Forum Geometricorum
Volume 8 (2008) 13–25.

FORUM GEOM
ISSN 1534-1178

Angles, Area, and Perimeter Caught in a Cubic

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Abstract. The main goal of this paper is to establish sharp bounds for the angles and for the side ratios of any triangle of known area and perimeter. Our work is also related to the well known isoperimetric inequality.

1. Isosceles triangles sharing area and perimeter

Suppose we wish to determine all isosceles triangles, if any, of area 3 and perimeter 10 – a problem that is a bit harder than the corresponding well known problem for rectangles!

Let x be the length of the base and y the length of the two equal sides, $x < 2y$. Then the height of the isosceles triangles we wish to determine is equal to $\sqrt{y^2 - \frac{x^2}{4}}$. Thus $x+2y = 10$ while $\frac{x}{2}\sqrt{y^2 - \frac{x^2}{4}} = 3$. Hence $\frac{x}{2}\sqrt{\left(5 - \frac{x}{2}\right)^2 - \frac{x^2}{4}} = 3$, which leads to $5x^3 - 25x^2 + 36 = 0$. The positive roots of this cubic are $x_1 \approx 1.4177$ and $x_2 \approx 4.6698$, so that $y_1 \approx 4.2911$ and $y_2 \approx 2.6651$. Thus there are just two isosceles triangles of area 3 and perimeter 10 (see Figure 1).

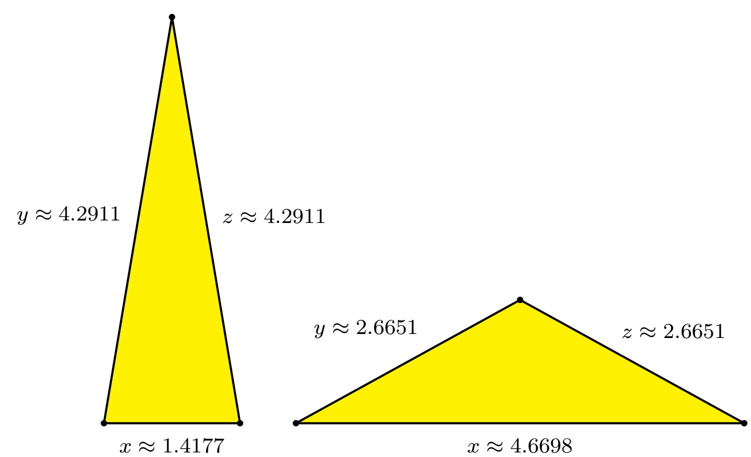

Figure 1. The two isosceles triangles of area 3 and perimeter 10

Are there always isosceles triangles of area A and perimeter P? A complete answer is provided by the following lemma and theorem.

Lemma 1. *Let x be the base of an isosceles triangle with given area A and perimeter P. Then*

$$2Px^3 - P^2x^2 + 16A^2 = 0. \tag{1}$$

Publication Date: January 22, 2008. Communicating Editor: Paul Yiu.

Proof. Working as in the above special case, we obtain $y = \frac{P-x}{2}$ and $\frac{x}{2}\sqrt{y^2 - \frac{x^2}{4}} = A$; substituting the former condition into the latter, we arrive at (1). $\square$

Theorem 2. *There are exactly two distinct isosceles triangles of area A and perimeter P if and only if $P^2 > 12\sqrt{3}A$. There is exactly one if and only if $P^2 = 12\sqrt{3}A$ and the triangle is equilateral. The vertex angles $\phi_1 < \phi_2$ of these two isosceles triangles also satisfy $\phi_1 < \frac{\pi}{3} < \phi_2$.*

Proof. Let $f(x)$ be the cubic in (1). We first show that it has at most two distinct positive roots. Indeed the existence of three distinct positive roots would yield, by Rolle's theorem, two distinct positive roots for $f'(x) = 6Px^2 - 2P^2x$; but the roots of $f'(x)$ are $x = \frac{P}{3}$ and $x = 0$.

Notice now that $f''(x) = 12Px - 2P^2$, hence $f''(0) = -2P^2 < 0$ and $f''(\frac{P}{3}) = 2P^2 > 0$. So f has a positive local maximum of $16A^2$ at $x = 0$ and a local minimum at $x = \frac{P}{3}$ (Figure 2). It is clear that f has two distinct positive roots $x_1 < \frac{P}{3} < x_2$ if and only if $f(\frac{P}{3}) < 0$; but $f(\frac{P}{3}) = -\frac{P^4}{27} + 16A^2$, so $f(\frac{P}{3}) < 0$ is equivalent to $P^2 > 12\sqrt{3}A$.

