Neither Newton nor Leibnitz:

The Pre-History of Calculus in Medieval Kerala. A Colloquium and Several Lectures

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Dedication

•The lifelong work of K. V. Sharma remains the main source and inspiration for all studies of the history of mathematics and astronomy in Kerala. My own first exposure to this subject was through the late Prof. S. Parameswaran of Kerala University. These talks are dedicated to them.

Colloquium


INDIA
States and Union Territories
N
PAKISTAN
CHINA (TIBET)
JAMMU & KASHMIR
HIMACHAL PRADESH
PUNJAB
HARYANA
UTTARANCHAL
ARUNACHAL PRADESH
NEPAL
SIKKIM
BHUTAN
RAJASTHAN
Jaipur
UTTAR PRADESH
Lucknow
BIHAR
Patna
ASSAM
NAGALAND
MANIPUR
MIZORAM
MEGHALAYA
TRIPURA
MYANMAR
GUJARAT
MADHYA PRADESH
Bhopal
JHARKHAND
WEST BENGAL
CHHATTISGARH
ORISSA
BANGLADESH
DIU
DAMAN
DADAR & NAGAR HAVELI
MAHARASHTRA
Mumbai (Bombay)
Hyderabad
ANDHRA PRADESH
YANAM (Pondicherry)
BAY OF BENGAL
ARABIAN SEA
GOA
KARNATAKA
LAKSHADWEEP
MAHE (Pondicherry)
TAMILNADU
PONDICHERRY (Puducherry)
KARAIKAL (Pondicherry)
Port Blair
ANDAMAN & NICOBAR ISLANDS
KERALA
Thiruvananthapuram
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Map not to Scale


76° East Longitude
78°
Mangalore
Puttur
Bangalore
KARNATAKA
Hosur
Kasaragod
Madikeri
Hosdrug
Hunsur
Mysore
Nileswaram
Dharmapuri
12°
Cannanore
Kuthuparamba
Chamrajnagar
Tellicherry
Mahe
Manantavadi
MAHE
Salem
Badagara
Gudalur
Kaveri
NILGIRI HILLS
Erode
Calicut
Manjeri
Tiruppur
North
Tirur
Karur
Shoranur
Coimbatore
Ponnani
Chittur
TAMIL NADU
Guruvayur
Palghat
Pollachi
Trichur
Dindigul
Irinjalakuda
KERALA
Periyar
Anai Mudi +2,695 m
Alwaye
10° North Latitude
Ernakulam
CARDAMOM HILLS
Vembanad Lake
Cochin
Madurai
Arabian Sea
Kottayam
Kambam
Alleppey
Changanacheri
Mavelikara
Adur
Kovilpatti
Kayankulam
Punalur
Quilon
Tirunelveli
Anjengo
Nedumangad
Trivandrum
Tiruchchendur
Neyyattinkara
0 50 Miles
0 50 Kilometers
Nagercoil
Cape Comorin
Gulf of Mannar

Kerala

•Kerala is a small coastal state of India: thirty two million people on 15,000 square miles most of whom speak Malayalam.

•It is far away from the heartland of India, a blessing in the times of invasions, famine and pestilence of the late middle ages 1300-1700 CE during which our story takes place.

•The current capital is Thiru-Ananta-Puram (Trivandrum). Ancient ports are at Kollam(Quilon), Kozhikode(Calicut) , Kodungalloor(Muziris) and Kochi (Cochin) and Varkala.

•Kerala was one of the wealthier parts of the ancient world; wealth from trade in luxury goods such as spices (black pepper, cardamom, nutmeg), silk,metal mirrors, cotton, dyes, sugar ($sarkara$).

•Rice was the staple food and the mainstay of the domestic economy. The English word rice comes from the Tamil and Malayalam word $Arissu$.

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Sources of Kerala History

•The text "Circumnavigation of the Eritrean Sea" (written in Greek around 1st century CE) describes this trade; many of the ports still exist.

•Marco Polo passed through Kerala on his way back from China in the early 1300's

•Ibn Batuta also passed through during his raucous adventure in India a generation later.

•Al-Biruni's book *India* (*Al-hind*, c. 1000 CE) is the source of much information on North India. He was a great scholar in astronomy, mathematics, literature and philosophy.

•We also can know of the history of this period early Malayalam literature :*History of Malayalam Literature* (in Malayalam,1940s) by Ulloor Paramesware Iyer, a great modern poet.

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Madhava of *Sangama-grama*

•Irinhalakkuta is today a small unremarkable town. There lived, during the time 1340-1425 CE, a mathematical genius named *Madhava* who discovered many of the basic ideas of calculus: the solution of transcendental equations by iteration, the infinite series for $\sin$, $\cos$ and $\arctan$, integration, integration of series term by term, tests of convergence of infinite series, approximation of transcendental numbers by continued fractions etc.

•Madhava belonged to the *Aryabhatta* school of astronomy (as opposed to the *Brahmagupta* school). There was astronomic work in Kerala as early as 4th century CE (*Vararuchi*), but *Madhava* made substantial breakthroughs.

•The *Aryabhateeyam* (composed in 499 CE) was an enormously influential text not only in India, but through translations, in the Arab world and indirectly through them in early Europe.

The Kerala School of Astronomy

•He also founded a lineage of astronomer-mathematicians which lasted till the early eighteenth century. His followers elaborated and refined his theories and composed hundreds of mathematical works.

•*Parameswara* (1360-1455) student of Madhava. Discovered *drk-ganita*, a mathematical model of astronomy based on observations. He made observations for a period of 55 years and had the most accurate observations of planetary motion before Tycho Brahe.Authored about 30 works in mathematics and astronomy

•*Damodara*(1410-1510) was *Paramswara*'s son and student. His works survive only in quotations in others.

•*Neelakanta Somayaji*(1444-1545) Student of Damodara. Author of *Tantra-Sangraha* a comprehensive treatise on astronomy and related mathematics as well as many other extremely influential books. His masterpiece was his commentary on the *Aryabhatteyam* which contains many results on calculus. *Grahapareeksaakrama* is a manual on how to make observations in astronomy using instruments of that time.

•*Jyeshtadeva* (c. 1500-1610) also a student of Damodara is the author of the *yuktibhasha* as well as *Drk-karana* on observations.

•*Achyuta-Pisharati*(c. 1550-1621) was a student of *Jyeshtadeva*. Discovered the technique of 'reduction to the ecliptic' ; author of *sphuta-nirnaya* and *Rasi-gola-sphuta-neeti*.

•*Melpathur Narayana-Bhatta-thiri* student of Achyuta was a mathematical linguist (vyakarana). His mastrepiece is *Prkriya-sarvawom* which sets forth an axiomatic system elaborating on the classical system of *Paanini*. But he is more famous for his devotional poem *Narayaneeyam* still sung at the temple were he worked, *Guruvayoor*.

•The lineage continues down to modern times, with new Sanskrit and Malayalam texts and commentaries written as late as the nineteen fifties. But the original research ends around the time of *Narayana*.

•Manuscripts of the Kerala school can be found all over India. A measure of their influence is that many imitation texts were written with the word Kerala inserted into the title to give them an aura of authenticity.

Early Results

•The *Aryabhateeyam* (499 CE) is a text on astronomy with chapters on spherical trigonometry *gola* and mathematics *ganita*. It contained a table of sines (*jya*). The word sin is derived from the latin *sinus* (meaning 'fold') which is a translation of *jaib* in Arabic, which in turn is a mis-transliteration of *jiva* (*jya*) the sanskrit word for chord. *Aryabhata*, the author of this work was famous throughout the ancient world. He is mentioned in Al-Biruni's India as the source of Arab trigonometry.

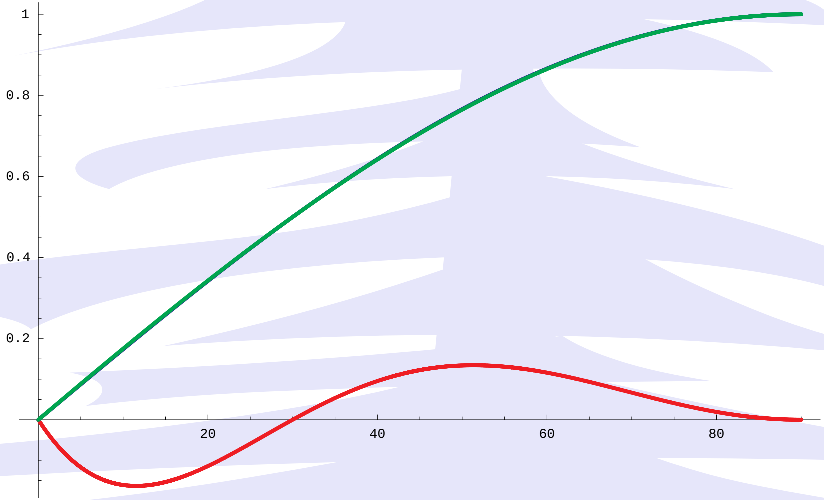
•Bhaskara's text *Leelavati* has a part called *Bijaganita* which is the source of algebra. Its Arab translations influenced *Al-Khwarizimi* who in turn is the source of the flowering of European algebra with Cardano and others.

