Chapter 1

Nonlinear Equations of the Atmospheric and the Oceanic Motions

There are usually two methods for predicting long-term weather and climate. First, by statistical methods, we can use the current climate, the historical record and numerical analysis to predict the future climate and the possible global climatic changes. Second, because air is compressible, and seawater is incompressible, by dynamical methods, we consider that the future status of climate is a consequence determined by the current status and the physical principles dominating these changes, thus we study equations and models describing the atmospheric and oceanic motions. Regarding weather prediction as an initial-boundary value problem in mathematical physics, we can establish numerical weather prediction models based on mathematical physical equation.

Numerical weather prediction is an outstanding applied research achievement of atmospheric science in the 20th century, of which theoretical foundation is the atmospheric dynamics. In 1922, Richardson introduced the concept of numerical weather prediction for the first time ([183]). His idea is that through solving the complete primitive equations governing the atmosphere motions numerically, one can simulate the evolution process of atmosphere, thus may predict weather quantitatively. Due to the weak calculation ability at that time, the dream of numerical weather prediction did not exist. Applying the long-wave theory and the scale-analysis theory established by Rossby and others, Charney set up a two-dimensional geostrophic model. With this model, he and his collaborators successfully made true 24-hour numerical weather prediction on the ENIAV computer of the Institute for Advanced Studies in Princeton for the first time. Along with the boom of atmosphere science and the enhancing of data dealing ability and numerical calculation ability of computer, researchers turn to numerical weather prediction by the primitive equation models from 1960s

([112,147,181,218]), greatly extend the time-range of numerical weather prediction. Afterward researchers started to make long-term numerical weather prediction, climate forecasting and numerical simulation of atmospheric circulation by some primitive equation models of the atmosphere and oceans.

To actualize long-term numerical weather prediction, climate prediction and numerical simulation of atmospheric circulation based on physical methods, the first thing is to establish some atmospheric and oceanic dynamical models, which are the nonlinear partial differential equations with initial-boundary value conditions which govern the atmospheric and oceanic motion. In this chapter, we mainly present basic and primitive equations and their boundary conditions which govern the atmospheric and oceanic motion. For more detail see [220], and also [84,145,162,205,211].

1.1 Basic Equations of the Atmospheric and the Oceanic Motions

1.1.1 *Basic Equations of the Atmosphere*

Regarding air and seawater as continuous media, one can use the Euler method to describe the atmospheric and oceanic motions. In the inertial coordinate frame (the coordinate axis is fixed with respect to the stellar), according to the Newton second law, the momentum conservation equation of the atmosphere is given by

$$\frac{\mathrm{d}_I \boldsymbol{V}_I}{\mathrm{d}t} = -\frac{1}{\rho}\mathrm{grad}_3 p + g_I + D,$$

where $\boldsymbol{V}_I$ is the absolute velocity of the atmosphere (velocity in the inertial coordinate frame), $\frac{\mathrm{d}_I \boldsymbol{V}_I}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}_I}{\partial t} + (\boldsymbol{V}_I \cdot \boldsymbol{\nabla}_3)\boldsymbol{V}_I$ is the absolute acceleration (acceleration in the inertial coordinate frame), ρ is the density of air, p is the atmospheric pressure, $-\frac{1}{\rho}\mathrm{grad}_3 p$ is the pressure-gradient force, g_I is the gravity, and D is a molecular viscous force (molecular friction force, dissipative force), which is a dissipative force caused by air internal friction or turbulent momentum transmission.

In general, researchers are concerned with the relative motions of the atmosphere to the earth. So taking a coordinate frame rotating together with the earth as a reference frame, researchers can observe atmospheric relative motions. Suppose that the angular velocity of rotation in the rotating coordinate frame is $\boldsymbol{\Omega}$ (that is the rotational angular velocity of the earth), $\boldsymbol{V}$ is the atmospheric relative velocity, $\dfrac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}$ is the atmospheric relative acceleration in the rotating coordinate frame, then

$$\boldsymbol{V}_I = \boldsymbol{V} + \boldsymbol{\Omega} \times \mathbf{r},$$

$$\frac{\mathrm{d}_I \boldsymbol{V}_I}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{V}_I}{\mathrm{d}t} + \boldsymbol{\Omega} \times \boldsymbol{V}_I,$$

where $\mathbf{r}$ is the radius vector. The proof of the second equation above appears in section 1.5 in [172]. According to the previous three equations, we get in the rotating coordinate frame **the atmospheric momentum conservation equation**

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\frac{1}{\rho}\mathrm{grad}_3 p + g - 2\boldsymbol{\Omega} \times \boldsymbol{V} + D, \tag{1.1.1}$$

where $g = g_I + \Omega^2 \mathbf{r}$ is commonly referred to gravity (Ω is the value of the earth rotation angular velocity), $-2\boldsymbol{\Omega} \times \boldsymbol{V}$ is the Coriolis force, $\Omega^2 \mathbf{r}$ is the inertial centrifugal force,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}_3$$

is the substantial derivative (often called the total derivative).

According to the mass conservation law, **the continuity equation** is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \mathrm{div}_3 \boldsymbol{V} = 0. \tag{1.1.2}$$

In general, when describing large-scale motions of the troposphere and the stratosphere, one may consider dry air as ideal gas, and can get the **atmospheric state equation**

$$p = R\rho T, \tag{1.1.3}$$

where the vaporation in the atmosphere is negligible, T means the temperature absolute term of the atmosphere, and $R = 287\ \mathrm{J{\cdot}kg^{-1}K^{-1}}$ is a gas constant of dry air.

According to the first law of thermodynamics, the **atmospheric thermodynamic equation** is given by

$$c_v \frac{\mathrm{d}T}{\mathrm{d}t} + p\frac{\mathrm{d}\frac{1}{\rho}}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t},$$

where $c_v = 718$ J·kg^{-1}K^{-1}, and $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is the quantity of heat per unit mass of air obtained from external environment per unit time. Applying (1.1.3), we have

$$R\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}\frac{p}{\rho}}{\mathrm{d}t} = \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}t} + p\frac{\mathrm{d}\frac{1}{\rho}}{\mathrm{d}t} = \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} + p\frac{\mathrm{d}\frac{1}{\rho}}{\mathrm{d}t}.$$

Combining the above two equations together, we get

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{1.1.4}$$

where $c_p = c_v + R$ is specific heat at constant pressure.

Equations (1.1.1)-(1.1.4) are called the **fundamental equations of dry air**, where the unknown functions are $\boldsymbol{V}$, ρ, p, and T in these equations. If D and $\frac{\mathrm{d}Q}{\mathrm{d}t}$ are fixed, equations (1.1.1)-(1.1.4) are self-closed.

When one has to consider vaporation in the air, the moist air state equation is

$$p = R\rho T(1 + cq), \tag{1.1.5}$$

where $q = \frac{\rho_1}{\rho}$ is the mixing ratio of water vapor in the air, and ρ_1 is the density of water vapor in the air. Here, c represents positive constant varying with context. $c = 0.618$ in (1.1.5). The thermodynamic equation of the moist atmosphere is

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT(1+cq)}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{1.1.6}$$

the conservation equation of the water vapor in the air is

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{\rho}W_1 + W_2, \tag{1.1.7}$$

where W_1 is the condensation ratio of steam per unit volume, and W_2 is the volume change ratio of unit mass steam due to horizontal and vertical diffusions. Equations (1.1.1), (1.1.2) and (1.1.5)-(1.1.7) are called the **equations of the moist atmospheric**.

1.1.2 *Basic Equations of the Oceans*

Suppose that there are massless source-sinks within the oceans. In the rotating coordinate frame, the equations of oceans consist of the following equations:

the momentum conservation equation

$$\rho\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\mathrm{grad}_3 p + \rho g - 2\rho\boldsymbol{\Omega}\times\boldsymbol{V} + D,$$

the continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\mathrm{div}_3\boldsymbol{V} = 0,$$

the state equation

$$\rho = f(T, S, p),$$

the thermodynamic equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q_1,$$

and the salinity conservation equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q_2,$$

where S is salinity, Q_1 is the heat source per unit mass seawater derive from the external environment in unit time, and Q_2 is the salt source per unit mass seawater derive from the external environment in unit time.

