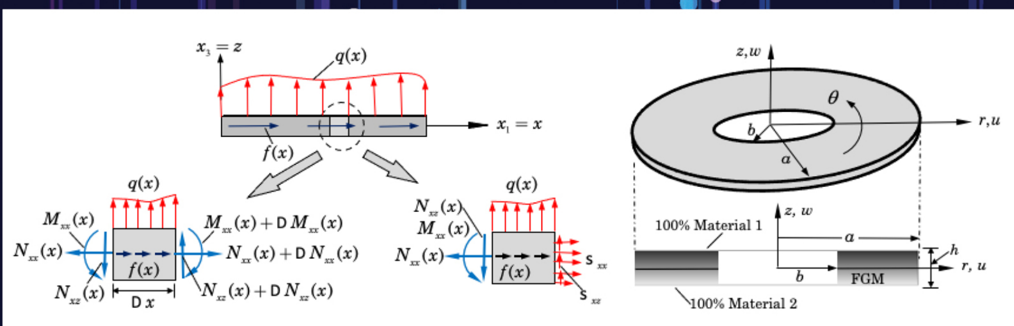
Theories and Analyses of Beams and Axisymmetric Circular Plates



J. N. Reddy

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To

My loving wife,

Aruna Reddy

The author has been very fortunate to have a companion and loving wife, who selflessly took care of his needs with care and affection. She does not know how much the author benefited professionally because of her time, kindness, and generosity. The author is more grateful to her than any words can express. He is ever thankful for everything she did, and hopes that her life is filled with peace and happiness even after he leaves this world.

Contents

Preface

The motivation for composing this book has come from the need to fill the gap in the literature and provide a comprehensive treatment of the classical and shear deformation theories of beams and axisymmetric circular plates in one volume. The book is a compendium of all related works by the author and his colleagues on the subject over his lifetime. The book contains detailed derivations of the governing equations, analytical solutions, variational solutions, and numerical solutions (FEM) of the classical and shear deformation theories of beams and axisymmetric circular plates. The readers and users will benefit to have such a comprehensive book available in the literature as a reference for finding the governing equations, analytical and numerical solutions of bending, vibration, and buckling for problems with various boundary conditions.

In this present book, classical and shear deformation theories are presented, accounting for through-thickness variation of two-constituent functionally graded material, modified couple stress (i.e., strain gradient), and the von Kármán nonlinearity. Analytical solutions of the linear theories and finite element analysis of linear and nonlinear theories are included.

Chapter 1 is devoted to a brief review of mechanics preliminaries that include vectors and tensors, summation convention, governing equations of solid mechanics, an introduction to functionally graded materials (FGMs), and the modified couple stress model. A reader familiar with these may skip this chapter but it is recommended that a casual walk through the chapter is beneficial to see the notation used. Chapter 2 deals with the concepts of work and energy, strain energy, and virtual work, and elements of the calculus of variations and variational principles of solid and structural mechanics. These ideas are useful in the development of variationally consistent theories of beams and plates and their solution by direct variational methods such as the Ritz and Galerkin methods.

The main thesis of the book begins with Chapter 3, which presents a detailed discussion of the classical beam theory (CBT), including kinematics, constitutive models, and governing equations of motion. The governing equations of plate strips (i.e., cylindrical bending of plates) are also discussed. The chapter also contains analytical and numerical solutions of the linearized equations. Analytical solutions include solutions by direct integration as well as the Navier method. The Ritz and other variational methods are introduced in this chapter and illustrated by their applications to CBT. Chapters 4 and 5 follow the same sequence of developments for first-order (TBT) and third-order (RBT) theories, respectively, of beams. A major feature of these chapters is the development of the algebraic relationships between the solutions of the

TBT and CBT and RBT and CBT. That is, if one has the analytical solution of a beam problem using the CBT, the relationships allow one to obtain the solutions of the same problem by the TBT and RBT models.

Chapters 6–8 are dedicated to the classical, first-order, and third-order theories of axisymmetric circular plates, following the same sequence of steps (i.e., derivation of equations, analytical and variational solutions, and relationships between classical and shear deformation theories).

Finally, finite element formulations and numerical solutions of beams and axisymmetric circular plates, respectively, are presented in Chapters 9 and 10. These chapters contain extensive theoretical results in the form of weak-form development, finite element models, tangent stiffness coefficient derivations, and numerical results for linear and nonlinear analysis of beams and circular plates. To keep the size of the book within reasonable limits, numerical results in these chapters are limited to static bending analysis.

The major feature of the book is the comprehensive treatment (within the scope of the book) of the subject matter. The readers never have to consult another source to follow the developments; although many references are provided mostly to acknowledge the developments, it is not necessary to read them to follow what is presented here. Of course, having a background in mechanics of materials, elasticity, and a first course on the finite element method would help. Historical notes are included in several places to make it interesting and derive some level of appreciation for those who have contributed to the subject matter covered herein. Few exercise problems are also included, but extensions and applications of the theories developed herein are possible. For such tasks, this book is an excellent reference to researchers.

As already stated, many of the results included herein were obtained during the course of the author's lifetime in collaboration with his students, postdocs, and colleagues around the world. For the reason of missing some, the names (a large number) of these individuals are not listed here. Instead, a list of papers coauthored with them on topics related to the subject matter are included at the back of the book. *The author is very appreciative of the friendship and collaboration of all these colleagues over the years.* The author is pleased to acknowledge the help of Dr. Eugenio Ruocco, Dr. Praneeth Nampally, Mr. Ho Yong Shin, and Ms. Alekhya Banki with the proofreading of the manuscript prior to its publication. A book of this nature, full of mathematical statements, is bound to have some typos and errors. The author requests the readers to send any comments and corrections to *jnreddy@tamu.edu*.

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Anyone who has never made a mistake has never tried anything new.
Albert Einstein

List of symbols used

The meaning of various symbols used in the book for some important quantities is defined in the following table. The list is not exhaustive (c_i and K_i are constants used at various places).

Symbol	Meaning
a	Outer radius of a circular or annular plate
a_{ij}	Coefficients of matrix $[A] = \mathbf{A}$
A	Area of cross section of a beam
$A_{xx}, A_{xz}, \ldots$	Axial and shear stiffness coefficients
b	Inner radius of an annular plate; width of a beam cross section
B_{xx}	Bending-stretching coupling stiffness
c_f	Modulus of elastic foundation per unit length
c_v, c_p	Specific heat at constant volume and pressure, respectively
$d\Gamma$	Surface element
dA	Area element ($dA = dxdy$)
$d\Omega$	Area element ($d\Omega = dxdy$) or volume element ($d\Omega = dxdydz$)
D_{xx}, D_{xz}	Bending and higher-order shear stiffness coefficients
$\bar{D}_{xx}$	Effective stiffness coefficient, $\bar{D}_{xx} = D_{xx} - \alpha F_{xx}$,
D^e_{xx}	Effective stiffness coefficient, $D^e_{xx} = D_{xx} + A_{xy}$, A_{xy} being the stiffness coefficient due to couple stress; also, $\hat{D}_{xx} = \bar{D}_{xx} - \alpha \bar{F}_{xx}$
D^*_{xx}	Effective stiffness coefficient, $D^*_{xx} = D_{xx}A_{xx} - B_{xx}B_{xx}$
$\hat{\mathbf{e}}_i$	Basis vector in the x_i-direction
$(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$	Basis vectors in the (r, θ, z) system
$(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$	Basis vectors in the (x, y, z) system
$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$	Basis vectors in the (x_1, x_2, x_3) system
E	Modulus of elasticity
E_1, E_2	Moduli of elasticity of a functionally graded structure or an orthotropic material
$E_{xx}, E_{yy}, \ldots$	Green strain components in rectangular Cartsian system; E_{xx} are the higher-order stiffness coefficient
$E_{rr}, E_{\theta\theta}, \ldots$	Green strain components in cyndrical coordinate system
$\mathbf{E}$	Green–Lagrange strain tensor
$\mathbf{f}$	Body force vector
f_x, f_y, f_z	Body force components in the x, y, and z directions
$F_{xx}, F_{rr}, \ldots$	Higher-order stress resultants
$F_i^\alpha, \mathbf{F}^\alpha$	Finite element force vectors
$\mathbf{F}$	Deformation gradient, $\mathbf{F} = (\boldsymbol{\nabla}\mathbf{x})^{\mathrm{T}}$
G	Shear modulus
h	Height of a beam or thickness of a plate; length of a finite element
$H_{xx}, H_{rr}, \ldots$	Higher-order stress resultants
I	Second moment of area, $I = bh^3/12$
$\mathbf{I}$	Unit second-order tensor
J	Determinant of $\mathbf{J}$ (Jacobian)
J_n	Bessel function of the first kind and of the nth order
$\mathbf{J}$	Jacobian (of transformation) matrix

Symbol	**Meaning**
k	Extensional spring constant
k_R	Rotational spring constant
K	Kinetic energy; bulk modulus
K_s	Shear correction coefficient
$\mathbf{K}$	Finite element stiffness matrix
$K_{ij}^{\alpha\beta}, \mathbf{K}^{\alpha\beta}$	Finite element stiffness submatrices
l, ℓ	Material length scale used couple stress model
L	Length of a beam
$\mathbf{m}$	Couple stress tensor
$\mathbf{M}$	Finite element mass matrix
$M_{ij}^{\alpha\beta}, \mathbf{M}^{\alpha\beta}$	Finite element mass submatrices
$\mathcal{M}_{xy}$	Couple stress
$M_{xx}, M_{rr}, \ldots$	Bending stress resultants
n	Index/exponent used in power-law model
$\hat{\mathbf{n}}$	Unit normal vector in the current configuration
n_i	ith component of the unit normal vector $\hat{\mathbf{n}}$
(n_x, n_y, n_z)	Components of the unit normal vector $\hat{\mathbf{n}}$
$N_{xx}, N_{rr}, \ldots$	Stretching stress resultants
$P_{xx}, P_{rr}, \ldots$	Higher-order stress resultants
q	Distributed transverse load per unit length
Q_{ij}	Plane stress-reduced elastic coefficients
r	Radial coordinate in the cylindrical polar system; $r = \lvert\mathbf{r}\rvert$
$\mathbf{r}$	Position vector in cylindrical coordinates, $\mathbf{x}$
(r, θ, z)	Cylindrical coordinate system
R	Outer radius of a circular plate
t	Time
$\mathbf{t}$	Stress vector; traction vector
$\mathbf{t}_i$	Stress vector on x_i-plane, $\mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_j$
T	Temperature
u	Axial displacement
$\mathbf{u}$	Displacement vector
u_r, u_θ, u_z	Components of a displacement vector $\mathbf{u}$ in a cylindrical coordinate system
u_x, u_y, u_z	Components of a displacement vector $\mathbf{u}$ in a rectangular Cartesian coordinate system
U	Strain energy of a body
U_0	Strain energy density of a body
v	Velocity, $v = \lvert\mathbf{v}\rvert$
(v_1, v_2, v_3)	Components of velocity vector $\mathbf{v}$ in (x_1, x_2, x_3) system
(v_r, v_θ, v_z)	Components of velocity vector $\mathbf{v}$ in (r, θ, z) system
$\mathbf{v}$	Velocity vector, $\mathbf{v} = \frac{D\mathbf{x}}{Dt}$
$\mathbf{v}_n$	Velocity vector normal to the plane (whose normal is $\hat{\mathbf{n}}$)
V	Potential energy due to external loads; shear force
V_1, V_2	Material volume fractions for functionally graded material
V_{eff}	Effective shear force
V_E	Work done by external forces ($= W_E$)
w	Transverse displacement component
W_E	External work done by forces
W_I	Internal work stored in the body
$\mathbf{x}$	Position vector in the current configuration
(x, y, z)	Rectangular Cartesian coordinates
(x_1, x_2, x_3)	Rectangular Cartesian coordinates (spatial)
(X_1, X_2, X_3)	Rectangular Cartesian coordinates (material)
Y_n	Bessel function of the second kind and of the nth order

