspicitur, quoties celeritas in C existat infinita, iudicium de ulteriori corporis motu esse suspendendum. Tam diu autem hoc tantum fiat, quoad ad motus curvilineos [p. 109] perveniamus; ex iisque enim, qui sint rectilinei, evidentius colligetur (762). Neque enim calculus, qui tum instituetur, obnoxius est huic incommodo, ut a hypothesi dissentiat; sed quaqua versus vis centripeta aequalis ponetur non refragante calculo.

## Scholion 3.

**273.** Semper autem, quando celeritas in *C* non est infinita magna, id quod accidit, quoties est $n+1$ numerus affirmativus, motum corporis integrum iudicio nostro poterimus cognoscere, etiamsi calculus sit insufficiens. Si enim celeritas in C est finita habensque directionem secundum *CB*, quam necessario habere debebit, fieri non potest, ut non in *CB* motum continuet. Simili autem modo hunc motum continuans a *C* recedet, quo ante in *AC* accesserat, atque in puncto quocunque Q eandem habebit celeritatem, quam ante in P puncto aeque dissito a C habebat, sicut ex § 251 intelligi potest. Perpetuo igitur motus reciprocos ex A in B et vicissim rediens corpus perficiet.

# PROPOSITIO 33.

## PROBLEMA.

**274.** *Centro C* (Fig. 26) *attrahente in ratione quacunque multiplicata distantiarum habeat corpus in D iam celeritatem datam; requiritur punctum A in recta CD producta, ex quo corpus descensum ad C inchoans, cum in D pervenerit, hanc ipsam acquirat celeritatem.* [p. 110]

## SOLUTIO.

Denotante ut supra n exponentem rationis multiplicatae, in qua sit vis centripeta, et f distantiam, in qua vis centripeta aequalis est vi gravitatas, sit $CD = b$, celeritas in $D$ debita altitudini $h$, et quaesita distantia $CA$ ponatur $= q$. Cum igitur hic $q$ idem denotet quod supra $a$, et $b$ idem quod $y$, et $h$ idem quod ibi $v$, habebitur ista aequatio $h = \frac{q^{n+1} - b^{n+1}}{(n+1) f^n}$. Ex qua fit $q^{n+1} = b^{n+1} + (n+1)hf^n$ atque $q = (b^{n+1} + (n+1)hf^n)^{\frac{1}{n+1}}$. Particulari autem casu, quo $n = -1$, habebitur $h = fl\frac{q}{b}$ hincque $q = e^{\frac{h}{f}} b$, ubi e est numerus, cuius logarithmus est unitas. Q. E. I.


Fig. 26.

## Corollarium 1.

**275.** Si vis centripeta est directe ut distantia, sit $n = 1$ eritque $q = \sqrt{(b^2 + 2fh)}$. Quae quantitas semper est finita, si modo $b$, $f$ et $h$ sunt tales. Simile evenit semper, dummodo $n + 1$ est numerus affirmativus. Atque etiam in casu $n = -1$ distantia $q$ nunquam sit infinita.

## Corollarium 2.

**276.** Si autem $n + 1$ numerus negativus, puta $-m$, ut sit $n = -m - 1$, erit $q = b\sqrt[m]{\frac{f^{m+1}}{f^{m+1} - mb^m h}}$, quae altitudo toties est infinita, quoties est $h = \frac{f^{m+1}}{mb^m}$, et si h est quantitas adhuc maior, [p. 111] sit $q$ negativa, seu potius infinito maior, vel etiam imaginaria. Ex quo intelligitur his casibus ne ex infinita quidem distantia corpus delapsum tantam in D acquirere posse celeritatem.

## Corollarium 3.

**277.** Manente $n+1$ numero negativo $-m$ et distantia puncto $A$ in infinitum erit $h=\frac{f^{m+1}}{mb^m}$. Atque distantia a centro C, in qua corpus, ex spatio infinito delapsum,celeritatem habet $\sqrt{h}$, erit $= f\sqrt[m]{\frac{f}{mh}}$.

## Corollarium 4.

**278.** Si vis centripeta reciproce proportionalis est distantiarum quadratis, erit $m = 1$. Quapropter fit $q=\frac{bf^2}{f^2-bh}$. Quando ergo est $h=\frac{f^2}{b}$, distantia AC, i. e. $q$, fit infinite magna.

## Corollarium 5.

**279.** Si hoc problema cum praecedente coniugatur, facile determinabitur motus corporis, quod ex $D$ descensum ad $C$ inchoat celeritate $\sqrt{h}$. Ex praecedente enim innotescit descensus corporis ex A factus, cuius cum sit descensus ex D celeritate $\sqrt{h}$ inceptus pars, vocetur $CP = y$, et celeritas, quam corpus in $P$ habet, debita sit altitudini $v$, erit $v=\frac{q^{n+1}-y^{n+1}}{(n+1)f^n}$ (264). [p. 112] Est autem $q^{n+1}=b^{n+1}+(n+1)hf^n$. Unde fit

$$v=\frac{b^{n+1}+(n+1)hf^n-y^{n+1}}{(n+1)f^n}=\frac{b^{n+1}-y^{n+1}}{(n+1)f^n}+h.$$

## Corollarium 6.

**280.** Expresso haec ipsius $v$, quando descensus ex $D$ incipitur celeritate altitudini $h$ debita, non differt ab ea, quae prodit, si descensus ex quiete fieret, nisi hoc, quod sit hac ipsa quantitate ipsa quantitate $h$ ubique maior.