Bhaskara's Rational Approximation to Sin

•Bhaskara had an interesting rational approximation for $\sin$ (with the angle measured in degrees):

$$\sin x = \frac{4x(180-x)}{40500 - x(180-x)}. \tag{1}$$

In the next slide we plot the sin (in blue) as well as Bhaskara's approximation to it (in green); they are so close that we can't tell them apart! So we plot the difference **magnified by a 100** on the same graph in red. Rational approximations such as this are much better than power series in representing transcendental functions.


1
0.8
0.6
0.4
0.2
20
40
60
80

The Circumference of the Circle

•It has always been of great interest to geometers and astronomers to relate the circumference of a circle to the its diameter.

•The basic method has been to inscribe or circumscribe a regular polygon. The problem then is to find the side of the polygon as a multiple of the diameter.

•Approximate formulae good enough for practical purposes had been known for a long time-$\pi \approx \frac{22}{7}$ is enough for most engineers.

•Archimedes of Syracuse (287-212 BCE) obtained the value $3\frac{10}{71} < \pi < 3\frac{10}{70}$ considering by considering a regular polygon of 91 sides.

•The *aaryabhatiiya* (499 CE) gives a value accurate to four decimal places: "The circumference of a circle of diameter 20,000 is 62832": or $\pi \approx 3.1416$.

•Bhaskara (1114-1185(?) CE) says that the circumference of a circle of diameter 1250 is 3927 by considering an inscribed regular polygon of 384 sides-correspond to $\pi \approx 3.14155$. Getting close!

•These days the value of π to a thousand decimals is just two clicks away if you have Mathematica! $\pi \approx =$ 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111 7450284102701938521105559644622948954930381964428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273724587006 6063155881748815209209628292540917153643678925903600113305305488204665213841469519415116094330572703657595919530921861173819326117931051185480744623799627495 6735188575272489122793818301194912983367336244065664308602139494639522473719070217986094370277053921717629317675238467481846766940513200056812714526356082778 5771342757789609173637178721468440901224953430146549585371050792279689258923542019956112129021960864034418159813629774771309960518707211349999998372978049951 0597317328160963185950244594553469083026425223082533446850352619311881710100031378387528865875332083814206171776691473035982534904287554687311595628638823537 8759375195778185778053217122680661300192787661119590921642019 9

Chapter Six of the Yuktibhasha

•What is new with the Kerala school is a convergent infinite process that can give the value of π to *arbitrary* accuracy. There were several such processes known to this school, we will study in detail two of them, explained in detail in the sixth chapter of the *Yuktibhasha*.

•There are two different approaches to calculating the circumference.

•The first will give an algebraic recursion relation-involving a square root- that converges to the exact value. In modern notation,

$$x_0 = 1, \quad x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n}, \quad \pi = 4\lim_{n\to\infty} 2^n x_n \tag{2}$$

•The second method-really a succession of improvements- goes much further. It starts as a way to avoid square roots in the calculation of the circumference.

•A finite series-whose terms depend on the number of terms in the series- is obtained which converges to the circumference as the number of terms grows. Again in our notation,

$$\pi = 4 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left[\frac{1}{1+\left(\frac{n}{N}\right)^2} \right]. \tag{3}$$

•We can recognize the sum as tending to $\int_0^1 \frac{dx}{1+x^2}$.

•Then this series is re-expressed in a way that the terms don't depend on the number of terms. Taking the limit this gives the fundamental infinite series

$$4D[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots] \tag{4}$$

for the circumference of a circle of diameter D.

•The integral was discovered in this context!

•Formulae such as

$$1 + 2 + 3 \cdots N = \frac{N(N+1)}{2} \tag{5}$$

$$1^2 + 2^2 + 3^2 \cdots N^2 = \frac{N(N+1)(2N+1)}{6} \tag{6}$$

$$1^3+2^3+3^3\cdots N^3=\frac{N^2(N+1)^2}{4} \tag{7}$$

for powers up to four were known.

•The key step was to realize that for large N (small steps in the rectification of the circle)

$$1^k+2^k+\cdots N^k \approx \frac{N^{k+1}}{k+1} \tag{8}$$

so that in the limit we can replace

$$\sum_{n=1}^{N}\left[\frac{n}{N}\right]^k \approx \frac{N}{k+1} \tag{9}$$

Quotation from the Tantra-Sangraha

•Of course this modern notation was not used.

•The language is tortured in the Yuktibhasha as the arguments gets harder and harder. The final result is quite simple and is expressed in an elegant poem quoted from the Tantra-Sangraha (by *Neelakanta Somayaji*, the result is attributed to Madhava though).

• *vyaase vaaridhi-nihate ruupahrte vyaasasaagaraabhihate thri-saradi-vishamasamkhyaa-bhaktam r.n.am svam pr.that kra-maal karyaat*

•K. V. Sharma's translation: "Multiply the diameter by four. Subtract from it and add to it alternately the quotients obtained by dividing four times the diameter to the odd numbers 3,5 etc. "

•This is not an absolutely convergent series; even when summed in the right order it is slowly converging. The commentator to the Yuktibhasha shows that summing 27 terms gives a value accurate to one (!)decimal place.

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Estimates of Error

•One can add corrections to the truncated sum which estimate the terms omitted

•In the first direction there is

$$C \approx 4D\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{(n+1)/2}{(n+1)^2+1}\right] \tag{10}$$

•Here is an even better formula (also attributed to Madhava in the *Kriyakumari*) for the correction to the finite sum:

$$\frac{\left(\frac{n+1}{2}\right)^2 + 1}{\left(\frac{n+1}{2}\right)\left[4\left(\frac{n+1}{2}\right)^2 + 1\right]} \tag{11}$$

Convergent Series for the Circumference

•Or, we can look for new series that converge.

•A result of Madhava when translated to modern language is

$$\pi = \sqrt{12}\left[1 - \frac{1}{3\times 3} + \frac{1}{5\times 3^2} - \frac{1}{7\times 3^3}\cdots\right] \tag{12}$$

•**Exercise** Prove this result by modern methods. Estimate the error if this series is stopped at the nth term.

•Madhava derived using this the result that the circumference of a circle of diameter 9^{11} is 2827433388233. He also derived a way to convert the radian to the degree.

•The Yuktibhasha also gives many rational approximations which have no parallel in modern mathematics. They are based on continued fractions and I have not been able to decipher them yet.

The Arctangent

•A poem of Madhava is quoted in the Yuktibhasha which gives the arc of the circle in terms of the ratio of jya ($\sin$) and the koti ($\cos$). (Remember that these quantities are proportional to the radius.)

•Based on a translation of K. V. Sharma: Multiply the jya by the trijya and divide the product by the koti. Multiply this by the square of the jya and divide by the square of the koti. We get a sequence of further results by repeatedly multiplying by the square of the jya and dividing by the square of the koti. Divide these in order by the odd numbers 1,3,5 and so on. Add the odd terms and subtract the even terms (preserving the order of the terms). This gives the dhanus (arc literally, bow) of these jya and koti. Here the smaller of the two sides should be taken as the jya as otherwise the result will be non-finite.

•If the jya is s and the koti is c and the trijya (radius) is R, we have

$$\frac{sR}{c} - \frac{1}{3}\frac{sR}{c}\left[\frac{s}{c}\right]^2 + \frac{1}{5}\frac{sR}{c}\left[\frac{s}{c}\right]^4 - \frac{1}{7}\frac{sR}{c}\left[\frac{s}{c}\right]^6 + \cdots \qquad (13)$$

•If we put $\frac{s}{c} = t$ as the tangent and measure the arc in units of the radius (as we would in modern notation) this is the infinite series for the arctangent:

$$t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \cdots \tag{14}$$

Obtained a couple of centuries before Gregory after whom this series is named!

•Madhava also obtained the infinite series for $\sin$.

Religion in Kerala

•The people of Kerala today belong to the three major religions: Christianity(20%) , Islam(20%) and Hinduism(60%).

•The proportions were different in the time we are speaking of: there was a small and ancient Christian church, founded by the Apostle St. Thomas himself if we are to believe in the legends.

•There was a tiny but vibrant jewish community. There were some converts into Islam along the coastal regions.

•But the vast majority of people followed the traditional religion of India known there simply as the 'Old Ways': the Sanaadhana Dharma. The Persians called the followers of this religion 'Hindus'- derived from their name for the Sindhu (Indus) river- which now is used even in India to describe them.

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The Hindu Religion-*Sanatana Dharma*

•Unlike modern religions (such as Buddhism, Christianity or Islam) Hinduism does not have a unique founder. Like Judaism it is a system of practices handed down from time immemorial.

•The basic spiritual texts are the four Veda,perhaps the oldest surviving texts of mankind. The word Veda simply means 'the knowledge'. These are supplemented in later times by the epics (puraana) (mainly the Ramaayana, MahaBhaaratha and the Bhaagavata) ; embedded in the epics are several important texts such as the Bhagavat Giita and the Yoga Vasishta. This classical literature is supplemented by the commentaries of saints the most important of whom is Sankara Achaarya.

•The Veda are a sort of encyclopedia of ancient knowledge. In addition to the hoary philosophy of the Upanishads, the Veda also contain the ancient rules of human behavior and of course, hymns and prayers.

