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Nonlinear realizations of Lie superalgebras

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ABSTRACT

For any decomposition of a Lie superalgebra \mathcal{G} into a direct sum $\mathcal{G} = \mathcal{H} \oplus \mathcal{E}$ of a subalgebra \mathcal{H} and a subspace \mathcal{E} , without any further restrictions on \mathcal{H} and \mathcal{E} , we construct a nonlinear realization of \mathcal{G} on \mathcal{E} . The result generalizes a theorem by Kantor from Lie algebras to Lie superalgebras. When \mathcal{G} is a differential graded Lie algebra, we show that it gives a construction of an associated L_∞ -algebra.

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1. Introduction

Representations of Lie algebras can be generalized to nonlinear realizations. This means that the elements in the Lie algebra are mapped to operators that are not necessarily linear, but constant, quadratic or of higher order. In many important applications, the operators act on a vector space which can be identified with a subspace of the Lie algebra itself, complementary to a subalgebra. One example is the conformal realization of the Lie algebra $\mathfrak{so}(2, D)$ on a D -dimensional vector space, based on the decomposition of $\mathfrak{so}(2, D)$ as a 3-graded Lie algebra $G = G_{-1} \oplus G_0 \oplus G_1$, where $G_0 = \mathfrak{so}(1, D - 1)$ and $G_{\pm 1}$ are D -dimensional subspaces. In this realization, the subalgebra $\mathfrak{so}(1, D - 1)$ acts linearly, whereas the two D -dimensional subspaces can be considered as consisting of constant and quadratic operators, respectively. In other examples, the linearly realized subalgebra is not the degree-zero subalgebra in a \mathbb{Z} -grading, but defined by being pointwise invariant under an involution.

For any decomposition of a Lie algebra G into a direct sum $G = H \oplus E$ of a subalgebra H and a complementary subspace E , there is formula for a nonlinear realization of G on E given by Kantor [10]. The conformal realization of a semisimple Lie algebra with a 3-grading $G = G_{-1} \oplus G_0 \oplus G_1$ is obtained from this formula in the special case where $H = G_0 \oplus G_1$ and $E = G_{-1}$. The corresponding application to a semisimple Lie algebra with a 5-grading $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$ leads to a *quasiconformal* realization if the subspaces $G_{\pm 2}$ are one-dimensional [7, 17]. In these cases the subspace E is actually also a subalgebra, but this need not be the case in the general formula. There are no restrictions on $[E, E]$, nor on $[H, E]$; the only requirement is $[H, H] \subseteq H$.

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In this paper, we generalize Kantor's formula from Lie algebras to Lie superalgebras. Also in the restriction to Lie algebras, our proof is very different from Kantor's, being purely algebraic, without references to homogeneous spaces for Lie groups.

We expect our generalization to be useful in applications to physics, in particular to models where a Lie superalgebra can be used to organize the field content or to encode the gauge structure. In such cases it might be interesting to investigate whether the Lie superalgebra also can be realized as a symmetry. We also expect the result to be relevant for applications of other related structures, such as Leibniz algebras, differential graded Lie algebras and L_∞ -algebras, for which a renewed interest has appeared recently in the context of gauge theories, see for example Refs. [2–4, 11, 14–16]. In fact, our framework illuminates the relations between these structures. In particular, our main result leads to the construction of an L_∞ -algebra associated to any differential graded Lie algebra.

The paper is organized as follows.

- We start in [Section 2](#) with an arbitrary vector space U_1 . We associate a \mathbb{Z} -graded Lie algebra U to it, from which we in turn construct the Lie algebra S of symmetric operators on U_1 . The \mathbb{Z} -graded Lie algebra U associated to a vector space U_1 was introduced in Ref. [9], but here we use a different recursive approach, following Refs. [17, 18].
- In [Section 3](#), we modify the construction: we then start with a vector space \mathcal{U}_1 that is equipped with a \mathbb{Z}_2 -grading, to which we associate a \mathbb{Z} -graded Lie superalgebra \mathcal{U} [18]. From \mathcal{U} we construct the Lie superalgebra \mathcal{S} of symmetric operators on \mathcal{U}_1 (where the symmetry is now actually a \mathbb{Z}_2 -graded symmetry).
- In [Section 4](#), we furthermore assume that the vector space \mathcal{U}_1 itself is a Lie superalgebra \mathcal{G} . This means that it is equipped with a Lie superbracket, consistent with the \mathbb{Z}_2 -grading already present in [Section 3](#). We show that it extends to a Lie superbracket on \mathcal{S} (different from the one defined in [Section 3](#)).
- In [Section 5](#), we still assume $\mathcal{U}_1 = \mathcal{G}$ but also that this Lie superalgebra decomposes into a direct sum $\mathcal{G} = \mathcal{H} \oplus \mathcal{E}$, where \mathcal{H} is a subalgebra. We show that it extends to a corresponding direct sum $\mathcal{S} = \mathcal{S}_{\mathcal{H}} \oplus \mathcal{S}_{\mathcal{E}}$. As our main result, [Theorem 5.4](#), we show that there is a Lie superalgebra homomorphism from \mathcal{G} to $\mathcal{S}_{\mathcal{E}}$. This result generalizes the main theorem in Ref. [10] from Lie algebras to Lie superalgebras.
- In [Section 6](#), we assume that $\mathcal{U}_1 = \mathcal{G}$ itself has a \mathbb{Z} -grading consistent with the \mathbb{Z}_2 -grading, and is equipped with a differential, turning it into a differential graded Lie algebra. As an example of an application of our main result, we use it in order to construct an L_∞ -algebra from \mathcal{G} , and show that the brackets agree with those given explicitly in Ref. [6].

2. The \mathbb{Z} -graded Lie algebra associated to a vector space

We start with an arbitrary vector space U_1 over some field of characteristic zero, from which we define vector spaces $U_0, U_{-1}, U_{-2}, \dots$ recursively by

$$U_{-p+1} = \text{Hom}(U_1, U_{-p+2}) \quad (2.1)$$

for $p = 1, 2, \dots$. Thus U_{-p+1} consists of all linear maps from U_1 to U_{-p+2} , and in particular $U_0 = \text{End } U_1$.

Let $A_p \in U_{-p+1}$, for some $p = 1, 2, \dots$, and let $x_1, x_2, \dots \in U_1$. Then $A_p(x_1) \in U_{-p+2}$ and if $p \geq 2$, this means that $A_p(x_1)(x_2) = (A_p(x_1))(x_2)$ is an element in U_{-p+3} . Continuing in this way, we finally find that $A(x_1)(x_2) \cdots (x_p)$ is an element in U_1 , which we may also write as $A(x_1, x_2, \dots, x_p)$. Thus we have a vector space isomorphism

$$U_{-p+1} = \text{Hom}(U_1, U_{-p+2}) \simeq \text{Hom}((U_1)^p, U_1) \quad (2.2)$$

and we may consider elements in U_{-p+1} not only as linear maps from U_1 to U_{-p+2} but also as linear maps from $(U_1)^p$ to U_1 , or as p -linear operators on U_1 . We will refer to elements in U_{-p+1}

simply as operators of order p , even for $p=0$, so that the elements in U_1 are considered as operators of order zero.

2.1. The Lie algebra U_{0-}

Next we let U_{0-} be the direct sum of the vector spaces defined in the previous section, $U_{0-} = U_0 \oplus U_{-1} \oplus U_{-2} \oplus \dots$. For any $A_p \in U_{-p+1}$ (where $p = 1, 2, \dots$) and any $x \in U_1$, we write

$$A_p \circ x = A_p(x), \quad x \circ A_p = 0. \quad (2.3)$$

We then define a map

$$\circ : U_{-p+1} \times U_{-q+1} \rightarrow U_{-(p+q-1)+1}, \quad (A_p, B_q) \mapsto A_p \circ B_q \quad (2.4)$$

for any $p, q = 1, 2, \dots$ recursively by

$$(A \circ B)(x) = A \circ B(x) + A(x) \circ B \quad (2.5)$$

and extend it to a bilinear operation on U_{0-} by linearity. For $p = q = 1$ this is the usual composition of (linear) maps,

$$(A_1 \circ B_1)(x) = A_1 \circ B_1(x) + A_1(x) \circ B_1 = A_1(B_1(x)), \quad (2.6)$$

where the last equality follows from (2.3) since $B_1(x)$ and $A_1(x)$ are elements in U_1 . We give two more examples,

$$\begin{aligned} (A_2 \circ B_1)(x_1, x_2) &= (A_2 \circ B_1)(x_1)(x_2) \\ &= (A_2 \circ B_1(x_1) + A_2(x_1) \circ B_1)(x_2) \\ &= (A_2 \circ B_1(x_1))(x_2) + A_2(x_1) \circ B_1(x_2) + A_2(x_1)(x_2) \circ B_1 \\ &= A_2(B_1(x_1))(x_2) + A_2(x_1)(B_1(x_2)) \\ &= A_2(B_1(x_1), x_2) + A_2(x_1, B_1(x_2)), \end{aligned} \quad (2.7)$$

$$\begin{aligned} (B_1 \circ A_2)(x_1, x_2) &= (B_1 \circ A_2)(x_1)(x_2) \\ &= (B_1 \circ A_2(x_1) + B_1(x_1) \circ A_2)(x_2) \\ &= (B_1 \circ A_2(x_1))(x_2) \\ &= B_1 \circ A_2(x_1)(x_2) + B_1(x_2) \circ A_2(x_1) \\ &= B_1 \circ A_2(x_1, x_2) = B_1(A_2(x_1, x_2)), \end{aligned} \quad (2.8)$$

which are easily generalized to

$$\begin{aligned} (A_p \circ B_1)(x_1, x_2, \dots, x_p) &= A_p(B_1(x_1), x_2, \dots, x_p) \\ &\quad + A_p(x_1, B_1(x_2), \dots, x_p) \\ &\quad + \dots + A_p(x_1, x_2, \dots, B_1(x_p)), \end{aligned} \quad (2.9)$$

$$(B_1 \circ A_p)(x_1, x_2, \dots, x_p) = B_1(A_p(x_1, x_2, \dots, x_p)). \quad (2.10)$$

In these examples, the subscripts of the operators indicate their orders (whereas the subscripts on elements x in U_1 are just labels used to distinguish them from each other).

