A New Approach to
Hilbert's Third Problem

BY TOMMY NEWTON

Diagram 1

Dr. David Benko

DAVID BENKO'S "NEW APPROACH TO HILBERT'S THIRD PROBLEM" HAS BEEN CALLED AN ELEGANT AND ELEMENTARY NEW SOLUTION TO ONE OF THE MOST CHALLENGING PROBLEMS IN MATHEMATICS. SO HOW DID THE WESTERN KENTUCKY UNIVERSITY MATH ASSISTANT PROFESSOR DO IT?
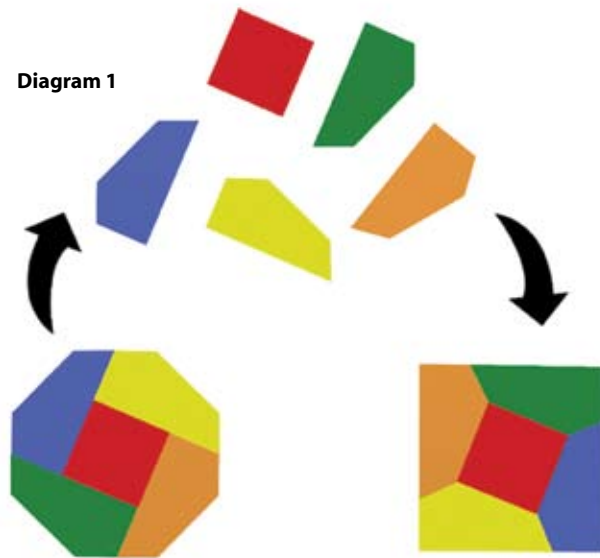
It could be a combination of the following factors:

• Having a father who is an international chess grandmaster and an eight-time winner of the U.S. Open Chess Championship.

• Having a mother who is a mathematics professor.

• As a teenager, solving problems in a Hungarian mathematics and science journal, and then submitting problems for others to solve.

• Winning mathematics and computer programming competitions.

Benko explains, "I started playing chess at age seven, and at age thirteen I started to write computer programs. I believe that chess and computer programming motivated me to discover the beauty of mathematics and to become a mathematician."

Dr. Benko has earned two doctorates in mathematics — in 2001 from the University of South Florida and in 2006 from the University of Szeged in Hungary (where he also received his bachelor's degree and two master's degrees). Benko, whose primary research interest is approximation theory, has been at WKU since 2004.

Dr. Benko found Hilbert's third problem in a book by Hungarian mathematician Miklós Laczkovich. David Hilbert, a German mathematician who is recognized as one of the most influential mathematicians of the nineteenth and early twentieth centuries, presented a list of twenty-three unsolved problems at the International Congress of Mathematicians in Paris in 1900. These problems have become some of the most well-known problems in mathematics.

Benko explains, "Hilbert's third problem is a geometry problem *in space*. First let us get acquainted with the same problem *on a plane*. If two polygons have the same area, it is always possible to decompose one of them into a finite number of polygons that can be rearranged — like a picture puzzle — to form the second polygon. This is called the Bolyai-Gerwien theorem (even though Wallace was the first one who proved it in 1807). For example, suppose we want to decompose an octagon into a square. By the foregoing theorem it is possible to achieve that. In fact, there are many ways to do that." One way is shown in Diagram 1.

Hilbert's third problem is a question concerning polyhedra. Polyhedra are solids with flat sides (e.g., a cube, or a pyramid). The formulation of Hilbert's third problem is simple: Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second? Based on research by Johann Carl Friedrich Gauss in 1844, Hilbert speculated that this is not always possible.

This was confirmed in 1902 by Max Dehn. He proved that the decomposition *cannot* be done in the case of the *regular tetrahedron* and the *cube*. In other words, the regular tetrahedron cannot be cut to smaller polyhedral pieces in such a way that by rearranging those pieces we get the cube! (see Diagram 2) A regular tetrahedron is a body which is formed by four equilateral triangles. (see the yellow body below.) However "his solution was complicated and hard to understand," Dr. Benko says. Over the past century, mathematicians have simplified Dehn's solution, by using formulas called "Dehn invariants."