•Although in its core Hinduism is not about Gods, but about a supreme existence of which we are all a part, much of the religious practice has to do with a multitude of Gods: each of which represent an aspect of this supreme reality.

•In the Veda, many of the Gods are identified with natural phenomena: the Sun, the Moon and the planets are minor Gods the major ones being Indra, Vishnu, Siva and so on.

•It is difficult to convey that there is an essence to Hinduism that lies beyond the Gods to those from another cultural and religious background: it often looks like a bewildering array of colorful, even scary images connected together by fantastic legends, much like the pre-Christian religions of Europe. However, abstract notions of the impersonal infinite are still quite familiar to Hindus and the abstractions of mathematics were often derived from this common religious background.

A Glimpse of Infinity

•As an example, here is a verse from the Isaavasya Upanishad of the Yajur Veda that many of us repeat daily even today:

purnamadah purnamidam purnaat purnamudachyate
purnasya purnaamadaya purnameva vashishyate

That is the Universe, This is the Universe,
The Universe arises from itself, it is said
If you subtract the Universe from the Universe
There remains the Universe, indeed.

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Namboothiri: The High Priests

•Most of the astronomer-mathematicians of Kerala were *Namboothiri*s, the highest ranking priests of the Hindu religion.

•They had the highest social status in the society, higher than the King.

•Although less than a half percent of the population, they controlled most of the wealth through land ownership.

•What they did not own out-right they controlled through the temple-trusts which were managed by them.

•The people who toiled in the fields (rice farming is very labor intensive) had no ownership of the land or its produce.

•Yet the enormous wealth that these rice plantations produced were entirely the product of their labor: without constant toil the land would have decayed out of over-cultivation in just a few years.

•At best, they were tenants who paid the landlords regularly for the privilege of cultivating the land.

•At worst they were the pariah [1] or pulaya who were in essence slaves: even the huts they lived in were owned by the brahmins.

•No worse than any other medieval society. This feudal system was finally dismantled in the mid-twentieth century under the leadership of E.M. S. Namboothitipad, the first Chief Minister (like Governor of a US state) who re-distributed the land to those worked in it.

[1] Pariah: (a sad word that Malayaalam has contributed to the English language) describes a caste at the very bottom of the social hierarchy.

How did the *namboothiris* Dominate ?

•The namboothiris are believed to have emigrated down the West coast of Indian starting with the fourth century CE. In any case by the ninth century every aspect of Kerala society came under their firm control.

•There are some sub-castes within the namboothiris who were later immigrants from *Thulu* region just to the North of Kerala. Madhava belonged to one of them: he was an *embranthiri*.

•The key to the dominance of the *namboothiri*s was that they were the keepers of the sacred Hindu scriptures, the *Veda*.

•It was their sacred duty to perform the vedic sacrifices to maintain the harmony of the universe.

•The title Somayaji denoted someone who had performed one of the most difficult and ancient of these rituals, the *Soma-yaga*.

•Being the only allowed priests, they controlled the temples which were also the only centers of learning, art and culture.

- Each village had a small standing army of *Nairs* who protected the temple.
- The temples and could not be taxed; indeed the King was dependent on them even to raise an army.

Simplicity, Dedication, Discipline

•The life of a *Namboothiri* was comfortable but by no means luxurious. There are no castles or opulent palaces in Kerala.

•They led a life of scholarship and spirituality.

•Food was strictly vegetarian. No alcohol or other intoxicants were allowed. Even strong tastes like onions were forbidden as it could inflame sexual appetite.

•Fasts on the eleventh day of each half of the lunar cycle .

•Servants were not allowed to cook food; it was the domain of the *nambothiri* women.

• Exercise was built into the spiritual practices; e.g., the prayer to the Sun God is an excellent aerboc workout. You can learn it in yoga classes in the US. They walked everywhere.

•They lived in large joint families (many unmarried adult brothers and sisters, the children of the oldest brother,the grandparents and sundry dependents)in a simple single story home.

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•The women rarely travelled outside: the Malayaalam word antarjanam for a Namboothiri woman means 'woman inside' (the house).

•The houses had one or two internal courtyards without a roof and was built either as a square or double square around them. (There is some resemblance to floor plan of the Roman Triclineum.)

•The roof was tiled, or in the older days a thatched with coconut leaves. This kind of roof had to be replaced annually. The floor was bare, just a mud: no form of cement or marble was used.

•There would be a shed some distance away for the cows and a sizeable stack of hay to feed them.

•Dress was also very simple: a piece of cotton cloth around the waist, a towel around the shoulders. A thread around the body indicating *brahmin* status. Even on the most formal occasions,men were bare chested. Marco Polo was aghast at that!

•Here is a picture of the High priest during the year 2003-2004 of the biggest temple in Kerala (Ayyappa kshetram): this was the typical dress of a *namboothiri* of that time.

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The Education of a Namboothiri

- The education of a Namboothiri boy was intense, deep and broad.
- He started at the latest by the age of eight and continued at least to age sixteen.
- The teacher was often a family member, an uncle or father or grandfather.
- Every morning started before sunrise (always 6:00 am in these tropical parts) with prayers to the Sun God, Suurya. After a short break for lunch it would continue till sundown.
- The center piece of the education was the learning of the Veda. No other caste had the right to learn the Veda.
- The Veda could not be written down, the entire corpus had to be memorized. Each family inherited a piece of the Veda assigned to it according to tribal succession laws and passed it on to the next generation.
- The smartest boys learned the hardest and most abstract kind of

knowledge: the Upanishads, about the nature of knowledge itself, that ultimate knowledge from which all else follows.

•Even the dumb ones at least had to memorize the Veda by rote without understanding its meaning. The chanting of the Veda is the ultimate duty of the Namboothiri. Each of them were in essence a walking library of ancient knowledge.

Error Correction by Redundancy

•To make sure that no error would creep into the oral transmission of the Veda, there were intricate error correction techniques built into this rote learning.

•You would learn the Veda not only as if read from left to right, but also in the reverse order. Which would make equal sense to someone who doesn't understand the ancient Sanskrit to begin with. Not only that you would memorize each verse by taking a syllable from the middle then one to the left then one to the right and so on.

•This redundancy as well as the redundancy in the large number of people who learned the Veda compensated for the volatility of human memory. Indeed the Veda are extremely remarkably well preserved: you can compare the Veda as recited by a Namboothiri brahmin to a Kashmir pundit. There would be no difference not only in literal content but also in the pronunciation and rhythm of the singing.

•There are greater disputes over Shakespeare's writing than over the text (*samhita*) of the Veda: in spite of the former being printed and much more recent. As the older generation who received this classical education die out, there is now the danger of entire branches of the Veda dying out with them. There is an ongoing project to record the chanting of the Veda before this happens.

•There were annual competitions in the recitation of the veda. Such a competeition (*anyonyam*) still continues but at a much smaller scale.
•The various city states and temples competed to attract well known scholars to stay in residence. In return the scholars were expected to compose some salutary verses honoring the local ruler (Prasasthi) which they completed with some grumbling.

•Secular subjects such as poetics, rhetoric, grammar, logic, astronomy (of which mathematics is a part), medicine occupied an important place in the education, but were considered distinctly inferior to the study of the Veda.

•Each person has a guru or teacher responsible for his overall education; although occasionally there might be more than one teacher when someone has expertise in several areas.

•In this respect advanced education today at the level graduate school holds a remarkable similarity to this ancient system. But the guru was often a relative: an uncle or ones father.

•The guru was held in the highest regard, indeed as a form of divinity.

•The guru was responsible for spiritual and moral development as well as education.In return the student was to obey and protect the guru for life.

•The word guru means literally 'heavy' or ponderous. In astronomy, guru is also the name of the planet that we call Jupiter in English: because of its ponderous motion across the sky with a period of twelve years. In mythology this guru represents the teacher of the Gods, a play on the meaning of the word.

•The schools or Madhams survived down to the early twentieth century. At this time the Namboothiris started to suffer from the lack of an English education. There was a popular reform movement which allowed the Namboothiris to adapt to the modern world.

•Now they have melted into the emerging vast Indian middle class as professionals: teachers, doctors, scientists and immigrants to the United States..

Lecture I

Kerala

•Kerala is one of the states in India, on the South West tip of the Indian peninsula.

•It has 3 percent of the total population and only one percent of the total area; thirty two million people in 15,000 square miles.

•Still larger than the population of Canada, and three times that of Greece or Portugal.

•Language spoken is Malayalam.

•The capital city of Trivandrum (Thiru-Ananantha-Puram) is to the South.

•There are ancient ports at Kochi (Cochin), Kozhikode(Calicut) and Kodungalloor.

•The town of Irinhalakuta (Sangamagrama) still exists. There is nothing remarkable about this town today.