The property $U_{-p+1} \circ U_{-q+1} \subseteq U_{-(p+q-1)+1}$ means that U_{0-} is a \mathbb{Z} -graded algebra with respect to \circ (with vanishing subspaces corresponding to positive integers). The following proposition says that this algebra furthermore is associative.

Proposition 2.1. *The vector space U_{0-} together with the bilinear operation \circ is an associative algebra.*

Proof. We will show that

$$((A_p \circ B_q) \circ C_r)(x) = (A_p \circ (B_q \circ C_r))(x) \quad (2.11)$$

for any triple of operators A_p, B_q, C_r of order $p, q, r \geq 1$, respectively, and any $x \in U_1$. We do this by induction over $p + q + r \geq 3$. When $p + q + r = 3$, we have $p = q = r = 1$, and the assertion follows by (2.6). Suppose now that it holds when $p + q + r = s$ for some $s \geq 3$, and set $p + q + r = s + 1$. We then have (omitting the subscripts)

$$\begin{aligned} (A \circ (B \circ C))(x) &= A \circ (B \circ C)(x) + A(x) \circ (B \circ C) \\ &= A \circ (B \circ C(x)) + A \circ (B(x) \circ C) + A(x) \circ (B \circ C) \\ &= (A \circ B) \circ C(x) + (A \circ B(x)) \circ C + (A(x) \circ B) \circ C \\ &= (A \circ B) \circ C(x) + (A \circ B)(x) \circ C \\ &= ((A \circ B) \circ C)(x), \end{aligned} \quad (2.12)$$

using the induction hypothesis in the third step, and the proposition follows by the principle of induction. \square

Note that the identity (2.11) is not satisfied when $r=0$ and $p, q \neq 0$. Then (omitting the subscripts and setting $C_0 = x$) we instead have the (right) Leibniz identity

$$(A \circ B) \circ x = A \circ (B \circ x) + (A \circ x) \circ B. \quad (2.13)$$

For any $A, B \in U_{0-}$, we now set

$$[[A, B]] = A \circ B - B \circ A \quad (2.14)$$

and we have the following obvious consequence of Proposition 2.1.

Corollary 2.2. *The vector space U_{0-} together with the bracket $[[\cdot, \cdot]]$ is a Lie algebra.*

2.2. Extending U_{0-} to U

Let $U_+ = U_1 \oplus U_2 \oplus \cdots$ be the free Lie algebra generated by the vector space U_1 (with the natural \mathbb{Z}_+ -grading) and set

$$U = U_{0-} \oplus U_+ = \cdots \oplus U_{-1} \oplus U_0 \oplus U_1 \oplus U_2 \oplus \cdots. \quad (2.15)$$

We will use the notation

$$U_{i+} = \bigoplus_{j \geq i} U_j, \quad U_{i-} = \bigoplus_{j \leq i} U_j \quad (2.16)$$

for any $i \in \mathbb{Z}$.

We use the same notation $[[\cdot, \cdot]]$ for the two Lie brackets on U_{0-} and U_+ , respectively, and we will now unify them into one Lie bracket on the whole of U , the direct sum of these two vector spaces. We thus have to define brackets $[[A, u]] = -[[u, A]]$ for any $A \in U_{0-}$ and $u \in U_+$. For $u = x \in U_1$ we set

$$[[A, x]] = A(x). \quad (2.17)$$

If $u \in U_{2+}$, then we may assume that $u = [[v, w]]$ for some $v, w \in U_+$. We then define recursively

$$[[A, [[v, w]]]] = [[[[A, v]], w]] - [[[[A, w]], v]] \quad (2.18)$$

and extend the bracket by linearity to the case when u is a sum of such terms $[[v, w]]$. In order to ensure that the definition is meaningful, we have to show that it respects the Jacobi identity in the sense that

$$[[A, [[[[u, v]], w]]]] = [[A, [[u, [[v, w]]]]]] - [[A, [[v, [[u, w]]]]]] \quad (2.19)$$

for any $A \in U_{0-}$ and $u, v, w \in U_+$. Indeed, we get

$$\begin{aligned}
 \llbracket A, \llbracket \llbracket u, v \rrbracket, w \rrbracket \rrbracket &= \llbracket \llbracket A, \llbracket u, v \rrbracket \rrbracket, w \rrbracket - \llbracket \llbracket A, w \rrbracket, \llbracket u, v \rrbracket \rrbracket \\
 &= \llbracket \llbracket \llbracket A, u \rrbracket, v \rrbracket, w \rrbracket - \llbracket \llbracket \llbracket A, v \rrbracket, u \rrbracket, w \rrbracket - \llbracket \llbracket \llbracket A, w \rrbracket, u \rrbracket, v \rrbracket \rrbracket + \llbracket \llbracket \llbracket A, w \rrbracket, v \rrbracket, u \rrbracket \rrbracket \\
 &= \llbracket \llbracket \llbracket A, u \rrbracket, w \rrbracket, v \rrbracket - \llbracket \llbracket \llbracket A, v \rrbracket, w \rrbracket, u \rrbracket - \llbracket \llbracket \llbracket A, w \rrbracket, u \rrbracket, v \rrbracket \rrbracket + \llbracket \llbracket \llbracket A, w \rrbracket, v \rrbracket, u \rrbracket \rrbracket \\
 &\quad + \llbracket \llbracket A, u \rrbracket, \llbracket v, w \rrbracket \rrbracket - \llbracket \llbracket A, v \rrbracket, \llbracket u, w \rrbracket \rrbracket \\
 &= \llbracket \llbracket A, \llbracket u, w \rrbracket \rrbracket, v \rrbracket - \llbracket \llbracket A, \llbracket v, w \rrbracket \rrbracket, u \rrbracket \\
 &\quad + \llbracket \llbracket A, u, \llbracket v, w \rrbracket \rrbracket - \llbracket \llbracket A, v \rrbracket, \llbracket u, w \rrbracket \rrbracket \\
 &= \llbracket A, \llbracket u, \llbracket v, w \rrbracket \rrbracket - \llbracket A, \llbracket v, \llbracket u, w \rrbracket \rrbracket \rrbracket
 \end{aligned} \tag{2.20}$$

using Jacobi identities like

$$\llbracket \llbracket A, w \rrbracket, \llbracket u, v \rrbracket \rrbracket = \llbracket \llbracket \llbracket A, w \rrbracket, u \rrbracket, v \rrbracket - \llbracket \llbracket \llbracket A, w \rrbracket, v \rrbracket, u \rrbracket. \tag{2.21}$$

Such Jacobi identities follow either (if $\llbracket A, w \rrbracket \in U_+$) by the fact that U_+ is a Lie algebra or (if $\llbracket A, w \rrbracket \in U_{0-}$) by the definition (2.18).

Proposition 2.3. *The vector space $U = U_{0-} \oplus U_+$ together with the bracket $\llbracket \cdot, \cdot \rrbracket$ is a Lie algebra.*

Proof. The Jacobi identities with either all three elements in U_{0-} or all three elements in U_+ are satisfied, by Corollary 2.2 and by the construction of U_+ as a free Lie algebra. Also the Jacobi identities with one element in U_{0-} and two elements in U_+ are satisfied, by the definition (2.18). It only remains to check the Jacobi identities with two elements $A, B \in U_{0-}$ and one element $u \in U_+$. Assuming that u is homogeneous with respect to the \mathbb{Z} -grading, $u \in U_k$, this can be done by induction over $k \geq 1$. For $k = 1$, we have

$$\begin{aligned}
 \llbracket \llbracket A, B \rrbracket, u \rrbracket &= \llbracket A, B \rrbracket(u) = (A \circ B - B \circ A)(u) \\
 &= A \circ B(u) + A(u) \circ B - B \circ A(u) - B(u) \circ A \\
 &= \llbracket A, B(u) \rrbracket - \llbracket B, A(u) \rrbracket = \llbracket A, \llbracket B, u \rrbracket \rrbracket - \llbracket B, \llbracket A, u \rrbracket \rrbracket.
 \end{aligned} \tag{2.22}$$

For $k \geq 2$, we may (as above), assume that $u = \llbracket v, w \rrbracket$, where $v, w \in U_{1+}$. Assuming furthermore (as the induction hypothesis) that

$$\begin{aligned}
 \llbracket \llbracket A, B \rrbracket, v \rrbracket &= \llbracket A, \llbracket B, v \rrbracket \rrbracket - \llbracket B, \llbracket A, v \rrbracket \rrbracket, \\
 \llbracket \llbracket A, B \rrbracket, w \rrbracket &= \llbracket A, \llbracket B, w \rrbracket \rrbracket - \llbracket B, \llbracket A, w \rrbracket \rrbracket,
 \end{aligned} \tag{2.23}$$

we get

$$\begin{aligned}
 \llbracket \llbracket A, B \rrbracket, u \rrbracket &= \llbracket \llbracket A, B \rrbracket, \llbracket v, w \rrbracket \rrbracket \\
 &= \llbracket \llbracket \llbracket A, B \rrbracket, v \rrbracket, w \rrbracket - \llbracket \llbracket \llbracket A, B \rrbracket, w \rrbracket, v \rrbracket \\
 &= \llbracket \llbracket A, \llbracket B, v \rrbracket \rrbracket, w \rrbracket - \llbracket \llbracket B, \llbracket A, v \rrbracket \rrbracket, w \rrbracket - \llbracket \llbracket A, \llbracket B, w \rrbracket \rrbracket, v \rrbracket + \llbracket \llbracket B, \llbracket A, w \rrbracket \rrbracket, v \rrbracket \\
 &= \llbracket A, \llbracket \llbracket B, v \rrbracket, w \rrbracket \rrbracket - \llbracket B, \llbracket \llbracket A, v \rrbracket, w \rrbracket \rrbracket - \llbracket A, \llbracket \llbracket B, w \rrbracket, v \rrbracket \rrbracket + \llbracket \llbracket B, \llbracket A, w \rrbracket, v \rrbracket \rrbracket \\
 &\quad - \llbracket \llbracket B, v \rrbracket, \llbracket A, w \rrbracket \rrbracket + \llbracket \llbracket A, v \rrbracket, \llbracket B, w \rrbracket \rrbracket + \llbracket \llbracket B, w \rrbracket, \llbracket A, v \rrbracket \rrbracket - \llbracket \llbracket A, w \rrbracket, \llbracket B, v \rrbracket \rrbracket \\
 &= \llbracket A, \llbracket B, \llbracket v, w \rrbracket \rrbracket \rrbracket - \llbracket B, \llbracket A, \llbracket v, w \rrbracket \rrbracket \rrbracket \\
 &= \llbracket A, \llbracket B, u \rrbracket \rrbracket - \llbracket B, \llbracket A, u \rrbracket \rrbracket,
 \end{aligned} \tag{2.24}$$

and it follows by induction that also these Jacobi identities are satisfied. \square

2.3. The Lie algebra S of symmetric operators

For $p \geq 0$, let S_{-p+1} be the subspace of U_{-p+1} consisting of elements $A_p \in U_{-p+1}$ such that $\llbracket A_p, u \rrbracket \subseteq U_{2+}$ for all $u \in U_{2+}$, and set

$$S = \bigoplus_{p \geq 0} S_{-p+1}. \quad (2.25)$$

Because of the \mathbb{Z} -grading, if $p \geq 2$ then S_{-p+1} consists of all operators A_p such that $\llbracket A, u \rrbracket = 0$ for all $u \in U_{2+}$, whereas $S_0 = U_0$ and $S_1 = U_1$. Furthermore, $S \oplus U_{2+}$ is the idealizer (or normaliser) of U_{2+} in U , and S can be identified with the quotient space obtained by factoring out U_{2+} from its idealizer in U .