Greek symbols

Symbol	Meaning
α	Angle; parameter in time approximations scheme; also, $\alpha = 4/3h^2$, h being the total height of the beam or plate
α_T	Coefficient of thermal expansion
β	Heat transfer coefficient (also other uses); also $\beta = 3\alpha = 4/h^2$
γ	Parameter in a time approximation scheme
Γ	Total boundary
δ	Dirac delta; variational symbol
δ_{ij}	Components of the unit tensor, $\mathbf{I}$ (Kronecker delta)
$\Delta, \boldsymbol{\Delta}$	Increment; generalized displacement vector
ϵ	Tolerance specified for nonlinear convergence
ε_{ij}	Infinitesimal strain components
ε_{ijk}	Alternating symbol
ζ, η	Natural (normalized) coordinate
θ	Angular coordinate in the cylindrical and spherical coordinate systems
λ	Lamé constant; eigenvalue
μ	Lamé constant
ν, ν_{ij}	Poisson's ratio; Poisson's ratios for an orthotropic material
ξ	Natural (normalized) coordinate
Π	Total potential energy
ρ	Mass density
$\boldsymbol{\sigma}$	Stress tensor
σ_{ij}	Components of the stress tensor in the rectangular coordinate system (x_1, x_2, x_3)
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}, \cdots$	Components of the stress tensor $\boldsymbol{\sigma}$ in the cylindrical coordinate system (r, θ, z)
τ	Shear stress
$\boldsymbol{\tau}$	Viscous stress tensor
ϕ	Angular coordinate in the spherical coordinate system
ϕ_i	Hermite cubic interpolation functions
ϕ_x, ϕ_r	Rotation functions
$\chi, \boldsymbol{\chi}$	Curvature and curvature tensor
ψ	Warping function; stream function
ψ_i	Lagrange interpolation functions
ω	Angular velocity
$\boldsymbol{\omega}$	Rotation vector
Ω	Domain of a problem; natural frequency
$\boldsymbol{\Omega}$	Spin tensor or skew symmetric part of the velocity
ω_i	Components of vorticity vector $\boldsymbol{\omega}$ in the rectangular coordinate system

Other symbols

Symbol	Meaning
$\boldsymbol{\nabla}$	Gradient operator (with respect to $\mathbf{X}$)
$\boldsymbol{\nabla}_x$	Gradient operator with respect to $\mathbf{x}$
∇^2	Laplace operator, $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$
[ ]	Matrix of components of the enclosed tensor
{ }	Column of components of the enclosed vector
$\cdot$	Symbol for the dot product or scalar product
$\times$	Symbol for the cross product or vector product

Table 1
Conversion factors

s = second; lb = pound; in = inch; ft = foot; hp = horse power;
kg = kilogram (= 10^3 grams); m = meter; mm = millimeter (10^{-3} m);
N = Newton; W = Watt; Pa = Pascal = N/m^2;
kN = 10^3 N; MN = 10^6 N; MPa = 10^6 Pa; GPa = 10^9 Pa

Quantity	US customary unit	SI equivalent
Mass	lb (mass)	0.4536 kg
Length	in	25.4 mm
	ft	0.3048 m
Density	lb/in^3	27.68×10^3 kg/m^3
Force	lb (force)	4.448 N
Pressure or stress	lb/in^2 (psi)	6.895 kN/m^2
Moment or torque	lb in	0.1130 Nm
Power	ft lb/s	1.356 W
	hp (550 ft lb/s)	745.7 W

Note:
Quotes by various people included in this book were found at different web sites; for example, visit:
http://naturalscience.com/dsqhome.html,
http://thinkexist.com/quotes/david˙hilbert/, and *http://www.yalescientific.org/*.

The historical notes included in various footnotes can be found at different websites, especially Wikipedia, https://en.wikipedia.org/.

This author is motivated to include the quotes for their wit and wisdom. The author cannot vouch for the accuracy of the quotes or the historical notes. The reason for the inclusion of the historical notes is to remind the readers that we are "standing on the shoulders" of many giants before us.

A few words of caution about the Wikipedia. Most of the references cited there belong to the authors who contributed to the subject matter, and they are neither authoritative nor original contributions to the subject; some selfish authors tried to promote their own work at the expense of not giving credit to the original contributors. In addition, the readers should be very careful in accepting what is found there as technically accurate. It is advised that the readers consult the papers and books by well-known researchers on the technical topic/subject.

If you are not willing to learn, no one can help you. If you are determined to learn, no one can stop you. Zig Ziglar

About the Author

J. N. Reddy, the O'Donnell Foundation Chair IV Professor in J. Mike Walker '66 Department of Mechanical Engineering at Texas A&M University, is a highly cited researcher, author of 24 textbooks and over 750 journal papers, and a leader in the applied mechanics field for nearly 50 years. He is known worldwide for his significant contributions to the field of applied mechanics through the authorship of widely used textbooks on mechanics of materials, continuum mechanics, linear and nonlinear finite element analyses, energy principles and variational methods, and composite materials and structures. His pioneering works on the development of shear deformation theories of beams, plates, and shells (that bear his name in the literature as *the Reddy third-order plate theory* and *the Reddy layerwise theory*), and nonlocal and non-classical continuum mechanics have had a major impact, and have led to new research developments and applications. Some of the ideas on shear deformation theories and penalty finite element models of fluid flows have been implemented into commercial finite element computer programs like Abaqus, NISA, and HyperXtrude (Altair).

Recent honors and awards include: the 2019 Timoshenko Medal from the American Society of Mechanical Engineers, 2018 Theodore von Kármán Medal from the Engineering Mechanics Institute of the American Society of Civil Engineers, the 2017 John von Neumann Medal from the U.S. Association of Computational Mechanics, the 2016 Prager Medal from the Society of Engineering Science, the 2016 Thomson Reuters IP and Science's Web of Science Highly Cited Researchers – Most Influential Minds, and the 2016 ASME Medal from the American Society of Mechanical Engineers. He is a member of the US National Academy of Engineering and foreign fellow of the Canadian Academy of Engineering, the Chinese Academy of Engineering, the Brazilian National Academy of Engineering, the Indian National Academy of Engineering, the Royal Academy of Engineering of Spain, the European Academy of Sciences, and the Academia Scientiarum et Artium Europaea (the European Academy of Sciences and Arts). For additional details, visit **http://mechanics.tamu.edu/**.

Teachers must make selfless efforts to make their own hard-earned knowledge and expertise accessible to motivated students. They must derive personal reward, not from demonstrating to students that they are experts, but from being able to explain complex ideas in a way that they can comprehend and ultimately master the ideas and utilize them in their professional life. I also believe that any measure of success must include giving back to the community.

There is no "complete" mathematical model of anything we study. We only try to "improve" on what we already know.

Junuthula Narasimha (J. N.) Reddy

1 Mechanics Preliminaries

Minds are like parachutes. They only function when they are open.
James Dewar

1.1 GENERAL COMMENTS

Engineers of all types contribute to science and technology for the benefit of mankind. They construct mathematical models, develop analytical and numerical approaches and methodologies, and design and manufacture various types of devices, systems, or processes. Mathematical models, engineering experiments, and numerical simulations constitute the three main pillars of scientific activity. Engineering analysis is an aid to designing systems for specific functionalities, and they involve (1) mathematical model development, (2) data acquisition by measurements, (3) numerical simulations, and (4) validation of the results in light of any experimental evidence. The most challenging task for engineers is to identify a suitable mathematical model of the system's behavior. It is in this connection this book is composed to provide interested readers with the theories and analyses of beams and circular plates. That is, we develop appropriate mathematical models (i.e., governing equations) for bending, buckling, natural vibration, and transient (to a limited extent) analyses of beams and axisymmetric circular plates. The book contains an up-to-date, relatively complete treatment of these specialized topics.

It is important to understand that all models, mathematical or experimental, are required to satisfy the laws of physics; beyond that, they are only approximate representations of the actual system or process. *There is no exact model of anything we study, and we only build on what we know to make them better for the intended purpose of the study.* In particular, continuum mechanics is not an exact science; as it stands now, it is not complete, and it will never be complete as we explore new phenomena. However, continuum mechanics is responsible for many advances in science and engineering, and we continue to build on it and make it better. Thus, the theories and analyses presented in this book for beams and axisymmetric circular plates form a basis for future developments.

This chapter is devoted to a review of preliminaries from engineering mechanics. The review includes: vectors and tensors, the definitions of the Green–Lagrange strain tensor, infinitesimal strain tensor, measures of stress, equations of elasticity, and stress–strain relations for plane stress problems, an introduction to functionally graded materials, and an introduction to the modified couple stress concept. These preliminaries are used in the coming

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chapters to develop the theories of beams and axisymmetric circular plates. Readers familiar with these may skip this chapter, but it is advised that they browse through the chapter to understand the notation used.