Mangalore
Puttur
76° East Longitude
Bangalore
78°
KARNATAKA
Hosur
Kasaragod
Madikeri
Hosdrug
Hunsur
Mysore
Nileswaram
Dharmapuri
12°
Cannanore
Kuthuparamba
Chamrajnagar
Tellicherry
Mahe
Manantavadi
MAHE
Salem
Badagara
Gudalur
Kaveri
NILGIRI HILLS
Erode
Calicut
Manjeri
Tiruppur
North
Tirur
Karur
Shoranur
Coimbatore
Ponnani
Chittur
TAMIL NADU
Guruvayur
Palghat
Pollachi
Trichur
Dindigul
Irinjalakuda
KERALA
Periyar
Anai Mudi +2,695 m
Alwaye
10° North Latitude
Ernakulam
CARDAMOM HILLS
Vembanad Lake
Cochin
Madurai
Kottayam
Kambam
Arabian Sea
Alleppey
Changanacheri
Mavelikara
Adur
Koyilpatti
Kayankulam
Punalur
Quilon
Triunelveli
Anjengo
Nedumangad
Trivandrum
Tiruchchendur
0
50 Miles
Neyyattinkara
Nagercoil
Gulf of Mannar
0
50 Kilometers
Cape Comorin

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The Kerala School of Astronomy and Mathematics

•In the centuries 1300-1600 of the Christian era there lived in villages around Irinhalakuta a group of great astronomer-mathematicians who seem to be the first to invent many ideas of calculus in common use today.

•The school was founded by Madhava of Sangamagrama in the fourteenth century of the Christian era (CE). His students and their students formed a continuous line (Parameswara, Neelakanta Somayaji, Jyeshtadeva....Melpathur Narayana Bhattathiri.) that lasted until the early seventeenth century.

•Madhava discovered the infinite series for $\arctan$ and $\sin$; also many methods for calculating the circumference of the circle.

•Other achievements: 'Cauchy test' of convergence, differentiation and integration term by term, the theorem that the area under a curve is its integral.

How Kerala came to the Attention of Europeans

•Kerala had trade with Europe via Arab intermediaries for centuries.

•Spices, silk, cotton, dyes, steel, mirrors were traded in the time of Christ.

•Jewish settlement since at least the time of Christ; St Thomas the Apostle himself said to be the founder the Malankara Orthodox Church.

•Marco Polo passed through Kerala on his way back from India in the early fourteenth century. His stories were not believed at first but turned out be quite accurate.

•Ibn Batuta also passed through Kerala during his adventures in India a generation later.

•Reports of the 'riches of the Indies' (esp. from Marco Polo) and enmity with the Arabs inspired a great exploratory project the Catholic nations of Spain and Portugal to find a trade route to India that did not rely on the Arabs.

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Direct Contact with Europeans 1500-1950

•Vasco de Gama discovered the sea route to India. Arrived in Calicut (Kozhikode) in 1495 CE.

•The Portugese were not successful as traders. There was a Dutch monopoly of trade to India for a hundred and fifty years.

•The French tried to get a foothold without success.

•The Dutch were finally defeated with the help of the British in the early eighteenth century. The Battle of Kolachel. The Dutch commander defected and trained Indian army in European techniques.

• The British established a small trade outpost in the village of Anchu Thengu.

•But they were more successful when they went around the tip of India to the East coast: first Madras then Bengal.

• Most of Kerala remained outside direct rule by British even up to Indian independence in 1947.

India is the Source of Much Ancient Mathematics

•Of most value to European visitors were methods of navigation, herbal medicine, spice cultivation.

•Astronomy and mathematics were not as practically useful, but were greatly valued by ancient scholars.

•Earlier Indian mathematics transmitted to Europe through Arabs. Decimal system: The 'Arab numerals' are called the 'Hindu numerals' by the Arabs.

•The $\sin$ was studied extensively in the Aryabhateeyam by Aryabhatta written around 500 CE. Spherical trigonometry was extensively developed. Translated to Arabic then to Latin. Transmission through Spain.

•Al-Biruni wrote extensively on Indian astronomy and mathematics in the tenth century CE. Brahmagupta's calculations of the eclipses made a big impression on him.

•Algebra was discovered in India. Bhaskara's text Bijaganitham (Indian name of algebra) influenced Al-Kwarizimi who in turn influeunced the Ars Magna.

•Axiomatic Linguistics (Vyakarana) was very developed in Sanskrit but also other Indian languages.

The Jesuits Arrived in early 1500s

- The Jesuit missionaries were the first Europeans to become proficient in Indian languages. Caldwell wrote the first comparative grammar of South Indian languages. Gundert created the first dictionary between a European language and an Indian language (German-Malayalam).
- Knowledge possibly transferred to Europe by Jesuit missionaries. Shrouded in mystery. Church records hold the key to unlocking this mystery.
- Could all this have been independently discovered in the West just at the time that European scholars mae first contact with India? Unlikely, but cannot be ruled out with existing evidence.

Ramanujan

•Many of the techniques are reminiscent of those made famous by Ramanujan many centuries later.(For example integration of infinite series term by term to get more transcendental series.)

•Was Ramanujan educated in this ancient tradition? His mother was well educated in the ancient Indian mathematics, although they were desperately poor.

•Ramanujan's hometown was Kumbhakonam, an ancient center of learning in the nearby state of Tamil Nadu. There were always contacts between Tamil and Kerala scholars. For example, Neelakanta Somayaji wrote a book in answer to questions posed to him by a Tamil scholar. Manuscripts related to the Kerala school have showed up all over India, many of them in Tamil Nadu.

Observational Astronomy

•Parameswara was the greatest observational astronomer, a precursor to Tycho Brache. He stressed the importance of checking celestial theories against observation (*Drk Granita*).

•*Drk*=to see (also means direction). *Ganita*=mathematical theory. Drk ganita is the theory based on what one observes.

•He made 50 years of meticulous observations from the sea shore. (He started at the age of 53 and continued till he died at the age of 103.)

•There is a legend that this activity caused him to be ostracised (*bhrashtu*) from the caste of high-priests (*Namboothiri*):he had to live with low caste fishermen on the beach for many years to make his observations possible. Hard to verify this story. See http://www.namboodiri.org for more information.

•Astrologers in Kerala still argue over whether to use his 'new math' or not.

Axiomatic Theory of Languages

• *Vyakarana* (Linguistics not mere Grammar) was an axiomatic theory of languages which anticipated many modern innovations. The dominant is that of Panini.

• Just as Euclid's geometry provided the standard model of an axiomatic system for the Greeks, Panini's *Vyakarana* provided such a model to ancient India. Ideas algebra (*bijaganita*)- the idea of manipulating a single symbol that can take many values-have their roots in *Vyakarana*.

• The Madhava school also contributed much to linguistics.

• Melpathur Narayana Bhattathiri's *Prakriya Sarvaswam* presents an elaboration of and an alternative theory to Panini.

• He was the last great intellect in this line. Wrote a famous obituary poem on the occassion of his Guru's death lamenting the death of knowledge itself.

Connection to Modern Culture

•As the society degenerated due to external attacks and infighting caused by inequities, mathematics suffered. Only the 'useful' sciences survived. Astronomy degenerated into astrology. Madhava's work was mostly forgotten until rediscovered by modern historians, most importantly K.V. Sharma.

•Ayurveda, the ancient system of medicine is flourishing to this day.

•Vyakarana came to be studied only for its literary value; Melpathur is famous for his devotional work Narayaneeyam not so much for his work on linguistics. Sanskrit was replaced by Malayalam in literature, which is thriving even now.

•The Ayurvedic and Poetic traditions of Kerala were in fact founded by the same group of geniuses who also were astronomers and mathematicians. They were accomplished poets; many lived to be over a hundred years old.

•Thachu Shastram (architecture) still used in building temples.

Lecture II

The Yuktibhasha

•The first textbook on calculus, written in Malayalam rather than Sanskrit.

•It is also unusual in providing proofs rather than just statements of results. Moreover,it is written in prose rather than poetry. In many ways it is the last book in the old tradition of India and the first of the modern tradition.

•The first four chapters are quite elementary mathematics.

•Chapter five describes some calculations on calendars. Also, the solution of the equation $ax - by = c$ for given positive integers a, b, c; x and y are to be found as integers. This methods $Kuttakaram$ was known to Brahmagupta (tenth century CE).

• The beginning of the new ideas are in chapter six on the circumference of a circle.

Chapter one of the Yuktibhasha

•The book starts with the intrepretation of the product of two integers as the area of a rectangle. It is shown how to find the area of figures made of straghtlines intersecting at right angles by divinding them up into rectangles.

•This part of the book appears to be a prose commentary to the Sanskrit poetic text $tantra - sangraha$.

•Division of numbers is introduced next.

•Then the notion of a square *varga* is introduced both as the product as two equal numbers and the area of a square.

•The formula $(a+b)^2 = a^2 + b^2 + 2ab$ is then proved by dividing up the square of side $a+b$ into rectangles of sides $a \times a, a \times b, b \times a$ and $b \times b$.

•After proving $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ by a similar method, the Pythagoras theorem is proved.

•This result is present in the ancient text *sulbasutra* on the con-

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struction of sacrificial altars. It is proved here in a way that should be familiar to a modern student of mathematics, by constructing squares whose sides are each of the sides of a right triangle.

• The hypotenuse is called $karna$; the short side is called $bhuja$ (literally arm) and the long side is called $kodi$. All this sets the terminology for more interesting constructions later.

• The square root $vargamula$ is defined as the inverse of the square.

Chapters two, three and four

•Chapter two is about ten standard problems of elementary mathematics. These ten are:

•Given any two quantities among the sum, difference, product, sum of the squares or difference of squares, determine the underlying numbers.

