# A Generalized N-Dimensional Pythagorean Theorem

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## Introduction

We all know the theorem Pythagoras demonstrated about right triangles, which when simply put in mathematical language, says $l_1^2 + l_2^2 = h^2$, where $l_{1,2}$ are the lengths of the two legs of the right triangle, and $h$ is the length of the hypotenuse. The French mathematician, J. P. de Gua de Malves, generalized in three dimensions what is now known as *de Gua's Theorem*, which says, "*The square of the area of the base (i.e., the face opposite the right trihedral angle) of a trirectangular tetrahedron is equal to the sum of the squares of the areas of its other three faces.*" A tetrahedron has four vertices, four facets, and six edges. A trirectangular tetrahedron has three of the six edges coming to one vertex as mutually orthogonal. De Gua's Theorem tells us, that since one vertex cuts three facets, the sum of the squares of the areas of the three facets that borders the *right* vertex, will equal the square of the area of the fourth facet.

Logically (or perhaps *illogically* per some people), we expect the statement of Pythagoras to be generalized even further to the higher dimensional analogs of right triangles. And here, I shall demonstrate a proof of the following theorem:

> Given a $N$-rectangular $N$-dimensional simplex, the square of the $(N-1)$-content of the facet opposing the right vertex will equal the sum of the squares of the $(N-1)$-contents of the remaining facets.

## The Theorem

The theorem cannot be easily understood without the proper terminologies. One of the most important terms in the statement of the theorem is *simplex*. As its name suggests, a simplex is the simplest $N$-dimensional hyper-solid that can be formed, or, it is the $N$-dimensional hyper-solid with the least $(N-1)$-facets. Some well known properties of simplex include:

- For an $N$-simplex, it has $\frac{N+1!}{(N-j)!(j+1)!}$ or $N$-Choose-$j$ $j$-dimensional constituents. For example, a 2-D simplex, or a triangle, has three $0$-D objects (or points or vertices), and three 1-D edges. For a 3-D simplex, there are four $0$-D vertices, six 1-D edges, and four 2-D facets.
- The content of a $N$-simplex can be given easily be the Cayley-Menger Determinant (see the next section for discussion).
- Every vertex is connected to every other vertex. And as a consequence, every vertex has $N$ edges converging on it.

The *content* can be thought of as the hyper-volume of a hyper-solid. It is the amount of the $N$-D space contained in the $N$-D object. And in our discussion, the term *facet* will be used to mean the $(N-1)$-dimensional objects that form the $N$-dimensional object in question.

The term $N$-*rectangular* defines the "right" vertex of the simplex. For a simplex to be $N$-rectangular, it means that there exist one vertex of the simplex such that all edges converging on it are pair-wise orthogonal, and such a vertex will be denoted as the *right* or *right angle* vertex of the simplex. To help establish a mental picture, a 2-dimensional 2-rectangular figure is the right triangle, and a 3-dimensional trirectangular figure is the object removed when you shave a corner off a cube. Analogous to that, any $N$-dimensional $N$-rectangular simplex is *one* corner of an $N$-cube. Furthermore, it is easily seen that the *non-opposing* facets (the *opposing* facet being the one opposite the right vertex) of a $N$-D $N$-R simplex are all themselves $(N-1)$-D $(N-1)$-R simplex.

## The Cayley-Menger Determinant

The *Cayley-Menger Determinant* gives a way of calculating the $N$-content of a $N$-simplex.

Define the $(N+1)$ by $(N+1)$ matrix B by: $\mathrm{B}_{lm} = \|v_l - v_m\|_2^2$
and define its companion matrix $\tilde{\mathrm{B}}$ by:

$$\tilde{\mathrm{B}} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \mathrm{B}_{11} & \cdots & \mathrm{B}_{1,N+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{B}_{N+1,1} & \cdots & \mathrm{B}_{N+1,N+1} \end{pmatrix}$$

where $v_\lambda$ are the $N+1$ vertices of the simplex, and $\|v_l - v_m\|_2$ denotes the two dimensional distance between the two vertices.

In that case, the following equation gives the content of the simplex:

$$V^2 = \frac{(-1)^{N+1}}{2^N (N!)^2} \det(\tilde{\mathrm{B}}) \tag{1}$$

Now that I am done witht the background stuff, we shall get on to the proof ...

## Proof

Let $\Gamma$ be the collection (or set) that contains all and only all of the facets of our $N$-rectangular $N$-simplex. Then $\Gamma$ contains $N+1$ objects, all of which are $(N-1)$-rectangular $(N-1)$-simplex. Let $\pi : \Gamma \to \mathrm{V}_\Gamma$ be a function that maps each $\gamma \in \Gamma$ to its corresponding content $V_\gamma \in \mathrm{V}_\Gamma$. Also, let $\alpha \in \Gamma$ denote the opposing facet.