1.2 BEAMS AND PLATES

Beams are structural members that have a ratio of length-to-cross-sectional dimensions very large, say, 10 to 100 or more and subjected to forces, both along and transverse to the length and moments that tend to rotate them about an axis perpendicular to their length. When all applied loads are along the length only, they are called *bars* (i.e., bars experience only tensile or compressive stresses and strains and no bending deformation). *Cables* (or ropes) may be viewed as a very flexible form of bars, which can only take tension and not compression. *Plates* are a two-dimensional version of beams, with plate in-plane dimensions much larger in order of magnitude than the thickness. Thus, plates are thin bodies subjected to forces, in the plane as well as in the direction normal to the plane and bending moments about either axis in the plane. Geometrically, plates can be used in different shapes: circular, rectangular, triangular, rhombic, or polygonal. Ancient Egyptians, Greeks, Indus valley civilizations, and Romans used beams and plates of various shapes in their temples, monumental buildings, and tombs. Because of their geometry and loads applied, the beams and plates are stretched and bent (by design, in infinitesimally small magnitudes) from their original shapes. Such members are known as *structural elements* and their study constitutes *structural mechanics*, which is a subset of solid mechanics. The difference between structural elements and three-dimensional solid bodies, such as solid blocks and spheres that have no restrictions on their geometric make up, is that the latter may change their original geometry, but they may not show significant "bending" deformation.

All deformable solids can be analyzed for stress and deformation using the elasticity equations. However, the original geometry, induced deformation, and stress fields can be predicted, for most practical engineering problems involving structural elements, with simplified theories in the place of the three-dimensional elasticity theory. Beams (including frames), plates, and shells are analyzed using structural theories that are derived from three-dimensional elasticity theory by making certain simplifying assumptions concerning the deformation (kinematics) and stress states in these members. The development of such theories dates back to Leonardo da Vinci[1], Galileo Galilei[2],

[1] In his "Codice Atlantico," Leonardo Da Vinci (1452–1519) made the first attempt known to us to correlate bending deflection and geometry for a beam.

[2] The book "Discorsi e dimostrazioni matematiche intorno a due nuove scienze attinenti la mecanica e i moti locali" by Galileo Galilei (1564–1642) is considered to be the first book devoted to structural mechanics.

Jacob Bernoulli[3], and Leonhard Euler[4]. The first one is *the Euler–Bernoulli beam theory*, a theory that is covered in all undergraduate mechanics of materials books. In the Euler–Bernoulli beam theory, the transverse shear strain is neglected, making the beam infinitely rigid in the transverse direction. The second one is popularly known as *the Timoshenko beam theory* [1, 2][5] which accounts for the transverse shear strain (γ_{xz}). In a recent paper, Elishakoff [3] pointed out that the beam theory that incorporates both the rotary inertia and shear deformation as is known presently, with shear correction coefficient included, should be referred to as the *Timoshenko–Ehrenfest beam theory* because the original paper published by Timoshenko had a coauthor by name Paul Ehrenfest. In view of the fact that many people have contributed to the development of shear deformation theories, Reddy [4] coined the phrase *first-order shear deformation theory*. Unfortunately, most people do not read original papers they cite, and errors in giving the due credit are propagated from one writing to the next (consult the article by Reddy and Srinivasa [5] for some misattributions and misnomers in mechanics).

All modern developments are dedicated to refinements to the above stated theories, by expanding the displacements in terms of higher-order terms and accounting for other non-classical continuum mechanics aspects (e.g., stress and strain gradient effects and material length scales). For example, a general higher-order theory is of the form

$$\mathbf{u} = u_x\,\hat{\mathbf{e}}_x + u_y\,\hat{\mathbf{e}}_y + u_z\,\hat{\mathbf{e}}_z, \tag{1.2.1}$$

where

$$u_x = \sum_{i=0}^{m} z^i \phi_x^{(i)}(x,t), \quad u_y = 0, \quad u_z = \sum_{i=0}^{p} z^i \psi_z^{(i)}(x,t). \tag{1.2.2}$$

Here $\phi_x^{(0)} = u$ and $\psi_z^{(0)} = w$ denote the midplane displacements along the x and z directions, respectively, and $\phi_x^{(i)}$ and $\psi_x^{(i)}$ are the higher-order terms, which can be mathematically interpreted as higher-order generalized displacements with the meaning

$$\phi_x^{(i)} = \frac{1}{(i)!}\left(\frac{\partial^i u_1}{\partial z^i}\right)_{z=0}, \qquad \psi_z^{(i)} = \frac{1}{(i)!}\left(\frac{\partial^i u_3}{\partial z^i}\right)_{z=0}. \tag{1.2.3}$$

[3] Jacob Bernoulli (1655–1705) was one of the many prominent Swiss mathematicians in the Bernoulli family. Jacob Bernoulli (1655–1705), along with his brother Johann Bernoulli (1667–1748), was one of the founders of calculus of variations. Jacob Bernoulli and Daniel Bernoulli (1700–1782) (son of Johann Bernoulli) are credited for initiating a beam theory.

[4] Leonhard Euler (1707–1783) was a pioneering Swiss mathematician and physicist who put forward the theory in 1750: "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes," *Leonhardi Euleri Opera Omnia Ser.* I, 14, 1744.

[5] Stephan Prokofyevich Timoshenko (1878–1972) was a Ukrainian, Russian, and American engineer and academician, who is considered to be the father of modern engineering mechanics.

For a general third-order beam theory, we have $m = 3$ and $p = 2$ in Eq. (1.2.2). The third-order beam theory of Reddy, derived from this third-order plate theory (see Reddy [6]–[9] and Heyliger and Reddy [10]), adopts a displacement field that is a special case of Eq. (1.2.1) and imposes zero transverse shear stress conditions on the bounding planes (i.e., top and bottom faces) of the beam to express the variables introduced with the higher order terms in terms of the variables that appear in the Euler–Bernoulli and Timoshenko beam theories.

In the remaining part of this chapter, we review some mathematical preliminaries involving calculus of vectors and tensors, and equations of solid mechanics that are useful in the sequel. The topic of vectors and tensors is in itself a major subject, and books are devoted to its treatment. Here we assume that the readers are sufficiently familiar with the subject, and we only review some useful concepts. The equations of solid mechanics include the strain–displacement relations, equations of motion in terms of stresses, and stress–strain relations. Other mechanics preliminaries needed in this book, such as the energy and variational principles (including the principles of virtual displacements and the minimum total potential energy, and Hamilton's principle), are presented in Chapter 2. The principle of virtual displacements plays a major role in the development of the governing equations of higher-order beam and plate theories presented in this book.

1.3 VECTORS AND TENSORS

1.3.1 VECTORS AND COORDINATE SYSTEMS

The elementary notion of a vector as being one with "magnitude" and "direction" is a geometric concept and applies to directed line segments. In the broader context, vectors can be quantities, such as functions and matrices, and satisfy the rules of vector addition and multiplication of a vector by a scalar. The terms "magnitude" and "direction" take different meaning in different contexts. Engineering examples of vectors are provided by displacements, velocities, forces, heat flux, and so on; and they are endowed with a direction and a magnitude. Note that entities like speed and temperature are scalars (i.e., they have only magnitudes but no directions). Stress as a measure of force per unit area is a vector (stress vector) whereas representation of a stress tensor requires not only a stress vector but also the specification of the area on which it acts. In written or typed material, a vector or tensor is denoted by a boldface letter, $\mathbf{A}$, such as used in this book, and its magnitude is denoted by $|\mathbf{A}|$ or just A.

We begin with an orthonormal Cartesian coordinate system (x_1, x_2, x_3) with the following orthonormal basis vectors:

$$\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} \qquad \text{or} \qquad \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}. \tag{1.3.1}$$

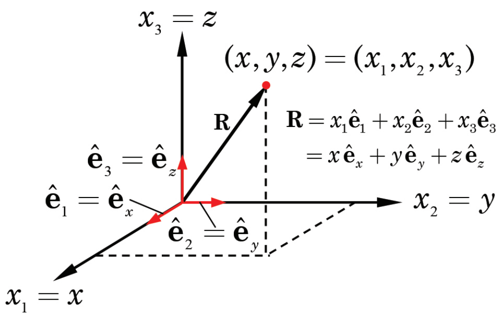

Figure 1.3.1: Rectangular Cartesian coordinates.

The associated Cartesian coordinates are denoted by $(x, y, z) = (x_1, x_2, x_3)$. The familiar rectangular Cartesian coordinate system is shown in Fig. 1.3.1. We shall always use right-handed coordinate systems.

We can represent any vector $\mathbf{A}$ in three-dimensional space as a linear combination of the orthonormal basis as

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3. \tag{1.3.2}$$

The vectors $A_1\hat{\mathbf{e}}_1$, $A_2\hat{\mathbf{e}}_2$, and $A_3\hat{\mathbf{e}}_3$ are called the *vector components* of $\mathbf{A}$, and A_1, A_2, and A_3 are called *scalar components* of $\mathbf{A}$ associated with the basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. Also, we use the notation $\mathbf{A} = (A_1, A_2, A_3)$ to denote a vector by its components. Thus, the position vector $\mathbf{R}$ can be written as $\mathbf{R} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$.

1.3.2 SUMMATION CONVENTION

Equation (1.3.2) can be expressed as

$$\mathbf{A} = \sum_{i=1}^{3} A_i\,\hat{\mathbf{e}}_i, \tag{1.3.3}$$

which can be shortened, by omitting the summation symbol, and understanding that summation over the range of the index is implied when an index is repeated, to

$$\mathbf{A} = A_i\,\hat{\mathbf{e}}_i. \tag{1.3.4}$$

The repeated index is called *dummy index* and thus can be replaced by *any other symbol that has not already been used.* Thus we can also write

$$\mathbf{A} = A_i\,\hat{\mathbf{e}}_i = A_m\,\hat{\mathbf{e}}_m, \text{ and so on.} \tag{1.3.5}$$

It is convenient at this time to introduce the Kronecker delta δ_{ij} and alternating symbol ε_{ijk} for representing the dot product and cross product of two orthonormal vectors in a right-handed basis system. We define the dot product $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ between the orthonormal basis vectors of a right-handed system as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \equiv \delta_{ij} = \begin{cases} 1, \text{ if } i = j, & \text{for any fixed value of } i, j \\ 0, \text{ if } i \neq j, & \text{for any fixed value of } i, j, \end{cases} \tag{1.3.6}$$

where δ_{ij} is called the *Kronecker delta symbol*. Similarly, we define the cross product $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$ for a right-handed system as

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \equiv \varepsilon_{ijk} \hat{\mathbf{e}}_k, \tag{1.3.7}$$

where

$$\varepsilon_{ijk} = \begin{cases} \;\;1, \text{ if } i, j, k \text{ are in cyclic order} \\ \qquad \text{and not repeated } (i \neq j \neq k), \\ -1, \text{ if } i, j, k \text{ are not in cyclic order} \\ \qquad \text{and not repeated } (i \neq j \neq k), \\ \;\;0, \text{ if any of } i, j, k \text{ are repeated.} \end{cases} \tag{1.3.8}$$

The symbol ε_{ijk} is called the *alternating symbol* or *permutation symbol*.