Since the equations above are too complex, one has to do some simplification. Generally, one takes **Boussinesq approximation**, that is, consider ρ in ρg and the state equation as unknown function, but ρ in other position as constant ρ_0. Moreover, we use the following approximation equation to replace the above state equation

$$\rho = \rho_0[1 - \beta_T(T - T_0) + \beta_S(S - S_0)],$$

where β_T and β_S are positive constants, and $T_0,\ S_0$ are the reference values of temperature and salinity, respectively. Thus, we get the equations of oceans as

$$\rho_0\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\mathrm{grad}_3 p + \rho g - 2\rho_0\boldsymbol{\Omega}\times\boldsymbol{V} + D, \tag{1.1.8}$$

$$\mathrm{div}_3\boldsymbol{V} = 0, \tag{1.1.9}$$

$$\rho = \rho_0[1 - \beta_T(T - T_0) + \beta_S(S - S_0)], \tag{1.1.10}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q_1, \tag{1.1.11}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q_2. \tag{1.1.12}$$

Remark 1.1.1. State equation (1.1.10) is an empirical equation, which appears in [212]. The more general form is

$$\rho = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \frac{p}{\rho_0 c_s^2}\right],$$

where c_s is a positive constant, and this equation appears in section 2.4.1 of [205].

1.2 Equations of the Atmosphere and the Oceans in the Sphere Coordinate Frame

1.2.1 *Equations of the Atmosphere in the Sphere Coordinate Frame*

The atmosphere moves on the rotating earth surface. To study the relative motion of the atmosphere, assuming that the earth surface is simulated by a sphere surface, we discuss the atmosphere motion in spherical coordinate system.

Let's deduce the basic atmospheric equations in spherical coordinate frame. Setting the center of earth as the origin of the spherical coordinate, $\theta(0 \le \theta \le \pi)$ denotes the co-latitude of earth (it mutually complement to latitude), $\varphi(0 \le \varphi \le 2\pi)$ denotes the longitude of earth, r denotes the distance between the center and point on the surface of the earth, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_r$ are the unit vectors in the directions of θ, φ, r respectively, $\boldsymbol{e}_\theta$ tends to the south along the longitude, $\boldsymbol{e}_\varphi$ tends to the east along the latitude, and $\boldsymbol{e}_r$ tends outward along the radius. Using differential geometry symbols, we have

$$\boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \boldsymbol{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}, \quad \boldsymbol{e}_r = \frac{\partial}{\partial r}.$$

According to the definition of velocity, the air velocity $\boldsymbol{V}$ is expressed as

$$\boldsymbol{V} = v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi + v_r\boldsymbol{e}_r,$$

where

$$v_\theta = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\dot{\theta},\ v_\varphi = r\sin\theta\frac{\mathrm{d}\varphi}{\mathrm{d}t} = r\sin\theta\dot{\varphi},\ v_r = \frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r}.$$

In spherical coordinate frame, the substantial derivative of any vector F is given by

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}[F(t+\Delta t, \theta(t+\Delta t), \varphi(t+\Delta t), r(t+\Delta t)) \\ &\quad -F(t, \theta(t), \varphi(t), r(t))] \\ &= \left(\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\varphi}\frac{\partial}{\partial\varphi} + \dot{r}\frac{\partial}{\partial r}\right)F \\ &= \left(\frac{\partial}{\partial t} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} + v_r\frac{\partial}{\partial r}\right)F.\end{aligned}$$

Since $\boldsymbol{\nabla}_3 = \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} + \boldsymbol{e}_r\frac{\partial}{\partial r}$ in spherical coordinate frame, the substantial derivative in spherical coordinate frame is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_3.$$

By direct calculation, we have

$$\frac{\mathrm{d}\boldsymbol{e}_\theta}{\mathrm{d}t} = \frac{v_\varphi \cot\theta}{r}\boldsymbol{e}_\varphi - \frac{v_\theta}{r}\boldsymbol{e}_r,$$

$$\frac{\mathrm{d}\boldsymbol{e}_\varphi}{\mathrm{d}t} = -\frac{v_\varphi \cot\theta}{r}\boldsymbol{e}_\theta - \frac{v_\varphi}{r}\boldsymbol{e}_r,$$

$$\frac{\mathrm{d}\boldsymbol{e}_r}{\mathrm{d}t} = -\frac{v_\varphi}{r}\boldsymbol{e}_\varphi + \frac{v_\theta}{r}\boldsymbol{e}_\theta.$$

Angular velocity of earth rotation is given by $\boldsymbol{\Omega} = -\Omega\sin\theta\boldsymbol{e}_\theta + \Omega\cos\theta\boldsymbol{e}_r$, so

$$-2\boldsymbol{\Omega}\times\boldsymbol{V} = 2\Omega\cos\theta v_\varphi\boldsymbol{e}_\theta + (-2\Omega\cos\theta v_\theta - 2\Omega\sin\theta v_r)\boldsymbol{e}_\varphi + 2\Omega\sin\theta v_\varphi\boldsymbol{e}_r.$$

Using

$$\mathrm{div}_3\boldsymbol{V} = \boldsymbol{\nabla}_3\cdot\boldsymbol{V} = \frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r},$$

and

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} &= \frac{\mathrm{d}(v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi + v_r\boldsymbol{e}_r)}{\mathrm{d}t}\\ &= \boldsymbol{e}_\theta\frac{\mathrm{d}v_\theta}{\mathrm{d}t} + \boldsymbol{e}_\varphi\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \boldsymbol{e}_r\frac{\mathrm{d}v_r}{\mathrm{d}t} + v_\theta\frac{\mathrm{d}\boldsymbol{e}_\theta}{\mathrm{d}t} + v_\varphi\frac{\mathrm{d}\boldsymbol{e}_\varphi}{\mathrm{d}t} + v_r\frac{\mathrm{d}\boldsymbol{e}_r}{\mathrm{d}t}.\end{aligned}$$

We rewrite equations (1.1.1)-(1.1.4) following the basic equations of atmosphere in spherical coordinate frame

$$\begin{aligned}
&\frac{\mathrm{d}v_\theta}{\mathrm{d}t} + \frac{1}{r}(v_r v_\theta - v_\varphi^2\cot\theta) = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + 2\Omega\cos\theta v_\varphi + D_\theta,\\
&\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{1}{r}(v_r v_\varphi + v_\theta v_\varphi\cot\theta) = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} - 2\Omega\cos\theta v_\theta - 2\Omega\sin\theta v_r + D_\varphi,\\
&\frac{\mathrm{d}v_r}{\mathrm{d}t} - \frac{1}{r}(v_\theta^2 + v_\varphi^2) = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega\sin\theta v_\varphi + D_r,\\
&\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}\right) = 0,\\
&c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t},\\
&p = R\rho T,
\end{aligned}$$

where $D = (D_\theta, D_\varphi, D_r)$ is viscosity term.

Because the thickness of the atmospheric layer to be studied (about 120 kilometers) is far less than the radius of earth $a \approx 6,371$ kilometers, we use a instead of previous r which appears as coefficient in the above equations.

For the large-scale motions, the item $\frac{2v_r}{r}$ in the mass conservation equation can be omitted. Thus we simplify the above equations as

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} + \frac{1}{a}(v_r v_\theta - v_\varphi^2 \cot\theta) = -\frac{1}{\rho a}\frac{\partial p}{\partial \theta} + 2\Omega\cos\theta v_\varphi + D_\theta, \tag{1.2.1}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{1}{a}(v_r v_\varphi + v_\theta v_\varphi \cot\theta) = -\frac{1}{\rho a\sin\theta}\frac{\partial p}{\partial \varphi} - 2\Omega\cos\theta v_\theta - 2\Omega\sin\theta v_r + D_\varphi, \tag{1.2.2}$$

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} - \frac{1}{a}(v_\theta^2 + v_\varphi^2) = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega\sin\theta v_\varphi + D_r, \tag{1.2.3}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_r}{\partial r}\right) = 0, \tag{1.2.4}$$

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{1.2.5}$$

$$p = R\rho T, \tag{1.2.6}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{v_\theta}{a}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial}{\partial\varphi} + v_r\frac{\partial}{\partial r}$.