Then our theorem says:

$$\sum_{\gamma \in \Gamma} V_\gamma^2 = 2V_\alpha^2$$

or, in a different form:

$$\sum_{\gamma \in \Gamma, \gamma \neq \alpha} V_\gamma^2 = V_\alpha^2 \tag{2}$$

Now, it is obvious that the content of a $j$-rectangular $j$-simplex can be written as a $j$-Integral over the $j$-orthogonal edges, and from which we arrive at the conclusion that let $\{e_i\}_1^j$ be the length of the orthogonal edges, we get

$$V = \frac{\prod_{i=1}^{j} e_i}{j!} \tag{3}$$

By combining equations 1, 2, and 3, and letting $v_\alpha$ be the right angle vertex the theorem is shown to be equivalent to the statement that:

$$\frac{(-1)^N}{2^{N-1}}\det(\tilde{\mathrm{B}}_\alpha) = (\prod_{i=1}^{N}\|v_i - v_\alpha\|_2^2)(\sum_{j=1}^{N}\frac{1}{\|v_i - v_\alpha\|_2^2}) \tag{4}$$

where $\{v_i\}_1^N$ are the other $N$ vertices of the simplex. And if we choose the coordinate system such that $v_\alpha$ is at the origin, and all the orthogonal edges are on the axis, we can simplify that equation to:

$$\frac{(-1)^N}{2^{N-1}}\det(\tilde{\mathrm{B}}_\alpha) = (\prod_{i=1}^{N}\|v_i\|^2)(\sum_{j=1}^{N}\frac{1}{\|v_i\|^2}) \tag{5}$$

and that

$$\begin{aligned}\mathrm{B}_{\alpha,(l,m)} &= \|v_l - v_m\|_2^2 \\ &= \begin{cases}\|v_l\|^2 + \|v_m\|^2 & ,l \neq m \\ 0 & ,l = m\end{cases}\end{aligned}$$

Therefore, proving the generalized pythagorean theorem is equivalent to showing that equation 5 is true in $N$-dimensional space.

Now, in the purely algebraic calculation, it is possible to isolate an individual term. Since the ordering of the edges should not affect the solution of the problem, we can apply the Axiom of Choice and single out the edge $v_1$. Now we can evaluate the determinant and list *all the terms that does not contain* $v_1$. By the Axiom of Choice, we can see that if all the terms on the left hand side of equation 5 that does not contain $v_1$ is equal to that on the right hand side, then the equation must hold.

Since the determinant is a multilinear operation on the row vector *and* the column vectors, the following is true:

$$\det(\tilde{\mathrm{B}}) = \det(\beta_0, \beta_1, \ldots, \beta_N)$$

where $\beta_0 = (0, 1, 1, 1, \ldots)$
$\beta_1 = (1, 0, \|v_1\|^2 + \|v_2\|^2, \|v_1\|^2 + \|v_3\|^2, \ldots, \|v_1\|^2 + \|v_N\|^2)$ etc.

So $\beta_1 = \beta_1' + \beta_1''$, where
$\beta_1' = (1, 0, \|v_2\|^2, \|v_3\|^2, \ldots, \|v_N\|^2)$ and
$\beta_1'' = (0, 0, \|v_1\|^2, \|v_1\|^2, \ldots)$

and therefore

$$\begin{aligned}\det(\tilde{\mathrm{B}}) &= \det(\beta_0, \beta_1' + \beta_1'', \ldots, \beta_N) \\ &= \det(\beta_0, \beta_1', \ldots, \beta_N) + \det(\beta_0, \beta_1'', \ldots, \beta_N)\end{aligned}$$

By the definition of $\beta_1''$, all the terms in the second determinant are either 0 or contains $\|v_1\|^2$. So in our effort to ignore all such terms, we can safely discard that.

Performing the similar operation on the column vectors gives us the following, that the determinant without any $v_1$ terms, denoted by $\det(\bar{\mathrm{B}})$ is:

$$\det(\bar{\mathrm{B}}) = \det\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \|v_2\|^2 & \|v_3\|^2 & \ldots & \|v_N\|^2 \\ 1 & \|v_2\|^2 & 0 & \|v_2\|^2 + \|v_3\|^2 & \ldots & \|v_2\|^2 + \|v_N\|^2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & \|v_N\|^2 & \|v_N\|^2 + \|v_2\|^2 & \ldots & \|v_N\|^2 + \|v_{(N-1)}\|^2 & 0 \end{pmatrix} \tag{6}$$

Anotherr property of determinants is that you can add or subtract one row from others without effecting the value of the determinant. Which we can write as

$$\begin{aligned}
\det(\bar{\mathrm{B}}) &= \det(\beta_0, \beta_1', \beta_2, \ldots, \beta_N) \\
&= \det(\beta_0, \beta_1', \beta_2 - \beta_1', \ldots, \beta_N - \beta_1') \\
&= \det \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \|v_2\|^2 & \|v_3\|^2 & \ldots & \|v_N\|^2 \\ 0 & \|v_2\|^2 & -\|v_2\|^2 & \|v_2\|^2 & \ldots & \|v_2\|^2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & \|v_N\|^2 & \|v_N\|^2 & \ldots & \|v_N\|^2 & -\|v_N\|^2 \end{pmatrix}
\end{aligned}$$

And doing it again with the column vectors, we get

$$\det(\bar{\mathrm{B}}) = \det \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \|v_2\|^2 & \|v_3\|^2 & \ldots & \|v_N\|^2 \\ 0 & \|v_2\|^2 & -2\|v_2\|^2 & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & \|v_N\|^2 & 0 & \ldots & 0 & -\|v_N\|^2 \end{pmatrix} \tag{7}$$

Which by simple inspection and some simple algebra can be demonstrated to be

$$\det(\bar{\mathrm{B}}) = (-1)^N 2^{N-1} (\prod_{i=2}^{N} \|v_i\|^2)$$

which is equal to the right hand side of equation 5 without the terms that contains $v_1$.

**Q. E. D.**