•Of course the simplest of these problems is when the sum and difference are given.

•The formula $\sqrt{\frac{(a^2+b^2)-(a^2-b^2)}{2}} = a$ solves the most complicated of these.

• These elementary chapters are useful to understand the notation and terminology. Mathematical formulas are traditionally expressed as poems that can be more easily memorized. Here they are stated in prose which is obviously a translation of Sanskrit poems from some previous text such as *tantrasangraha*.

•It is an old Indian tradition to start mathematics discussions with the most elementary of processes; the mathematics chapter of the *aryabha-*

teeyam (from 495 CE) starts with a listing of the powers of ten. Explicit calculations are stressed at all levels.

•Usually the discussion proceeds very rapidly merely stating results with no proofs or explanations. These are to be provided by the teacher. There are also commentaries in which the results are explained and further refinements are made.

•Some of the deepest texts are in fact commentaries (*bhashya*) on classic texts: for example the commentary by *Neelakanta Somayaji* to the *aryabhatteyam* is a foundational reference for calculus. The *yuktibhasha* is much kinder to reader, to the point of being pedantic: even minor results are explained in detail.

•Chapter three is about fractions. The usual rules of additions of fractions are established using geometric arguments.

•Subdivision of rectangles into equal parts is the basic trick.

•Chapter four is very brief and is on proportions. Given three quantities in the statement $a : b :: c : d$, find the fourth.

•Proportions are viewed geometrically in terms of similar triangles, as well as in terms of problems in everyday life such as cost of rice.

Ch. Five: *Kuttakaram*

•*Kuttakaram* is a method of solving integer equations $Ax - By = C$. The method is from the *Leelavathi* of Bhaskara (11th century CE).

•The problem arises in the preparation of calendars, one of the main practical uses of astronomy. A solar month can be thought of as one twelfth of a solar year. This is close but not quite the same as a lunar month.

•Certain religious observances are tied to the moon (e.g.,fasts on the eleventh day of each half-period of the moon) while others are tied to the Sun (e.g., the start of a new year on the day of the vernal equinox). Converting from one system to the other leads to these kinds of problems.

•The practical problems are compounded by the fact that different regions of India used different calendars.

•In Kerala a calendar marking the rebuilding of the town of *Kollam* was in wide use for civil purposes; the months are named as in the Greek calendar *Chingam* (Leo) being the first month.

•A completely different calendar was also in use mostly for religious observances: the *Kali* era starting on the day of the death of Krishna and the beginning of the 'modern' era of India according to mythology.

•The astronomers tried to avoid these problems by simply counting the number of revolutions of the Earth from a standard point (the first day of the *kali* era-mythically the day after Krishna's death) in time to denote each day.This day one is assumed to be a Friday.

•This would be a big but standard number (the *kali-dina-samkhya*, the kali-date-number) that could then be converted to whatever calendar you want: computing the day of the week, whatever system of months and years you use and what day of the phase of moon etc.

•Modern computer scientists use a similar tricks. In unix, the clock time is the number of milliseconds that have elapsed since 1 Jan 1970: the mythical beginning of the digital computer era. Is this the day that the last analogue computer was unplugged?

•This number is then converted to the date,month and year with corrections for leap years etc. built into the program. (The command 'time' in unix gives the number of milliseconds and 'date gives the latter.)

•For example, the year 1120 of the Kollam year corresponds to the year 5046 in the *kali* year, and 1949 in the Christian era.

This example is adapted from the commentary to the *Yuktibhasha* by *Akhileshara Iyer* and *Rama Varma* published by *Mangalodayam Press* of *Trissur* in the year 1952 CE.

So on 1/*Chingam*/1120 in the Kollam calendar, what is the day of the week?We need to find the kalidinasankhya of this date. Taken modulo seven it will give the day of the week; modulo twenty-eight it gives the lunar date and so on. We give just a fragment of this calcutaion to give a flavor.

The calculations proceeds in this way with several corrections added to produce the kali date number 1842853 for the required date. The mathematics is not interesting. It is interesting to check the accuracy of the astronomical observations for the ratio of the moon's revolution period to the Earth's rotation period; and to the period of revolution of the Earth around the Sun.

To find this number we start with number of solar months Earth since the beginning of the *kaliyuga* to the last day of the *kali* year 5045; this is $5045 \times 12 = 60540$.

To convert this into lunar months, a correction has to be applied since the lunar month is shorter than the solar month. This correction was known to be $1593320/51840000$ times the number of solar months. That is $60540 \times \frac{1593320}{51840000} = 1860$ in our case.

This ratio comes from measurements of the moon's period. It is expressed by the formula that that the length of a *yuga* (a mythical interval with no astronomical signifcance) is 4320000 solar years or 51840000 solar months; the moon makes 57753320 revolutions in this time. The difference between their number of revolutions is 5343320; the excess number of lunar months over solar months is 1593320.

On the first day of the *kali* year 5045, $60540 + 1860 = 62400$. To this we add four more months to take us to the first of *Chingam* from the beginning of the kali year 5046.

Linear Diophantine Equations

•In calendar calculations, the following problem arises: Find all integer solutions to $ax - by = c$ given a, b, c.

•Clearly we can reduce a, b, c until they have no common factor.

•Given one solution (z, u) to the equation $ax - by = c$, all other solutions are of the form $x = z + mb, y = u + ma$. The idea is to apply this transformation repeatedly to reduce the equations to one with small coefficients, which is then solved 'by hand'; or it is clear that there is no solution..

•The method is called *kuttaka* after the iron rod that is used to pound grain into smaller chunks: the method 'pounds' on the numbers until they becomes small.

An example

•For example, solve $195x - 221y = 65$.

•$221 = 195 + 26$. So $195x_1 - 26y = 65$ where $x_1 = x - y$.

•$195 = 7 \times 26 + 13$. So $13x_1 - 26y_1 = 65$ where $y_1 = y - 7x_1$.

•Cancelling common factors $x_1 - 2y_1 = 5$ which has the solution $x_1 = 7, y_1 = 1$.

•Backtracking, $x = 57, y = 50$ is a solution. All others follow by the above transformation.

•**Exercise**: Write a computer program (in a symbolic algebra language like Mathematica or in a numerical language like Fortran or C) to solve the equation $ax - by = c$ given a, b, c using the above idea; or determine that it does not have a solution.

Lecture III

The Circle and Sin

•Sin and other related functions are these days thought of in terms of triangles. In India they were more closely associated with the circle. In fact the Indian version of the sin was a quantity with the dimensions of length.

•We will follow the discussion in *The Golden Age of Indian Mathematics* by S. Parameswaran pub by Swadeshi Science in Kochi, India.

•Imagine a circle of center O, andOS a line to a point on the circumference of the circle. Draw a perpendicular to OC intersecting the circle at A and B. The chord AB intersects OS at M.

•The sanskrit word *jya* means chord. AB is called *samasta-jya* the full-chord while AM is the ardha-jya, or half-chord. In practice the *ardha-jya* appears much more often and by default jya came to mean this half-chord. It is associated to the arc of the circle SA. Thus the jya as well as the arc are lengths proportional to the radius of the circle.

•The radius is chosen for convenience.

•The line MS is called the *saram* or 'arrow'. The picture of a bow and arrow with a chord stretching across is clear.

•The hypotenuse of a right triangle was called $karna$-literally, 'ear'. The other two sides are called $bhuja$-arm- and $kodi$.

•The length OM is also called *koti-jya*. It corresponds to $\cos$.

•The chord is also often called *jiva* (Sanskrit names have cases and have to adapted according to the context). For example, the formula for the jya of the sum of two arcs is called *jiveparasparanayaya*.

•Aryabhatta's work was famous in the Arab world. (Al-Biruni refers to him as 'Aryabhajos'.) In translations $jiva$ became confused with the Arab word $jaib$ which means 'fold' or 'bay'. Arabic is written without vowels and you have to know how to read the word from the context.

•Translations to Latin rendered *jaib* as sinus which mean fold. Then sinus got abbreviated to $\sin$. Along the way the radius was set to unity so that $\sin$ became a dimensionless quantity; also the arc came to measured in radians rather than degrees.

•Something similar happened with π. The Indian texts speak of the circumference of a circle of a given radius, it being understood that the circumference is proportional to the radius, which is then chosen for convenience. It was Euler who introduced the now standard notation of π for the ratio of the circumference to the diameter. •Bhaskara gives a rational approximation to $\sin$. It is equivalent to the formula

$$\sin x = \frac{4x(180-x)}{40500 - x(180-x)}. \tag{15}$$

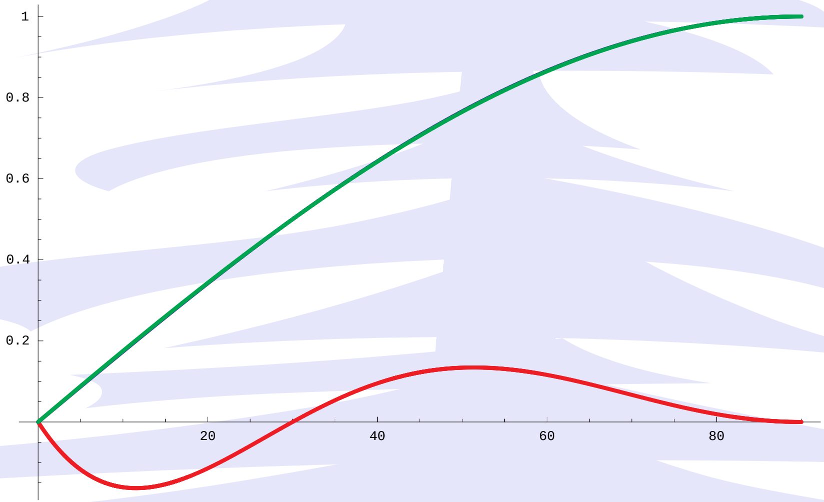
when x is in the first quadrant. The remaining cases can be brought to this by the symmetries of the $\sin$. The error is never worse than 0.01.