In an orthonormal basis, the scalar product $\mathbf{A} \cdot \mathbf{B}$ and vector product $\mathbf{A} \times \mathbf{B}$ can be expressed in the index form using the Kronecker delta symbol δ_{ij} and alternating symbol ε_{ijk} as

$$\mathbf{A} \cdot \mathbf{B} = (A_i \hat{\mathbf{e}}_i) \cdot (B_j \hat{\mathbf{e}}_j) = A_i B_j \delta_{ij} = A_i B_i, \tag{1.3.9}$$

$$\mathbf{A} \times \mathbf{B} = (A_i \hat{\mathbf{e}}_i) \times (B_j \hat{\mathbf{e}}_j) = A_i B_j \varepsilon_{ijk} \hat{\mathbf{e}}_k. \tag{1.3.10}$$

The Kronecker delta and the permutation symbol are related by the identity, known as the ε-δ *identity*:

$$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}. \tag{1.3.11}$$

Then the length of a vector in an orthonormal basis can be expressed as $A = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_i A_i} = \sqrt{A_1^2 + A_2^2 + A_3^2}$. Similarly, we have $R^2 = x_i x_i$.

1.3.3 STRESS VECTOR AND STRESS TENSOR

Consider the equilibrium of an element of a continuum acted upon by forces. The surface force acting on a small element of area in a continuous medium depends not only on the magnitude of the area but also upon the orientation of the area. It is customary to denote the direction of a plane area by means of a unit vector drawn normal to that plane (see Fig. 1.3.2). To fix the direction of the normal, we assign a *sense of travel* along the contour of the boundary of the plane area in question. The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the sense of travel along the boundary curve or contour (see Fig. 1.3.2). Let the unit normal vector be given by $\hat{\mathbf{n}}$. Then the area can be denoted by $\mathbf{s} = s\hat{\mathbf{n}}$.

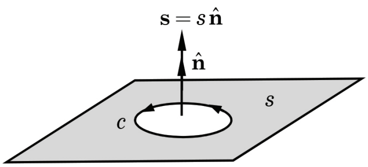

Figure 1.3.2: Plane area as a vector. Unit normal vector and sense of travel are shown.

If we denote by $\Delta\mathbf{F}(\hat{\mathbf{n}})$ the force on an elemental area $\hat{\mathbf{n}}\Delta s = \Delta\mathbf{s}$ located at the position $\mathbf{r}$ (see Fig. 1.3.3), the *stress vector* is defined as

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta s \to 0} \frac{\Delta\mathbf{F}(\hat{\mathbf{n}})}{\Delta s}. \tag{1.3.12}$$

We see that the stress vector is a point function of the unit normal $\hat{\mathbf{n}}$, which denotes the orientation of the surface Δs. The component of $\mathbf{t}$ that is in the direction of $\hat{\mathbf{n}}$ is called the *normal stress*. The component of $\mathbf{t}$ that is normal to $\hat{\mathbf{n}}$ (or in the plane) is called a *shear* stress.

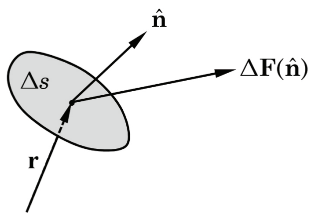

Figure 1.3.3: Force on an area element.

At a fixed point $\mathbf{r} = \mathbf{x}$ for each given unit vector $\hat{\mathbf{n}}$, there is a stress vector $\mathbf{t}(\hat{\mathbf{n}})$ acting on the plane normal to $\hat{\mathbf{n}}$. To establish a relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$ and introduce the stress tensor, we now set up an infinitesimal tetrahedron in Cartesian coordinates, as shown in Fig. 1.3.4.

If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$, and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are Δs_1, Δs_2, Δs_3, and Δs, respectively, we have by Newton's second law for the mass inside the tetrahedron:

$$\mathbf{t}\Delta s - \mathbf{t}_1\Delta s_1 - \mathbf{t}_2\Delta s_2 - \mathbf{t}_3\Delta s_3 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}, \tag{1.3.13}$$

where Δv is the volume of the tetrahedron, ρ is the density, $\mathbf{f}$ is the body

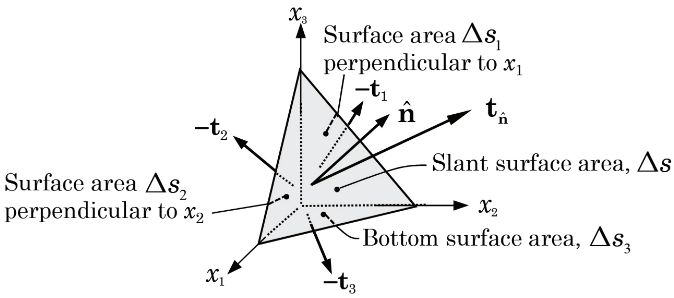

Figure 1.3.4: Tetrahedral element in Cartesian coordinates.

force per unit mass, and $\mathbf{a}$ is the acceleration. Since the total vector area of a closed surface is zero, we have

$$\Delta s\hat{\mathbf{n}} - \Delta s_1\hat{\mathbf{e}}_1 - \Delta s_2\hat{\mathbf{e}}_2 - \Delta s_3\hat{\mathbf{e}}_3 = \mathbf{0}. \tag{1.3.14}$$

It follows that

$$\Delta s_1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\Delta s, \quad \Delta s_2 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\Delta s, \quad \Delta s_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\Delta s. \tag{1.3.15}$$

The volume of the element Δv can be expressed as

$$\Delta v = \frac{\Delta h}{3}\Delta s, \tag{1.3.16}$$

where Δh is the perpendicular distance from the origin to the slant face.

Substitution of Eqs. (1.3.15) and (1.3.16) in Eq. (1.3.13) and dividing throughout by Δs reduces it to

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a}-\mathbf{f}).$$

In the limit when the tetrahedron shrunk to a point (to obtain the relation at a point), $\Delta h \to 0$, we are left with

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_i)\mathbf{t}_i, \tag{1.3.17}$$

which can be displayed as

$$\mathbf{t} = \hat{\mathbf{n}}\cdot\left(\hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3\right). \tag{1.3.18}$$

The terms in the parenthesis are to be treated as a dyad, called *stress dyad* or *stress tensor* $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3. \tag{1.3.19}$$

The stress tensor is a point property of the medium that is independent of the unit normal vector $\hat{\mathbf{n}}$. Thus, we have[6]

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \quad (t_i = n_j \sigma_{ji}) \tag{1.3.20}$$

and the dependence of $\mathbf{t}$ on $\hat{\mathbf{n}}$ has been explicitly displayed. Equation (1.3.20) is known as *Cauchy's formula*, and $\boldsymbol{\sigma}$ is termed the *Cauchy stress tensor*.

It is useful to resolve the stress vectors $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ into their orthogonal components. We have

$$\mathbf{t}_i = \sigma_{i1}\hat{\mathbf{e}}_1 + \sigma_{i2}\hat{\mathbf{e}}_2 + \sigma_{i3}\hat{\mathbf{e}}_3 = \sigma_{ij}\hat{\mathbf{e}}_j \tag{1.3.21}$$

for $i = 1, 2, 3$. Hence, the Cauchy stress tensor can be expressed in the rectangular Cartesian system using the summation notation as

$$\boldsymbol{\sigma} = \hat{\mathbf{e}}_i \mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{1.3.22}$$

The component σ_{ij} represents the stress (force per unit area at a point) on a plane perpendicular to the ith coordinate and in the jth coordinate direction (see Fig. 1.3.5). The stress vector $\mathbf{t}$ represents the vectorial stress on a plane whose normal coincides with $\hat{\mathbf{n}}$.

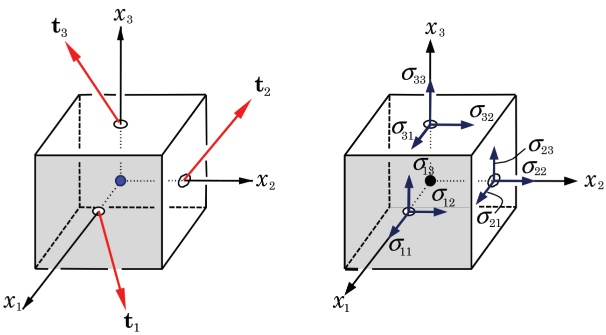

Figure 1.3.5: Definition of stress components in Cartesian rectangular coordinates.

1.3.4 THE GRADIENT OPERATOR

Let us denote a scalar field by $\phi = \phi(\mathbf{x}) = \phi(x_1, x_2, x_3)$, where $\mathbf{x} = (x_1, x_2, x_3)$ is a position vector of a typical point in space. The differential change in ϕ is given by

$$d\phi = \frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3. \tag{1.3.23}$$

[6]In some books, $\boldsymbol{\sigma}$ is defined to be the transpose of that defined in Eq. (1.3.19); see Reddy [11]. This is because Eq. (1.3.17) can be expressed, in view of the fact that $\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i$ is a scalar quantity that can be placed on the other side of the vector $\mathbf{t}_i$, making Eq. (1.3.18) to become $\mathbf{t} = (\mathbf{t}_1\hat{\mathbf{e}}_1 + \mathbf{t}_2\hat{\mathbf{e}}_2 + \mathbf{t}_3\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{n}}$ and $\mathbf{t}(\hat{\mathbf{n}}) = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}}$.