We will refer to elements in S_{-p+1} as *symmetric* operators of order p , and a linear combination of symmetric operators will also be called a symmetric operator, even if it is not homogenous with respect to the \mathbb{Z} -grading. Note that we consider all elements in U_0 as symmetric operators of order one, and even all elements in U_1 as symmetric operators of order zero.

It follows easily by the Jacobi identity that if A is a symmetric operator of order one or higher, then $A \circ x$ is a symmetric operator as well, for any $x \in U_1$.

The operators in U of order two or higher included in S are indeed precisely those that are symmetric in the following sense. If $A_2 \in S_{-1}$ and $x, y \in U_1$ (so that $\llbracket x, y \rrbracket \in U_2$), then

$$\begin{aligned} 0 &= \llbracket A_2, \llbracket x, y \rrbracket \rrbracket = \llbracket \llbracket A_2, x \rrbracket, y \rrbracket - \llbracket \llbracket A_2, y \rrbracket, x \rrbracket \\ &= A_2(x)(y) - A_2(y)(x) = A_2(x, y) - A_2(y, x), \end{aligned} \quad (2.26)$$

so that $A_2(x, y) = A_2(y, x)$. It is straightforward to show that generally, the condition $\llbracket A_p, u \rrbracket = 0$ for all $u \in U_{2+}$ is equivalent to the condition that

$$A_p(x_1, \dots, x_p) = A_p(y_1, \dots, y_p), \quad (2.27)$$

where (y_1, \dots, y_p) is any permutation of (x_1, \dots, x_p) . We write this (as usual) as

$$A_p(x_1, \dots, x_p) = A_p(x_{(1)}, \dots, x_{(p)}), \quad (2.28)$$

where the right hand side denotes $1/p!$ times the sum of $A(y_1, \dots, y_p)$ over all permutations (y_1, \dots, y_p) of (x_1, \dots, x_p) .

For any symmetric operator A_p there is a unique corresponding map $U_1 \rightarrow U_1$ (non-linear if $p \neq 1$) given by $x \mapsto A(x, x, \dots, x)$. In order to characterize a symmetric operator A_p it is thus sufficient to set $x_1 = x_2 = \dots = x_p$ in $A(x_1, \dots, x_p)$.

In particular for symmetric operators, it is convenient to replace the bilinear operation \circ on U_{0-} by another one, which differs from \circ by normalization. We define a bilinear operation \bullet on U_{0-} by

$$A_p \bullet B_q = \frac{p! \, q!}{(p+q-1)!} A_p \circ B_q \quad (2.29)$$

for $A_p \in U_{-p+1}$ and $B_q \in U_{-q+1}$. If A_p and B_q are symmetric operators, we then get

$$(A_p \bullet B_q)(x, x, \dots, x) = p A_p(B_q(x, x, \dots, x), x, \dots, x). \quad (2.30)$$

Since $A_p \circ B_q$ is an operator of order $p+q-1$, the linear map $\phi : U_{0-} \rightarrow U_{0-}$ given by

$$\phi(A_p) = \frac{1}{p!} A_p \quad (2.31)$$

satisfies

$$\phi(A) \bullet \phi(B) = \phi(A \circ B) \quad (2.32)$$

for any two operators A and B and thus the two algebras obtained by equipping the vector space U_{0-} with \circ and \bullet , respectively, are isomorphic to each other.

We now extend the bilinear operation \bullet from U_{0-} to U_{1-} . First we set

$$A_p \bullet x = p(A_p \circ x) = p A_p(x), \quad x \bullet A_p = 0 \quad (2.33)$$

for $A_p \in U_{-p+1}$ (where $p = 1, 2, \dots$) and $x \in U_1$. Thus the definition (2.29) is still valid if we allow one of A_p and B_q to be an operator of order zero, that is, an element in U_1 . For example, we have

$$(A_p \bullet y)(x_1, \dots, x_{p-1}) = pA_p(y, x_1, \dots, x_{p-1}), \quad (2.34)$$

whereas

$$(A_p \circ y)(x_1, \dots, x_{p-1}) = A_p(y, x_1, \dots, x_{p-1}). \quad (2.35)$$

Second, we set

$$x \bullet y = 0 \quad (2.36)$$

for $x, y \in U_1$ in order to close U_{1-} under \bullet . This makes the operation \bullet really different from \circ (not only up to normalization), since we kept $x \circ y$ undefined.

For any $A, B \in S$, we set

$$[[A, B]] = A \bullet B - B \bullet A. \quad (2.37)$$

It follows that the vector space S equipped with this bracket is a Lie algebra isomorphic to the quotient algebra obtained by factoring out U_{2+} from the idealizer of U_{2+} in U . Moreover, if U_1 is n -dimensional, it is straightforward to show that S is isomorphic to the Lie algebra W_n of formal vector fields $\sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$, where f_i are formal power series in n variables x_1, \dots, x_n .

3. Generalization from Lie algebras to Lie superalgebras

We will now repeat the steps in the preceding section in a more general case. Instead of starting with an arbitrary vector space U_1 we now start with an arbitrary \mathbb{Z}_2 -graded vector space \mathcal{U}_1 . Thus \mathcal{U}_1 can be decomposed into a direct sum $\mathcal{U}_1 = \mathcal{U}_1^{(0)} \oplus \mathcal{U}_1^{(1)}$ of two subspaces $\mathcal{U}_1^{(0)}$ and $\mathcal{U}_1^{(1)}$. Like for any \mathbb{Z}_2 -graded vector space, these subspaces (and their elements) are said to be *even* and *odd*, respectively. This leads to a corresponding decomposition

$$\mathcal{U}_{p+1} = \mathcal{U}_{p+1}^{(0)} \oplus \mathcal{U}_{p+1}^{(1)} \quad (3.1)$$

of each vector space \mathcal{U}_{p+1} , by refining (2.1) to

$$\begin{aligned} \mathcal{U}_{p+1}^{(0)} &= \text{Hom}(\mathcal{U}_1^{(0)}, \mathcal{U}_{-p+2}^{(0)}) \oplus \text{Hom}(\mathcal{U}_1^{(1)}, \mathcal{U}_{-p+2}^{(1)}), \\ \mathcal{U}_{p+1}^{(1)} &= \text{Hom}(\mathcal{U}_1^{(0)}, \mathcal{U}_{-p+2}^{(1)}) \oplus \text{Hom}(\mathcal{U}_1^{(1)}, \mathcal{U}_{-p+2}^{(0)}). \end{aligned} \quad (3.2)$$

Now, let $\mathcal{U}_+ = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \dots$ be the free Lie superalgebra generated by \mathcal{U}_1 (with the natural \mathbb{Z} -grading) and set

$$\mathcal{U} = \mathcal{U}_{0-} \oplus \mathcal{U}_+ = \dots \oplus \mathcal{U}_{-1} \oplus \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \dots. \quad (3.3)$$

We thus have a \mathbb{Z}_2 -graded vector space $\mathcal{U} = \mathcal{U}^{(0)} \oplus \mathcal{U}^{(1)}$. If $u \in \mathcal{U}_{0-}^{(i)}$ for $i = 0, 1$, we use the notation $|u| = i$ for the \mathbb{Z}_2 -degree of u .

We can now repeat the steps in the preceding section, carrying over notation and terminology in a straightforward way. The formulas will however differ from those in the preceding section by factors of powers of (-1) , where we (without loss of generality) have to assume that the elements in \mathcal{U} that appear are homogeneous with respect to the \mathbb{Z}_2 -grading.

Thus, we equip \mathcal{U}_{0-} with an associative bilinear operation \circ , from which we define a Lie superbracket $[[\cdot, \cdot]]$ on \mathcal{U}_{0-} . In these definitions, we modify (2.5) to

$$(A \circ B)(x) = A \circ B(x) + (-1)^{|B||x|} A(x) \circ B \quad (3.4)$$

and (2.14) to

$$[[A, B]] = A \circ B - (-1)^{|A||B|} B \circ A. \quad (3.5)$$

When we then unify the brackets on \mathcal{U}_{0-} and \mathcal{U}_+ to one on the whole of \mathcal{U} , we keep the definition $[[A, x]] = A(x)$ in (2.17), but modify (2.18) to

$$[[[A, v], w]] = [[[[A, v], w]]] - (-1)^{|v||w|} [[[[A, w], v]]]. \quad (3.6)$$

We do not give the proofs here, since they differ from those given in the preceding section only by factors of powers of (-1) .