•To prove this, make the approximation that $\sin$ (with the angle expressed in degrees) is the ratio of two quadratic polynomials in the range $0 < x < 180$ degrees. Use the symmetry $x \to 180 - x$ of the sine to see that it must have the form

$$f(x) = \frac{a + bx(180-x)}{p + qx(180-x)}. \tag{16}$$

Then use the known values at $x = 0, 30, 90$ to determine the three constants. (An overall constant cancels out.)

•In the next slide we plot the sin (in blue) as well as Bhaskara's approximation to it (in green); they are so close that we can't tell them apart! So we plot the difference **magnified by a 100** on the same graph in red. Rational approximations such as this are much better than power series in representing transcendental functions.


1
0.8
0.6
0.4
0.2
20
40
60
80

The Circumference of the Circle

•It has always been of great interest to geometers and astronomers to relate the circumference of a circle to the its diameter.

•The basic method has been to inscribe or circumscribe a regular polygon. The problem then is to find the side of the polygon as a multiple of the diameter.

•Approximate formulae good enough for practical purposes had been known for a long time-$\pi \approx \frac{22}{7}$ is enough for most engineers.

•Archimedes of Syracuse (287-212 BCE) obtained the value $3\frac{10}{71} < \pi < 3\frac{10}{70}$ by considering a regular polygon of 91 sides.

•The *aaryabhatiiya* (499 CE) gives a value accurate to four decimal places: "The circumference of a circle of diameter 20,000 is 62832": or $\pi \approx 3.1416$.

•Bhaskara (1114-1185(?) CE) says that the circumference of a circle of diameter 1250 is 3927 by considering an inscribed regular polygon of 384 sides-correspond to $\pi \approx 3.14155$. Getting close!

• These days the value of π to a thousand decimals is just two clicks away if you have Mathematica! $\pi \approx =$

3.1415926535897932384626433832795028841971693993751058209749445923078164062862 0899862803482534211706798214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196442881097566593344612847564823378678316527120190914564856692346034861045432664821339360726024914127372458700660631558817488152092096282925409171536436789259036001133053054882046652138414695194151160943305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912983367336244065664308602139494639522473719070217986094370277053921717629317675238467481846766940513200056812714526356082778577134275778960917363717872146844090122495343014654958537105079227968925892354201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859502445945534690830264252230825334468503526193118817101000313783875288658753320838142061717766914730359825349042875546873115956286388235378759375195778185778053217122680661300192787661119590921642019 9

Chapter Six of the Yuktibhasha

•What is new with the Kerala school is a convergent infinite process that can give the value of π to *arbitrary* accuracy. There were several such processes known to this school, we will study in detail two of them, explained in detail in the sixth chapter of the $Yuktibhasha$.

•There are two different approaches to calculating the circumference.

•The first will give an algebraic recursion relation-involving a square root- that converges to the exact value. In modern notation,

$$x_0 = 1, \quad x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n}, \quad \pi = 4 \lim_{n\to\infty} 2^n x_n \tag{17}$$

•The second method-really a succession of improvements- goes much further. It starts as a way to avoid square roots in the calculation of the circumference.

•A finite series-whose terms depend on the number of terms in the series- is obtained which converges to the circumference as the number of terms grows. Again in our notation,

$$\pi = 4 \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left[\frac{1}{1+\left(\frac{n}{N}\right)^2} \right]. \tag{18}$$

•We can recognize the sum as tending to $\int_0^1 \frac{dx}{1+x^2}$.

•Then this series is re-expressed in a way that the terms don't depend on the number of terms. Taking the limit this gives the fundamental infinite series

$$4D[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots] \tag{19}$$

for the circumference of a circle of diameter D.

•The integral was discovered in this context!

•Formulae such as

$$1 + 2 + 3 \cdots N = \frac{N(N+1)}{2} \tag{20}$$

$$1^2 + 2^2 + 3^2 \cdots N^2 = \frac{N(N+1)(2N+1)}{6} \tag{21}$$

$$1^3 + 2^3 + 3^3 \cdots N^3 = \frac{N^2(N+1)^2}{4} \tag{22}$$

for powers up to four were known.

•The key step was to realize that for large N (small steps in the rectification of the circle)

$$1^k + 2^k + \cdots N^k \approx \frac{N^{k+1}}{k+1} \tag{23}$$

so that in the limit we can replace

$$\sum_{n=1}^{N} \left[\frac{n}{N}\right]^k \approx \frac{N}{k+1} \tag{24}$$

Quotation from the Tantra-Sangraha

•Of course this modern notation was not used.

•The language is tortured in the Yuktibhasha as the arguments gets harder and harder. The final result is quite simple and is expressed in an elegant poem quoted from the Tantra-Sangraha (by *Neelakanta Somayaji*, the result is attributed to Madhava though).

• *vyaase vaaridhi-nihate ruupahrte vyaasasaagaraabhihate thri-saradi-vishamasamkhyaa-bhaktam r.n.am svam pr.that kra-maal karyaat*

•K. V. Sharma's translation: "Multiply the diameter by four. Subtract from it and add to it alternately the quotients obtained by dividing four times the diameter to the odd numbers 3,5 etc. "

•This is not absolutely convergent series; even when summed in the right order it is slowly converging. The commentator to the Yuktibhasha shows that summing 27 terms gives a value accurate to one (!)decimal place.

Estimates of Error

•One can add corrections to the truncated sum which estimate the terms omitted

•In the first direction there is

$$C \approx 4D\left[1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{n}\mp\frac{(n+1)/2}{(n+1)^2+1}\right] \tag{25}$$

•Here is an even better formula (also attributed to Madhava in the *Kriyakumari*) for the correction to the finite sum:

$$\frac{\left(\frac{n+1}{2}\right)^2+1}{\left(\frac{n+1}{2}\right)\left[4\left(\frac{n+1}{2}\right)^2+1\right]} \tag{26}$$

Convergent Series for the Circumference

•Or, we can look for new series that converge.

•A result of Madhava when translated to modern language is

$$\pi = \sqrt{12}\left[1 - \frac{1}{3\times 3} + \frac{1}{5\times 3^2} - \frac{1}{7\times 3^3}\cdots\right] \tag{27}$$

•**Exercise** Prove this result by modern methods. Estimate the error if this series is stopped at the nth term.

•Madhava derived using this the result that the circumference of a circle of diameter 9^{11} is 2827433388233. He also derived a way to convert the radian to the degree.

•The Yuktibhasha also gives many rational approximations which have no parallel in modern mathematics. They are based on continued fractions and I have not been able to decipher them yet.

The Arctangent

•A poem of Madhava is quoted in the Yuktibhasha which gives the arc of the circle in terms of the ratio of jya ($\sin$) and the koti ($\cos$). (Remember that these quantities are proportional to the radius.)

•Based on a translation of K. V. Sharma: Multiply the jya by the trijya and divide the product by the koti. Multiply this by the square of the jya and divide by the square of the koti. We get a sequence of further results by repeatedly multiplying by the square of the jya and dividing by the square of the koti. Divide these in order by the odd numbers 1,3,5 and so on. Add the odd terms and subtract the even terms (preserving the order of the terms). This gives the dhanus (arc literally, bow) of these jya and koti. Here the smaller of the two sides should be taken as the jya as otherwise the result will be non-finite.

•If the jya is s and the koti is c and the trijya (radius) is R, we have

$$\frac{sR}{c} - \frac{1}{3}\frac{sR}{c}\left[\frac{s}{c}\right]^2 + \frac{1}{5}\frac{sR}{c}\left[\frac{s}{c}\right]^4 - \frac{1}{7}\frac{sR}{c}\left[\frac{s}{c}\right]^6 + \cdots \qquad (28)$$

•If we put $\frac{s}{c} = t$ as the tangent and measure the arc in units of the radius (as we would in modern notation) this is the infinite series for the arctangent:

$$t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \cdots \tag{29}$$

Obtained a couple of centuries before Gregory after whom this series is named!

•Madhava also obtained the infinite series for $\sin$.

Lecture IV

An Algebraic Recursion formulae for π

The essence of the first argument is to circumscribe the circle in regular polygons: an idea going back to Archimedes,but improved here to give arbitrary accuracy. The end result will be a remarkable recursion relation involving square roots which if repeated, gives increasingly accurate approximations for the circumference.

We will give a more or less literal translation of the text of the *Yuktibhasha*, in san serif font, interspersed by a commentary (in roman font) which will give the argument in modern algebraic notation.