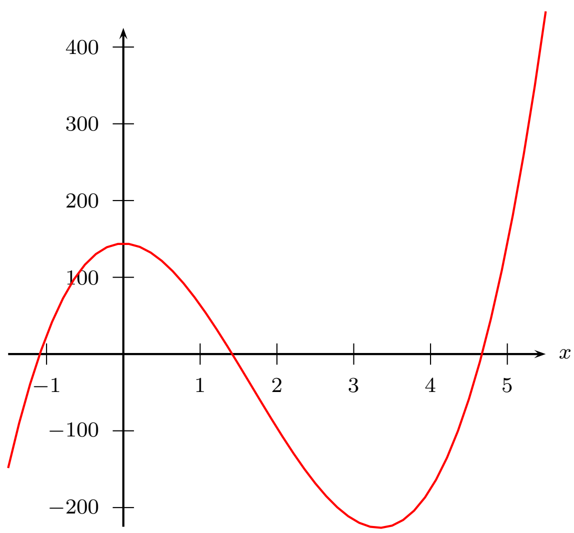

Figure 2. $2Px^3 - P^2x^2 + 16A^2$ for $A = 3$ and $P = 10$

Moreover, $f(\frac{P}{3}) = 0$ if and only if $P^2 = 12\sqrt{3}A$, implying that $f(x) = 0$ has precisely one ('tangential') positive solution if and only if $P^2 = 12\sqrt{3}A$. As it turns out, the cubic is then equivalent to $(3x - P)^2(6x + P) = 0$, and its unique positive solution corresponds to the equilateral triangle of side $\frac{P}{3}$.

As also noticed in [1], the vertex angles ϕ_1 and ϕ_2 of the two isosceles triangles of area A and perimeter P (that correspond to the positive roots x_1 and x_2 of (1)) do satisfy the inequalities $\phi_1 < \frac{\pi}{3} < \phi_2$. These inequalities follow from $x_1 < \frac{P}{3} < x_2$ since, in every triangle, the greater angle is opposite the greater side: indeed in every isosceles triangle of perimeter P, base x, vertex angle ϕ, and sides

$y = z$, the inequality $x < \frac{P}{3}$ implies $y = z > \frac{P}{3}$, so that $y = z > x$; therefore $\frac{\pi-\phi}{2} > \phi$, thus $\phi < \frac{\pi}{3}$. In a similar fashion one can prove that $x > \frac{P}{3}$ implies $\phi > \frac{\pi}{3}$. □

Remark. That the cubic in (1) can have at most two distinct positive roots may also be derived algebraically. Indeed, the existence of three distinct positive roots x_1, x_2, x_3 would imply that the cubic may be written as $c(x-x_1)(x-x_2)(x-x_3)$, with $c(x_1x_2 + x_2x_3 + x_3x_1)$ being the *positive* coefficient of the first power of x. That would contradict the fact that the cubic being analyzed has zero as the coefficient of the first power of x.

2. The isoperimetric inequality for arbitrary triangles

We have just seen that the inequality $P^2 \geq 12\sqrt{3}A$ holds for every isosceles triangle, with equality precisely when the triangle is equilateral. We will prove next that this *isoperimetric* inequality ([5, p.85], [3, p.42]) holds for every triangle.

First we notice that for every scalene triangle BCD, there exists an isosceles triangle ECD with BE parallel to CD (see Figure 3). Let ℓ be the line through B parallel to CD and F be the symmetric reflection of D with respect to ℓ. Let E and G be the points of ℓ on CF and DF, respectively. Clearly, $EG \| CD$ and $|FG| = |DG|$ imply $|FE| = |CE|$. Moreover, triangles FGE and DGE are congruent by symmetry, therefore $|FE| = |DE|$. We conclude that triangle ECD is isosceles with $|CE| = |DE|$.

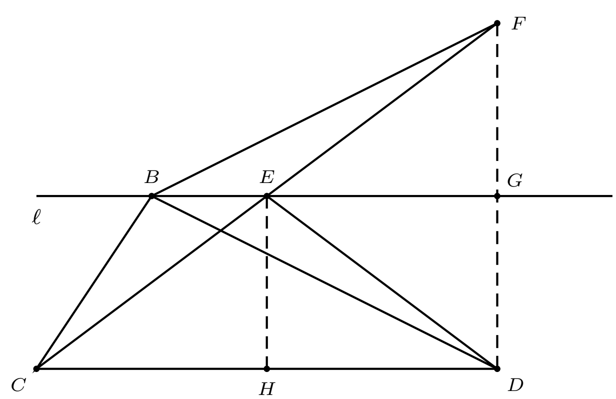

Figure 3. Reduction to the case of an isosceles triangle

It follows immediately from $BE \| CD$ that ΔECD and ΔBCD have equal areas. Less obviously, the perimeter of ΔECD is *smaller* than that of ΔBCD : $|CD| + |DE| + |EC| = |CD| + |FE| + |EC| = |CD| + |FC| < |CD| + |FB| + |BC| = |CD| + |DB| + |BC|$, with the last equality following from symmetry and the congruency of ΔFGB and ΔDGB.

So, given an arbitrary scalene triangle BCD of area A and perimeter P, there exists an isosceles triangle ECD of area A and perimeter $Q < P$. Since $Q^2 \geq$

$12\sqrt{3}A$, it follows that $P^2 > 12\sqrt{3}A$, so the isoperimetric inequality for triangles has been proven.

We invite the reader to use this geometrical technique to derive the isoperimetric inequality for quadrilaterals ($P^2 \geq 16A$ for every quadrilateral of area A and perimeter P), and possibly for other n-gons as well.

It should be mentioned here that the standard proof of the isoperimetric inequality for triangles (see for example [2, p.88]) relies on Heron's area formula (which we essentially derive later through a generalization of (1) for arbitrary triangles) and the arithmetic-geometric-mean inequalilty.