In fact, the modifications made here are actually generalizations, since the Lie superalgebra \mathcal{U} reduces to the original Lie algebra U in the special case where \mathcal{U}_1 has a trivial odd subspace, $\mathcal{U}_1^{(1)} = 0$. Thus, starting with a vector space \mathcal{U}_1 , we can decompose it in different ways into a direct sum of an even and an odd subspace, which lead to different associated \mathbb{Z} -graded Lie superalgebras \mathcal{U} . The decomposition where \mathcal{U}_1 is considered as an even vector space (coinciding with its even subspace) leads to the associated \mathbb{Z} -graded Lie algebra described in the preceding section. But we can also consider it as an odd vector space. Only in this case the \mathbb{Z} -grading of the Lie superalgebra is *consistent*, which means that $\mathcal{U}_i \subseteq \mathcal{U}_{(0)}$ if i is even and $\mathcal{U}_i \subseteq \mathcal{U}_{(1)}$ if i is odd.

The Lie superalgebra \mathcal{S} , constructed from \mathcal{U} in the same way as S is constructed from U , now consists of operators with a \mathbb{Z}_2 -graded symmetry, rather than purely symmetric ones. However, for simplicity we will still refer to them as symmetric operators. Generalizing the notation (2.28), we denote \mathbb{Z}_2 -graded symmetry with angle brackets rather than ordinary parentheses, so that

$$A_p(x_1, \dots, x_p) = A_p(x_{\langle 1}, \dots, x_{p\rangle}) \quad (3.7)$$

if $A_p \in \mathcal{S}$, where the right hand side denotes $1/p!$ times the sum of $(-1)^\varepsilon A(y_1, \dots, y_p)$ over all permutations (y_1, \dots, y_p) of (x_1, \dots, x_p) , where ε is the number of transpositions of two odd elements.

3.1. Leibniz algebras

In the next section we will assume that the \mathbb{Z}_2 -graded vector space \mathcal{U}_1 is a Lie superalgebra. Before that, we will briefly give another example of a case where \mathcal{U}_1 is an algebra. In many such cases, identities for elements in this algebra can be reformulated as identities for elements in the associated \mathbb{Z} -graded Lie superalgebra \mathcal{U} , including the bilinear operation of the algebra as an element in \mathcal{U}_{-1} .

A (left) *Leibniz algebra* is an algebra \mathcal{U}_1 where the bilinear operation \odot satisfies the (left) Leibniz identity

$$x \odot (y \odot z) = (x \odot y) \odot z + y \odot (x \odot z). \quad (3.8)$$

If we now consider \mathcal{U}_1 as a \mathbb{Z}_2 -graded vector space with trivial even subspace and let Θ be the element in \mathcal{U}_{-1} associated to \odot by

$$x \odot y = \Theta(x, y) = \Theta(x)(y) = [[[\Theta, x], y]] \quad (3.9)$$

then the Leibniz identity (3.9) is equivalent to the condition

$$[[\Theta, \Theta]] = 0. \quad (3.10)$$

(Since \mathcal{U}_1 is odd, \mathcal{U}_{-1} is odd as well, and the condition $[[\Theta, \Theta]] = 0$ is not trivially satisfied, but equivalent to $\Theta \circ \Theta = 0$). Indeed, by the Jacobi identity (keeping in mind that Θ, x, y, z are all odd),

$$\begin{aligned} [[\Theta, \Theta]](x, y, z) &= [[[[[[\Theta, \Theta], x], y], z]] \\ &= 2[[[[\Theta, [\Theta, x]], y], z]] \\ &= 2[[[\Theta, [[\Theta, x], y]], z]] + 2[[[[\Theta, y], [\Theta, x]], z]] \\ &= 2[[[\Theta, [[\Theta, x], y]], z]] + 2[[[[\Theta, y], [[\Theta, x], z]], z]] - 2[[[[\Theta, x], [[\Theta, y], z]], z]] \\ &= 2(x \odot y) \odot z + 2y \odot (x \odot z) - 2x \odot (y \odot z). \end{aligned} \quad (3.11)$$

Now let $\langle \Theta \rangle$ be the one-dimensional subspace of \mathcal{U}_{-1} spanned by Θ . Since $[\![\Theta, \Theta]\!] = 0$, the subspace $\langle \Theta \rangle \oplus \mathcal{U}_{0+}$ of \mathcal{U} is a subalgebra. This Lie superalgebra can also be considered as a differential graded Lie algebra \mathcal{U}_{0+} with a differential $[\![\Theta, \cdot]\!]$. Thus any Leibniz algebra gives rise to a differential graded Lie algebra [2, 11, 14]. In Section 6 we will see how in turn any differential graded Lie algebra gives rise to an L_∞ -algebra.

4. The case when \mathcal{U}_1 is a Lie superalgebra \mathcal{G}

We now assume not only that \mathcal{U}_1 is a \mathbb{Z}_2 -graded vector space, but furthermore that \mathcal{U}_1 is a Lie superalgebra \mathcal{G} with a bracket $[\cdot, \cdot]$. We extend the bracket to the whole of \mathcal{U}_{1-} recursively by

$$[A, B] \bullet x = [A, B \bullet x] + (-1)^{|x||B|}[A \bullet x, B]. \quad (4.1)$$

We recall that any operator A_p of order p is defined by its action on \mathcal{U}_1 , and that $A_p \bullet x = pA_p(x)$. If $B = y$ is an element in \mathcal{U}_0 , that is, an operator of order zero, then $y \bullet x = 0$, so that

$$[A, y] \bullet x = [A, y \bullet x] + (-1)^{|x||y|}[A \bullet x, y] = (-1)^{|x||y|}[A \bullet x, y]. \quad (4.2)$$

Proposition 4.1. *The \mathbb{Z}_2 -graded vector space \mathcal{U}_{1-} together with the bracket $[\cdot, \cdot]$ is a Lie superalgebra.*

Proof. We will show that the Jacobi identity

$$[[A, B], C] = [A, [B, C]] + (-1)^{|B||C|}[[A, C], B] \quad (4.3)$$

is satisfied for any triple of operators A, B, C of order p, q, r , respectively, by induction over $p + q + r \geq 0$. When $p + q + r = 0$, we have $p = q = r = 0$ and the Jacobi identity is satisfied since \mathcal{G} is a Lie superalgebra. If we assume that it is satisfied when $p + q + r = s$ for some $s \geq 0$ and set $p + q + r = s + 1$ we then get

$$\begin{aligned} [[A, B], C] \bullet x &= [[A, B], C \bullet x] + (-1)^{|C||x|}[[A, B] \bullet x, C] \\ &= [A, [B, C \bullet x]] + (-1)^{|B|(|C|+|x|)}[[A, C \bullet x], B] \\ &\quad + (-1)^{|C||x|}[[A, B \bullet x], C] + (-1)^{|B||x|+|C||x|}[[A \bullet x, B], C] \\ &= [A, [B, C \bullet x]] + (-1)^{|B|(|C|+|x|)}[[A, C \bullet x], B] \\ &\quad + (-1)^{|C||x|}[A, [B \bullet x, C]] + (-1)^{|B||x|+|C||x|}[A \bullet x, [B, C]] \\ &\quad + (-1)^{|B||C|}[[A, C], B \bullet x] + (-1)^{|B||C|+|B||x|+|C||x|}[[A \bullet x, C], B] \\ &= [A, [B, C] \bullet x] + (-1)^{|B|(|C|+|x|)}[[A, C] \bullet x, B] \\ &\quad + (-1)^{|B||x|+|C||x|}[A \bullet x, [B, C]] \\ &\quad + (-1)^{|B||C|}[[A, C], B \bullet x] \\ &= [A, [B, C]] \bullet x + (-1)^{|B||C|}[[A, C], B] \bullet x \end{aligned} \quad (4.4)$$

using the induction hypothesis in the second and third steps, and the proposition follows by the principle of induction. \square

It is furthermore easy to see that this Lie algebra is \mathbb{Z} -graded, but the \mathbb{Z} -grading is different from the one that is respected by $[\![\cdot, \cdot]\!]$ (on the subspace \mathcal{U}_{0-}). We have

$$[\mathcal{U}_{-p+1}, \mathcal{U}_{-q+1}] \subseteq \mathcal{U}_{-(p+q)+1} \quad (4.5)$$

so the relevant \mathbb{Z} -degree of an operator is just (the negative of) its order.

We will now show that the subspace \mathcal{S} of the Lie superalgebra \mathcal{U} closes under the bracket (4.1) and thus form a subalgebra.

Proposition 4.2. *If $A, B \in \mathcal{S}$, then $[A, B] \in \mathcal{S}$ as well.*

Proof. Since all operators of order zero or one are included in \mathcal{S} , and because of the \mathbb{Z} -grading (4.5), we can assume that both A and B are of order one or higher, so that $[A, B]$ is of order two or higher.

We have to show that $[[[A, B], u]] = 0$ for any $u \in U_{2+}$. We first show this for $u \in U_{2+}$, and in particular when $u = [x, y]$ for $x, y \in U_1$. Thus we have to show that

$$([A, B] \bullet x) \bullet y - (-1)^{|x||y|}([A, B] \bullet y) \bullet x = 0 \quad (4.6)$$

under the assumption that

$$(A \bullet x) \bullet y - (-1)^{|x||y|}(A \bullet y) \bullet x = 0 \quad (4.7)$$

and

$$(B \bullet x) \bullet y - (-1)^{|x||y|}(B \bullet y) \bullet x = 0. \quad (4.8)$$

The first term in (4.6) is equal to

$$\begin{aligned} [A, B \bullet x] \bullet y + (-1)^{|B||x|}[A \bullet x, B] \bullet y &= [A, (B \bullet x) \bullet y] \\ &\quad + (-1)^{|y|(|B|+|x|)}[A \bullet y, B \bullet x] \\ &\quad + (-1)^{|B||x|}[A \bullet x, B \bullet y] \\ &\quad + (-1)^{|B||x|+|B||y|}[(A \bullet x) \bullet y, B]. \end{aligned} \quad (4.9)$$

Now the second and third term on the right hand side cancel the corresponding contributions from the second term in (4.6). Furthermore, the first and fourth term cancel the corresponding contributions from the second term in (4.6) by (4.7) and (4.8). When $u \in U_k$ for $k \geq 3$, we can assume $u = [x, v]$, where $v \in U_{2+}$. If we then assume that $[[[A, B], v]] = 0$ (as induction hypothesis), we get

$$[[[A, B], [x, v]]] = [[[[A, B], x], v]] = [[A, B] \circ x, v], \quad (4.10)$$

which is proportional to

$$[[A, B] \bullet x, v] = [[A, B \bullet x], v] + (-1)^{|B||x|}[[A \bullet x, B], v]. \quad (4.11)$$

Now, since A and B are symmetric, $A \bullet x$ and $B \bullet x$ are symmetric as well, and the proposition can be proven by induction. \square

For $A_p \in \mathcal{S}_{-p+1}$ and $B_q \in \mathcal{S}_{-q+1}$, considering the operator $[A_p, B_q]$ as a linear map $(\mathcal{U}_1)^{p+q} \rightarrow \mathcal{U}_1$, we have

$$[A_p, B_q](x_1, \dots, x_{p+q}) = [A_p(x_1, \dots, x_p), B_q(x_{p+1}, \dots, x_{p+q})]. \quad (4.12)$$

The next proposition says that the identity (4.1) can be generalized in the sense that $x \in U_1$ can be replaced by any $C \in U_{1-}$. We omit the proof since the steps are the same as in Proposition 4.1.