1.2.2 *Equations of the Oceans in the Sphere Coordinate Frame*

Suppose that the velocity of seawater is $\boldsymbol{V} = (u, v, w)$, and u, v, w are the velocity of seawater respectively in the direction of θ, φ, r. In spherical coordinate frame, the equations of the oceans under Boussinesq approximation are

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{a}(wu - v^2\cot\theta) = -\frac{1}{\rho_0 a}\frac{\partial p}{\partial\theta} + 2\Omega\cos\theta v + D_u, \tag{1.2.7}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{a}(wv + uv\cot\theta) = -\frac{1}{\rho_0 a\sin\theta}\frac{\partial p}{\partial\varphi} - 2\Omega\cos\theta u - 2\Omega\sin\theta w + D_v, \tag{1.2.8}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} - \frac{1}{a}(u^2 + v^2) = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} - \frac{\rho}{\rho_0}g + 2\Omega\sin\theta v + D_w, \tag{1.2.9}$$

$$\frac{1}{a\sin\theta}\frac{\partial u\sin\theta}{\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v}{\partial\varphi} + \frac{\partial w}{\partial r} = 0, \tag{1.2.10}$$

$$\rho = \rho_0[1 - \beta_T(T - T_0) + \beta_S(S - S_0)], \tag{1.2.11}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q_1, \tag{1.2.12}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q_2, \tag{1.2.13}$$

where $D = (D_u, D_v, D_w)$ is viscosity term.

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{u}{a}\frac{\partial}{\partial \theta} + \frac{v}{a\sin\theta}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial r}.$$

1.3 Equations of the Atmosphere in Atmospheric Pressure Coordinate Frame

The basic equations of atmosphere motions are so complicated that researcher is not able to solve them numerically or theoretically at present. Therefore, researchers have to omit the minor and medium scale factors, and simplify the basic equations of atmosphere motions reasonably in order to achieve numerical weather prediction. As the vertical scale of the atmosphere is far smaller than horizontal scale, the most natural simplification method is to adopt the **hydrostatic approximation**, that is, substituting the hydrostatic equilibrium equation

$$\frac{\partial p}{\partial r} = -\rho g$$

for the vertical momentum conservation equation. The hydrostatic equilibrium equation demonstrates the equilibrium relationship between the vertical pressure-gradient force and the gravity. It's in conformity with the weather observation data of the large-scale atmosphere, and also the theoretical analysis.

Here we use the scale analysis to interpret briefly the rationality of the hydrostatic approximation. For large-scale atmosphere motions, the horizontal characteristic length scale of the motion is $L \approx O(10^6)$, the vertical characteristic length scale of the motion is $D \approx O(10^4)$, the characteristic scale of horizontal velocity is $U \approx O(10^1)$, the characteristic scale of vertical velocity is $W \approx O(10^{-2})$, $\varOmega \approx O(10^{-4})$, and the characteristic scale of atmospheric pressure is $P \approx O(10^5)$. Thus we know, in the vertical momentum equation, except $-\frac{1}{\rho}\frac{\partial p}{\partial r} \approx O(10^1)$, $-g \approx O(10^1)$, scale of other terms is all less than $O(10^{-3})$. So we can replace the vertical momentum conservation equation by the hydrostatic equilibrium equation.

According to the hydrostatic equilibrium equation, we know that the pressure p is a monotonic decreasing function of r, that is, the mapping $(\theta, \varphi, r; t) \to (\theta, \varphi, p; t)$ is one-to-one. Thus, we substitute a pressure coordinate system $(\theta, \varphi, p; t)$ (also called the isobaric surface coordinate frame) for the coordinate frame $(\theta, \varphi, r; t)$. Introducing a new pressure coordinate frame $(\theta^*, \varphi^*, p; t^*)$, we have

$$t^* = t,\ \theta^* = \theta,\ \varphi^* = \varphi,\ p = p(\theta, \varphi, r; t).$$

Now, let's deduce the form of the atmospheric equations in the new pressure coordinate frame $(\theta^*, \varphi^*, p; t^*)$. Firstly, in the pressure coordinate frame, the substantial derivative of any vector F is

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t^*} &= \left(\frac{\partial}{\partial t^*} + \dot{\theta^*}\frac{\partial}{\partial \theta^*} + \dot{\varphi^*}\frac{\partial}{\partial \varphi^*} + \dot{p}\frac{\partial}{\partial p}\right) F \\ &= \left(\frac{\partial}{\partial t^*} + \dot{\theta}\frac{\partial}{\partial \theta^*} + \dot{\varphi}\frac{\partial}{\partial \varphi^*} + \dot{p}\frac{\partial}{\partial p}\right) F. \end{aligned}$$

To obtain a new form of the momentum equation in the new pressure coordinate frame, we only compute a new form of the force here. In meteorology, usually substitute height $z = r - a$ for r, thus, the original coordinate is expressed as a function of the new coordinates $t = t^*,\ \theta = \theta^*,\ \varphi = \varphi^*,\ z = r - a = z(\theta^*, \varphi^*, p; t^*)$. So we have

$$p = p(\theta, \varphi, a + z(\theta^*, \varphi^*, p; t^*); t).$$

Differentiating the above function with respect to the variable p, we have

$$1 = \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial r}{\partial p} = \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial z}{\partial p},$$

where, to distinguish derivatives in the two coordinate frames, we use $\dfrac{\tilde{\partial} p}{\tilde{\partial} r}$ to indicate the derivative of p with respect to r in the original coordinate frame, $\dfrac{\partial z}{\partial p}$ to indicate the derivative of z with respect of p in the new coordinates frame. The following symbols in this section are defined similarly. Taking differential quotient of θ^* and φ^* in the above relationship of p, we have

$$0 = \frac{\tilde{\partial} p}{\tilde{\partial} \theta}\frac{\partial \theta}{\partial \theta^*} + \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial r}{\partial \theta^*} = \frac{\tilde{\partial} p}{\tilde{\partial} \theta} + \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial r}{\partial \theta^*},$$

$$0 = \frac{\tilde{\partial} p}{\tilde{\partial} \varphi}\frac{\partial \varphi}{\partial \varphi^*} + \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial r}{\partial \varphi^*} = \frac{\tilde{\partial} p}{\tilde{\partial} \varphi} + \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial r}{\partial \varphi^*}.$$

Combining the above two equations with the hydrostatic equilibrium equation, we obtain

$$-\frac{1}{\rho a}\frac{\tilde{\partial} p}{\tilde{\partial} \theta} = -\frac{1}{a}\frac{\partial \Phi}{\partial \theta^*}, \quad -\frac{1}{\rho a \sin\theta}\frac{\tilde{\partial} p}{\tilde{\partial} \varphi} = -\frac{1}{a \sin\theta^*}\frac{\partial \Phi}{\partial \varphi^*},$$

where $\Phi = gz$ is generally called the geopotential. With the equation $1 = \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial z}{\partial p}$ and the hydrostatic equilibrium equation, we have

$$p\frac{\partial \Phi}{\partial p} = -RT.$$

Therefore, we get the new form of momentum equations in the new pressure coordinate frame as follows

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t^*} - \frac{1}{a}v_\varphi^2 \cot\theta^* = -\frac{1}{a}\frac{\partial \Phi}{\partial \theta^*} + 2\Omega\cos\theta^* v_\varphi + D_\theta, \tag{1.3.1}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t^*} + \frac{1}{a}v_\theta v_\varphi \cot\theta^* = -\frac{1}{a\sin\theta^*}\frac{\partial \Phi}{\partial \varphi^*} - 2\Omega\cos\theta^* v_\theta + D_\varphi, \tag{1.3.2}$$

$$p\frac{\partial \Phi}{\partial p} = -RT. \tag{1.3.3}$$

According to the principle that Coriolis force does no work, we omit the term $-2\Omega\sin\theta^* v_r$. Similarly, because the scale of v_r is very small to the large scale atmospheric motions, we also omit $\frac{1}{a}v_r v_\theta$, $\frac{1}{a}v_r v_\varphi$.