3. Newton's parametrization

Turning now to our main goal, namely the relations among a triangle's area, perimeter, and angles, we first find an expression for the sides of a triangle in terms of its area, perimeter, and *one* angle. To achieve this, we simply generalize Newton's derivation of the formula $x = \frac{P}{2} - \frac{2A}{P}$, expressing a right triangle's hypotenuse in terms of its area and perimeter; this work appeared in Newton's *Universal Arithmetick*, *Resolution of Geometrical Questions*, Problem III, p. 57 ([6, p.103]).

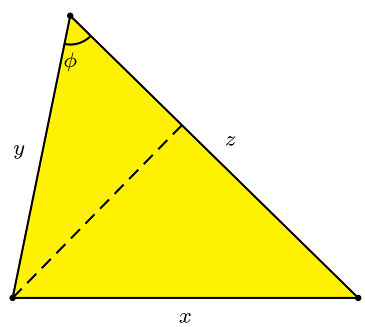

Figure 4. Toward 'Newton's parametrization'

Observe (as in Figure 4) that $A = \frac{1}{2}zy\sin\phi$, so $y^2 = Py - xy - \frac{2A}{\sin\phi}$; moreover, the law of cosines yields $y^2 = Px + Py - xy + \frac{2A\cos\phi}{\sin\phi} - \frac{P^2}{2}$. It follows that

$$x = x(\phi) = \frac{P}{2} - \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right), \tag{2}$$

extending Newton's formula for $0 < \phi < \pi$. Of course we need to have $\frac{P^2}{A} > 4\left(\frac{1+\cos\phi}{\sin\phi}\right)$ for x to be positive, so we need the condition $s(\phi) > 0$, where

$$s(\phi) = \frac{P^2\sin\phi}{4(1+\cos\phi)} - A. \tag{3}$$

Once x is determined, y and z are easily determined via $yz = \frac{2A}{\sin\phi}$ and $y+z = \frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)$: they are the roots of the quadratic $t^2 - \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)t + \frac{2A}{\sin\phi} = 0$, provided that $h(\phi) \geq 0$, where

$$h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi} \tag{4}$$

is the discriminant; that is, $y = y(\phi)$ and $z = z(\phi)$ are given by

$$z,\ y = \left(\frac{P}{4} + \frac{A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right) \pm \frac{1}{2}\sqrt{\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}}. \tag{5}$$

Putting everything together, and observing that x, y, z as defined in (2) and (5) above do satisfy the triangle inequality and are the sides of a triangle of area A and perimeter P, we arrive at the following result.

Theorem 3. *The pair of conditions $s(\phi) > 0$ and $h(\phi) \geq 0$, where $s(\phi) = \frac{P^2\sin\phi}{4(1+\cos\phi)} - A$ and $h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}$, is equivalent to the existence of a triangle of area A, perimeter P, sides $x(\phi), y(\phi), z(\phi)$ as given in* (2), (5) *above, and angle ϕ between the sides y, z; that triangle is isosceles with vertex angle ϕ if and only if $h(\phi) = 0$.*

Figures 5 and 6 below offer visualizations of the three sides' parametrizations by the angle ϕ and of the two functions essential for the 'triangle conditions' of Theorem 3, respectively.

The 'vertical' intersections of $y(\phi)$ and $z(\phi)$ with each other in Figure 5 occur at $\phi \approx 0.33166 \approx 19.003°$ and $\phi \approx 2.13543 \approx 122.351°$: those are the positive roots of $h(\phi) = 0$, which are none other than the vertex angles of the two isosceles triangles in Figure 1. There are also intersections of $x(\phi)$ with $z(\phi)$ at $\phi \approx 1.40485 \approx 80.492°$ and of $x(\phi)$ with $y(\phi)$ at $\phi \approx 0.50305 \approx 28.822°$; which are again associated, via side renaming as needed and with ϕ being a *base* angle, with the isosceles triangles of Figure 1.

As we see in Figure 6, s and h cannot be simultaneously positive outside the interval defined by the two largest roots of h ($\phi \approx 0.33166$ and $\phi \approx 2.13543$): this fact remains true for arbitrary A and P and is going to be of central importance in what follows.

4. Angles 'bounded' by area and perimeter

We are ready to state and prove our first main result.

Theorem 4. *In every non-equilateral triangle of area A and perimeter P every angle ϕ must satisfy the inequality $\phi_1 \leq \phi \leq \phi_2$, where $\phi_1 < \frac{\pi}{3} < \phi_2$ are the vertex angles of the two isosceles triangles of area A and perimeter P; specifically,*

$$\arccos\left(\frac{P^2 - 2Px_1 - x_1^2}{P^2 - 2Px_1 + x_1^2}\right) \leq \phi \leq \arccos\left(\frac{P^2 - 2Px_2 - x_2^2}{P^2 - 2Px_2 + x_2^2}\right),$$

