the ordinary problem of the calculus of variations. It is used as a point of departure for a brief but illuminating chapter on the functional calculus, which is the real motive of the entire book, as explained above. The ideas of Volterra are first outlined, including the notion of a derivative of a functional expression. Then the concept of a linear functional expression is introduced, and some of the results of Hadamard, Fréchet and others are expounded. An interesting discussion of this part is given by Carathéodory in his review mentioned above.

The third book consists of a discussion of the final conditions of all types, for the general case of a "free" extremum. This discussion is more closely traditional, perhaps, though it is by no means slavish in following the methods of previous writers in detail. The second variation, the conditions of Jacobi, the fundamental Weierstrass theory, the methods of Hilbert and Kneser, are discussed fully; and such other problems as the case of discontinuous solutions (Carathéodory) and the existence of an absolute extremum (Osgood, Hilbert) are given in satisfactory completeness.

In all, the treatment is certainly well planned and well balanced. Due emphasis is given to generalizations, but the simplest forms of the problem predominate, and the treatment is therefore not inordinately complicated. The work of many authors is presented in a thoroughly digested form, and in a manner which is at once comprehensive and comprehensible; the student is given a well-rounded view of the entire subject. The special interests of the author are limited to their proper proportions as compared with the work of others.

Hadamard's treatise has already affected the development of the calculus of variations and that of the functional calculus; its influence on future developments should be profound.

E. R. HEDRICK.

Les Principes de l'Analyse mathématique. Par PIERRE BOUTROUX. Exposé historique et critique. Tome premier. Paris, Hermann, 1914. 8vo. xi + 547 pp.

THIS work is designed for those who desire a comprehensive view of mathematics, for the purpose of becoming acquainted with its intrinsic significance, and its historical evolution. The object of the book is to exhibit the facts of mathematics, rather than the methods of demonstration or discovery, from the most elementary to some of the most complex. It is intended to be serviceable to students of mathematics and equally to students of philosophy who desire to be armed with some knowledge of mathematics.

The fundamental basis of an enterprise of this kind is of course the first thing one wishes to find. This we do not discover explicitly stated in the tome under consideration, but in another place where three chapters of this work are printed under the title: "L'objet et la méthode de l'analyse mathématique"* the purpose becomes more evident, particularly as part of the article cited is from the second tome, not yet published. In a different place† the author states his view rather succinctly, in these terms: "The progress of analysis appears to us, not as a continuous evolution, . . . but as an endless duel in which the human mind, which seeks to subdue rebellious matter, gains more and more the advantage, thanks to its suppleness in continually modifying or enlarging its point of view and in fashioning new weapons." The mind meets a problem or class of problems, and whether it is able to solve them completely or not, it at least constructs an algebra of some kind, with which it is able to reach certain definite conclusions. Thus the unsolvability of algebraic equations of order above the fourth led to the theory of groups, which, though a most happy invention, for the discovery of the logical and algebraical nature of the connections between laws or mathematical functions, is, nevertheless, like the algebra of the sixteenth and seventeenth centuries, of limited power, neither the end nor the center of mathematical analysis. In like manner the theory of functions far from becoming elucidated as the years roll by, has become unfathomable, and we shall never succeed in getting it entirely within the limits of our science. We can set up a stable theory of functions only by restricting (and in an arbitrary way for the most part) our conception of functional correspondence.

We find accordingly that the two books of this tome are called respectively, Statement of the facts, and Construction. Under the first heading one finds: numbers, magnitudes, figures, and combinatory calculus. Under the second are grouped algebraic calculus, calculus of functions, and alge-

* *Revue de Métaphysique et de Morale*, May, 1913.
† Same, January, 1913.

braic geometry. We are told that the next tome will contain analytic geometry, imaginary quantities, series, logic of mathematics, and infinitesimal calculus. In the reference* given we find the reason for these divisions, as follows. The scientist was at first content to gather facts. He looked around him and perceived the harmonious properties of the world of numbers and the world of geometric figures, and also of measurable magnitudes, the synthesis of quantity and figure, the union of arithmetic and geometry. But with the diffusion of algebra he ceased to be contemplative, and became constructive. From simple pieces he built more and more elaborate designs until he became so charmed with his creations that he left to the physicist entirely the application and interpretation of his theories. In the end his principal aim ceased to be the acquisition of new facts but became the perfecting of methods and processes. The ideal of this phase of mathematics is the algebraico-logical synthesis, by which one would start from a few premises and deduce with no further reference to the world of facts the whole of mathematics. In this attempt however the mathematician is turned back upon himself, and his work begins to consist in the more minute analysis of his notions and the discernment of subtilities that previously had escaped his attention. Thus we pass from the algebraico-logical synthesis to analysis in the proper sense of the term. For example, we no longer look upon an ellipse as made up of an infinity of infinitesimal particles called points. An ellipse is a law, the law of a locus. This law is pregnant with the properties of the ellipse and a sufficient analysis of it will bring out all the characteristic properties of the ellipse. From this point of view we find the tangent and the curvature, or the area and the length. This is what we find in the infinitesimal calculus when we leave out the infinity. But the object of the analysis is ever new construction.