The differentials dx_1, dx_2, and dx_3 are components of $d\mathbf{x}$. Since $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$, we can write

$$\begin{aligned} d\phi &= \hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} \cdot \hat{\mathbf{e}}_1\, dx_1 + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} \cdot \hat{\mathbf{e}}_2\, dx_1 + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3} \cdot \hat{\mathbf{e}}_3\, dx_3 \\ &= (dx_1\, \hat{\mathbf{e}}_1 + dx_2\, \hat{\mathbf{e}}_2 + dx_3\, \hat{\mathbf{e}}_3) \cdot \left(\hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3} \right) \\ &= d\mathbf{x} \cdot \left(\hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3} \right). \end{aligned} \tag{1.3.24}$$

Let us now denote the magnitude of $d\mathbf{x}$ by $ds \equiv |d\mathbf{x}|$. Then $\hat{\mathbf{e}} = d\mathbf{x}/ds$ is a unit vector in the direction of $d\mathbf{x}$, and we have

$$\left(\frac{d\phi}{ds} \right)_{\hat{\mathbf{e}}} = \hat{\mathbf{e}} \cdot \left(\hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3} \right). \tag{1.3.25}$$

The derivative $(d\phi/ds)$ is called the *directional derivative* of ϕ, and it is the *rate of change* of ϕ with respect to distance. Because the magnitude of this vector is equal to the maximum value (by being along the vector $d\mathbf{x}$) of the directional derivative, it is called the *gradient vector* and is denoted by $\boldsymbol{\nabla}\phi$:

$$\text{grad } \phi = \boldsymbol{\nabla}\phi \equiv \hat{\mathbf{e}}_1 \frac{\partial \phi}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial \phi}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial \phi}{\partial x_3}. \tag{1.3.26}$$

It is important to note that whereas the gradient operator $\boldsymbol{\nabla}$ has some of the properties of a vector, it does not have them all, because it is an operator. For instance, $\boldsymbol{\nabla} \cdot \mathbf{A}$ is a scalar, called the divergence of vector $\mathbf{A}$, whereas $\mathbf{A} \cdot \boldsymbol{\nabla}$ is a scalar *differential operator*. Thus, the del operator does not commute in this sense. The dot product of del operator with a vector is called the *divergence of a vector* and is denoted by

$$\boldsymbol{\nabla} \cdot \mathbf{A} \equiv \text{div}\mathbf{A} = \frac{\partial A_i}{\partial x_i}. \tag{1.3.27}$$

If we take the divergence of the gradient vector, we have

$$\text{div}(\text{grad } \phi) \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi = \nabla^2 \phi. \tag{1.3.28}$$

The notation $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ is called the *Laplace operator*. In the Cartesian rectangular coordinate system, this reduces to the form

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial x_i \partial x_i}. \tag{1.3.29}$$

The curl of a vector is defined as the del operator operating on a vector by means of the cross product [the ith component of $(\boldsymbol{\nabla} \times \mathbf{A})$ is $\frac{\partial A_k}{\partial x_j}\, \varepsilon_{jki}$]:

$$\text{curl}\, \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} \times \hat{\mathbf{e}}_k A_k = \frac{\partial A_k}{\partial x_j} (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) = \frac{\partial A_k}{\partial x_j}\, \varepsilon_{jki}\, \hat{\mathbf{e}}_i. \tag{1.3.30}$$

We also note that the gradient of a vector, $\boldsymbol{\nabla}\mathbf{A}$, is a dyad (i.e., second-order tensor) because it has two base vectors to represent it: $\boldsymbol{\nabla}\mathbf{A} = \hat{\mathbf{e}}_j \frac{\partial A_i}{\partial x_j} \hat{\mathbf{e}}_i = \frac{\partial A_i}{\partial x_j} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i$. One should make note of the order of the base vectors[7]. The transpose of $\boldsymbol{\nabla}\mathbf{A}$ is to interchange the basis vectors $(\boldsymbol{\nabla}\mathbf{A})^{\mathrm{T}} = \frac{\partial A_i}{\partial x_j} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$.

Useful expressions for the integrals of the gradient, divergence, and curl of a vector can be established between volume integrals and surface integrals[8]. Let Ω denote a region in space surrounded by the closed surface Γ. Let $d\Gamma$ be a differential element of surface and $\hat{\mathbf{n}}$ the unit outward normal, and let $d\Omega$ be a differential volume element. The following integral relations between volume and surface integrals (or between area integrals and line integrals) are proven to be useful in the coming chapters. In three dimensions, these relations involve the gradient, curl, and divergence of field variables. The specific forms are presented here.

Gradient theorem

$$\int_\Omega \boldsymbol{\nabla}\, \phi \, d\Omega = \oint_\Gamma \hat{\mathbf{n}}\phi \, d\Gamma \quad \left[\int_\Omega \hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i} \, d\Omega = \oint_\Gamma \hat{\mathbf{e}}_i n_i \phi \, d\Gamma \right]. \tag{1.3.31}$$

Curl theorem (also known as Kelvin–Stokes' theorem[9])

$$\int_\Omega \boldsymbol{\nabla} \times \mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}} \times \mathbf{A} \, d\Gamma \quad \left[\int_\Omega \varepsilon_{ijk} \hat{\mathbf{e}}_k \frac{\partial A_j}{\partial x_i} \, d\Omega = \oint_\Gamma \varepsilon_{ijk} \hat{\mathbf{e}}_k n_i A_j \, d\Gamma \right]. \tag{1.3.32}$$

Divergence theorem (also known as Green-Gauss's theorem[10]),

$$\int_\Omega \boldsymbol{\nabla} \cdot \mathbf{A} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}} \cdot \mathbf{A} \, d\Gamma \quad \left[\int_\Omega \frac{\partial A_i}{\partial x_i} \, d\Omega = \oint_\Gamma n_i A_i \, d\Gamma \right]. \tag{1.3.33}$$

The three theorems can be expressed in a single equation as

$$\int_\Omega \boldsymbol{\nabla} * \mathbf{F} \, d\Omega = \oint_\Gamma \hat{\mathbf{n}} * \mathbf{F} \, d\Gamma, \tag{1.3.34}$$

---

[7] In some books the gradient operator $\boldsymbol{\nabla}$ is defined, different from that in Eq. (1.3.26), as one with the backward operation: $\boldsymbol{\nabla}\mathbf{A} = (\partial \mathbf{A}/\partial x_j)\hat{\mathbf{e}}_j = (\partial A_i/\partial x_j)\, \hat{\mathbf{e}}_i \, \hat{\mathbf{e}}_j$.

[8] The notion of surface integrals was introduced by Joseph-Louis Lagrange (1736–1813) in 1760 and again in 1811 in the second edition of his *Mécanique Analytique* in more general terms. He discovered the divergence theorem in 1762.

[9] Named after Lord Kelvin (1824–1907) and George Stokes (1819–1903).

[10] Carl Friedrich Gauss (1777–1855) used surface integrals while working on the gravitational attraction of an elliptical spheroid in 1813, when he proved special cases of the divergence theorem. George Green (1793–1841) proved special cases of the theorem in 1828 in "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism."

where * is a gradient, curl, or divergence operation, and the field variable is necessarily be a vector or tensor field when * denotes curl or divergence operation.

The forms of a typical vector and its gradient, curl, and divergence in the cylindrical coordinate system (see Fig. 1.3.6) are presented here for a ready reference when we study circular plates.

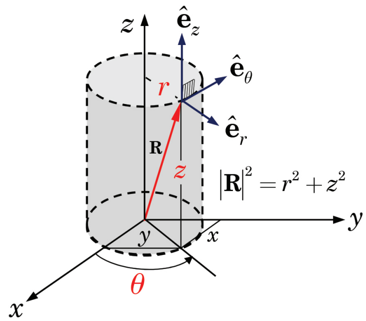

Figure 1.3.6: Cylindrical coordinate system.

Cylindrical coordinate system (r, θ, z)

$$\text{Position vector:}\ \ \mathbf{R} = r\,\hat{\mathbf{e}}_r + z\,\hat{\mathbf{e}}_z = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z, \tag{1.3.35}$$

$$\text{Relation to } (x, y, z):\quad x = r\,\cos\theta,\quad y = r\,\sin\theta,\quad z = z, \tag{1.3.36}$$

$$\hat{\mathbf{e}}_r = \cos\theta\ \hat{\mathbf{e}}_x + \sin\theta\ \hat{\mathbf{e}}_y,\quad \hat{\mathbf{e}}_\theta = -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_y,\ \ \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z, \tag{1.3.37}$$

$$\begin{aligned}\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} &= -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\theta,\\ \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} &= -\left(\cos\theta\ \hat{\mathbf{e}}_x + \sin\theta\ \hat{\mathbf{e}}_y\right) = -\hat{\mathbf{e}}_r.\end{aligned} \tag{1.3.38}$$

All other derivatives of the base vectors of the cylindrical coordinate system are zero. A typical vector $\mathbf{u}$ (such as the displacement), which is a function of the coordinates, can be expressed in the cylindrical coordinate system in terms of its components (u_r, u_θ, u_z) as

$$\mathbf{u} = u_r\,\hat{\mathbf{e}}_r + u_\theta\,\hat{\mathbf{e}}_\theta + u_z\,\hat{\mathbf{e}}_z. \tag{1.3.39}$$

Then $\boldsymbol{\nabla}$ and its various operations on $\mathbf{u}$ are given by

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z},\quad \nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + r\frac{\partial^2}{\partial z^2}\right], \tag{1.3.40}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r}\left[\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + r\frac{\partial u_z}{\partial z}\right], \tag{1.3.41}$$

$$\boldsymbol{\nabla}\times\mathbf{u}=\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{e}}_r+\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\hat{\mathbf{e}}_\theta+\frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right]\hat{\mathbf{e}}_z, \tag{1.3.42}$$

$$\begin{aligned}\boldsymbol{\nabla}\mathbf{u}=&\frac{\partial u_r}{\partial r}\,\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r+\frac{\partial u_\theta}{\partial r}\,\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta+\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r+\frac{\partial u_z}{\partial r}\,\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z+\frac{\partial u_r}{\partial z}\,\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r\\&+\frac{1}{r}\left(u_r+\frac{\partial u_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\,\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z+\frac{\partial u_\theta}{\partial z}\,\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta+\frac{\partial u_z}{\partial z}\,\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z.\end{aligned} \tag{1.3.43}$$