Proposition 4.3. *For any $A, B, C \in U_{1-}$, we have*

$$[A, B] \bullet C = [A, B \bullet C] + (-1)^{|B||C|}[A \bullet C, B]. \quad (4.13)$$

4.1. Multiple brackets involving the identity map

The identity map on \mathcal{U}_1 is an even symmetric operator of order one. We denote it simply by 1, so that $1 \bullet x = 1(x) = x$ and

$$[A, 1] \bullet x = [A, 1 \bullet x] + [A \bullet x, 1] = [A, x] + [A \bullet x, 1]. \quad (4.14)$$

We now generalize this notation and, for any integer $k \geq 1$ write

$$[A, k] = [\cdots [[A, 1], 1], \dots, 1] \quad (4.15)$$

where the identity map 1 appears k times on the right hand side. We will furthermore from now on use multibrackets to denote nested brackets (for any elements in any Lie superalgebra) and write (4.15) as

$$[A, k] = [A, 1, 1, \dots, 1]. \quad (4.16)$$

Note that $[A, i, j] = [A, i + j]$.

Proposition 4.4. *If $A \in \mathcal{S}$, then $[A, k] \in \mathcal{S}$ as well.*

Proof. By induction, using that $[A, k, 1] = [A, k + 1]$, it suffices to show this in the case when $k = 1$, which is a special case of [Proposition 4.2](#). \square

For $A_p \in \mathcal{S}_{-p+1}$, considering $[A_p, q]$ as linear map $\mathcal{U}_1^{p+q} \rightarrow \mathcal{U}_1$, we have

$$[A_p, q](x_1, \dots, x_{p+q}) = [A_p(x_{(x_1, \dots, x_p)}, x_{p+1}, \dots, x_{p+q})]. \quad (4.17)$$

In calculations with multiple brackets involving the identity map, we will need the rules in the next proposition. They are more or less obvious when reformulated in the notation (4.17) and also straightforward to prove rigorously in the more compact notation that we have demonstrated here. However, since the calculations are rather lengthy, and we have already given similar proofs, we omit this one.

Proposition 4.5. *Let A and B be operators and $n \geq 1$ an integer. Then we have*

$$[A, B, n] = \sum_{k=0}^n \binom{n}{k} [A, k, [B, n - k]] \quad (4.18)$$

and

$$\begin{aligned} [A, n] \bullet B &= \sum_{i+j=n-1} [A, i, B, j] + [A \bullet B, n] \\ &= \sum_{i+j=n-1} \binom{n}{j+1} [A, i, [B, j]] + [A \bullet B, n]. \end{aligned} \quad (4.19)$$

In the summations in (4.19), the summation variables i and j take all non-negative integer values (such that $i + j = n - 1$), and we set $[A, 0] = A$ for any operator A . Also in all summations below, the summation variables are allowed to be zero, unless otherwise stated.

5. Main theorem

Suppose that the Lie superalgebra $\mathcal{U}_1 = \mathcal{G}$ decomposes into a direct sum $\mathcal{G} = \mathcal{H} \oplus \mathcal{E}$ of a subalgebra \mathcal{H} and a subspace \mathcal{E} . For any $x \in \mathcal{G}$, we write $x = x_{\mathcal{H}} + x_{\mathcal{E}}$, where $x_{\mathcal{H}} \in \mathcal{H}$ and $x_{\mathcal{E}} \in \mathcal{E}$. Since \mathcal{H} is a subalgebra, we thus have

$$[a_{\mathcal{H}}, b_{\mathcal{H}}] = [a_{\mathcal{H}}, b_{\mathcal{H}}]_{\mathcal{H}} \quad (5.1)$$

and $[a_{\mathcal{H}}, b_{\mathcal{H}}]_{\mathcal{E}} = 0$ for any $a, b \in \mathcal{G}$. Then this decomposition of $\mathcal{U}_1 = \mathcal{G}$ extends to a decomposition of the Lie superalgebra \mathcal{S} into a corresponding direct sum $\mathcal{S} = \mathcal{S}_{\mathcal{H}} \oplus \mathcal{S}_{\mathcal{E}}$, where $\mathcal{S}_{\mathcal{H}}$ is a subalgebra. For any $A \in \mathcal{U}_{1-}$, we define $A_{\mathcal{H}}$ and $A_{\mathcal{E}}$ recursively by

$$A_{\mathcal{H}} \bullet x = (A \bullet x)_{\mathcal{H}}, \quad A_{\mathcal{E}} \bullet x = (A \bullet x)_{\mathcal{E}} \quad (5.2)$$

for any $x \in \mathcal{G}$. It follows immediately that $A = A_{\mathcal{H}} + A_{\mathcal{E}}$, and also that if $A \in \mathcal{S}$, then $A_{\mathcal{H}} \in \mathcal{S}$ and $A_{\mathcal{E}} \in \mathcal{S}$ as well. We let $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{E}}$ be the subspaces spanned by all $A_{\mathcal{H}}$ and $A_{\mathcal{E}}$, respectively, such that $A \in \mathcal{S}$. We then have the following proposition, which can be proven in the same way as [Proposition 4.1](#), by induction over the sum of the orders of A and B .

Proposition 5.1. *For any $A, B \in \mathcal{S}$, we have*

$$[A_{\mathcal{H}}, B_{\mathcal{H}}] = [A_{\mathcal{H}}, B_{\mathcal{H}}]_{\mathcal{H}}. \quad (5.3)$$

Thus the subspace $\mathcal{S}_{\mathcal{H}}$ of the Lie superalgebra \mathcal{S} is a subalgebra.

The next proposition says that the identity (5.2) can be generalized in the sense that $x \in \mathcal{U}_1$ can be replaced by any $B \in \mathcal{S}$. Again, it can be proven in the same way as [Proposition 4.1](#), by induction over the order of B .

Proposition 5.2. *For any $A, B \in \mathcal{S}$ we have*

$$A_{\mathcal{H}} \bullet B = (A \bullet B)_{\mathcal{H}}, \quad A_{\mathcal{E}} \bullet B = (A \bullet B)_{\mathcal{E}} \quad (5.4)$$

We thus obtain the following generalization of [Proposition 4.5](#) by projecting all outermost brackets on \mathcal{E} .

Proposition 5.3. *Let A and B be operators and $n \geq 1$ an integer. Then we have*

$$[A, B, n]_{\mathcal{E}} = \sum_{k=0}^n \binom{n}{k} [A, k, [B, n-k]]_{\mathcal{E}} \quad (5.5)$$

and

$$\begin{aligned} [A_p, n]_{\mathcal{E}} \bullet B_q &= \sum_{i+j=n-1} [A_p, i, B_q, j]_{\mathcal{E}} + [A_p \bullet B_q, n]_{\mathcal{E}} \\ &= \sum_{i+j=n-1} \binom{n}{j+1} [A_p, i, [B_q, j]]_{\mathcal{E}} + [A_p \bullet B_q, n]_{\mathcal{E}}. \end{aligned} \quad (5.6)$$

Proof. This follows directly from [Propositions 4.5](#) and [5.2](#). □

In particular, when $n = 1$ we have the identity

$$[A, 1]_{\mathcal{E}} \bullet B = [A, B]_{\mathcal{E}} + [A \bullet B, 1]_{\mathcal{E}}, \quad (5.7)$$

which we will use below (in the case where A and B are operators of order zero, so that the second term vanishes).

For any $a \in \mathcal{G}$ and any integer $p \geq 0$, we define $a(p) \in \mathcal{S}_{1-p}$ recursively by

$$a(0) = a_{\mathcal{E}} \quad (5.8)$$

and

$$a(p) = \frac{1}{p!} [a, p]_{\mathcal{E}} - \sum_{q+r=p-1} \frac{1}{(r+2)!} [a(q), r+1]_{\mathcal{E}} \quad (5.9)$$

for $p \geq 1$. In particular, we have

$$a(1) = [a, 1]_{\mathcal{E}} - \frac{1}{2}[a_{\mathcal{E}}, 1]_{\mathcal{E}}, \quad (5.10)$$

and thus

$$a(1) \bullet x = [a, x]_{\mathcal{E}} - \frac{1}{2}[a_{\mathcal{E}}, x]_{\mathcal{E}}. \quad (5.11)$$

For example,

$$\begin{aligned} a(2)(x_1, x_2) &= \frac{1}{2!}[a, x_{\langle 1, x_2 \rangle}]_{\mathcal{E}} - \frac{1}{2!}[[a, x_{\langle 1 \rangle}]_{\mathcal{E}}, x_2]_{\mathcal{E}} \\ &\quad - \frac{1}{3!}[a_{\mathcal{E}}, x_{\langle 1, x_2 \rangle}]_{\mathcal{E}} + \frac{1}{2!2!}[[a_{\mathcal{E}}, x_{\langle 1 \rangle}]_{\mathcal{E}}, x_2]_{\mathcal{E}}. \end{aligned} \quad (5.12)$$

For any $a \in \mathcal{G}$ we also define

$$\tilde{a}(p) = \frac{1}{p!}[a, p] - \sum_{q+r=p} \frac{1}{(r+1)!}[a(q), r]. \quad (5.13)$$

This is however not a recursive definition, since it is $a(q)$, not $\tilde{a}(q)$, that appears in the second term, and $r+1 \geq 1$ is replaced by $r \geq 0$. Note also that the bracket is not projected on \mathcal{E} . In fact, $\tilde{a}(p)$ is projected on \mathcal{H} , since

$$\tilde{a}(0) = [a, 0] - [a(0), 0] = a - a(0) = a - a_{\mathcal{E}} = a_{\mathcal{H}} \quad (5.14)$$

and

$$\begin{aligned} \tilde{a}(p) &= \frac{1}{p!}[a, p] - \sum_{q+s=p-1} \frac{1}{(s+2)!}[a(q), s+1] - a(p) \\ &= \frac{1}{p!}[a, p] - \sum_{q+s=p-1} \frac{1}{(s+2)!}[a(q), s+1] \\ &\quad - \frac{1}{p!}[a, p]_{\mathcal{E}} + \sum_{q+r=p-1} \frac{1}{(r+2)!}[a(q), r+1]_{\mathcal{E}} \\ &= \frac{1}{p!}[a, p]_{\mathcal{H}} - \sum_{q+r=p-1} \frac{1}{(r+2)!}[a(q), r+1]_{\mathcal{H}} \end{aligned} \quad (5.15)$$

for $p \geq 1$. It follows that

$$[\tilde{a}(p), \tilde{b}(q)]_{\mathcal{E}} = 0 \quad (5.16)$$

for any a, b and $p, q \geq 0$.