Now we will see how to construct a circle from a square. Imagine a square whose side you may choose. We want to know the circumference of the circle whose diameter is the side of the square. Through the center of this imagined square draw East-West and North-South lines. Now we have four squares. Then from the center of the large square draw a line to the corner. This will be the hypotenuse[2] (*karna*). Imagine it to be in the 'fire corner' (top right corner, while facing East).Draw another hypotenuse from the South radius to the East radius This contains the circle centered at the center of the square.

The circle is being thought of as drawn by a thread stretched from the center; the word for radius *suutraagra* literally means 'thread-end'. This radius always has a specified direction, a vector in modern language. The radius of the circle in the sense of a magnitude or scalar is called *vyaasaardha* or half-diameter. We will keep this distinction in the translation, when needed.

Now, for any triangle, imagine (like a carpenter making a beam for a roof) the largest of the three sides to touch the ground and the two other sides meeting directly overhead. Hang a weight from the meeting point by a thread. This thread is called a *plumbline*. The side on the ground is called the *base* (literally, *bhoomi*, the Earth). The two pieces

[2] The sides of a triangle-*thryasha*-are the *karna*, the longest side which we will translate as the *hypotenuse*; the *kodi* which we will translate as the *long side*; and the *bhuja*, the *arm* or short side.

of the base from the point where the plumbline touches the base are called the *segments* ($Aaabaadha$).

Here, the hypotenuse from the center to the corner is imagined to be the base. The East radius and the South-half of the East side are the sides. Half of the hypotenuse from the East radius is the plumbline. The same way there is a triangle with the South-thread and the East-half of the South side as sides. The base being the same as before. This way, from a square, are two triangles made.

(Think of) the segment touching the corner as the denominator; the side from the corner to the East radius is the numerator. What remains of the hypotenuse after taking out the half-diameter from the base is the desired-multiple (of a similarity of triangles, see below). The result is measured from the corner in each direction to make two points and cut out the corner so formed. Then there is an octagon. Take out double this desired-multiple from the side of the original square. What remains is the side of the octagon.

The idea here is to construct a regular octagon circumscribing the circle and to find its side. A similarity is thought of as a standard-measure *pramaanam* P, its standard-multiple *pramanaphalam* p, the desired-measure *iccha* X, and the desired-multiple *icchaaphalam* x, related by $x = \frac{p}{P}X$. This is usually applied to similar triangles: the standard-measure and standard-multiple are two (usually known) sides of one of these triangles and the desired-measure and desired-result are the corresponding sides of the other.

If we work out the geometry (more detail below) we are being asked to measure out a distance equal to $\sqrt{2}(2-\sqrt{2})$ units from the corner in each direction; the side of the small square being $\sqrt{2}$ units and that of the original square being $2\sqrt{2}$ units. So the side of the octagon is $2\sqrt{2}-2\times\sqrt{2}(2-\sqrt{2})=2(2-\sqrt{2})$ units. This would be $(\sqrt{2}-1)$ times the diameter. The perimeter of the octagon is then $8(\sqrt{2}-1)\sim 3.3$ times the diameter. Next we will construct the sixteen-gon circumscribing the circle and find its side.

Then, the square root of the sum of the squares of the radius to the midpoint of an octagon-side and half the octagon-side is the hypotenuse from the center to the octagon-corner.(Using the Pythagoras theorem.) Imagine a plumbline to the triangle with this as base. This will fall from the midpoint of the octagon-side to the hypotenuse. There are segments into which the base is divided by this plumbline. The two sides of the triangle are the radius and half the octagon-side. The difference of the square of these segments is equal to the difference of these two sides.For, the square of the plumbline can be got from the sides and the segments in two different ways.(The Pythagoras theorem again.) Therefore, if we divide the difference of the square of the sides by the hypotenuse we get the difference of the segments, since the hypotenuse is the sum of the segments. The difference of squares divided by the sum is the difference.

Then, subtracting from the hypotenuse the difference of segments and halving we get the smaller segment. Think of this segment as the new standard-measure. Half the octagon-side is the standard-multiple. Take from the hypotenuse the radius to get the desired-multiple. There is now a smaller pair of segments. The hypotenuse to these segments is half the octagon-side. Again, form the triangle and measure from the corners to find the two points that form the triangle. Cut out the triangle to get a sixteen-gon. Its side will be twice the above desired-multiple taken away from the half-octagon-side.

The argument that constructed the sixteen-gon can be used to make the thirtytwo-gon; and so on by doubling and doubling ; finding the perimeter as the number of corners becomes infinite we get close to a circle. Think of this as a circle. This circle will have as diameter the side of the original square.

Let us translate this into modern mathematical notation. As in the text above draw a square circumscribing a circle, so that its diameter is the side of the square. Draw the bisectors dividing the square into four smaller squares. It is enough to focus on the upper right hand quadrant. Let O be the center of the circle, E and S the point of contact of the circle with the square. ('East' and 'South'). Let the remaining vertex of the quadrant be called X_0. Draw the hypotenuse of the square OX_0. Let it meet the circle at Z_0. Let the tangent at Z_1 meet the line EX_0 at X_1.

Draw the perpendicular from E to the hypotenuse OX_0; let them meet at Y_0. Then the triangles EY_0X_0 and $X_1Z_0X_0$ are similar. Hence we have

$$X_1X_0 = \frac{EX_0}{Y_0X_0} Z_0X_0 = \sqrt{2}(OX_0 - OZ_0) = \sqrt{2}(\sqrt{2} - 1)OE$$

so that

$$EX_1 = EX_0 - X_1X_0 = [\sqrt{2} - 1]OE.$$

Now EX_1 and X_1Z_1 are two of the half-sides of a regular octagon circumscribing the circle. Its perimeter is therefore $8\times 2EX_1 = 8(\sqrt{2}-1)(2OE)$. This gives the approximation $\pi \sim 8(\sqrt{2}-1)$.

We now repeat the construction by drawing the hypotenuse OX_1 of the octagon which meets the circle at Z_1; and the perpendicular EY_1 to this line. Also, the perpendicular to the hypotenuse at Z_1 meeting the side of the square at X_2. Again, EX_2 and X_2Z_1 are two of the half-sides of a sixteen-gon circumscribing the circle. The perimeter of the sixteen-gon, $16 \times 2EX_2$, gives a better approximation to the circumference of the circle. Let us find EX_2 in terms of EX_1. Clearly it is enough to find X_2X_1 since $EX_2 = EX_1 - X_2X_1$. Considering the similar triangles EY_1X_1 and $X_2Z_1X_1$ gives again

$$X_2X_1 = \frac{EX_1}{Y_1X_1} Z_1X_1.$$

Thinking of the perpendicular EY_1 in terms of the two right triangles to which it is a side gives

$$OE^2 - OY_1^2 = EX_1^2 - Y_1X_1^2$$

or

$$OE^2 - EX_1^2 = OY_1^2 - Y_1X_1^2.$$

Since $OX_1 = OY_1 + Y_1X_1$,

$$\frac{OE^2 - EX_1^2}{OX_1} = OY_1 - Y_1X_1.$$

Also,

$$\begin{aligned} Y_1X_1 &= \frac{1}{2}[OX_1 - (OY_1 - Y_1X_1)] = \frac{1}{2}\left[OX_1 - \frac{OE^2 - EX_1^2}{OX_1}\right] \\ &= \frac{1}{2OX_1}[OX_1^2 - OE^2 + EX_1^2] = \frac{EX_1^2}{OX_1}. \end{aligned}$$

In the last step we use the Pythagoras theorem on the triangle OEX_1. Putting this in,

$$\begin{aligned} EX_2 &= EX_1 - X_2X_1 = EX_1 - EX_1\frac{OX_1 - OZ_1}{Y_1X_1} \\ &= EX_1 - \frac{(OX_1 - OE)OX_1}{EX_1} = \frac{EX_1^2 - OX_1^2 + OE \times OX_1}{EX_1} \end{aligned}$$

$$= \frac{OE \times OX_1 - OE^2}{EX_1}$$

So we have the recursion relation

$$EX_2 = \frac{OE\sqrt{OE^2 + EX_1^2} - OE^2}{EX_1} \tag{30}$$

This argument can be repeated by constructing a new hypotenuse OX_2, a new perpendicular EY_2 to it and so on. After n steps we will get a polygon with 4×2^n sides; a half side of the polygon will be given by the same recursion relation

$$EX_{n+1} = \frac{OE\sqrt{OE^2 + EX_n^2} - OE^2}{EX_n}. \tag{31}$$

The perimeter of this polygon will be $4 \times 2^n \times 2EX_n$ to be compared with the circumference of the circle $2\pi OE$. Thus the approximation to $\frac{\pi}{4}$ is $2^n \frac{EX_n}{OE}$. As the number of sides grow the perimeter will tend to the circumference of the circle from above: it will always be an overestimate as the polygon lies outside the circle.