1.4 REVIEW OF THE EQUATIONS OF SOLID MECHANICS

1.4.1 GREEN–LAGRANGE STRAIN TENSOR

For most part, the measure of strain in solid mechanics is the Green–Lagrange strain tensor[11] defined by (see Reddy [11])

$$\mathbf{E}=\tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{u})+(\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}+(\boldsymbol{\nabla}\mathbf{u})\cdot(\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right], \tag{1.4.1a}$$

where $\mathbf{u}(\mathbf{X},t)$ is the displacement vector of a material particle occupying location $\mathbf{X}$ in the reference configuration (and the same material particle occupies a location $\mathbf{x}=\mathbf{X}+\mathbf{u}$ in the deformed body), and $\boldsymbol{\nabla}$ is the gradient operator with respect to $\mathbf{X}$:

$$\boldsymbol{\nabla}=\hat{\mathbf{E}}_1\frac{\partial}{\partial X_1}+\hat{\mathbf{E}}_2\frac{\partial}{\partial X_2}+\hat{\mathbf{E}}_3\frac{\partial}{\partial X_3}=\hat{\mathbf{E}}_i\frac{\partial}{\partial X_i}, \tag{1.4.1b}$$

where $(\hat{\mathbf{E}}_1,\hat{\mathbf{E}}_2,\hat{\mathbf{E}}_3)$ are the unit base vectors in the coordinate system (X_1,X_2,X_3). Clearly, the last term in Eqs. (1.4.1a) is nonlinear in the displacement gradients. In terms of the displacement components (u_1,u_2,u_3) referred to the rectangular coordinates (X_1,X_2,X_3), we have (see Fig. 1.4.1)

$$E_{ij}=\tfrac{1}{2}\left(\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}+\frac{\partial u_m}{\partial X_i}\frac{\partial u_m}{\partial X_j}\right), \tag{1.4.2}$$

where the summation on repeated (or dummy) index m over the range of $m=1,2,3$ is implied.

---

[11] There are several measures of strains. The most commonly used strain measures are: the Cauchy–Green deformation tensor, $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$; the Green–Lagrange strain tensor, $2\mathbf{E}=\mathbf{C}-\mathbf{I}$; and the Euler–Almansi strain tensor, $2\mathbf{e}=\mathbf{I}-\mathbf{F}^{-\mathrm{T}}\cdot\mathbf{F}^{-1}$. Here $\mathbf{F}$ denotes the deformation gradient, $\mathbf{F}^{\mathrm{T}}=\boldsymbol{\nabla}\mathbf{x}$, and $\mathbf{I}$ is the unit second order tensor (see Reddy [11] for details). George Green (1793–1841) was a British mathematical physicist and well-known for Cauchy–Green tensor and Green's theorem. Joseph-Louis Lagrange (1736–1813) was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to analysis, number theory, and classical and celestial mechanics. Leonhard Euler was succeeded by Lagrange as the director of mathematics at the Prussian Academy of Sciences in Berlin.

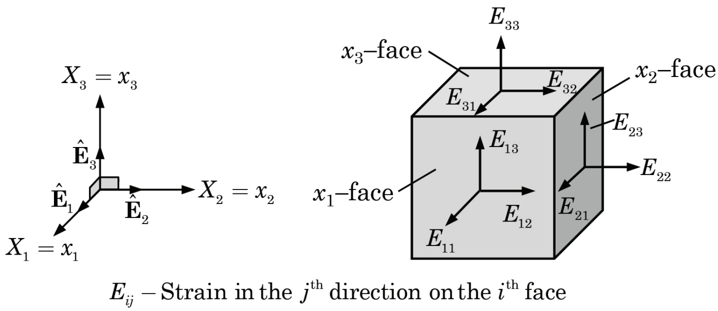

Figure 1.4.1: Notation used for the Green strain components in the rectangular Cartesian coordinate system.

In expanded notation, the Green strain tensor components referred to the rectangular Cartesian coordinate system (X_1, X_2, X_3) in the reference (undeformed) configuration in terms of the displacement components (u_1, u_2, u_3) are given by

$$\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial X_1} + \tfrac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right],\\
E_{22} &= \frac{\partial u_2}{\partial X_2} + \tfrac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right],\\
E_{33} &= \frac{\partial u_3}{\partial X_3} + \tfrac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right],
\end{aligned} \tag{1.4.3a}$$

$$\begin{aligned}
2E_{12} &= \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2},\\
2E_{13} &= \frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3},\\
2E_{23} &= \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\,.
\end{aligned} \tag{1.4.3b}$$

The components E_{11}, E_{22}, and E_{33} are the normal (i.e., extensional) strains, and E_{12}, E_{23}, and E_{13} are the shear strains.

By definition, the Green–Lagrange strain tensor is symmetric, $E_{ij} = E_{ji}$. It is the measure often used in the large deformation analysis. It is a strain measure that is “energetically conjugate” to the second Piola–Kirchhoff stress tensor introduced in Section 1.4.2. As we shall see shortly, we will consider a special case of $\mathbf{E}$ that is suitable for small strains but accounts for moderately large rotations, as experienced in beams and plates.

The Green–Lagrange strain tensor components in the cylindrical coordinate system ($r = X_1, \theta = X_2, z = X_3$; see Fig. 1.4.2) are given by

$$
\begin{aligned}
E_{rr} &= \frac{\partial u_r}{\partial r} + \tfrac{1}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2\right],\\
E_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2\right.\\
&\quad \left. - \frac{2}{r^2}u_\theta\frac{\partial u_r}{\partial \theta} + \frac{2}{r^2}u_r\frac{\partial u_\theta}{\partial \theta} + \left(\frac{u_\theta}{r}\right)^2 + \left(\frac{u_r}{r}\right)^2\right],\\
E_{zz} &= \frac{\partial u_z}{\partial z} + \tfrac{1}{2}\left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right],
\end{aligned}
\tag{1.4.4a}
$$

$$
\begin{aligned}
2E_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial \theta} + \frac{1}{r}\frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial \theta}\\
&\quad + \frac{1}{r}\frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \theta} + \frac{u_r}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\frac{\partial u_r}{\partial r},\\
2E_{rz} &= \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right),\\
2E_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\frac{\partial u_r}{\partial z} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\frac{\partial u_\theta}{\partial z}\\
&\quad + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\frac{\partial u_z}{\partial z} - \frac{u_\theta}{r}\frac{\partial u_r}{\partial z} + \frac{u_r}{r}\frac{\partial u_\theta}{\partial z}.
\end{aligned}
\tag{1.4.4b}
$$

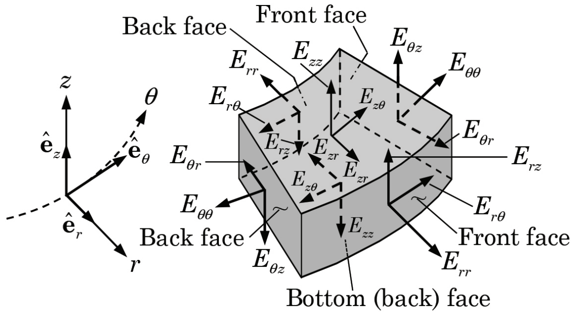

Figure 1.4.2: Notation used for the Green strain components on a volume element in the cylindrical coordinate system (r, θ, z); see Fig. 1.3.6. As indicated in Fig. 1.4.1, $E_{\xi\eta}$ is the strain on the plane perpendicular to ξ-coordinate and in the η-coordinate direction, where ξ and η take on the symbols r, θ, and z. The notation shown here for strains follows that of the stress components shown in Fig. 1.3.5.

When the displacement gradients are small (say less than 1%), that is, $|\boldsymbol{\nabla}\mathbf{u}| << 1$,

$$\frac{\partial u_i}{\partial X_j} << 1, \quad \left(\frac{\partial u_i}{\partial X_j}\right)^2 \approx 0, \text{ for any } i \text{ and } j,$$

we may neglect the nonlinear terms in the definition of the Green strain tensor $\mathbf{E}$ and obtain the linearized strain tensor $\boldsymbol{\varepsilon}$, called the *infinitesimal strain tensor* ($X_i \approx x_i$):

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right]; \quad \varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.4.5}$$

In expanded form in the rectangular coordinate system (x, y, z), the infinitesimal strain tensor components are

$$\begin{gathered}\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \\ 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad 2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \quad 2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}.\end{gathered} \tag{1.4.6}$$

If one presumes that the strains are small and but rotations about the y-axis of the material lines transverse to the x-axis are moderately large, that is, the squares and products of $\partial u_z/\partial x$ and $\partial u_z/\partial y$ are not negligible but squares and products of $\partial u_x/\partial x$, $\partial u_y/\partial y$, and $\partial u_z/\partial z$ are negligible, the strains resulting from the Green strain tensor components are known as the Föppl–von Kármán strains[12]

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right)^2, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\frac{\partial u_z}{\partial y}\right)^2, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \tag{1.4.7a}$$

$$\begin{gathered}2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}, \quad 2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \\ 2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}.\end{gathered} \tag{1.4.7b}$$

[12]They are named after August Föppl and Theodore von Kármán. August Otto Föppl (1854–1924) was a professor of Technical Mechanics and Graphical Statics at the Technical University of Munich, Germany. Theodore von Kármán (1881–1963) was a Hungarian-American mathematician, aerospace engineer, and physicist. He received his doctorate under the guidance of Ludwig Prandtl at the University of Göttingen, Germany in 1908. He was invited to the United States by Robert A. Millikan to advise California Institute of Technology (Caltech) engineers on the design of a wind tunnel. In 1930, he accepted the directorship of the Guggenheim Aeronautical Laboratory at the California Institute of Technology (GALCIT). His contributions include: theories of non-elastic buckling and supersonic aerodynamics. He made additional contributions to elasticity, vibration, heat transfer, and crystallography.

In the cylindrical coordinates, the von Kármán nonlinear strains are

$$
\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} + \tfrac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2, &\quad \varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2,\\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} + \tfrac{1}{2}\left(\frac{\partial u_z}{\partial z}\right)^2, & 2\varepsilon_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \theta},\\
2\varepsilon_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, & 2\varepsilon_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\frac{\partial u_r}{\partial z}.
\end{aligned} \tag{1.4.8}
$$

1.4.2 THE SECOND PIOLA–KIRCHHOFF STRESS TENSOR

The Cauchy[13] stress tensor $\boldsymbol{\sigma}$ (sometimes called "true stress") introduced in Eqs. (1.3.19) and (1.3.22) is the most natural and physical measure of the state of stress at a point in the deformed body and measured as the force in the deformed body per unit area of the deformed body. Since the geometry of the deformed body is not known (and yet to be determined), the governing equations must be written in terms of the known reference configuration[14], say, configuration at $t = 0$. This need gives rise to come up with a measure of stress that can be calculated using the known reference configuration. One such measure is the second Piola–Kirchhoff stress tensor[15], which is a measure of the *transformed internal force (from the deformed to the undeformed body) per undeformed area*. It is a mathematical entity introduced for the convenience of calculating stresses in a deformed solid.