We are now ready to formulate and prove our main theorem.

Theorem 5.4. *The map*

$$\mathcal{G} \rightarrow \mathcal{S}_{\mathcal{E}}, \quad a \mapsto \sum_{p=0}^{\infty} a(p) \quad (5.17)$$

is a Lie superalgebra homomorphism.

Proof. We will show that

$$\llbracket \sum_{p=0}^{\infty} a(p), \sum_{q=0}^{\infty} b(q) \rrbracket = \sum_{r=0}^{\infty} [a, b](r) \quad (5.18)$$

for any $a, b \in \mathcal{G}$. The left hand side is equal to

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \llbracket a(p), b(q) \rrbracket = \sum_{r=0}^{\infty} \sum_{p+q=r+1} \llbracket a(p), b(q) \rrbracket. \quad (5.19)$$

Thus it suffices to show that

$$\sum_{p+q=r+1} \llbracket a(p), b(q) \rrbracket = [a, b](r) \quad (5.20)$$

for $r = 0, 1, 2, \dots$. We will do this by induction. When $r = 0$, the left hand side in (5.20) equals

$$\begin{aligned} \sum_{p+q=1} \llbracket a(p), b(q) \rrbracket &= \llbracket a(0), b(1) \rrbracket + \llbracket a(1), b(0) \rrbracket \\ &= \llbracket a_{\mathcal{E}}, [b, 1]_{\mathcal{E}} \rrbracket - \frac{1}{2} \llbracket a_{\mathcal{E}}, [b_{\mathcal{E}}, 1]_{\mathcal{E}} \rrbracket \\ &\quad + \llbracket [a, 1]_{\mathcal{E}}, b_{\mathcal{E}} \rrbracket - \frac{1}{2} \llbracket [a_{\mathcal{E}}, 1]_{\mathcal{E}}, b_{\mathcal{E}} \rrbracket \\ &= -(-1)^{|a||b|} [b, 1] \bullet a_{\mathcal{E}} + \frac{1}{2} (-1)^{|a||b|} [b_{\mathcal{E}}, 1]_{\mathcal{E}} \bullet a_{\mathcal{E}} \\ &\quad + [a, 1]_{\mathcal{E}} \bullet b_{\mathcal{E}} - \frac{1}{2} [a_{\mathcal{E}}, 1]_{\mathcal{E}} \bullet b_{\mathcal{E}} \\ &= -(-1)^{|a||b|} [b, a_{\mathcal{E}}] + \frac{1}{2} (-1)^{|a||b|} [b_{\mathcal{E}}, a_{\mathcal{E}}]_{\mathcal{E}} \\ &\quad + [a, b_{\mathcal{E}}]_{\mathcal{E}} - \frac{1}{2} [a_{\mathcal{E}}, b_{\mathcal{E}}]_{\mathcal{E}} \\ &= [a, b_{\mathcal{E}}]_{\mathcal{E}} + [a_{\mathcal{E}}, b]_{\mathcal{E}} - [a_{\mathcal{E}}, b_{\mathcal{E}}]_{\mathcal{E}}, \end{aligned} \quad (5.21)$$

where we have used (5.7) and (5.10), whereas the right hand side equals $[a, b]_{\mathcal{E}}$. Thus the right hand side minus the left hand side equals

$$[a, b]_{\mathcal{E}} - [a, b_{\mathcal{E}}]_{\mathcal{E}} - [a_{\mathcal{E}}, b]_{\mathcal{E}} + [a_{\mathcal{E}}, b_{\mathcal{E}}]_{\mathcal{E}} = [a - a_{\mathcal{E}}, b - b_{\mathcal{E}}]_{\mathcal{E}} = [a_{\mathcal{H}}, b_{\mathcal{H}}]_{\mathcal{E}} = 0. \quad (5.22)$$

In the induction step, we need to study

$$\sum_{m+n=k} \llbracket a(m), b(n) \rrbracket = \sum_{m+n=k} \left(a(m) \bullet b(n) - (-1)^{|a||b|} b(n) \bullet a(m) \right) \quad (5.23)$$

for some $k \geq 2$ and show that this the expression equals

$$[a, b](k-1) \quad (5.24)$$

under the assumption that

$$\sum_{m+n=s} \llbracket a(m), b(n) \rrbracket = [a, b](s-1) \quad (5.25)$$

for $s = 1, \dots, k-1$.

We will first study the first term in the summand on the right hand side of (5.23), and a particular part of it. Its counterpart, the corresponding part of the second term in the summand, is then obtained by interchanging a and b , and multiplying with $(-1)^{|a||b|}$. In the summations in (5.26) and (5.27) below where the summation variables add up to $m-1$, the sum should be read as zero if $m=0$.

We have

$$\begin{aligned}
a(m) \bullet b(n) &= \left(\frac{1}{m!} [a, m]_{\mathcal{E}} - \sum_{p+q=m-1} \frac{1}{(q+2)!} [a(p), q+1]_{\mathcal{E}} \right) \bullet b(n) \\
&= \frac{1}{m!} [a, m]_{\mathcal{E}} \bullet b(n) - \sum_{p+q=m-1} \frac{1}{(q+2)!} [a(p), q+1]_{\mathcal{E}} \bullet b(n) \\
&= \frac{1}{m!} [a \bullet b(n), m]_{\mathcal{E}} + \frac{1}{m!} \sum_{p+q=m-1} \binom{m}{q+1} [[a, p], [b(n), q]]_{\mathcal{E}} \\
&\quad - \sum_{p+q=m-1} \frac{1}{(q+2)!} [a(p) \bullet b(n), q+1]_{\mathcal{E}} \\
&\quad - \sum_{p+q=m-1} \frac{1}{(q+2)!} \sum_{r+s=q} \binom{q+1}{s+1} [[a(p), r], [b(n), s]]_{\mathcal{E}} \\
&= \sum_{p+q=m-1} \frac{1}{p!(q+1)!} [[a, p], [b(n), q]]_{\mathcal{E}} \\
&\quad - \sum_{p+q=m-1} \frac{1}{(q+2)!} [a(p) \bullet b(n), q+1]_{\mathcal{E}} \\
&\quad - \sum_{p+r+s=m-1} \frac{1}{(r+s+2)!} \binom{r+s+1}{s+1} [[a(p), r], [b(n), s]]_{\mathcal{E}}.
\end{aligned} \tag{5.26}$$

The contribution to the sum (5.23) from the last term in (5.26), and its counterpart, is

$$\begin{aligned}
&\sum_{m+n=k} \left(- \sum_{p+r+s=m-1} \frac{1}{(r+s+2)!} \binom{r+s+1}{s+1} [[a(p), r], [b(n), s]]_{\mathcal{E}} \right. \\
&\quad \left. + (-1)^{|a||b|} \sum_{q+r+s=n-1} \frac{1}{(r+s+2)!} \binom{r+s+1}{r+1} [[b(q), s], [a(m), r]]_{\mathcal{E}} \right) \\
&= \sum_{p+q+r+s=k-1} \left(- \frac{1}{(r+s+2)!} \binom{r+s+1}{s+1} [[a(p), r], [b(q), s]]_{\mathcal{E}} \right. \\
&\quad \left. + (-1)^{|a||b|} \frac{1}{(r+s+2)!} \binom{r+s+1}{r+1} [[b(q), s], [a(p), r]]_{\mathcal{E}} \right) \\
&= - \sum_{p+q+r+s=k-1} \frac{1}{(r+s+2)!} \left(\binom{r+s+1}{s+1} + \binom{r+s+1}{r+1} \right) [[a(p), r], [b(q), s]]_{\mathcal{E}} \\
&= - \sum_{p+q+r+s=k-1} \frac{1}{(r+s+2)!} \binom{r+s+2}{r+1} [[a(p), r], [b(q), s]]_{\mathcal{E}} \\
&= - \sum_{p+q+r+s=k-1} \frac{1}{(r+1)!} \frac{1}{(s+1)!} [[a(p), r], [b(q), s]]_{\mathcal{E}}.
\end{aligned} \tag{5.27}$$