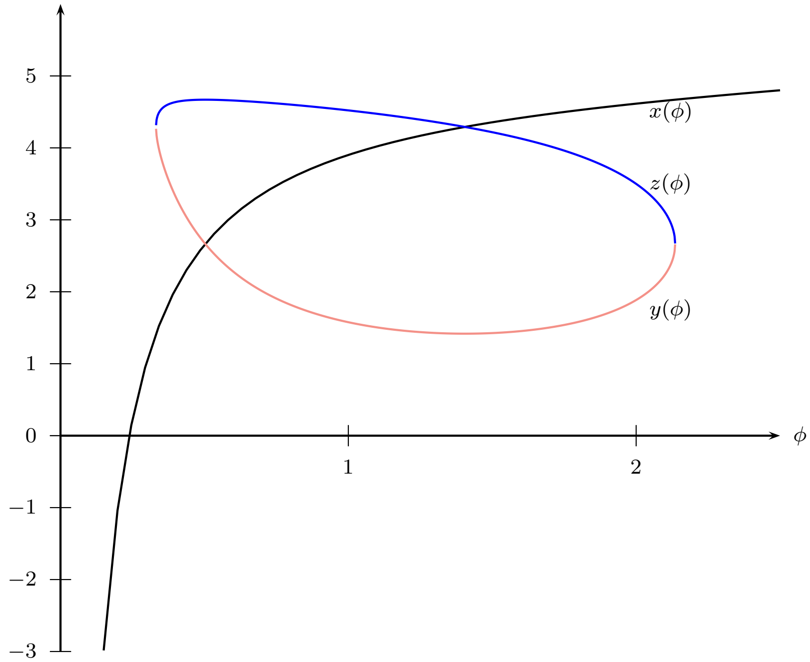

Figure 5. The triangle's three sides parametrized by ϕ for $19.003^\circ = 0.33166 \leq \phi \leq 2.13543 = 122.351^\circ$ at $A = 3$, $P = 10$

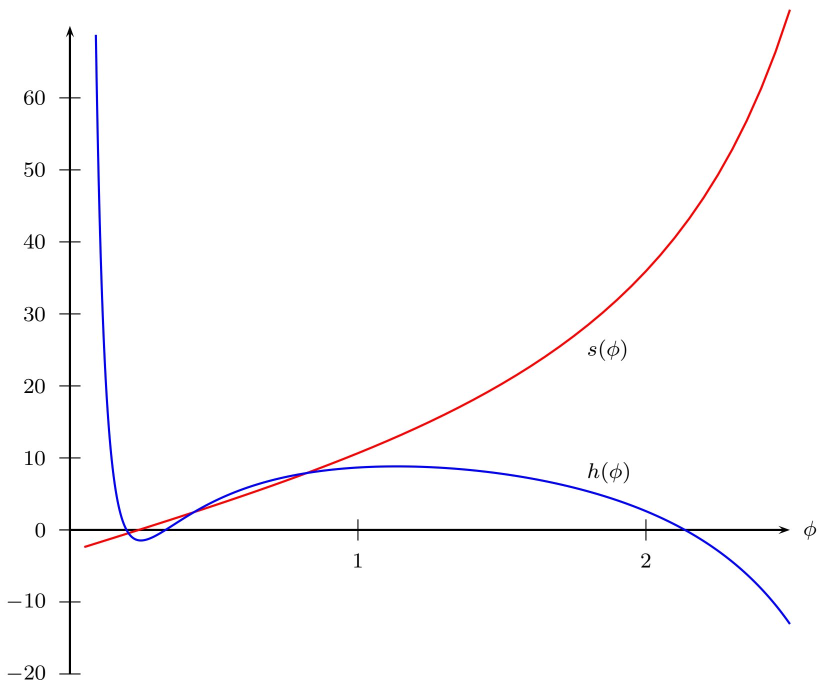

Figure 6. $s(\phi)$ and $h(\phi)$ for $0.1 \leq \phi \leq 2.3$ at $A = 3$, $P = 10$

where $x_1 < \frac{P}{3} < x_2$ *are the positive roots of* $2Px^3 - P^2x^2 + 16A^2 = 0$.

Proof. As we have seen in Lemma 1, the cubic (1) yields the base x of each of the two isosceles triangles of area A and perimeter P; and the formula above for the vertex angle ϕ of an isosceles triangle follows from $x^2 = 2y^2 - 2y^2\cos\phi$ (law of cosines) and $y = \frac{P-x}{2}$.

So it suffices to show that the inequality $\phi_1 \le \phi \le \phi_2$ is equivalent to the pair of conditions $s(\phi) > 0$ and $h(\phi) \ge 0$, where $s(\phi)$ and $h(\phi)$ are defined as in Theorem 3; for this, we need four lemmas.

Lemma 5. *For some* ψ *in* $(0,\phi_1)$, $s(\psi) = 0$.

Proof. Notice that $\lim_{\phi\to 0^+} s(\phi) = -A < 0$. On the other hand, the existence of an isosceles triangle with vertex angle ϕ_1 guarantees that $s(\phi_1) > 0$ (Theorem 3). By the continuity of s on $(0,\pi)$, there must exist ψ such that $0 < \psi < \phi_1$ and $s(\psi) = 0$. □

Lemma 6. *The function* s *is strictly increasing on* $(0,\pi)$ *and, for* $\phi \ge \phi_1$, $s(\phi) > 0$.

Proof. Since the derivative $s'(\phi) = \frac{P^2}{4(1+\cos\phi)}$ is positive on $(0,\pi)$, s is strictly increasing; it follows that $s(\phi) \ge s(\phi_1) > 0$ for $\phi \ge \phi_1$. □

Lemma 7. *For* $\phi > \phi_2$, $h(\phi) < 0$.