Next, let's deduce a new form of the mass conservation equation in the new pressure coordinate frame. According to the hydrostatic equilibrium equation, we have $\rho = -\frac{1}{g}\frac{\tilde{\partial} p}{\tilde{\partial} r}$. Substituting this equality in (1.2.4), we have

$$\frac{\tilde{\mathrm{d}}\frac{\tilde{\partial} p}{\tilde{\partial} r}}{\tilde{\mathrm{d}}t} + \frac{\tilde{\partial} p}{\tilde{\partial} r}\left(\frac{1}{a\sin\theta}\frac{\tilde{\partial} v_\theta \sin\theta}{\tilde{\partial}\theta} + \frac{1}{a\sin\theta}\frac{\tilde{\partial} v_\varphi}{\tilde{\partial}\varphi} + \frac{\tilde{\partial} v_r}{\tilde{\partial} r}\right) = 0. \tag{1.3.4}$$

With the definition of substantial derivative in the original coordinate

$$\frac{\tilde{\mathrm{d}}}{\tilde{\mathrm{d}}t} = \frac{\tilde{\partial}}{\tilde{\partial} t} + \frac{v_\theta}{a}\frac{\tilde{\partial}}{\tilde{\partial}\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\tilde{\partial}}{\tilde{\partial}\varphi} + v_r\frac{\tilde{\partial}}{\tilde{\partial} r},$$

and

$$\frac{\tilde{\mathrm{d}}p}{\tilde{\mathrm{d}}t} = \frac{\tilde{\partial} p}{\tilde{\partial} t} + \frac{v_\theta}{a}\frac{\tilde{\partial} p}{\tilde{\partial}\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\tilde{\partial} p}{\tilde{\partial}\varphi} + v_r\frac{\tilde{\partial} p}{\tilde{\partial} r},$$

we obtain

$$\frac{\tilde{\mathrm{d}}\frac{\tilde{\partial} p}{\tilde{\partial} r}}{\tilde{\mathrm{d}}t} = \frac{\tilde{\partial}\frac{\tilde{\mathrm{d}}p}{\tilde{\mathrm{d}}t}}{\tilde{\partial} r} - \frac{\tilde{\partial} v_\theta}{\tilde{\partial} r}\frac{\tilde{\partial} p}{a\tilde{\partial}\theta} - \frac{\tilde{\partial} v_\varphi}{\tilde{\partial} r}\frac{\tilde{\partial} p}{a\sin\theta\tilde{\partial}\varphi} - \frac{\tilde{\partial} v_r}{\tilde{\partial} r}\frac{\tilde{\partial} p}{\tilde{\partial} r}. \tag{1.3.5}$$

With the relationship between the pressure coordinate frame and the original coordinate frame, we have

$$\frac{\tilde{\partial}}{\tilde{\partial} r} = \frac{\tilde{\partial} p}{\tilde{\partial} r}\frac{\partial}{\partial p},\ \frac{\tilde{\partial}}{\tilde{\partial}\theta} = \frac{\partial}{\partial\theta^*} + \frac{\tilde{\partial} p}{\tilde{\partial}\theta}\frac{\partial}{\partial p},\ \frac{\tilde{\partial}}{\tilde{\partial}\varphi} = \frac{\partial}{\partial\varphi^*} + \frac{\tilde{\partial} p}{\tilde{\partial}\varphi}\frac{\partial}{\partial p}.$$

Thus,

$$
\begin{aligned}
&\frac{\tilde{\partial}\dfrac{\tilde{\mathrm{d}}p}{\tilde{\mathrm{d}}t}}{\tilde{\partial}r} = \frac{\tilde{\partial}\dot{p}}{\tilde{\partial}r} = \frac{\tilde{\partial}p}{\tilde{\partial}r}\frac{\partial\dot{p}}{\partial p},\\
&-\frac{\tilde{\partial}v_\theta}{\tilde{\partial}r}\frac{\tilde{\partial}p}{a\tilde{\partial}\theta} + \frac{\tilde{\partial}p}{\tilde{\partial}r}\frac{\tilde{\partial}v_\theta\sin\theta}{a\sin\theta\tilde{\partial}\theta}\\
&= -\frac{\tilde{\partial}p}{\tilde{\partial}r}\left(\frac{\tilde{\partial}p}{a\tilde{\partial}\theta}\frac{\partial v_\theta}{\partial p} - \frac{\tilde{\partial}v_\theta\sin\theta}{a\sin\theta\tilde{\partial}\theta}\right)\\
&= -\frac{\tilde{\partial}p}{\tilde{\partial}r}\left[\frac{\tilde{\partial}p}{a\tilde{\partial}\theta}\frac{\partial v_\theta}{\partial p} - \frac{1}{a\sin\theta}\left(\frac{\partial}{\partial\theta^*} + \frac{\tilde{\partial}p}{\tilde{\partial}\theta}\frac{\partial}{\partial p}\right)v_\theta\sin\theta\right]\\
&= \frac{\tilde{\partial}p}{\tilde{\partial}r}\left(\frac{\partial v_\theta\sin\theta^*}{a\sin\theta^*\partial\theta^*}\right),\\
&-\frac{\tilde{\partial}v_\varphi}{\tilde{\partial}r}\frac{\tilde{\partial}p}{a\sin\theta\tilde{\partial}\varphi} + \frac{\tilde{\partial}p}{\tilde{\partial}r}\frac{\tilde{\partial}v_\varphi}{a\sin\theta\tilde{\partial}\varphi}\\
&= -\frac{\tilde{\partial}p}{\tilde{\partial}r}\left(\frac{\tilde{\partial}p}{a\sin\theta\tilde{\partial}\varphi}\frac{\partial v_\varphi}{\partial p} - \frac{\tilde{\partial}v_\varphi}{a\sin\theta\tilde{\partial}\varphi}\right)\\
&= -\frac{\tilde{\partial}p}{\tilde{\partial}r}\left[\frac{\tilde{\partial}p}{a\sin\theta\tilde{\partial}\varphi}\frac{\partial v_\varphi}{\partial p} - \frac{1}{a\sin\theta}\left(\frac{\partial}{\partial\varphi^*} + \frac{\tilde{\partial}p}{\tilde{\partial}\varphi}\frac{\partial}{\partial p}\right)v_\varphi\right]\\
&= \frac{\tilde{\partial}p}{\tilde{\partial}r}\left(\frac{\partial v_\varphi}{a\sin\theta^*\partial\varphi^*}\right).
\end{aligned}
$$

In the process of verifying the first equality, we have used the relationship $p = \dfrac{\tilde{\mathrm{d}}p}{\tilde{\mathrm{d}}t} = \dfrac{\mathrm{d}p}{\mathrm{d}t^*}$. Combining the above three equations together, we deduce the continuity equation in the pressure coordinate frame from (1.3.4) and (1.3.5)

$$\frac{\partial\dot{p}}{\partial p} + \frac{1}{a\sin\theta^*}\left(\frac{\partial v_\theta\sin\theta^*}{\partial\theta^*} + \frac{\partial v_\varphi}{\partial\varphi^*}\right) = 0, \tag{1.3.6}$$

the thermodynamic equation in the pressure coordinate frame is

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t^*} - \frac{RT}{p}\dot{p} = \frac{\mathrm{d}Q}{\mathrm{d}t^*}. \tag{1.3.7}$$

The equations (1.3.1)-(1.3.3), (1.3.6) and (1.3.7) are known as **the dry atmospheric equations in pressure coordinate frame,** where

$$\frac{\mathrm{d}}{\mathrm{d}t^*} = \frac{\partial}{\partial t^*} + \dot{\theta}^*\frac{\partial}{\partial\theta^*} + \dot{\varphi}^*\frac{\partial}{\partial\varphi^*} + \dot{p}\frac{\partial}{\partial p}.$$

With definitions of the substantial derivative and the hydrostatic equilibrium equation in pressure coordinate frame, we get the vertical velocity given by

$$\begin{aligned} v_r = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial t^*} + \dot{\theta}^* \frac{\partial z}{\partial \theta^*} + \dot{\varphi}^* \frac{\partial z}{\partial \varphi^*} + \dot{p}\frac{\partial z}{\partial p} \\ &= \frac{\partial z}{\partial t^*} + \frac{v_\theta}{a}\frac{\partial z}{\partial \theta^*} + \frac{v_\varphi}{a \sin\varphi^*}\frac{\partial z}{\partial \varphi^*} + \dot{p}\frac{\partial z}{\partial p} \\ &= \frac{\partial z}{\partial t^*} + \frac{v_\theta}{a}\frac{\partial z}{\partial \theta^*} + \frac{v_\varphi}{a \sin\varphi^*}\frac{\partial z}{\partial \varphi^*} - \frac{\dot{p}}{\rho g}. \end{aligned}$$

1.4 Equations of the Atmosphere in the Topography Coordinate Frame

In the practical case, researchers sometimes need to consider the variation of topography of the earth surface. As the earth surface is not an isobaric surface in this situation, we usually can't take the pressure coordinate frame, otherwise it's difficult to give a reasonable lower boundary condition. Therefore, we take the following topography coordinate $(\theta, \varphi, \pi; t)$, that is

$$t = t^*,\ \theta = \theta^*,\ \varphi = \varphi^*,\ \pi = \pi\left(\frac{p}{p_s}\right),$$

where $p_s(\theta^*, \varphi^*; t^*)$ denotes pressure of the earth surface, and π is a strictly monotonic function of $\dfrac{p}{p_s}$. Here $\pi(1)$ denotes the earth surface, and $\pi(0)$ denotes the upper boundary of atmosphere. Here, we suppose $\pi = \zeta = \dfrac{p}{p_s}$.