Taking all terms into account, we get

$$\begin{aligned}
\sum_{m+n=k} \llbracket a(m), b(n) \rrbracket &= \sum_{m+n=k} (a(m) \bullet b(n) - (-1)^{ab} b(n) \bullet a(m)) \\
&= \sum_{q+r+s=k-1} \frac{1}{r!(s+1)!} \llbracket [a, r], [b(q), s] \rrbracket_{\mathcal{E}} \tag{a} \\
&\quad - \sum_{p+q+r=k-1} \frac{1}{(r+2)!} [a(p) \bullet b(q), r+1]_{\mathcal{E}} \\
&\quad - \sum_{p+q+r+s=k-1} \frac{1}{(r+s+2)!} \binom{r+s+1}{s+1} \llbracket [a(p), r], [b(q), s] \rrbracket_{\mathcal{E}} \\
&\quad - (-1)^{ab} \sum_{p+r+s=k-1} \frac{1}{s!(r+1)!} \llbracket [b, s], [a(p), r] \rrbracket_{\mathcal{E}} \\
&\quad + (-1)^{ab} \sum_{p+q+r=k-1} \frac{1}{(r+2)!} [b(q) \bullet a(p), r+1]_{\mathcal{E}} \\
&\quad + (-1)^{ab} \sum_{p+q+r+s=k-1} \frac{1}{(r+s+2)!} \binom{r+s+1}{r+1} \llbracket [b(q), s], [a(p), r] \rrbracket_{\mathcal{E}} \\
&= \sum_{q+r+s=k-1} \frac{1}{r!(s+1)!} \llbracket [a, r], [b(q), s] \rrbracket_{\mathcal{E}} \tag{b} \\
&\quad + \sum_{p+r+s=k-1} \frac{1}{s!(r+1)!} \llbracket [a(p), r], [b, s] \rrbracket_{\mathcal{E}} \tag{5.28} \\
&\quad - \sum_{p+q+r=k-1} \frac{1}{(r+2)!} \llbracket [a(p), b(q)], r+1 \rrbracket_{\mathcal{E}} \\
&\quad - \sum_{p+q+r+s=k-1} \frac{1}{(r+1)!} \frac{1}{(s+1)!} \llbracket [a(p), r], [b(q), s] \rrbracket_{\mathcal{E}} \\
&= - \sum_{p+q=k-1} [\tilde{a}(p), \tilde{b}(q)]_{\mathcal{E}} + \sum_{p+q=k-1} \frac{1}{p! q!} \llbracket [a, p], [b, q] \rrbracket_{\mathcal{E}} \tag{c} \\
&\quad - \sum_{p+q=k-2} \frac{1}{(q+2)!} [a, b](p), q+1]_{\mathcal{E}} \\
&= \frac{1}{(k-1)!} \sum_{p+q=k-1} \binom{k-1}{q} \llbracket [a, p], [b, q] \rrbracket_{\mathcal{E}} \tag{d} \\
&\quad - \sum_{p+q=k-2} \frac{1}{(q+2)!} [a, b](p), q+1]_{\mathcal{E}} \\
&= \frac{1}{(k-1)!} \llbracket [a, b], k-1 \rrbracket_{\mathcal{E}} - \sum_{p+q=k-2} \frac{1}{(q+2)!} [a, b](p), q+1]_{\mathcal{E}} \tag{e} \\
&= [a, b](k-1).
\end{aligned}$$

Here we have used (5.26) in (a). In (b) we have used the definition of $\llbracket \cdot, \cdot \rrbracket$ and (5.27). The second and the fifth term on the right hand side of (a) go into the third term of the right hand side of (b), whereas the third and sixth term of (a) go into the fourth term of the right hand side of (b), by (5.27). In (c) we have used the definition (5.13) of $\tilde{a}(p)$ and $\tilde{b}(q)$, and the induction hypothesis. In (d) we have used (5.16) and in (e) we have used Proposition 5.3. The theorem now follows by the principle of induction. \square

Considering $a(p) = a_p$ as a linear map $\mathcal{G}^p \rightarrow \mathcal{E}$, we have

$$a_p(x_1, \dots, x_p) = \sum_{k=0}^p \sum \frac{1}{m_1!} \frac{(-1)}{(m_2 - m_1 + 1)!} \cdots \frac{(-1)}{(p - m_k + 1)!} \times [\cdots [a, x_{(1, \dots, x_{m_1}]_{\mathcal{E}}, x_{m_1+1}, \dots, x_{m_2}]_{\mathcal{E}}, x_{m_2+1}, \dots, x_{m_k}]_{\mathcal{E}}, x_{m_k+1}, \dots, x_p]_{\mathcal{E}}, \quad (5.29)$$

where the inner sum goes over all k -tuples of integers (m_1, \dots, m_k) such that

$$0 \leq m_1 < m_2 < \cdots < m_k < p. \quad (5.30)$$

If $m_1 = 0$ (and $k > 0$), the factor in the second and third line should be read as

$$\times [\cdots [a_{\mathcal{E}}, x_{(1, \dots, x_{m_2}]_{\mathcal{E}}, x_{m_2+1}, \dots, x_{m_k}]_{\mathcal{E}}, x_{m_k+1}, \dots, x_p]_{\mathcal{E}}. \quad (5.31)$$

If $k = 0$, the inner sum in (5.29) should be read as

$$\frac{1}{p!} [a, x_{(1, \dots, x_p)]_{\mathcal{E}}. \quad (5.32)$$

Here $x_1, \dots, x_p \in \mathcal{G}$, but since \mathcal{E} is a subspace of \mathcal{G} , we can as well assume $x_1, \dots, x_p \in \mathcal{E}$ and consider $a(p) = a_p$ as a linear map $\mathcal{E}^p \rightarrow \mathcal{E}$.

6. Getzler's theorem

As an example of an application, we end this paper by proving a theorem which says that any differential graded Lie algebra (a Lie superalgebra with a consistent \mathbb{Z} -grading and a differential) gives rise to an L_{∞} -algebra (a generalization of a differential graded Lie algebra including also higher brackets [12, 13]). Combined with the result described in Section 3.1 that any Leibniz algebra gives rise to a differential graded Lie algebra, it leads to the conclusion that any Leibniz algebra gives rise to an L_{∞} -algebra [11, 15, 16]. The theorem has already been proven in at least two different ways in the literature. It follows from the results in Ref. [8] by Fiorenza and Manetti, and has been proven more directly in Ref. [6] by Getzler. Here we follow Getzler's formulation of the it, and prove it using our main result, [Theorem 5.4](#).

Suppose that the Lie superalgebra \mathcal{G} has a consistent \mathbb{Z} -grading, $\mathcal{G} = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}^{(i)}$. Then this \mathbb{Z} -grading induces a \mathbb{Z} -grading on each subspace \mathcal{S}_i of \mathcal{S} , and thus a \mathbb{Z} -grading of \mathcal{S} , different from the one that \mathcal{S} comes with by construction,

$$\mathcal{S}_i = \bigoplus_{j \in \mathbb{Z}} \mathcal{S}_i^{(j)}, \quad \mathcal{S}^{(j)} = \bigoplus_{i \in \mathbb{Z}} \mathcal{S}_i^{(j)}. \quad (6.1)$$

The two \mathbb{Z} -gradings form together a $(\mathbb{Z} \times \mathbb{Z})$ -grading,

$$\mathcal{S} = \bigoplus_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{S}_i^{(j)} = \bigoplus_{i \in \mathbb{Z}} \mathcal{S}_i = \bigoplus_{j \in \mathbb{Z}} \mathcal{S}^{(j)}. \quad (6.2)$$

If there is an element $Q \in \mathcal{S}_{0-}^{(-1)}$ such that $\llbracket Q, Q \rrbracket = 0$, then \mathcal{G} together with Q constitutes an L_{∞} -algebra. The element Q can then be decomposed as a sum of elements $Q_p \in \mathcal{S}_{-p+1}$, for $p = 1, 2, \dots$, each of which can be considered as a linear map $\mathcal{G}^p \rightarrow \mathcal{G}$, called a p -bracket. Following Ref. [6], we use curly brackets for the p -brackets, $Q_p(x_1, \dots, x_p) = \{x_1, \dots, x_p\}$. The condition $\llbracket Q, Q \rrbracket = 0$ decomposes into infinitely many identities for these p -brackets, similar to the usual Jacobi identity.

We note that there are different conventions for L_∞ -algebras. The fact that we consider the p -brackets as elements $Q_p \in \mathcal{S}_{-p+1}^{(-1)}$ means that we use the convention where they are graded symmetric and have degree -1 .

Theorem 6.1. [6] *Let $L = \bigoplus_{i \in \mathbb{Z}} L_i$ be a differential graded Lie algebra with differential δ of degree -1 and bracket $[\cdot, \cdot]$. Let D be the linear operator on L which equals δ on L_1 but vanishes on L_i for $i \neq 1$. Then the subspace $\bigoplus_{i \geq 1} L_i$ is an L_∞ -algebra with p -brackets given by*

$$\{x\} = \delta x - Dx \quad (6.3)$$

for $p = 1$ and

$$\{x_1, x_2, \dots, x_p\} = -\frac{1}{(p-1)!} B_{p-1}^- [Dx_{\langle 1}, x_2, \dots, x_p \rangle] \quad (6.4)$$

for $p = 2, 3, \dots$ where B_n^- are the Bernoulli numbers ($B_n^- = -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \dots$ for $n = 1, 2, 3, 4, 5, \dots$).

In Ref. [6] there appears to be a sign error that we have here corrected by inserting a minus sign on the right hand side of (6.4) [15]. The occurrence of Bernoulli numbers in this context was observed in Ref. [1], and it was also shown in Ref. [5] that they similarly show up in extended geometry, encoding the gauge structure of generalized diffeomorphisms.