Proof. Recall that $h(\phi) = \left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 - \frac{8A}{\sin\phi}$. By L'Hospital's rule, we have $\lim_{\phi\to\pi} \frac{1+\cos\phi}{\sin\phi} = \lim_{\phi\to\pi} \frac{-\sin\phi}{\cos\phi} = 0$; it follows that $\lim_{\phi\to\pi^-} h(\phi) = \frac{P^2}{4} - \lim_{\phi\to\pi^-} \frac{8A}{\sin\phi} = -\infty$. Suppose $h(\phi) \ge 0$ for some $\phi > \phi_2$. Then $h(\phi_3) = 0$ for some $\phi_3 > \phi_2$ because h is continuous on $(0,\pi)$ and $\lim_{\phi\to\pi^-} h(\phi) = -\infty$. At the same time, $s(\phi_3) > 0$ (Lemma 6). Then by Theorem 3, there exists a third isosceles triangle of area A and perimeter P, which is impossible. □

Lemma 8. *There is no* ϕ *in* $(0,\pi)$ *for which* $h(\phi) = h'(\phi) = 0$.

Proof. Suppose $h(\phi) = h'(\phi) = 0$ for some ϕ in $(0,\pi)$. It follows that

$$\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 = \frac{8A}{\sin\phi} \quad \text{and} \quad \frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right) = \frac{2P\cos\phi}{1+\cos\phi}.$$

Squaring the latter and dividing it by the former expression we get $P^2 = \frac{2A(1+\cos\phi)^2}{\sin\phi\cos^2\phi}$. Substituting this expression for P^2 into $\left(\frac{P}{2} + \frac{2A}{P}\left(\frac{1+\cos\phi}{\sin\phi}\right)\right)^2 = \frac{8A}{\sin\phi}$ we arrive at the equation $\frac{A(1+\cos\phi)^2}{2\sin\phi\cos^2\phi} + \frac{2A(1+\cos\phi)}{\sin\phi} + \frac{2A\cos^2\phi}{\sin\phi} = \frac{8A}{\sin\phi}$, which reduces to $(\cos\phi - 1)(2\cos\phi - 1)(2\cos^2\phi + 5\cos\phi + 1) = 0$. The only roots in $(0,\pi)$ are given by $\phi = \frac{\pi}{3}$ and $\phi = \arccos\left(\frac{-5+\sqrt{17}}{4}\right)$. It is easy to see that $h'(\phi) < 0$ for $\phi > \frac{\pi}{2}$, so $\arccos\left(\frac{-5+\sqrt{17}}{4}\right)$ is an extraneous solution. Moreover, $\phi = \frac{\pi}{3}$ turns

$P^2 = \frac{2A(1+\cos\phi)^2}{\sin\phi\cos^2\phi}$ into $P^2 = 12\sqrt{3}A$, contradicting the fact that the given triangle was assumed to be non-equilateral. We conclude that $h(\phi) = h'(\phi) = 0$ is impossible. $\square$

Completing the proof of Theorem 4.

Claim(a) *For $\phi_1 \leq \phi \leq \phi_2$, $s(\phi) > 0$ and $h(\phi) \geq 0$, with $h(\phi) > 0$ for $\phi_1 < \phi < \phi_2$.*

Recall from Lemma 6 that $s(\phi) > 0$ for $\phi \geq \phi_1$. So it remains to establish $h(\phi) \geq 0$ for $\phi_1 \leq \phi \leq \phi_2$. We will argue by contradiction.

Of course $h(\phi_1) = h(\phi_2) = 0$. Notice that $h(\phi) = 0$ for $\phi_1 < \phi < \phi_2$ is impossible for this would imply (by Theorem 3) the existence of a third isosceles triangle of area A and perimeter P. If $h(\phi_3) < 0$ for some ϕ_3 strictly between ϕ_1 and ϕ_2 then continuity of h, together with the impossibility of $h(\phi) = 0$ for $\phi_1 < \phi < \phi_2$, implies $h(\phi) < 0$ for *all* angles strictly between ϕ_1 and ϕ_2. But we already know from Lemma 7 that $h(\phi) < 0$ for all angles greater than ϕ_2. It follows that h has a local maximum at $\phi = \phi_2$, so that $h(\phi_2) = h'(\phi_2) = 0$, contradicting Lemma 8.

Recalling the statement immediately before Lemma 5, we see that the proof of Theorem 4 will be completed by establishing

Claim(b) *At least one of the conditions $s(\phi) > 0$ and $h(\phi) \geq 0$ fails when either $\phi < \phi_1$ or $\phi > \phi_2$.*

Of course the failure of $h(\phi) \geq 0$ for $\phi > \phi_2$ has been established in Lemma 7, so we only need to show either $s(\phi) \leq 0$ or $h(\phi) < 0$ for $\phi < \phi_1$.

Lemma 5 asserts that there exists ψ in $(0, \pi)$ such that $\psi < \phi_1$ and $s(\psi) = 0$. Consider now an arbitrary $\phi < \phi_1$. If $\phi \leq \psi$ then by Lemma 6 $s(\phi) \leq s(\psi) = 0$, so we only need to pay attention to the possibility $\phi_1 > \phi > \psi$ and $s(\phi) > 0$. In that case we show below that $h(\phi) < 0$, arguing by contradiction.