Diagram 2
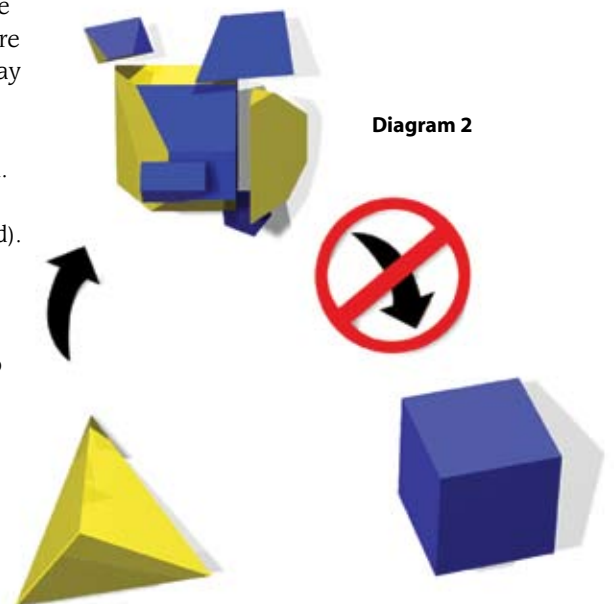
The negative answer to Hilbert's question is quite a surprise since the planar version of the same problem is true. Benko adds, "Proving the Bolyai-Gerwien theorem is much easier than solving Hilbert's third problem. It is easier to prove that something *can be done* than to prove that something *cannot be done*. On the plane we just have to find a strategy for decomposing polygons into each other, and we are done. But how do you prove that the regular tetrahedron *cannot* be decomposed into the cube? Perhaps you try to make a decomposition and it does not work. Then you try another one and it does not work either. Then you try it a hundred different ways and none of them works. Does this prove that it really cannot be done? No. Maybe there is another way which you did not try!"

Benko further says, "I looked at Hilbert's third problem in the book and I wanted to see if I could prove it myself without reading the proof. It took me about a month, but I came up with a solution."

Dr. Benko then asked other professors, and also checked the mathematics literature about his solution. It turns out that his approach is similar to that of a French mathematician, Raoul Bricard, who published a paper in 1896. In his paper Bricard claimed to have proved that the regular tetrahedron cannot be decomposed into the cube. However there was a gap in Bricard's argument which nullified the whole proof. No one, even to this day, has been able to fix this gap.

In his article submitted to the *American Mathematical Monthly*, Dr. Benko writes that his method has been overlooked for a century. "This proof is completely elementary. Since it uses no linear algebra, it could even be presented in a high school math club," he writes. Dr. Benko's new solution to Hilbert's third problem is based on an idea which he calls the method of "integer measures." The solution uses a number theoretic property of the rational numbers.

In explaining how he arrived at the solution, Benko draws several examples. "Sometimes when I have an idea or theorem in mathematics, I create a story to describe it. I have friends who are not mathematicians and I want to explain the problem in a way they can understand," he says. Two examples that proved helpful in his solution are "the pearl problem" and "the ladder problem."