## Scholion.

**281.** Quod ad tempora attinet, quibus in quavis hypothesi vis centripetae spatium $AC$ (Fig. 25) eiusque partes absolvuntur, ea facile ex cognitis celeritatibus cognoscentur. Pro generali quidem valore litterae $n$ tempus non potest in terminis finitis exhiberi, quippe tempus per $AP$ invenitur.

$$=\int\frac{dx\sqrt{(n+1)}f^n}{\sqrt{(a^{n+1}-(a-x)^{n+1})}},$$

quae quantitas universaliter neque integrari neque ad cognitarum curvarum quadraturas potest reduci. Attamen in variis ipsius $n$ casibus satis concinne exprimi potest, quamobrem missis generalibus praecipuos casus speciales sequentibus propositionibus sumus complexuri.

## PROPOSITIO 34.

### PROBLEMA.

**282.** *Si fuerit vis centripeta distantiis a centro C* (Fig. 27) *proportionalis et corpus ex A in C usque delabatur,* [p. 113] *determinari oporteat tempus, quo corpus quamque huius spatii partem absolvat.*

### SOLUTIO.

Positis $AC = a$ et distantia a centro $C$, in qua vis centrpeta aequalis est vi gravitatis, $= f$, sit spatii quaevis portio $CP = y$ et celeritas in $P$ debita altitudini $v$. Erit ergo tempus, quo spatium $CP$ absolvitur, $\int \frac{dy}{\sqrt{v}}$; negligo hic fractionem $\frac{1}{250}$, quia haec tempori in minutis secundis cognoscendo inservit et, cum libuerit, potest adiungi. Est vero ex prop. 32 (264) facto $n = 1$ $v = \frac{a^2-y^2}{2f}$, ergo $\sqrt{v} = \frac{\sqrt{(a^2-y^2)}}{\sqrt{2f}}$.

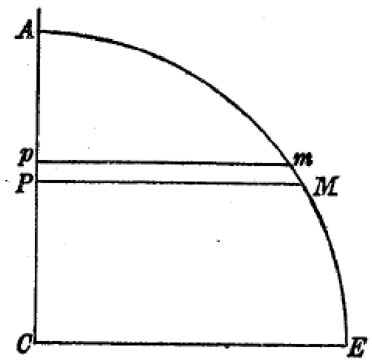

Fig. 27.

Ex quo fit tempus per $PC = \int \frac{dy\sqrt{2f}}{\sqrt{(a^2-y^2)}} = \frac{\sqrt{2f}}{a}\int \frac{ady}{\sqrt{(a^2-y^2)}}$.
Super AC construatur circuli quadrans AME, in eoque ducantur applicatae CE, PM. Quo facto erit, ut constat, arcus $EM = \int \frac{ady}{\sqrt{(a^2-y^2)}}$. Quamobrem tempus per PC fiet = $\frac{EM.\sqrt{2f}}{a}$. Tempus igitur totius descensus per AC erit = $\frac{AME.\sqrt{2f}}{a}$. Hinc erit tempus descensus per $AP = \frac{AM.\sqrt{2f}}{a}$. Ex his igitur tempus descensus per quamvis spatii percursi portionem innotescit, idque in minutes secundis , si hae expressiones per 250 dividantur et longitudo $f$ in partibus millesimis pedis Rhenani exhibeatur. Q. E. I.

## Corollarium 1.

**283.** Denotet 1 : π rationem diametri ad peripheriam, erit 2AME : a = π : 1 et $\frac{AME}{a} = \frac{\pi}{2}$.
Hanc ob rem erit tempus descensus per AC = $\frac{\pi}{2}\sqrt{2f}$ . [p. 114] Id quod non pendet ab altitudine lapsu percursa $a$, sed quantacunque haec sit, eundem valorem retinet. Omnia igitur corpora, quae ad hoc centrum delabuntur, aequalibus temporibus eo pervenient.

## Scholion.

**284.** Sequitur haec temporum aequalitas ex ipsa celeritas expressione $\frac{\sqrt{(a^2-y^2)}}{\sqrt{2f}}$, in qua $a$ et $y$ unam dimensionem habere censenda sunt. Quoties enim hoc evenit, tempora, quibus quaecunque spatia a percurruntur, debebunt inter se esse aequalia. (46).

## Corollarium 2.

**285.** Si praeterea aliud sit huiusmodi centrum virium, sed diversa praeditum efficacia, ita ut distantia, in qua centripeta vis aequalis est gravitati, sit $F$, erunt tempora descensuum ad utrumque centrum inter se ut $\sqrt{f}$ ad $\sqrt{F}$. Sed efficaciae ipsae hoc casu tenent inversam rationem distantiarum $f : F$; sunt enim ut vires, quas haec centra exercent in aequalibus distantiis. Quapropter tempora descensuum ad diversa huiusmodi virium centra sunt in ratione reciproca subduplicata efficaciarum. Quae quidem ratio in omnibus similibus centris virium locum tenet, si spatia percursa sunt inter se aequalia, ut in sequenti docebitur.