Then, let's deduce the form in the new topography coordinate $(\theta, \varphi, \zeta; t)$ of the equations of the atmosphere in the pressure coordinate frame $(\theta^*, \varphi^*, p; t^*)$, which appear in the above section. First, in the topography coordinate frame, the substantial derivative of any vector F is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\varphi}\frac{\partial}{\partial \varphi} + \dot{\zeta}\frac{\partial}{\partial \zeta}\right) F = \left(\frac{\partial}{\partial t} + \dot{\theta}^*\frac{\partial}{\partial \theta} + \dot{\varphi}^*\frac{\partial}{\partial \varphi} + \dot{\zeta}\frac{\partial}{\partial \zeta}\right) F,$$

where

$$\dot{\zeta} = \frac{p_s \dot{p} - p\dot{p}_s}{p_s^2}.$$

Applying the relationship between the pressure coordinate frame and the topography coordinate frame, we have

$$\frac{\bar{\partial}}{\bar{\partial} p} = \frac{\bar{\partial}\zeta}{\bar{\partial} p}\frac{\partial}{\partial \zeta} = \frac{1}{p_s}\frac{\partial}{\partial \zeta}, \tag{1.4.1}$$

$$\frac{\bar{\partial}}{\bar{\partial}\theta^*} = \frac{\partial}{\partial\theta} + \frac{\bar{\partial}\zeta}{\bar{\partial}\theta^*}\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial\theta} - \frac{\zeta}{p_s}\frac{\partial p_s}{\partial\theta}\frac{\partial}{\partial\zeta}, \tag{1.4.2}$$

$$\frac{\bar{\partial}}{\bar{\partial}\varphi^*} = \frac{\partial}{\partial\varphi} + \frac{\bar{\partial}\zeta}{\bar{\partial}\varphi^*}\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial\varphi} - \frac{\zeta}{p_s}\frac{\partial p_s}{\partial\varphi}\frac{\partial}{\partial\zeta}. \tag{1.4.3}$$

To distinguish derivatives in the two coordinate frames, we use here $\frac{\bar{\partial}\zeta}{\bar{\partial}p}$ to denote the derivative of ζ with respect to p in the original coordinate, $\frac{\partial}{\partial\zeta}$ to denote in the new coordinate. The following symbols are defined in a similar way. With (1.4.1) and equation $p\frac{\partial\Phi}{\partial p} = -RT$, we get

$$\zeta\frac{\partial\Phi}{\partial\zeta} = -RT. \tag{1.4.4}$$

We obtain the momentum equation in the new topography coordinate frame. Here we only give the form of the pressure-gradient force. Using (1.4.2), (1.4.3) and (1.4.4), we have

$$-\frac{1}{a}\frac{\bar{\partial}\Phi}{\bar{\partial}\theta^*} = -\frac{\partial\Phi}{a\partial\theta} + \frac{\zeta}{ap_s}\frac{\partial p_s}{\partial\theta}\frac{\partial\Phi}{\partial\zeta} = -\frac{\partial\Phi}{a\partial\theta} - \frac{RT}{ap_s}\frac{\partial p_s}{\partial\theta},$$

$$-\frac{1}{a\sin\theta^*}\frac{\bar{\partial}\Phi}{\bar{\partial}\varphi^*} = -\frac{\partial\Phi}{a\sin\theta\partial\varphi} + \frac{\zeta}{a\sin\theta p_s}\frac{\partial p_s}{\partial\varphi}\frac{\partial\Phi}{\partial\varphi} = -\frac{\partial\Phi}{a\sin\theta\partial\varphi} - \frac{RT}{a\sin\theta p_s}\frac{\partial p_s}{\partial\varphi}.$$

According to the above two equations, we get the horizontal momentum equation in the new topography coordinate frame as

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{1}{a}v_\varphi^2\cot\theta = -\left(\frac{\partial\Phi}{a\partial\theta} + \frac{RT}{ap_s}\frac{\partial p_s}{\partial\theta}\right) + 2\Omega\cos\theta v_\varphi + D_\theta, \tag{1.4.5}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{1}{a}v_\theta v_\varphi\cot\theta = -\left(\frac{\partial\Phi}{a\sin\theta\partial\varphi} + \frac{RT}{a\sin\theta p_s}\frac{\partial p_s}{\partial\varphi}\right) - 2\Omega\cos\theta v_\theta + D_\varphi. \tag{1.4.6}$$

Next, let's deduce the mass conservation equation in the new topography coordinate frame. With $\dot{\zeta} = \frac{p_s\dot{p} - p\dot{p}_s}{p_s^2}$, we have

$$\dot{p} = \dot{\zeta}p_s + \zeta\dot{p}_s.$$

Applying (1.4.1) and the above equation, we get

$$\frac{\bar{\partial}\dot{p}}{\bar{\partial}p} = \frac{\bar{\partial}\dot{\zeta}p_s}{\bar{\partial}p} + \frac{\bar{\partial}\zeta\dot{p}_s}{\bar{\partial}p} = \frac{\partial\dot{\zeta}}{\partial\zeta} + \frac{\dot{p}_s}{p_s} + \frac{\zeta}{p_s}\frac{\partial\dot{p}_s}{\partial\zeta}. \tag{1.4.7}$$

With (1.4.2) and (1.4.3), we have

$$\begin{aligned}
&\frac{1}{a\sin\theta^*}\left(\frac{\partial v_\theta \sin\theta^*}{\partial\theta^*}+\frac{\partial v_\varphi}{\partial\varphi^*}\right)\\
=&\frac{1}{a\sin\theta}\left[\left(\frac{\partial v_\theta\sin\theta}{\partial\theta}-\frac{\zeta}{p_s}\frac{\partial p_s}{\partial\theta}\frac{\partial v_\theta\sin\theta}{\partial\zeta}\right)+\left(\frac{\partial v_\varphi}{\partial\varphi}-\frac{\zeta}{p_s}\frac{\partial p_s}{\partial\varphi}\frac{\partial v_\varphi}{\partial\zeta}\right)\right]\\
=&\frac{1}{a\sin\theta}\left(\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{\partial v_\varphi}{\partial\varphi}\right)-\frac{\zeta}{a\sin\theta p_s}\left(\frac{\partial p_s}{\partial\theta}\frac{\partial v_\theta\sin\theta}{\partial\zeta}+\frac{\partial p_s}{\partial\varphi}\frac{\partial v_\varphi}{\partial\zeta}\right).
\end{aligned} \tag{1.4.8}$$

Since $p_s(\theta^*,\varphi^*;t^*)=p_s(\theta,\varphi;t)$, we get the following system by the definition of substantial derivative in the topography coordinate frame

$$\dot{p_s}=\frac{\partial p_s}{\partial t}+\frac{v_\theta}{a}\frac{\partial p_s}{\partial\theta}+\frac{v_\varphi}{a\sin\theta}\frac{\partial p_s}{\partial\varphi}, \tag{1.4.9}$$

$$\frac{\zeta}{p_s}\frac{\partial\dot{p_s}}{\partial\zeta}=\frac{\zeta}{p_s}\left(\frac{\partial p_s}{a\partial\theta}\frac{\partial v_\theta}{\partial\zeta}+\frac{\partial p_s}{a\sin\theta\partial\varphi}\frac{\partial v_\varphi}{\partial\zeta}\right). \tag{1.4.10}$$