In the pearl problem, Dr. Benko says two people want to make a pearl necklace and are given threads of varying lengths and colors. They have identical sets of threads, meaning that the lengths of threads having the same color are the same in both sets. However the threads are in different positions (see Diagram 3). Each person has to place pearls on the threads in such a way that the number of pearls are the same on threads which have the same color. A possible way is demonstrated on Diagram 4. (Observe that the number of pearls is 4 on both red threads, the number of pearls is 2 on both blue threads, and so on.) "Can we always do this even if we have, say, 100 threads? The answer is: yes we can," Benko explains.
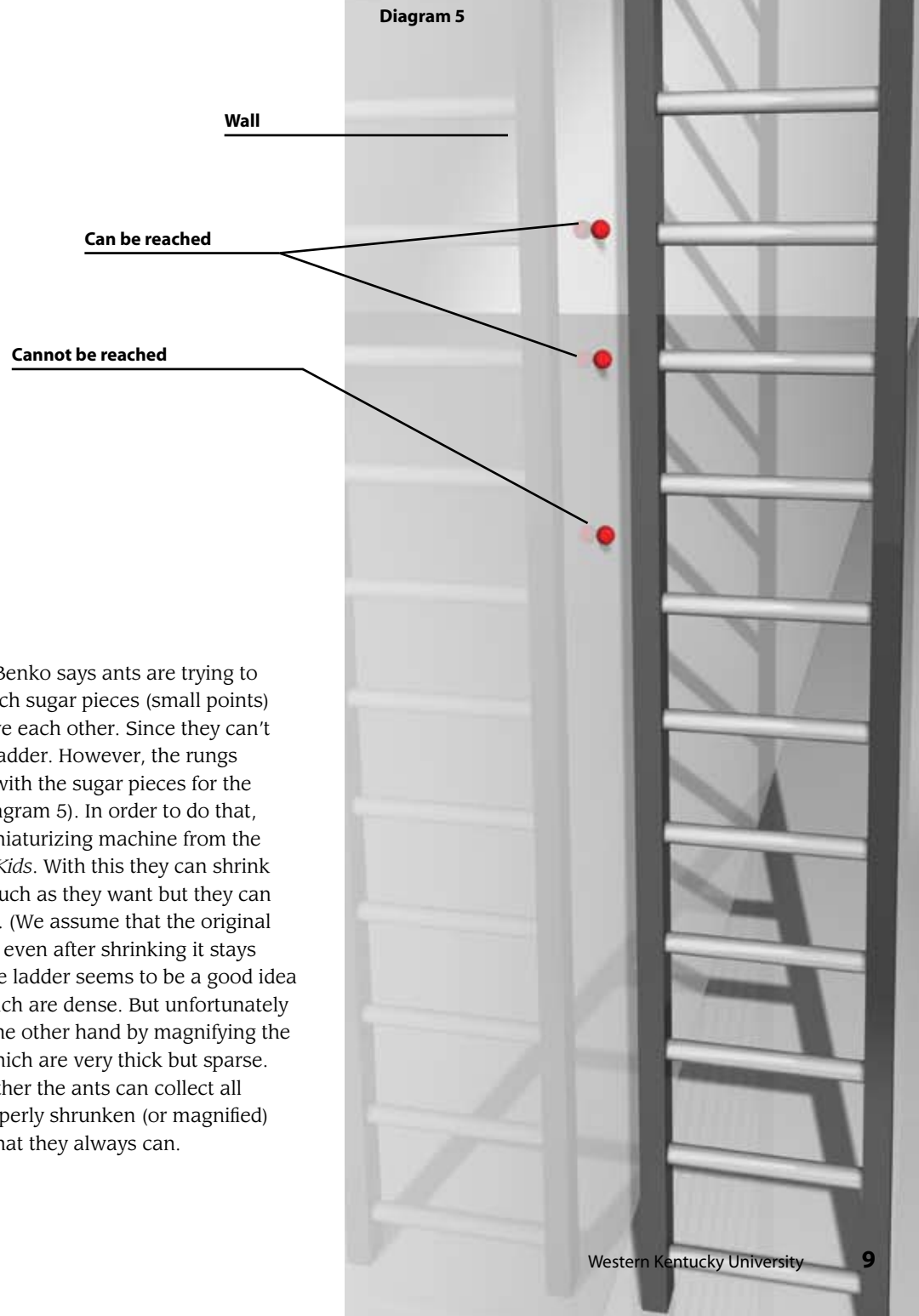
Diagram 3

Diagram 4
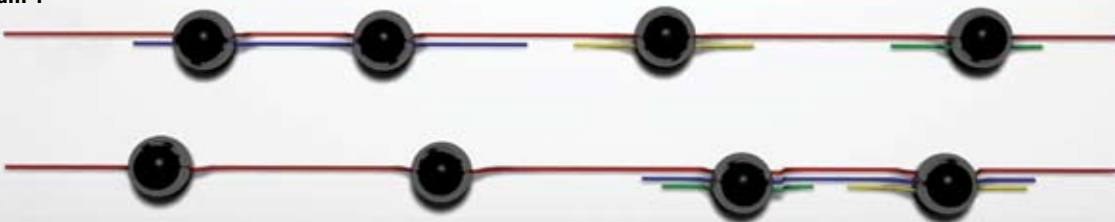
I looked at Hilbert's third problem in the book and I wanted to see if I could prove it myself without reading the proof. It took me about a month, but I came up with a solution.