The Green strain tensor $\mathbf{E}$ can be shown to be the dual (or energetically conjugate) to the second Piola–Kirchhoff stress tensor $\mathbf{S}$ (see Reddy [11]) in the sense that the strain energy density stored in an elastic body is equal to the product of $\mathbf{E}$ and $\mathbf{S}$ and it is invariant (i.e., independent of the coordinate system used). The second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ are related according to

$$
\mathbf{S}^{\mathrm{T}} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}^{\mathrm{T}}\cdot\mathbf{F}^{-\mathrm{T}}, \quad \boldsymbol{\sigma}^{\mathrm{T}} = \frac{1}{J}\mathbf{F}\cdot\mathbf{S}^{\mathrm{T}}\cdot\mathbf{F}^{\mathrm{T}}. \tag{1.4.9}
$$

where $\mathbf{F}$ is the deformation gradient defined by [11]

$$
\mathbf{F} = (\boldsymbol{\nabla}\mathbf{x})^{\mathrm{T}} = \mathbf{I} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}. \tag{1.4.10}
$$

[13] Baron Augustin-Louis Cauchy (1789–1857) was a French mathematician, engineer, and physicist who made pioneering contributions to mathematical analysis and continuum mechanics.

[14] We shall use the term *configuration* to mean the simultaneous position of all material points of a body for any fixed time.

[15] Gustav Robert Kirchhoff (1824–1887) was a German physicist who contributed to the fundamental understanding of electrical circuits, spectroscopy, black-body radiation by heated objects, and theoretical mechanics.

and J is the determinant of $\mathbf{F}$, called the *Jacobian of the motion*. The deformation gradient $\mathbf{F}$, in general, involves both stretch and rotation. In Eq. (1.4.10), $\mathbf{I}$ denotes the second-order identity tensor (i.e., $\mathbf{I} = \delta_{ij}\,\hat{\mathbf{E}}_i\hat{\mathbf{E}}_j$; see Fig. 1.4.1).

1.4.3 EQUATIONS OF MOTION

The principle of balance of linear momentum as applied to a deformed solid continuum and expressed in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$ gives

$$\boldsymbol{\nabla}_x \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{1.4.11}$$

where $\boldsymbol{\nabla}_x$ is the gradient operator with respect to the spatial coordinate $\mathbf{x}$ occupied by the material particle X which was at location $\mathbf{X}$ in the undeformed body (i.e., the displacement vector is $\mathbf{u} = \mathbf{x} - \mathbf{X}$); $\mathbf{f}$ is the body force vector measured per unit deformed volume; and ρ is the mass per unit deformed volume. Equation (1.4.10) is not useful for the analysis of large deformation because there the measure of the stress is the second Piola–Kirchhoff stress tensor, $\mathbf{S}$. Therefore, we express the equation of motion in terms of the second Piola–Kirchhoff stress tensor $\mathbf{S}$ as

$$\boldsymbol{\nabla} \cdot [\mathbf{S} \cdot (\mathbf{I} + \boldsymbol{\nabla}\mathbf{u})] + \hat{\mathbf{f}} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{1.4.12}$$

where $\boldsymbol{\nabla}$ is the gradient operator with respect to the material coordinate $\mathbf{X}$, ρ_0 is the mass density measured in the undeformed body and $\hat{\mathbf{f}}$ is the body force per unit volume in the undeformed body. Clearly, the equations of motion expressed in terms of the second Piola–Kirchhoff stress tensor are nonlinear, and this (spatial) nonlinearity is in addition to any nonlinearity that may come from the strain–displacement relations and constitutive relations.

1.4.4 STRESS–STRAIN RELATIONS

Due to the smallness of the thickness dimension in beams and plates, the normal stress in the thickness direction, namely, σ_{zz}, is assumed to be small and negligible compared to the in-plane stresses. More importantly, in the case of an orthotropic material with different moduli in the material 1 and 2 directions (i.e., planes of material symmetry), the shear stresses are assumed to be only function of their respective shear strains $\sigma_{ij} = G_{ij}2\varepsilon_{ij}$ (no sum on repeated subscripts) for $i \neq j = 1, 2, 3$. Then the 3-D constitutive equations resulting from the application of Hooke's law[16] must be modified to account for

[16]Robert Hooke (1635–1703) was an English scientist and architect and recently called "England's Leonardo." Hooke's law states that the force (F) needed to elongate or compress a spring by some distance (x) is linearly proportional to the distance, $F = kx$, where k is the proportionality constant which is characteristic of the spring stiffness.

this fact. The stress–strain relations obtained are termed plane-stress-reduced constitutive equations, which are adopted for beams, plates, and shells, whose thickness is very small compared to the other dimensions.

Here, we assume that the beam or plate material is characterized as orthotropic with respect to the (x, y, z) system (i.e., the material coordinates coincide with the coordinates used to describe the governing equations). Then we have

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha_x \, \Delta T \\ \varepsilon_{yy} - \alpha_y \, \Delta T \\ 2\varepsilon_{xy} \end{Bmatrix},$$
$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{Bmatrix}, \tag{1.4.13}$$

where Q_{ij} are the *plane stress-reduced elastic stiffness coefficients*; α_x and α_y are the coefficients of thermal expansion along the x and y directions, respectively; and $\Delta T = T - T_0$ is the temperature increment from a reference state T_0. The elastic coefficients Q_{ij} are related to the six independent engineering constants (E_1, E_2, ν_{12}, G_{12}, G_{13}, G_{23}) as follows:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}},$$
$$Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}. \tag{1.4.14}$$

Note that ν_{21} is computed from the following reciprocal relationship implied by the symmetry of elasticity tensor (see Reddy [8] for details):

$$\nu_{21} = \nu_{12} \, \frac{E_2}{E_1}. \tag{1.4.15}$$

1.5 FUNCTIONALLY GRADED STRUCTURES

1.5.1 BACKGROUND

Functionally graded materials (FGMs) are characterized by the variation in composition of two or more materials gradually over surface or volume, resulting in a composite material that has desired properties. An FGM can be designed for specific functionality and application. Most structures found in nature – from sea shells, trees and plants, to organs of living bodies – are multi-material graded structures, formed over millions of years, to satisfy certain functionalities. In the modern times, the man-made FGMs were proposed (see [12] and [13]) as thermal barrier materials for applications in space planes, space structures, nuclear reactors, turbine rotors, flywheels, and gears, to name only a few. As conceived and manufactured today, these materials are isotropic and non-homogeneous. In general, all the multi-phase materials, in which the material properties are varied gradually in a predetermined manner, fall into the category of functionally gradient materials. As stated before, the functionally gradient material characteristics are present in most structures found in nature, and perhaps, a better understanding of the highly complex form of materials in nature will help us in synthesizing new materials (the science of so called "biomimetics"). Such property enhancements endow FGMs with material properties such as resilience to fracture. FGMs promise attractive applications in a wide variety of wear coating and thermal shielding problems such as gears, cams, cutting tools, high temperature chambers, furnace liners, turbines, micro-electronics, and space structures.

A large number journal papers dealing with functionally graded beams and plates have appeared in the literature and a critical review of these papers is not a focus of this introduction to FGM structures [14] (also see, e.g., [15]–[41] and references therein). A majority of these works considered two-constituent FGM structures, and typically the material variation is considered through the thickness of beams, plates, and shell structures. The works of Praveen and Reddy [19] and Reddy [22] have also considered the von Kármán nonlinearity in functionally graded plates.

With the progress of technology and fast growth of the use of nanostructures, FGMs have found potential applications in micro and nano scales in the form of shape memory alloy thin films [42], atomic force microscopes (AFMs) [43], electrically actuated actuators [44], and micro switches [45], to name a few. The von Kármán nonlinearity may have significant contribution to the response of micro- and nano-scale devices such as biosensors and AFMs [46].

A typical FGM represents a particulate composite with a prescribed distribution of volume fractions of constituent phases. In the case of beams, plates, and shells, the material properties are assumed to vary continuously through the thickness. The effective properties of macroscopic homogeneous beams, plates, and shells are derived from the microscopic heterogeneous material distributions using homogenization techniques [14, 47, 48]. Several models, like the rule of mixtures [19, 22], Hashin–Shtrikman type bounds [49], Mori-Tanaka scheme [48, 50], and self-consistent schemes [51] are available in the literature for determination of the bounds for the effective properties. Voigt scheme and the Mori–Tanaka scheme [50] have been generally used for the study of FGM plates and structures by researchers [35, 52].

1.5.2 MORI–TANAKA SCHEME

For those parts of the graded microstructure that have a well-defined continuous matrix and discontinuous reinforcement, the overall properties and local fields can be closely predicted by Mori–Tanaka estimates. The assumption of spherical particles embedded in a matrix is considered. The primary matrix phase is assumed to be reinforced by spherical particles of secondary phase.

Mori and Tanaka [48, 50] derived a method to calculate the average internal stress in the matrix of a material. This has been reformulated by Benveniste [53] for use in the computation of the effective properties of composite materials. According to the Mori–Tanaka scheme, the effective elastic properties of the FGM can be expressed as

$$\frac{K-K_1}{K_2-K_1}=\frac{1-V_1}{1+V_1\frac{K_2-K_1}{K_1+\frac{4}{3}G_1}}\,,\quad \frac{G-G_1}{G_2-G_1}=\frac{1-V_1}{1+V_1\frac{G_2-G_1}{G_1+f_1}}\,,\tag{1.5.1}$$

where

$$f_1=\frac{G_1(9K_1+8G_1)}{6(K_1+2G_1)}\,,\quad V_2=1-V_1,\tag{1.5.2}$$

in which K and G are bulk modulus and shear modulus, respectively, and V is the volume fraction of the material (the subscript 1 and 2 refer to materials 1 and 2, respectively). The bulk modulus K and shear modulus G are related to Young's modulus E and Poisson's ratio ν, by the following equations:

$$E=\frac{9KG}{3K+G}\,,\qquad \nu=\frac{3K-2G}{2(3K+G)}\,.\tag{1.5.3}$$

1.5.3 VOIGT SCHEME: RULE OF MIXTURES

There are two rule of mixture models to describe the effective mechanical properties of a composite comprising two elastically isotropic constituent phases: the Voigt and Reuss models [54]. The Voigt model corresponds to axial loads and the Reuss model to transverse loads.