Substituting (1.4.7) and (1.4.8) into (1.3.6), and applying (1.4.9) and (1.4.10), we deduce the continuity equation in the new topography coordinate frame as

$$\frac{\partial p_s}{\partial t}=-\frac{\partial p_s\dot{\zeta}}{\partial\zeta}-\frac{1}{a\sin\theta}\left(\frac{\partial p_s v_\theta\sin\theta}{\partial\theta}+\frac{\partial p_s v_\varphi}{\partial\varphi}\right). \tag{1.4.11}$$

According to $\dot{p}=\dot{\zeta}p_s+\zeta\dot{p_s}$, we obtain the thermodynamic equation in the new topography coordinate frame

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t}-\frac{RT}{\zeta p_s}(\dot{\zeta}p_s+\zeta\dot{p_s})=\frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{1.4.12}$$

We denote equations (1.4.4)-(1.4.6), (1.4.11) and (1.4.12) as **the dry atmospheric equations in topography coordinate frame,** where

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{\theta}\frac{\partial}{\partial\theta}+\dot{\varphi}\frac{\partial}{\partial\varphi}+\dot{\zeta}\frac{\partial}{\partial\zeta}=\frac{\partial}{\partial t}+\frac{v_\theta}{a}\frac{\partial}{\partial\theta}+\frac{v_\varphi}{a\sin\theta}\frac{\partial}{\partial\varphi}+\dot{\zeta}\frac{\partial}{\partial\zeta}.$$

When studying the atmospheric motions below specific barometric altitude (supposing the upper bound of atmospheric pressure is $p=p_0$, where p_0 is a positive constant), researchers use the modified topography coordinate frame $(\theta,\varphi,\boldsymbol{\pi};t)$, where

$$t=t^*,\ \theta=\theta^*,\ \varphi=\varphi^*,\ \boldsymbol{\pi}=\boldsymbol{\pi}(\eta),\ \eta=\frac{p-p_0}{p_s-p_0}.$$

With the above methods, we can also deduce the atmospheric equations in the modified topography coordinate frame.

1.5 Equations of the Atmosphere and the Oceans in Local Rectangular Coordinate Frame under β-Plane Approximation

For the oceanic motions, researchers sometimes are concerned with some local areas containing no polar region. One can simplify the local earth surface into a flat plane, then choose the local rectangular coordinate frame. In the same way, studying the properties of local atmospheric motions, we can also choose local rectangular coordinate frame. Now let's discuss the oceanic equations in local rectangular coordinate frame.

Let O as a point on the sea level (its co-latitude is θ_0), the forward direction of x-axis point to the south, y-axis to the east, z-axis vertically upward, $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ in the three-coordinate direction of local rectangular coordinate frame $\{O; x, y, z\}$ are respectively unit constant vectors in x, y, z axis directions. According to the definition of velocity, the velocity $\boldsymbol{V}$ is expressed as

$$\boldsymbol{V} = u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z,$$

where

$$u = \frac{\mathrm{d}x}{\mathrm{d}t}, \ v = \frac{\mathrm{d}y}{\mathrm{d}t}, \ w = \frac{\mathrm{d}z}{\mathrm{d}t}.$$

In the local rectangular coordinate frame, the total derivative of any vector F is

$$\begin{aligned}&\frac{\mathrm{d}F}{\mathrm{d}t}\\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[F(t+\Delta t, x(t+\Delta t), y(t+\Delta t), z(t+\Delta t)) - F(t, x(t), y(t), z(t))]\\ &= \left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)F = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)F.\end{aligned}$$

As in the local rectangular coordinate frame, the gradient operator $\boldsymbol{\nabla}_3 = \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z}$, the total derivative in the local rectangular coordinate frame is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}_3.$$

To acquire the oceanic equations in the local rectangular coordinate system, we have to find the approximate form of Coriolis force $-2\boldsymbol{\Omega}\times\boldsymbol{V}$ in the local rectangular coordinate frame. In the sphere coordinate frame,

$$-2\boldsymbol{\Omega} \times \boldsymbol{V} = 2\Omega\cos\theta v\boldsymbol{e}_\theta + (-2\Omega\cos\theta u - 2\Omega\sin\theta w)\boldsymbol{e}_\varphi + 2\Omega\sin\theta v\boldsymbol{e}_r.$$

Because $2\Omega\sin\theta w \ll 2\Omega\cos\theta u$, $2\Omega\sin\theta v \ll g$, in general situations, one can omit $2\Omega\sin\theta w$, $2\Omega\sin\theta v$. Thus, we just have to find the approximate forms of $2\Omega\cos\theta v$, $2\Omega\cos\theta u$ in the local rectangular coordinate frame, that is, we should deal with the Coriolis parameter $f = 2\Omega\cos\theta$. Using the Taylor expansion of $f = 2\Omega\cos\theta$ at θ_0, we get

$$f = 2\Omega\cos\theta = 2\Omega[\cos\theta_0 - (\theta-\theta_0)\sin\theta_0 - (\theta-\theta_0)^2\frac{\cos\theta_0}{2!} + (\theta-\theta_0)^3\frac{\sin\theta_0}{3!} + \cdots].$$

When $\theta - \theta_0$ is small enough, there is

$$\theta - \theta_0 \approx \frac{x}{a},$$

where a is the earth radius. When $\frac{y}{a} \ll 1$, we take

$$f \approx f_0 = 2\Omega\cos\theta_0,$$

which means, the Coriolis parameter f can be considered as a constant. So, when $\frac{y}{a} < 1$, and $(\frac{y}{a})^2 \ll 1$. So we can take

$$f \approx 2\Omega\cos\theta_0 - x\frac{2\Omega\sin\theta_0}{a} = f_0 - \beta_0 x.$$

Here $\beta_0 = \dfrac{2\Omega\sin\theta_0}{a}$ is the origin value of Rossby parameter $\beta = \dfrac{2\Omega\sin\theta}{a}$ in the local rectangular coordinate frame. The above equation is usually called β-plane approximation. The Coriolis parameter can be considered as a linear function of x in the local rectangular coordinate frame.

Remark 1.5.1. If O is a point on the sea level (its latitude is $\phi_0 = \frac{\pi}{2} - \theta_0$), the forward direction of x-axis point to the east, y-axis to the north, z-axis vertically upward, then the β-plane approximation is given by

$$f = 2\Omega\sin\phi \approx 2\Omega\sin\phi_0 + y\frac{2\Omega\cos\phi_0}{a} = f_0 + \beta_0 y.$$

After taking Boussinesq approximation (except buoyancy term and density in the state equation, densities of other position are all considered constant) and β-plane approximation, we get oceanic equations in the local rectangular coordinate frame

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + f_\beta v + D_u, \tag{1.5.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - f_\beta u + D_v, \tag{1.5.2}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g + D_w, \tag{1.5.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.5.4}$$

$$\rho = \rho_0[1 - \beta_T(T - T_0) + \beta_S(S - S_0)], \tag{1.5.5}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q_1, \tag{1.5.6}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q_2, \tag{1.5.7}$$

where $f_\beta = f_0 - \beta_0 x$, $D = (D_u, D_v, D_w)$ is the viscosity term, and

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

In the same way, one can obtain the equations of the dry atmosphere (without hydrostatic approximation) in the local rectangular coordinate frame

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_\beta v + D_u, \tag{1.5.8}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f_\beta u + D_v, \tag{1.5.9}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + D_w, \tag{1.5.10}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0, \tag{1.5.11}$$

$$p = R\rho T, \tag{1.5.12}$$

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{1.5.13}$$

where $f_\beta = f_0 - \beta_0 x$, $D = (D_u, D_v, D_w)$ is the viscosity term.

1.6 Equations of the Atmosphere and the Oceans under Satification Approximation

The inhomogeneous heating of sun to earth causes noticeable density variance of the atmosphere and oceans, thus the density statification in the atmosphere and oceans. One observational property of statification is that, in general, the large-scale atmosphere and oceans are always gravity stable, that is, the lighter fluids are always above the heavier fluids. The atmospheric density is almost monotonic decreasing with the height, and the ocean density is almost monotonic increasing with the depth. The existence of stable statification in the atmosphere and oceans, which results in the inhibition of vertical motion, contributes to the formation of the almost horizontal motions.