"Having mined the simple notions like those of algebraic relations or the classical geometric figures, the mathematician attacks the fundamental principles upon which repose the algebraico-logical construction.

"He seeks for example, how one may present and particularize the axioms of geometry in order to simplify so far as possible those he is forced to admit without demonstration, and to reduce their number to a minimum.

* *Revue de Métaphysique et de Morale*, May, 1913.

"He analyzes the notion of continuity in order to discover what is essential in it for his purposes, and what hypotheses it is necessary but sufficient to make, regarding an aggregate that is discontinuous, whether of points or of numbers, in order to be able to reason about this aggregate as he does about the continuum.

"Then penetrating to the heart of mathematics, the analyst attacks the general notion of function (law of correspondence between magnitudes) such as he conceives it to be a priori. He searches out how he ought to determine this notion, and in what measure it must be limited, in order to subject the function to his methods of calculation and to represent it quantitatively by algebraic expressions. He sees thus the possibility of generalizing and extending considerably the algebraic technique of his predecessors."

The first tome of this interesting work is pleasant reading. It is quite elementary on the whole and the references to parts of the next tome lead one to suppose that some of the more philosophical notions will appear there. There are some inaccuracies noticeable. For example we remark the misuse of the term *associativity* with regard to involution, page 11. Division by zero, page 48, is left a little hazy as to whether the sign ∞ is intended to be included as a number or not. The term *rationnel* in the last note on page 81, does not seem to be what the author intends to say. On page 137 the series for π as well as the decimal, are inaccurate. On page 138 at the bottom, the 2 in front of each radical should be omitted. On page 139 the continued fraction represents $4/\pi$. The figure on page 249 is badly drawn. On page 331 the formula for the second root, x'', needs a 4. However these are slips, and we need not dwell on them. The fundamentals that come out here and there are the vital part. Such as, page 44, "The world of numbers is for us essentially a class of abstract elements, about which we suppose nothing, save that they are subject to certain definite operations." On page 145, "Every relative number is defined by an ordinary number (rational or irrational, called the absolute value of the relative number) and by a sign $+$ or $-$." On page 275, "Algebra is a rule, or more exactly, the totality of rules, according to which one carries out certain transformations or combinations called algebraic; in general these combinations are those defined by the fundamental operations of arithmetic, but nothing prevents our imagining others." We have however probably said sufficient to show that there is an interest in the book for everyone. JAMES BYRNIE SHAW.

Principes de la Théorie des Fonctions entières d'Ordre infini. By OTTO BLUMENTHAL. Paris, Gauthier-Villars, 1910. vi + 147 pp.

Entire functions of finite order as well as certain classes of entire functions of infinite order have been treated by a number of mathematicians in recent years. In this volume of the Borel series of monographs on the theory of functions, Blumenthal considers the *general* entire function of infinite order, so that the book forms a natural and satisfactory sequel to Borel's own Leçons sur les Fonctions entières of the same series. The interest of the results obtained lies in their generality rather than in their applicability to special entire functions not before treated. These results are in large measure original with Blumenthal although a similar range of ideas had been earlier developed by Kraft (Dissertation, Göttingen, 1903).

It is by the aid of the notion of function-type (fonction-type) that Blumenthal is enabled to overcome the inherent difficulties of the problems which arise. Let $\nu(x)$ be a function which increases to $+\infty$ with x, and let $\mu(x) \geqq \nu(x)$ be a like function whose rate of increase is governed by an inequality

$$\mu(x') \leqq \mu(x)^{1+\epsilon(x)}, \quad x' = x^{1+\frac{1}{\mu(x)}\epsilon(x)}$$

where $\epsilon(x)$ is a decreasing infinitesimal. Then $\mu(x)$ is a *function-type adjoint to* $\nu(x)$ *and* $\epsilon(x)$ if the inequality

$$\mu(x) \leqq \nu(x)^{1+\delta}$$

holds for any $\delta > 0$ and an infinite number of values of x. If $\nu(x)$ is given it is clear that $\mu(x)$ yields a measure of the increase of $\nu(x)$, and at the same time possesses a certain regularity of increase (croissance typique).

The fundamental theorem concerning function-types is that corresponding to any given $\nu(x)$ a function-type $\mu(x)$ adjoint to $\nu(x)$ and some infinitesimal $\epsilon(x)$ may be found. The proof first given by Blumenthal (pages 24–31) contains an error. It is assumed that for any given increasing function $w(x)$ an infinitesimal $\epsilon(x)$ (not the $\epsilon(x)$ of the th orem) can be found such