Voigt scheme has been adopted in most analysis of FGM structures [18, 22, 25, 35, 37, 38, 39, 41]. The advantage of the Voigt model is the simplicity of implementation and the ease of computation. According to Voigt scheme, the effective property P of the composite of two phases is the weighted average of the properties of the constituent phases:

$$P(z)=P_1\,V_1(z)+P_2\,V_2(z),\quad V_2(z)=1-V_1(z),\tag{1.5.4}$$

where P_1 and P_2 represent the constituent material properties (e.g., modulus, conductivity, and so on) of materials 1 and 2, respectively; and, V_1 and V_2 represent the volume fractions of materials 1 and 2, respectively, which may vary with respect to thickness coordinate z.

1.5.4 EXPONENTIAL MODEL

The exponential model, which is often employed in fracture studies, is based on the formula (see [29, 30])

$$P(z)=P_1\,\exp\left[-\alpha\left(\frac{1}{2}-\frac{z}{h}\right)\right],\quad \alpha=\log\left(\frac{P_1}{P_2}\right).\tag{1.5.5}$$

1.5.5 POWER-LAW MODEL

The variation of properties through the thickness is considered to be either exponential (called E-FGM), as given in Eq. (1.5.5), or based on a power series (called P-FGM), as presented in Eqs. (1.5.4) and (1.5.6), which covers most of the existing analytical models.

The volume fractions of materials 1 and 2, V_1 and V_2 can be expressed in the form of power law as (see Fig. 1.5.1)

$$V_1(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n, \quad V_2(z) = 1 - V_1(z). \tag{1.5.6}$$

where n is the volume fraction exponent (termed here as the *power-law index*). Then the property P as a function of the thickness coordinate z is given by Eq. (1.5.4). Fig. 1.5.2 shows the variation of the volume fraction of ceramic,

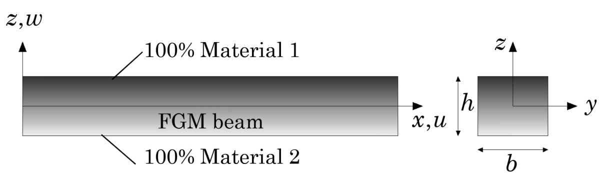

Figure 1.5.1: Geometry of a through-thickness functionally graded beam.

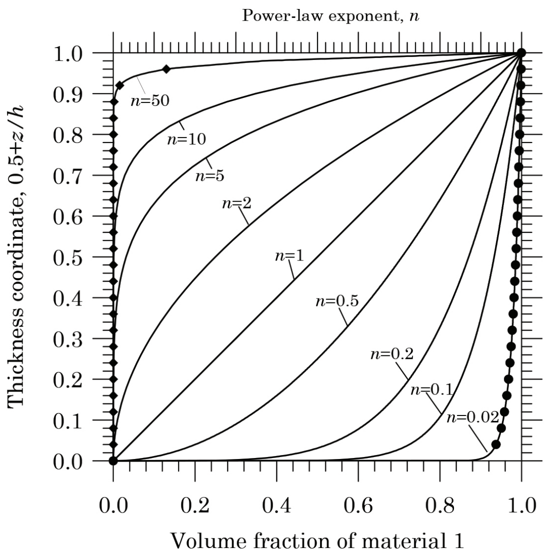

Figure 1.5.2: Volume fraction of material 1, V_1, through the beam thickness for various values of power-law index, n.

V_1, through the beam thickness for various values of the power-law index n. Note that the volume fraction $V_1(z)$ decreases with increasing value of n. The Power law is most popular because of its simplicity and algebraic nature.

Equations (1.5.4) and (1.5.6) can be combined to express a typical material property variation through the beam height or plate thickness, h, as

$$P(z) = (P_1 - P_2)\, V_1(z) + P_2, \quad V_1(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n. \tag{1.5.7}$$

1.6 MODIFIED COUPLE STRESS EFFECTS

1.6.1 BACKGROUND

The increasing demand for safe, lightweight, and environmentally acceptable structures has increased the need to investigate new structural configurations, including cellular or architected beams and plates. Such structures offer higher load-bearing capacity compared to their conventional counter parts. Computational models that take into account all architectural details are prohibitively expensive, requiring nonlocal continuum theories which account for the structural details, without homogenizing the structure, are needed. The *modified couple stress theory* of Mindlin [55], Koiter [56], and Toupin [57], and the *strain gradient theory* of [58]–[61] provide examples of such nonlocal theories. A more complete review of the early developments can be found in the paper of Srinivasa and Reddy [62]. The strain gradient theory is a more general form of the modified couple stress theory and the relationship between the modified couple stress theory and the strain gradient theory can be found in the work of Reddy and Srinivasa [63]. In recent years a number of attempts have been made to bring microstructural length scales into the continuum description of beams and plates. Such models are useful in determining the structural response of micro and nano devices made of a variety of new materials that require the consideration of small material length scales over which the neighboring secondary constituents interact, especially when the spatial resolution is comparable to the size of the secondary constituents. Examples of such materials are provided by nematic elastomers, carbon nanotube composites [64], and CNT-reinforced environment-resistant coatings [65].

Microstructure-dependent theories are developed for the Bernoulli-Euler beam by Park and Gao [66, 67], for the shear deformable beams and plates by Ma, Gao, and Reddy [68]–[70], for the third-order theory of plates for bending and vibration by Aghababaei and Reddy [71], and for vibration and buckling of shear deformable beams by Araujo dos Santos and Reddy [72, 73]. In the last two decades, Reddy and his colleagues [68]–[85] have published a large number of papers dealing with linear and nonlinear bending of classical and first- and third-order shear deformable beams and plates using the modified couple stress theory. Some of these works have accounted for the the von Kármán nonlinearity and functionally graded (through the thickness) materials. The von Kármán nonlinearity may have significant contribution to the response of beam-like elements used in micro- and nano-scale devices such as, biosensors and AFMs [46, 86].

1.6.2 THE STRAIN ENERGY FUNCTIONAL

Let $\mathbf{u}$ denote the displacement vector of an arbitrary point in the beam or plate. The rotation vector $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega} = \tfrac{1}{2}\left(\boldsymbol{\nabla} \times \mathbf{u}\right). \tag{1.6.1}$$

Physically, $\boldsymbol{\omega}$ denotes the macro-rotation at a point of the continuum. The curvature tensor $\boldsymbol{\chi}$, which represents the rate of change of the rotation, is defined as (assumed to be small):

$$\boldsymbol{\chi} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{\omega} + (\boldsymbol{\nabla}\boldsymbol{\omega})^{\mathrm{T}}\right]. \tag{1.6.2}$$

The modified couple stress theory is based on the hypothesis that the rate of change of macro-rotations cause additional stresses, called *couple stresses*, in the continuum. The modified couple stress tensor $\mathbf{m}$ is related to the curvature tensor $\boldsymbol{\chi}$ through the constitutive relations [55]:

$$\mathbf{m} = 2G\ell^2\,\boldsymbol{\chi}, \tag{1.6.3}$$

where ℓ is the length scale parameter (sometimes denoted by l) and G is the shear modulus.

According to the modified couple stress theory, the strain energy potential of an elastic beam of length a or circular plate of radius a can be expressed as (see Section 2.2 for the concept of strain energy)

$$U = \frac{1}{2}\int_A \left[\int_0^a \left(\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \mathbf{m} : \boldsymbol{\chi}\right) dx\right] dA, \tag{1.6.4}$$

where A is the area of cross section (for beam we set $dA = dydz$ and for circular plate we take $dx = dr$ and $dA = rd\theta dz$), $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the simplified Green–Lagrange strain tensor, $\mathbf{m}$ is the deviatoric part of the symmetric couple stress tensor, and $\boldsymbol{\chi}$ is the symmetric curvature tensor defined in Eq. (1.6.2). In the coming chapters, these relations will be specialized to various beam and plate theories.

1.7 CHAPTER SUMMARY

In this chapter, beginning with a short discussion of vectors and tensors and the introduction of the Cauchy stress vector and Cauchy stress tensor, measures of Green strain tensor, infinitesimal strain tensor and the von Kármán strain tensor components are reviewed. The definition of the second Piola–Kirchhoff stress tensor is introduced, but for small strains (as is the case with the present study), it is indistinguishable from the Cauchy stress tensor. Then the equations of motion of a deformable solid are presented, and stress–strain relations for a linear elastic material are summarized.

In addition, an introduction to two-constituent functionally graded materials is presented and various models of material gradation are reviewed. Then the modified couple stress theory is briefly visited, and the pertinent equations are summarized. Overall, the contents of this chapter will be utilized in the coming chapters.

There are other nonlocal models [87, 88, 89, 92]. Among them the stress gradient model (especially the differential model) of Eringen [89]–[93] has been used to study beams and circular plates, and the topic is not included in this book. Interested readers may consult [94]–[101] and references therein.

SUGGESTED EXERCISES

1.1 Establish the relations in Eqs. (1.3.41)–(1.3.43).

1.2 Verify the relations in Eqs. (1.4.4a) and (1.4.4b)

1.3 Establish the equation of motion in Eq. (1.4.12):

$$\boldsymbol{\nabla} \cdot [\mathbf{S} \cdot (\mathbf{I} + \boldsymbol{\nabla}\mathbf{u})] + \mathbf{f} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2},$$

where $\boldsymbol{\nabla}$ is the gradient operator with respect to the material coordinate $\mathbf{X}$, ρ_0 is the mass density measured in the undeformed body and $\hat{\mathbf{f}}$ is the body force per unit volume of the undeformed body.

1.4 Establish the relations in Eq. (1.4.14) beginning with the strain–stress relations (Hooke's law) for the plane stress case (i.e., $\sigma_3 \equiv \sigma_{33} = 0$):

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}.$$

1.5 If the displacement vector is given by

$$\mathbf{u}(x_1, x_2, x_3) = [u(x_1) + x_3\, \phi(x_1)]\, \hat{\mathbf{e}}_1 + w(x_1)\, \hat{\mathbf{e}}_3,$$

determine the components of $\boldsymbol{\omega}$ and $\boldsymbol{\chi}$.

If your theory is found to be against the second law of thermodynamics, I give you no hope; there is nothing for it but to collapse in deepest humiliation. Arthur Eddington

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