Basing on the existence of stable statification in the atmosphere, researchers can think that the large-scale atmospheric motions are that the oscillations occur near the average state $((v_\theta, v_\varphi, \dot{p}, \Phi, T) = (0, 0, 0, \bar{\Phi}(p), \bar{T}(p))$, also called standard state), where $\bar{\Phi}(p)$ and $\bar{T}(p)$ satisfy the hydrostatic equilibrium relationship, that is,

$$p\frac{\partial \bar{\Phi}(p)}{\partial p} = -R\bar{T}(p).$$

Next, let's deduce the equations for perturbation states near the average state $(0, 0, 0, \bar{\Phi}(p), \bar{T}(p))$ from (1.3.1)–(1.3.3), (1.3.6) and (1.3.7).
Suppose

$$(v_\theta, v_\varphi, \dot{p}) = (0 + v'_\theta, 0 + v'_\varphi, 0 + \dot{p}'),$$

$$\Phi = \bar{\Phi}(p) + \Phi', \quad T = \bar{T}(p) + T'.$$

The equations (1.3.1)-(1.3.3) and (1.3.6) are written as

$$\frac{\mathrm{d}v'_\theta}{\mathrm{d}t} - \frac{1}{a}v'^2_\varphi \cot\theta^* = -\frac{1}{a}\frac{\partial \Phi'}{\partial \theta^*} + 2\Omega\cos\theta^* v'_\varphi + D_\theta, \tag{1.6.1}$$

$$\frac{\mathrm{d}v'_\varphi}{\mathrm{d}t} + \frac{1}{a}v'_\theta v'_\varphi \cot\theta^* = -\frac{1}{a\sin\theta^*}\frac{\partial \Phi'}{\partial \varphi^*} - 2\Omega\cos\theta^* v'_\theta + D_\varphi, \tag{1.6.2}$$

$$p\frac{\partial \Phi'}{\partial p} = -RT', \tag{1.6.3}$$

$$\frac{\partial \dot{p}'}{\partial p} + \frac{1}{a\sin\theta^*}\left(\frac{\partial v'_\theta \sin\theta^*}{\partial \theta^*} + \frac{\partial v'_\varphi}{\partial \varphi^*}\right) = 0. \tag{1.6.4}$$

Substituting $T = \bar{T}(p) + T'$ into equation (1.3.7), we have

$$c_p\frac{\mathrm{d}T'}{\mathrm{d}t} + c_p\frac{\partial \bar{T}(p)}{\partial p}\dot{p} - \frac{R(\bar{T}(p) + T')}{p}\dot{p} = \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

As $|T - T'| \ll \dfrac{p}{R\dot{p}}$, we take $\dfrac{R(\bar{T}(p) + T')}{p}\dot{p} \approx \dfrac{R\bar{T}(p)}{p}\dot{p}$. Thus

$$c_p\frac{\mathrm{d}T'}{\mathrm{d}t} + \left(c_p\frac{\partial \bar{T}(p)}{\partial p} - \frac{R\bar{T}(p)}{p}\right)\dot{p} = \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

If $\bar{T}(p)$ satisfies $R\left(\dfrac{R\bar{T}(p)}{C_p} - p\dfrac{\partial \bar{T}(p)}{\partial p}\right) = C^2$, where C is a positive constant, then

$$c_p\frac{\mathrm{d}T'}{\mathrm{d}t} - \frac{c_p C^2}{pR}\dot{p} = \frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{1.6.5}$$

Getting rid of "$'$" and "$*$", introducing viscosity to the equation, and letting $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = \mu_2 \Delta T + \nu_2 \dfrac{\partial}{\partial p}[(\dfrac{gp}{R\bar{T}})^2 \dfrac{\partial T}{\partial p}] + F(\theta, \varphi, p)$, we write equations (1.6.1)-(1.6.5) as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{v} + \omega \frac{\partial \boldsymbol{v}}{\partial p} + f \boldsymbol{k} \times \boldsymbol{v} + \operatorname{grad} \varPhi - \mu_1 \Delta \boldsymbol{v} - \nu_1 \frac{\partial}{\partial p} \left[\left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial \boldsymbol{v}}{\partial p} \right] = 0, \tag{1.6.6}$$

$$\operatorname{div} \boldsymbol{v} + \frac{\partial \omega}{\partial p} = 0, \tag{1.6.7}$$

$$\frac{\partial \varPhi}{\partial p} + \frac{bP}{p} T = 0, \tag{1.6.8}$$

$$\frac{R^2}{C^2} \left(\frac{\partial T}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{v}} T + \omega \frac{\partial T}{\partial p} \right) - \frac{R}{p} \omega - \mu_2 \Delta T - \nu_2 \frac{\partial}{\partial p} \left[\left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial T}{\partial p} \right] = F, \tag{1.6.9}$$

where $\omega = \dot{p}$,

$$\begin{aligned}
\boldsymbol{\nabla}_{\boldsymbol{v}} \widetilde{\boldsymbol{v}} = & \left(\frac{v_\theta}{a} \frac{\partial \widetilde{v_\theta}}{\partial \theta} + \frac{v_\varphi}{a \sin\theta} \frac{\partial \widetilde{v_\theta}}{\partial \varphi} - \frac{v_\varphi \widetilde{v_\varphi}}{a} \cot\theta \right) \boldsymbol{e}_\theta \\
& + \left(\frac{v_\theta}{a} \frac{\partial \widetilde{v_\varphi}}{\partial \theta} + \frac{v_\varphi}{a \sin\theta} \frac{\partial \widetilde{v_\varphi}}{\partial \varphi} + \frac{v_\varphi \widetilde{v_\theta}}{a} \cot\theta \right) \boldsymbol{e}_\varphi, \\
\operatorname{grad} \varPhi = & \frac{\partial \varPhi}{a \partial \theta} \boldsymbol{e}_\theta + \frac{1}{a \sin\theta} \frac{\partial \varPhi}{\partial \varphi} \boldsymbol{e}_\varphi, \\
\Delta \boldsymbol{v} = & \left(\Delta v_\theta - \frac{2\cos\theta}{a^2 \sin^2\theta} \frac{\partial v_\varphi}{\partial \varphi} - \frac{v_\theta}{a^2 \sin^2\theta} \right) \boldsymbol{e}_\theta \\
& + \left(\Delta v_\varphi + \frac{2\cos\theta}{a^2 \sin^2\theta} \frac{\partial v_\theta}{\partial \varphi} - \frac{v_\varphi}{a^2 \sin^2\theta} \right) \boldsymbol{e}_\varphi, \\
\operatorname{div} \boldsymbol{v} = & \operatorname{div}(v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi) = \frac{1}{a \sin\theta} \left(\frac{\partial v_\theta \sin\theta}{\partial \theta} + \frac{\partial v_\varphi}{\partial \varphi} \right), \\
\boldsymbol{\nabla}_{\boldsymbol{v}} T = & \frac{v_\theta}{a} \frac{\partial T}{\partial \theta} + \frac{v_\varphi}{a \sin\theta} \frac{\partial T}{\partial \varphi}, \\
\Delta T = & \frac{1}{a^2 \sin\theta} \left[\frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 T}{\partial \varphi^2} \right],
\end{aligned}$$

where $\boldsymbol{v} = v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi$, and $\widetilde{\boldsymbol{v}} = \widetilde{v_\theta} \boldsymbol{e}_\theta + \widetilde{v_\varphi} \boldsymbol{e}_\varphi$. In the atmosphere science, the equations (1.6.6)–(1.6.9) are also called the dry atmospheric primitive equations.