The failure of $h(\phi) < 0$ implies, in the presence of $s(\phi) > 0$, that $h(\phi) > 0$: indeed $h(\phi) = 0$ and $s(\phi) > 0$ would yield a third isosceles triangle of area A and perimeter P, again by Theorem 3. The same argument applies in fact to all angles between ψ and ϕ_1. But we have already established through Claim(a) the strict positivity of h for all angles between ϕ_1 and ϕ_2. We conclude that h has a local minimum at $\phi = \phi_1$, so that $h(\phi_1) = h'(\phi_1) = 0$, contradicting Lemma 8. This completes the proof of Theorem 4.

Having completed the proof of Theorem 4, let us provide an example: the bases of the two isosceles triangles of area 3 and perimeter 10 (Figure 1) have already been computed as the positive roots of the cubic $5x^3 - 25x^2 + 36 = 0$; it follows then that *all* angles of *every* triangle of area 3 and perimeter 10 must be between about $19.003°$ and $122.351°$, the angles shown in Figure 5.

Remark. It can be shown that ϕ_1 and ϕ_2 are the two largest roots of

$$(P^2 \sin\phi + 4A + 4A\cos\phi)^2 - 32P^2 A \sin\phi = 0$$

in $(0, \pi)$, and that they also satisfy the equation

$$\sin\phi_2 \left(1 + \sin\frac{\phi_1}{2}\right)^2 = \sin\phi_1 \left(1 + \sin\frac{\phi_2}{2}\right)^2 .$$

5. Heron's curve

Theorem 4 establishes bounds for the angles of every triangle of given area and perimeter; appealing to the law of sines, we see that it also yields bounds for the ratio of any two sides. Determining *sharp* bounds for side ratios relies on some machinery we develop next.

Instead of looking for isosceles triangles $(z = y)$ of area A and perimeter P, let us now look for triangles of area A and perimeter P where two sides have ratio r ($\frac{z}{y} = r$); without loss of generality, we may assume $r > 1$. (Observe here - as in fact noticed through Figure 5 and related discussion - that $r > 1$ does not rule out the possibilities $x = z$ (with $r \approx 3.0268$ at $A = 3$, $P = 10$) or $x = y$ (with $r \approx 1.7522$ at $A = 3$, $P = 10$).) Extending the procedure of Lemma 1 to arbitrary triangles, from $y^2 - x_1^2 = r^2y^2 - x_2^2$ and $x = x_2 \pm x_1$ (Figure 7) we find that

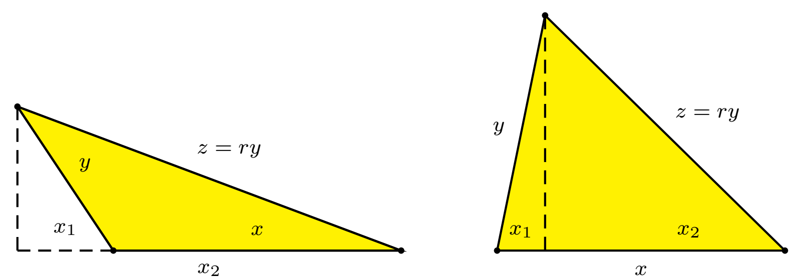

Figure 7. The case of an arbitrary triangle

$x_1 = \pm\frac{(1-r^2)y^2+x^2}{2x}$. In view of $\frac{x}{2}\sqrt{y^2 - x_1^2} = A$ and $y = \frac{P-x}{r+1}$, further algebraic manipulation leads to an equation that generalizes the isosceles triangle's cubic (1):

$$8rPx^3+4(r^2-3r+1)P^2x^2-4(1-r)^2P^3x+(1-r)^2P^4+16(1+r)^2A^2 = 0. \quad (6)$$

Appealing to Rolle's theorem as in the case of the isosceles triangle, we see that this cubic cannot have more than two positive roots. Indeed one of the derivative's roots, $\left(\frac{-(r^2-3r+1)-\sqrt{r^4-r^2+1}}{6r}\right)P$, is negative since $|r^2 - 3r + 1| < \sqrt{r^4 - r^2 + 1}$ for $r > 1$.

Unlike the case of the isosceles triangle, however, the isoperimetric inequality $P^2 > 12\sqrt{3}A$ does not guarantee the existence of two positive roots. So there can be *at most* two triangles of area A and perimeter P satisfying the condition $\frac{z}{y} = r > 1$.

Setting $x = P - y - z$ and $r = \frac{z}{y}$ in the cubic (6) leads to

$$P^4 - 4P^3(y + z) + 4P^2(y^2 + 3yz + z^2) - 8Pyz(y + z) + 16A^2 = 0, \quad (7)$$

which can be shown to be equivalent to Heron's area formula. The graph of this curve for $A = 3$ and $P = 10$ (Figure 8) illustrates the fact established above by (6): for every pair of A and P, there can be at most two triangles of area A and perimeter P satisfying $\frac{z}{y} = r > 1$. Indeed, the three unbounded regions shown in Figure 8 correspond to $x < 0$ (first quadrant), $y < 0$ (second quadrant), and $z < 0$ (fourth quadrant), hence it is only the boundary of the bounded region that corresponds to triangles of area 3 and perimeter 10; clearly, this boundary that we call *Heron's curve* (Figure 9) may be intersected by any line at most twice.