Diagram 5

Wall

Can be reached

Cannot be reached

In the ladder problem, Benko says ants are trying to climb a slippery wall to reach sugar pieces (small points) spaced along the wall above each other. Since they can't climb the wall, they use a ladder. However, the rungs of the ladder must line up with the sugar pieces for the ants to reach them (see Diagram 5). In order to do that, they borrow Szalinski's miniaturizing machine from the movie *Honey, I Shrunk the Kids*. With this they can shrink or magnify the ladder as much as they want but they can use the machine only once. (We assume that the original ladder is extremely long so even after shrinking it stays long enough.) Shrinking the ladder seems to be a good idea since we can get rungs which are dense. But unfortunately they will be very thin. On the other hand by magnifying the ladder we can get rungs which are very thick but sparse. So it is not clear at all whether the ants can collect all the sugar pieces with a properly shrunken (or magnified) ladder. Yet, the answer is that they always can.
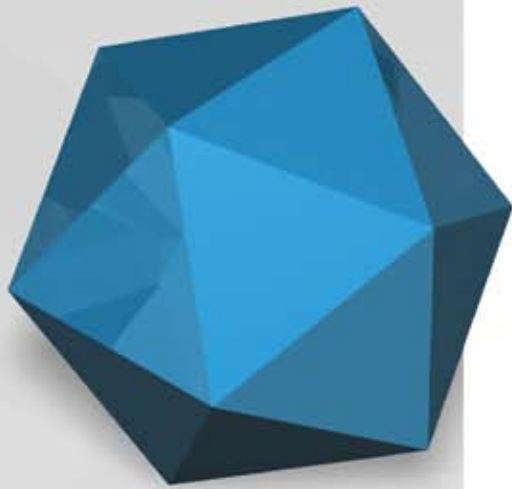
Seemingly, the pearl and ladder problems have nothing to do with Hilbert's third problem. But the fact is that by using the ladder problem, one can find a solution for the pearl problem. And then using the pearl problem, one can solve Hilbert's third problem.

The method of Dr. Benko's solution to Hilbert's third problem can also be applied to get new solutions to some other problems. For example: a rectangle with dimensions m and n can be tiled with finitely many squares if and only if m/n is a rational number. (Tiling means a covering without any gaps.) This theorem was first proved by Dehn in 1903. To demonstrate the statement, let us suppose that m is the square root of two and n is one. Since m/n is now the square root of two — which is not rational — our rectangle *cannot* be tiled with squares, even if we are allowed to use one million small squares.

"We talked about things in geometry which could be done and things which could not be done. Let me tell you two more problems in mathematics which have shocking solutions." Benko says.

In 1925, Alfred Tarski posed his famous circle-squaring problem where he asked if it was possible to cut a circle into finitely many pieces and reassemble them to get a square of equal area. In this problem the pieces can be arbitrary sets — not just polygons. In 1990 Laczkovich gave an affirmative answer to the problem. His decomposition used about 10^{50} different pieces. (The number of atoms in the universe is estimated to be 10^{80}, so 10^{50} is an extremely large number!) Further, these pieces were so strangely shaped — mathematicians call them non-measurable sets — that they could not be cut out with scissors.

An even more puzzling result is the "doubling the ball paradox," discovered by Stefan Banach and Alfred Tarski in 1924. They proved that one can decompose a ball into some pieces, and then reassemble the pieces into *two* balls of the same size! So we can buy a melon of ten pounds, decompose it into some pieces and then reassemble those pieces to get *two melons* of ten pounds each — at least theoretically. Because of this paradox some people reject a fundamental axiom of mathematics called the "axiom of choice" — an axiom which was used in the proof.

"I would say mathematics and real life are not always the same," Benko muses. "If you want to decompose the melon into those pieces, you cannot do it with a knife. But mathematically it can be done."

And as David Benko learned in his youth, that's the beauty of mathematics. ■