For the ocean equations (1.5.1)-(1.5.7) in Boussinesq approximation, ignoring salinity, taking $\rho = \rho_0[1 - \beta_T(T - T_0)]$, and introducing viscosity, we have the following equations

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + f_\beta v + \nu\Delta u,\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - f_\beta u + \nu\Delta v,\\
\frac{\mathrm{d}w}{\mathrm{d}t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g + \nu\Delta w,\\
\frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \kappa\Delta\rho.
\end{aligned}$$

$(u, v, w, p, \rho) = (0, 0, 0, \bar{p}(z), \bar{\rho}(z))$ is a solution of the above equations, which is always denoted as an average state (or a standard state). Where $\bar{p}(z)$ and $\bar{\rho}(z)$ satisfy the hydrostatic equilibrium relationship, that is,

$$\frac{\partial \bar{p}(z)}{\partial z} = -g\bar{\rho}(z).$$

Researchers can consider motions near the average state $(0, 0, 0, \bar{p}(z), \bar{\rho}(z))$. Next, let's deduce equations for motions perturbed near the average state $(0, 0, 0, \bar{p}(z), \bar{\rho}(z))$. Supposing

$$(u, v, w) = (0 + u', 0 + v', 0 + w'), \quad p = \bar{p}(z) + p', \quad \rho = \bar{\rho}(z) + \rho',$$

we get the equations of oceans as follows

$$\frac{\mathrm{d}u'}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + f_\beta v' + \nu\Delta u', \tag{1.6.10}$$

$$\frac{\mathrm{d}v'}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} - f_\beta u' + \nu\Delta v', \tag{1.6.11}$$

$$\frac{\mathrm{d}w'}{\mathrm{d}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g + \nu\Delta w', \tag{1.6.12}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{1.6.13}$$

$$\frac{\mathrm{d}\rho'}{\mathrm{d}t} + \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}w = \kappa\Delta\rho' + \kappa\frac{\mathrm{d}^2\bar{\rho}}{\mathrm{d}z^2}. \tag{1.6.14}$$

If we take the static approximation, that is, substituting $\dfrac{\partial P'}{\partial Z} = \rho' g$ for (1.6.12), and the equations (1.6.10)–(1.6.14) are called the ocean primitive equations in the rectangular coordinate frame.

1.7 Boundary Conditions

The atmospheric and oceanic motions can not only be described by the equations listed above, but also are under the influence of boundary. So, in this section we discuss the boundary conditions of the atmosphere and oceans.

1.7.1 *The Lower Boundary Conditions of the Atmosphere*

In the pressure coordinate frame, if the lower interface of the atmosphere $p = P$ (the approximate value of earth surface, seen as a constant) is considered as ideal rigid body, and it is a material surface, then the normal velocity of the air is zero, that is,

$$\dot{p}|_{p=P} = 0. \tag{1.7.1a}$$

In the same way, taking the spherical coordinate frame, and setting $z = r - a$, z standing for the elevation, where a is the earth radius, we can assume that the lower boundary condition of the atmosphere is

$$v_r|_{z=0} = 0. \tag{1.7.1b}$$

In the topography coordinate frame, the lower boundary condition of the atmosphere is

$$\dot{\zeta}|_{\zeta=1} = 0, \tag{1.7.1c}$$

where $\zeta = \dfrac{p}{p_s}$. If the viscosity of the lower boundary surface $p = P$ is considered, then the velocity is zero, that is,

$$v_\theta|_{p=P} = 0,\ v_\varphi|_{p=P} = 0,\ \dot{p}|_{p=P} = 0. \tag{1.7.2a}$$

In the spherical coordinate frame, if the viscosity of the lower boundary surface $z = 0$ is considered, then the lower boundary conditions of the atmosphere are

$$v_\theta|_{z=0} = 0,\ v_\varphi|_{z=0} = 0,\ v_r|_{z=0} = 0. \tag{1.7.2b}$$

In the same way, in the topography coordinate frame, the lower boundary conditions of the atmosphere are

$$v_\theta|_{\zeta=1} = 0,\ v_\varphi|_{\zeta=1} = 0,\ \dot{\zeta}|_{\zeta=1} = 0. \tag{1.7.2c}$$

(1.7.1a)-(1.7.2c) are generally called **kinematic boundary conditions**.

In the local rectangular coordinate frame, if the vertical motion of the lower boundary surface is caused by the force of landform $z = h(x, y)$, then the lower boundary condition is given by

$$w|_{z=h(x,y)} = \frac{\mathrm{d}h}{\mathrm{d}t} = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}.$$

In the spherical coordinate frame, if the lower boundary surface is the surface of the ocean, of which pressure is $p_0(\theta, \varphi, t)$, then the lower boundary condition of pressure p is given by

$$p(\theta, \varphi, z, t)|_{z=0} = p_0(\theta, \varphi, t). \tag{1.7.3a}$$

If the lower boundary surface has the topography $h_s(\theta, \varphi)$, of which pressure is $p_s(\theta, \varphi, t)$, the lower boundary condition of pressure p is

$$p(\theta, \varphi, z, t)|_{z=h_s(\theta,\varphi)} = p_s(\theta, \varphi, t). \tag{1.7.3b}$$

(1.7.3a) and (1.7.3b) are denoted as **dynamic boundary conditions**.

In the pressure coordinate frame, the thermodynamic condition of the lower boundary surface $p = P$ can be briefly written as

$$\frac{\partial T}{\partial p}|_{p=P} = -\alpha_s(T - T_s),$$

where α_s is a parameter about turbulent thermal conductivity, which relies on the properties of the lower surface, and T_s is the reference temperature of the lower surface.

In the pressure coordinate frame, for Φ in (1.3.3), the geometrical condition of the lower boundary surface is

$$\Phi|_{p=P} = \Phi_s(\theta, \varphi, t).$$

1.7.2 *The Upper Boundary Conditions of the Atmosphere*

First, in the real atmosphere, the upper boundary ($z \to +\infty$) should satisfy

$$\lim_{z\to+\infty} p = 0.$$

Second, since the total energy of vertical air column per unit section is bounded, we get condition

$$\int_0^{+\infty} \left(\frac{v_\theta^2 + v_\varphi^2}{2} + c_v T + gz\right) \rho \mathrm{d}z < +\infty.$$

Thus,

$$\rho v_\theta^2, \rho v_\varphi^2, \rho T, \rho z \to 0, \text{ as } z \to +\infty. \tag{1.7.4}$$

(1.7.4) is generally called a physical boundary condition.

In the practical case, researchers always use homogeneous atmosphere models, such as barotropic models (shallow water models), two-dimensional quasi-geostrophic models and multi-dimensional quasi-geostrophic models. In these models, researchers divide atmosphere into some layers, which are incompressible homogeneous fluids, where the upper surface of each layer is free surface, denoted by $z = h(x, y, t)$ in the local rectangular coordinate frame. The boundary condition on that free surface is given by

$$w|_{z=h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y},$$

where w is velocity in the direction of z, and u, v are respectively velocities in the direction of x, y. For the interface of two layers $z = h(x, y, t)$, the boundary condition on the interface is given by

$$w_i|_{z=h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + u_i\frac{\partial h}{\partial x} + v_i\frac{\partial h}{\partial y},$$

where $i = 1, 2$. This condition is called a kinematic condition of interface.

1.7.3 *The Boundary Conditions of the Oceans*

In the local rectangular coordinate frame, boundary conditions of the sea level are given by

$$\frac{\partial \boldsymbol{v}}{\partial z}|_{z=0} = h\tau,\ w|_{z=0} = 0, \tag{1.7.5}$$

$$\frac{\partial T}{\partial z}|_{z=0} = -\alpha(T - T^*), \tag{1.7.6}$$

where $\boldsymbol{v}$ is the horizontal velocity, w denotes velocity in the direction of z, h is the depth of ocean (supposed as constant), τ is the wind stress of the sea level, and T^* is the sea water parameter temperature at the sea level. The first equation of (1.7.5) indicates the effect of wind stress of the ocean surface to the sea water, and (1.7.6) indicates the energy alternation at the sea level. The boundary conditions of the lateral ocean can be written as

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \times \boldsymbol{n} = 0, \tag{1.7.7}$$

$$\frac{\partial T}{\partial \boldsymbol{n}} = 0. \tag{1.7.8}$$

Here $\boldsymbol{n}$ denotes the external normal vector of the lateral ocean. (1.7.7) indicates that the normal component of lateral horizontal velocity is zero, and horizontal velocity here is no-ship at the same time. (1.7.8) indicates that there is no lateral heat flux. The boundary conditions of the bottom ocean surface can be written as

$$\frac{\partial \boldsymbol{v}}{\partial z}|_{z=-h} = 0, w|_{z=-h} = 0, \frac{\partial T}{\partial z} = 0.$$