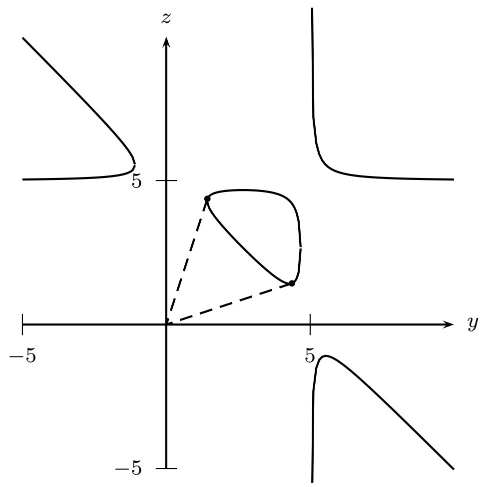

Figure 8. Graph of (7) for $A = 3$ and $P = 10$

Rather predictably, in view of its symmetry about $z = y$, the triangles corresponding to Heron's curve's intersections with (for example) $z = 2y$ and $z = \frac{y}{2}$ (see Figure 9) are mirror images of each other (about the third side x's perpendicular bisector); so it suffices to restrict our computations to $r > 1$, sticking to our initial assumption. These triangles are found by first solving the cubic (6) when $r = 2$ and are approximately $\{3.0077, 2.3307, 4.6615\}$ and $\{4.5977, 1.8007, 3.6015\}$; they are associated with parametrizing angles of about $33.529°$ and $112.315°$, respectively.

6. Side ratios 'bounded' by area and perimeter

We present now the following companion to Theorem 4.

Theorem 9. *In every non-equilateral triangle of area A and perimeter P, the ratio r of any two sides must satisfy the inequality $r_1 \le r \le r_2$, where $r_1 < 1 < r_2$, $r_1 r_2 = 1$ are the positive roots of the sextic*

$$32P^4A^2(2r^6 - 3r^4 - 3r^2 + 2) - P^8r^2(r-1)^2 + 6912A^4r^2(r+1)^2 = 0. \quad (8)$$

Proof. Figures 8 and 9 (and the discussion preceding them) make it clear that not all lines $z = ry$ intersect Heron's curve: such intersections (corresponding to triangles of area A and perimeter P satisfying $\frac{z}{y} = r$) occur only at $r = 1$ and a varying

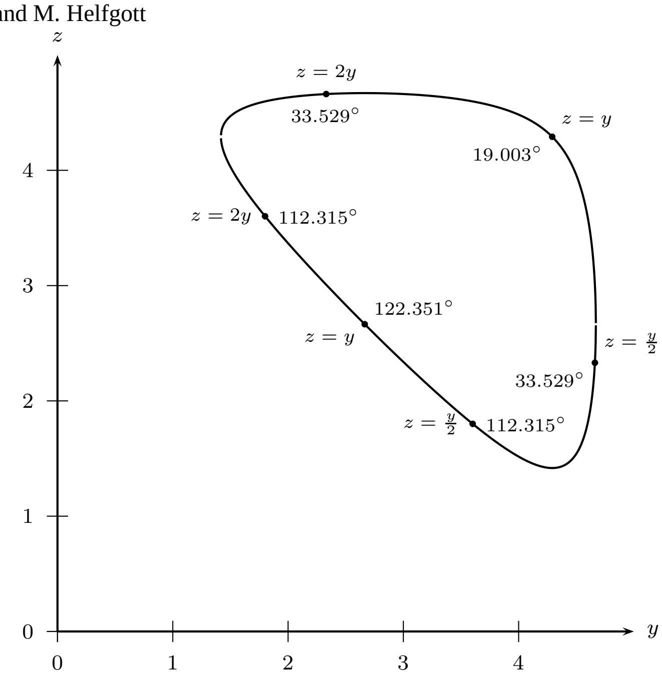

Figure 9. Heron's curve for $A = 3$ and $P = 10$

interval around it depending on A and P by way of (6). To establish sharp bounds for such 'intersecting' r, we observe that these bounds are none other than the slopes of the lines *tangent* to Heron's curve; in the familiar case $A = 3$, $P = 10$, these tangent lines are shown in Figure 8. But a line $z = ry$ is tangent to Heron's curve if and only if there is precisely one triangle of area A and perimeter P satisfying $\frac{z}{y} = r$; that is, if and only if the cubic (6) has a double root.

It is well known (see for example [4, p.91]) that the cubic $ax^3 + bx^2 + cx + d$ has a double root if and only if

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = 0.$$

(The reader may arrive at this 'tangential' condition independently, arguing as in the proof of Theorem 2.) So we may conclude that the slopes of the two lines tangent to Heron's curve and passing through the origin are the positive roots of the polynomial $S(r) = -64P^2(r+1)^2Q(r)$, where $Q(r)$ is the sixth degree polynomial in (8).