Proof. Let \mathcal{G}_{-1} be a one-dimensional vector space spanned by an element Θ and set $\mathcal{G}_i = L_i$ for $i = 0, 1, 2, \dots$. Then

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \dots \quad (6.5)$$

is a consistently \mathbb{Z} -graded Lie superalgebra, where the bracket in the subalgebra $\bigoplus_{i \geq 0} L_i$ of L is extended by $[\Theta, \Theta] = 0$ and $[\Theta, x] = \delta x$ for $x \in \mathcal{G}_0, \mathcal{G}_1, \dots$. Furthermore, set $\mathcal{H} = \mathcal{G}_{-1} \oplus \mathcal{G}_0$ and $\mathcal{E} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots$. Then \mathcal{H} is a subalgebra of \mathcal{G} and $\mathcal{G} = \mathcal{H} \oplus \mathcal{E}$. We can thus use Theorem 5.4, which in particular says that the element $Q = \sum_{p=0}^\infty \Theta(p)$ in \mathcal{S} satisfies $[[Q, Q]] = 0$, since $[\Theta, \Theta] = 0$ in \mathcal{G} . Also, it follows by the construction of $\Theta(p)$ and the \mathbb{Z} -grading that $\Theta(p) \in \mathcal{S}^{(-1)}$, since $\Theta \in \mathcal{S}^{(-1)}$ and the identity map $1 \in \mathcal{S}^{(0)}$. If we write $\Theta(p) = Q_p$, it thus follows that \mathcal{E} together with the element $Q = \sum_{p=1}^\infty Q_p$ in \mathcal{S} (note that $Q_0 = \Theta(0) = \Theta_{\mathcal{E}} = 0$) is an L_∞ -algebra, with the p -brackets

$$\{x_1, \dots, x_p\} = Q_p(x_1, \dots, x_p). \quad (6.6)$$

The right hand side here is given by (5.29) with $a = \Theta$, that is

$$\begin{aligned} Q_p(x_1, \dots, x_p) &= \sum_{k=0}^p \sum \frac{1}{m_1!} \frac{(-1)}{(m_2 - m_1 + 1)!} \cdots \frac{(-1)}{(p - m_k + 1)!} \\ &\times [[\cdots [[\Theta, x_{\langle 1}, \dots, x_{m_1}]_{\mathcal{E}}, x_{m_1+1}, \dots, x_{m_2}]_{\mathcal{E}}, x_{m_2+1}, \dots, \\ &\quad \dots, x_{m_k}]_{\mathcal{E}}, x_{m_k+1}, \dots, x_p]_{\mathcal{E}}, \end{aligned} \quad (6.7)$$

where the inner sum goes over all k -tuples of integers (m_1, \dots, m_k) such that

$$0 \leq m_1 < m_2 < \cdots < m_k < p. \quad (6.8)$$

It remains to show that (6.7) equals the expressions on the right hand side of (6.3) and (6.4) when $p = 1$ and $p \geq 2$, respectively. When $p = 1$, we indeed get

$$\Theta(x) = [\Theta, x]_{\mathcal{E}} - \frac{1}{2} [\Theta_{\mathcal{E}}, x]_{\mathcal{E}} = [\Theta, x]_{\mathcal{E}} = (\delta x)_{\mathcal{E}} = \delta x - Dx. \quad (6.9)$$

When $p \geq 2$, all terms in (6.7) with $m_1 = 0$ are zero, since $\Theta_{\mathcal{E}} = 0$. Furthermore, all the subscripts \mathcal{E} but the first one can be removed, since \mathcal{E} is a subalgebra in this case. Also, when $m_1 \geq 2$ even the first subscript \mathcal{E} can be removed. Thus, for $p \geq 2$ we have

$$\begin{aligned}
Q_p(x_1, \dots, x_p) &= \sum_{k=0}^p \sum \frac{1}{1!} \frac{(-1)}{n_1!} \frac{(-1)}{(n_2 - n_1 + 1)!} \cdots \frac{(-1)}{(p - n_k + 1)!} \\
&\quad \times [\cdots [[\Theta, x_{\langle 1 \rangle_{\mathcal{E}}}, x_2, \dots, x_{n_1}], x_{n_1+1}, \dots, x_{n_2}], x_{n_2+1}, \dots, \\
&\quad \dots, x_{n_k}], x_{n_k+1}, \dots, x_p] \\
&+ \sum_{k=0}^p \sum \frac{1}{n_1!} \frac{(-1)}{(n_2 - n_1 + 1)!} \cdots \frac{(-1)}{(p - n_k + 1)!} \\
&\quad \times [\cdots [[\Theta, x_{\langle 1 \rangle}, \dots, x_{n_1}], x_{n_1+1}, \dots, x_{n_2}], x_{n_2+1}, \dots, \\
&\quad \dots, x_{n_k}], x_{n_k+1}, \dots, x_p]
\end{aligned} \tag{6.10}$$

where the inner sums go over all k -tuples of integers (n_1, \dots, n_k) such that

$$1 < n_1 < n_2 < \cdots < n_k < p. \tag{6.11}$$

Since we have removed the subscripts \mathcal{E} , the factor in the second and third line of each term above is actually independent of the choice of n_1, n_2, \dots, n_k , and also of the integer k . Setting

$$C_p = \sum_{k=0}^p \sum \frac{1}{n_1!} \frac{(-1)}{(n_2 - n_1 + 1)!} \cdots \frac{(-1)}{(p - n_k + 1)!}, \tag{6.12}$$

where the term with $k=0$ should be read as $1/p!$, we thus get

$$\begin{aligned}
Q_p(x_1, \dots, x_p) &= -C_p [\Theta, x_{\langle 1 \rangle_{\mathcal{E}}}, x_2, \dots, x_p] \\
&\quad + C_p [\Theta, x_{\langle 1 \rangle}, x_2, \dots, x_p] \\
&= -C_p ([\delta x_{\langle 1 \rangle}, x_2, \dots, x_p] - [Dx_{\langle 1 \rangle}, x_2, \dots, x_p]) \\
&\quad + C_p [\delta x_{\langle 1 \rangle}, x_2, \dots, x_p] \\
&= C_p [Dx_{\langle 1 \rangle}, x_2, \dots, x_p].
\end{aligned} \tag{6.13}$$

Now C_p can also be written

$$C_p = - \sum_{j=1}^p \sum \frac{(-1)}{(m_1 + 1)!} \cdots \frac{(-1)}{(m_j + 1)!} \tag{6.14}$$

where the inner sum goes over all k -tuples of positive integers (m_1, \dots, m_j) such that $m_1 + \cdots + m_j = p - 1$. Written this way, it is easily shown (by induction, using recursion formulas for the Bernoulli numbers) that

$$C_p = - \frac{1}{(p-1)!} B_{p-1}^- \tag{6.15}$$

and we arrive at (6.4). □

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References

- [1] Bering, K. (2007). On non-commutative Batalin-Vilkovisky algebras, strongly homotopy Lie algebras and the Courant bracket. *Commun. Math. Phys.* 274(2):297–341. [hep-th/0603116]. DOI: [10.1007/s00220-007-0278-3](https://doi.org/10.1007/s00220-007-0278-3).
- [2] Bonezzi, B., Hohm, O. (2020). Leibniz gauge theories and infinity structures. *Commun. Math. Phys.* 377(3): 2027–2077. DOI: [10.1007/s00220-020-03785-2](https://doi.org/10.1007/s00220-020-03785-2). [1904.11036].
- [3] Bonezzi, R., Hohm, O. (2021). Duality hierarchies and differential graded Lie algebras. *Commun. Math. Phys.* 382(1):277–315. DOI: [10.1007/s00220-021-03973-8](https://doi.org/10.1007/s00220-021-03973-8). [1910.10399].
- [4] Borsten, L., Kim, H., Saemann, C. EL_∞ -algebras, generalized geometry, and tensor hierarchies. DOI: [10.48550/arXiv.2106.00108](https://doi.org/10.48550/arXiv.2106.00108). [2106.00108]
- [5] Cederwall, M., J. Palmkvist, J. (2019). L_∞ algebras for extended geometry from Borchers superalgebras. *Commun. Math. Phys.* 369:721–760. [1804.04377]. DOI: [10.1007/s00220-019-03451-2](https://doi.org/10.1007/s00220-019-03451-2).
- [6] Getzler, E. *Higher Derived Brackets*. DOI: [10.48550/arXiv.1010.5859](https://doi.org/10.48550/arXiv.1010.5859). [1010.5859].
- [7] Günaydin, M., Koepsell, K., Nicolai, H. (2001). Conformal and quasiconformal realizations of exceptional Lie groups. *Commun. Math. Phys.* 221:57–76. DOI: [10.1007/PL00005574](https://doi.org/10.1007/PL00005574). [hep-th/0008063].
- [8] Fiorenza, D., Manetti, M. (2007). L-infinity structures on mapping cones. *ANT.* 1(3):301–330. [math/0601312]. DOI: [10.2140/ant.2007.1.301](https://doi.org/10.2140/ant.2007.1.301).
- [9] Kantor, I. (1970). Graded Lie algebras. *Trudy Sem. Vect. Tens. Anal.* 15:227–266.
- [10] Kantor, I. (2001). On a vector field formula for the Lie algebra of a homogeneous space. *J. Algebra* 235(2): 766–782. DOI: [10.1006/jabr.2000.8504](https://doi.org/10.1006/jabr.2000.8504).
- [11] Kotov, A., Strobl, T. (2020). The embedding tensor, Leibniz–Loday algebras, and their higher gauge theories. *Commun. Math. Phys.* 376(1):235–258. [1812.08611]. DOI: [10.1007/s00220-019-03569-3](https://doi.org/10.1007/s00220-019-03569-3).
- [12] Lada, T., Markl, M. (1995). Strongly homotopy Lie algebras. *Commun. Algebra* 23(6):2147–2161. [hep-th/9406095]. DOI: [10.1080/00927879508825335](https://doi.org/10.1080/00927879508825335).
- [13] Lada, T., Stasheff, J. (1993). Introduction to SH Lie algebras for physicists. *Int. J. Theor. Phys.* 32(7): 1087–1104. [hep-th/9209099]. DOI: [10.1007/BF00671791](https://doi.org/10.1007/BF00671791).
- [14] Lavau, S. (2019). Tensor hierarchies and Leibniz algebras. *J. Geom. Phys.* 144:147–189. [1708.07068]. DOI: [10.1016/j.geomphys.2019.05.014](https://doi.org/10.1016/j.geomphys.2019.05.014).
- [15] Lavau, S., Palmkvist, J. (2020). Infinity-enhancing of Leibniz algebras. *Lett Math Phys.* 110(11):3121–3152. [1907.05752]. DOI: [10.1007/s11005-020-01324-7](https://doi.org/10.1007/s11005-020-01324-7).
- [16] Lavau, S., Stasheff, J. *From Differential Crossed Modules to Tensor Hierarchies*. DOI: [10.48550/arXiv.2003.07838](https://doi.org/10.48550/arXiv.2003.07838). [2003.07838].
- [17] Palmkvist, J. (2006). A realization of the Lie algebra associated to a Kantor triple system. *J. Math. Phys.* 47(2):023505. [math/0504544]. DOI: [10.1063/1.2168690](https://doi.org/10.1063/1.2168690).
- [18] Palmkvist, J. (2010). Three-algebras, triple systems and 3-graded Lie superalgebras. *J. Phys. A: Math. Theor.* 43(1):015205. [0905.2468]. DOI: [10.1088/1751-8113/43/1/015205](https://doi.org/10.1088/1751-8113/43/1/015205).