It may not be obvious but Q, and therefore S as well, must have precisely two positive roots, as they ought to. This relies on the following facts (which imply a total of *four* real roots for Q): the leading coefficient of Q is positive and its highest power is even, so $\lim_{r\to\pm\infty} Q(r) = +\infty$; $Q(-1) = -4P^8 - 64P^4A^2 < 0$; $Q(0) = 64P^4A^2 > 0$; $Q(1) = -64A^2(P^4 - (12\sqrt{3})^2A^2) < 0$; $Q(\frac{1}{r}) = \frac{Q(r)}{r^6}$ for $r \neq 0$, so that r is a root of Q if and only if $\frac{1}{r}$ is. $\square$

In the familiar example of $A = 3$ and $P = 10$, the two positive roots of S are $r_1 \approx 0.3273$ and $r_2 \approx 3.0551$. As pointed out above, these two roots are inverses

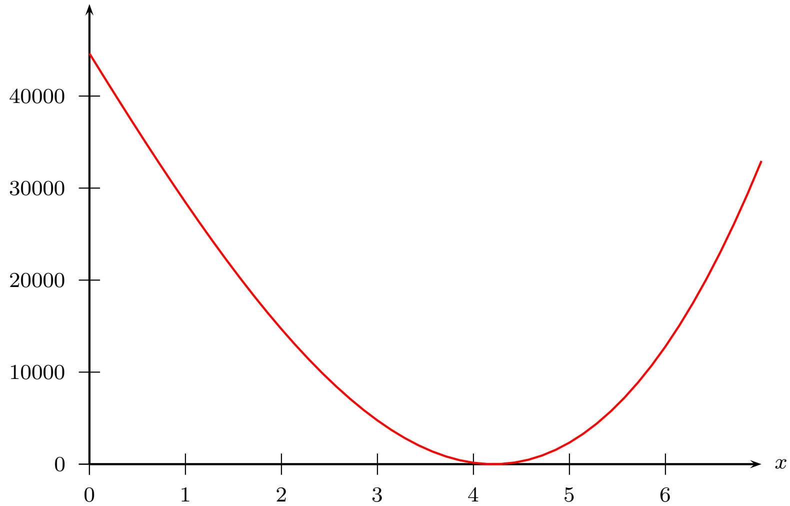

Figure 10. Graph of (6) for $A = 3$, $P = 10$, and $r \approx 3.0551$

of each other: this is geometrically justified by the fact that the two roots are the slopes of the two tangent lines in Figure 8, which are of course mirror images of each other about the diagonal $z = y$. Moreover, r_1 and r_2 lead to the *same* (modulo a factor) cubic in (6).

We conclude that the side ratios of every triangle of area 3 and perimeter 10 must be between approximately 0.3273 and 3.0551. To obtain the unique (modulo reflection) triangle of area 3 and perimeter 10 where these ratios are realized, we need to determine its third side x. It is the double root of the cubic (6) for r equal to approximately 3.0551 (Figure 10). It turns out that x equals approximately 4.2048.

The triangle is now fully determined through $y \approx \frac{10-4.2048}{3.0551+1} \approx 1.4291$ and $z \approx 3.0551 \times 1.4291 \approx 4.366$ (upper 'corner' in Figure 9). The angle-parameter (between sides y and z) at that 'corner' is now easy to find as $\arccos\left(\frac{y^2+z^2-x^2}{2yz}\right) \approx 74.079°$. The triangle obtained, approximately $\{4.2048, 4.3661, 1.4291\}$ (see Figure 11), is the furthest possible from being isosceles - or rather the furthest possible from being equilateral! - among all triangles of area 3 and perimeter 10.

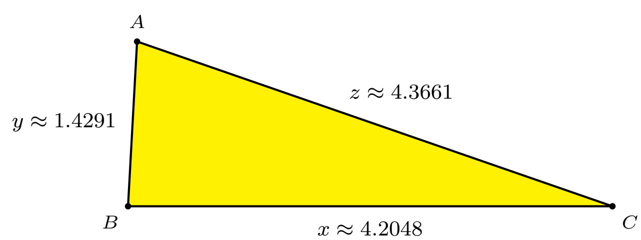

Figure 11. The unique extreme-side-ratio triangle of area 3 and perimeter 10

Our findings are confirmed in Figure 12 by a graph of $\frac{z(\phi)}{y(\phi)}$, where $z(\phi)$ and $y(\phi)$ are the Newton parametrizations of sides z and y in (5). That graph shows a maximum value of about 3.055 for $\frac{z(\phi)}{y(\phi)}$ with ϕ approximately equal to $1.293 \approx 74.08°$:

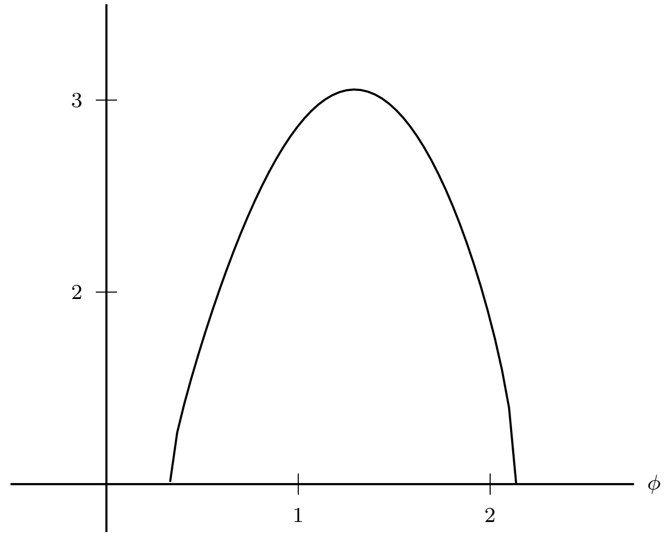

Figure 12. $\frac{z(\phi)}{y(\phi)}$ for $19.003° \approx 0.33166 \leq \phi \leq 2.13543 \approx 122.351°$, $A = 3$ and $P = 10$

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