It is convenient to set $x_n = \frac{EX_n}{OE}$. This leads to the recursion formula

$$x_0 = 1, \quad x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n}, \quad z_n = 2^n x_n.$$

where z_n is the nth approximation to $\frac{\pi}{4}$. Or, put another way

$$z_0 = 1, \quad z_{n+1} = 2\frac{4^n}{z_n}\left\{\sqrt{1+\frac{z_n^2}{4^n}}-1\right\}.$$

This gives the sequence of values (easily calculated with a standard calculator that can extract square roots)

$$z_1 = 0.82842712474619009760337744841\,94,$$

$$z_2 = 0.78793122685731402461758017033062,$$

$$z_3 = 0.78602959631147606568549314034102,$$

$$z_4 = 0.78555590748561421134655212674908,$$

$$z_5 = 0.78543759229224161477680340699333,$$

to be compared with the ‘exact’ value

$$\frac{\pi}{4} = 0.785398163397448309615660845819 88\cdots .$$

Assuming that you can calculate the square roots accurately, the error in this formula can be shown to be:

$$\left| \frac{z_{n+1} - z_n}{z_n} \right| \sim 4^{-n} \frac{\pi^2}{4^3}.$$

Thus this gives an algebraic recursion relation for π that converges exponentially fast. The difficulty of course, lies in extracting the square roots accurately. Calculating each square root itself involves an infinite recursion so the promised exponential convergence is hard to realize in practice.

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The Recursion as a Dynamical System

Given a map $f : C \to C$ of the complex plane to itself and an initial point x_0, it is possible to define a dynamical system See John Milnor by the recursion relation

$$x_{n+1} = f(x_n). \tag{32}$$

Our example is of course the case $f(x) = \frac{\sqrt{1+x^2}-1}{x}$, $x_0 = 1$ which converges to 0. what would have happened if we chose a different initial condition? Would it still converge to zero? Does the limit $\lim_{n\to\infty} 2^n x_n$ exist for any choice of x_0 and if so what is it as a function of x_0?

If we look through the derivation of the recursion earlier we will see that angle between the East-radius and the nearest corner of the regular polygon is being halved at each step of the iteration. Moreover half the side of the polygon is the tangent of this angle multiplied by

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the radius. Hence

$$x_n = \tan\theta_n, \quad \theta_{n+1} = \frac{1}{2}\theta_n. \tag{33}$$

is just as good a way of thinking of the iteration. Moreover, at the beginning of the iteration when we have a square, $\theta_0 = \frac{\pi}{4}$ which is why $x_0 = 1$. Since

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \tag{34}$$

we have

$$x_n = \frac{2x_{n+1}}{1-x_{n+1}^2}. \tag{35}$$

Inverting this, which involves solving a quadratic equation,

$$x_{n+1}^2 x_n + 2x_{n+1} - x_n = 0 \tag{36}$$

we get the recursion relation above:

$$x_{n+1} = \frac{-2 \pm \sqrt{4+4x^2}}{2x_n} \Rightarrow x_{n+1} = \frac{\pm\sqrt{1+x_n^2}-1}{x_n}. \tag{37}$$

We choose the root that gives a positive value for x_n, as it is supposed to be the half-length of a side of the regular polygon.

Thus the change of variables $x_n = \tan\theta_n$ has reduced our recursion relation to something very simple:

$$\theta_{n+1} = \frac{1}{2}\theta_n. \tag{38}$$

There is no doubt that this converges: for any choice of initial θ_0, $\theta_n = 2^{-n}\theta_0$ which tends to zero. Moreover, $2^n x_n = 2^n \tan\theta_n \approx 2^n\theta_n$ since for large enough n, the angle will become small enough. Thus the sequence always converges to zero and the limit $\lim_{n\to\infty} 2^n x_n$ does always exist:

$$x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n} \Rightarrow \lim_{n\to\infty} 2^n x_n = \arctan x_0. \tag{39}$$

Thus this recursion relation is a way to calculate the value of arc-tangent.

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Lecture VI

Sociology of Kerala

•The people of Kerala today belong to the three major religions: Christianity(20%) , Islam(20%) and Hinduism(60%).

•The proportions were different in the time we are speaking of: there was a small and ancient Christian church, founded by the Apostle St. Thomas himself if we are to believe in the legends.

•There was a tiny but vibrant jewish community. There were some converts into Islam along the coastal regions.

•But the vast majority of people followed the traditional religion of India known there simply as the 'Old Ways': the Sanaadhana Dharma. The Persians called the followers of this religion 'Hindus'- derived from their name for the Sindhu (Indus) river- which now is used even in India to describe them.

The Hindu Religion-*Sanatana Dharma*

•Unlike modern religions (such as Buddhism, Christianity or Islam) Hinduism does not have a unique founder. Like Judaism it is a system of practices handed down from time immemorial.

•The basic spiritual texts are the four Veda,perhaps the oldest surviving texts of mankind. The word Veda simply means 'the knowledge'. These are supplemented in later times by the epics (puraana) (mainly the Ramaayana, MahaBhaaratha and the Bhaagavata) ; embedded in the epics are several important texts such as the Bhagavat Giita and the Yoga Vasishta. This classical literature is supplemented by the commentaries of saints the most important of whom is Sankara Achaarya.

•The Veda are a sort of encyclopedia of ancient knowledge. In addition to the hoary philosophy of the Upanishads, the Veda also contain the ancient rules of human behavior and of course, hymns and prayers.

•Although in its core Hinduism is not about Gods, but about a supreme existence of which we are all a part, much of the religious practice has to do with a multitude of Gods: each of which represent an aspect of this supreme reality.

•In the Veda, many of the Gods are identified with natural phenomena: the Sun, the Moon and the planets are minor Gods the major ones being Indra, Vishnu, Siva and so on.

•It is difficult to convey that there is an essence to Hinduism that lies beyond the Gods to those from another cultural and religious background: it often looks like a bewildering array of colorful, even scary images connected together by fantastic legends, much like the pre-Christian religions of Europe. However, abstract notions of the impersonal infinite are still quite familiar to Hindus and the abstractions of mathematics were often derived from this common religious background.

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A Glimpse of Infinity

- As an example, here is a verse from the Isaavasya Upanishad of the Yajur Veda that many of us repeat daily even today:

purnamadah purnamidam purnaat purnamudachyate
purnasya purnaamadaya purnameva vashishyate
That is the Universe, This is the Universe,
The Universe arises from itself, it is said
If you subtract the Universe from the Universe
What remains is the Universe itself.

Namboothiri: The High Priests

•Most of the astronomer-mathematicians of Kerala were *Namboothiri*s, the highest ranking priests of the Hindu religion.

•They had the highest social status in the society, higher even than the King.

•Although less than a half percent of the population, they controlled most of the wealth through land ownership.

•What they did not own out-right they controlled through the temple-trusts which were managed by them.

•The people who toiled in the fields (rice farming is very labor intensive) had no ownership of the land or its produce.

•Yet the enormous wealth that these rice plantations produced were entirely the product of their labor: without constant toil the land would have decayed out of over-cultivation in just a few years.

•At best, they were tenants who paid the landlords regularly for the privilege of cultivating the land.

•At worst they were the pariah [3] or pulaya who were almost slaves: even the huts they lived in were owned by the brahmins.

•Although terrible by modern standards, Kerala at that time was no worse than any other medieval society.

•This feudal system was finally dismantled in the mid-twentieth century under the leadership of E.M. S. Namboothiripad, the first Chief Minister (similar to the Governor of a US state) who re-distributed the land to those worked in it.

[3] Pariah: (a sad word that Malayaalam has contributed to the English language) describes a caste at the very bottom of the social hierarchy.

How did the *namboothiris* Dominate ?

•The namboothiris are believed to have emigrated down the West coast of Indian starting with the fourth century CE. In any case by the ninth century every aspect of Kerala society came under their firm control.

•There are some sub-castes within the namboothiris who were later immigrants from *Thulu* region just to the North of Kerala. Madhava belonged to one of them: he was an *embranthiri*.

•The key to the dominance of the *namboothiri*s was that they were the keepers of the sacred Hindu scriptures, the *Veda*.

•It was their sacred duty to perform the vedic sacrifices to maintain the harmony of the universe.

•The title Somayaji denoted someone who had performed one of the most difficult and ancient of these rituals, the *Soma-yaga*.

•*Neelakanta Somayaji*, the author of the important text $tantra - sangraha$ held this title.

•Being the only allowed priests, they controlled the temples which were also the only centers of learning, art and culture.

•Each village had a small standing army of *Nairs* who protected the temple.

•The temples and could not be taxed; indeed the King was dependent on them even to raise an army.

Simplicity, Dedication, Discipline

- The life of a *Namboothiri* was comfortable but by no means luxurious. There are no castles or opulent palaces in Kerala.
- They led a life of scholarship and spirituality.
- Food was strictly vegetarian, no meat, fish or eggs were allowed.
- Dairy products were allowed:milk,butter, ghee (melted butter) and yogurt. Cheeses were unknown in the hot climate of Kerala.
- The staple was rice, balanced by a huge variety of vegetarian dishes which were lightly spiced. All six taste were represented in each meal to provide balance.
- No alcohol or other intoxicants were allowed. Even strong tastes like onions were forbidden as it could inflame sexual appetite.
- Fasts on the eleventh day of each half of the lunar cycle .
- Servants were not allowed to cook food; it was the domain of the *nambothiri* women.
- Exercise was built into the spiritual practices; e.g., the prayer to

the Sun God is an excellent aerobic workout. You can learn it in yoga classes in the US. They walked everywhere.

•They lived in large joint families (many unmarried adult brothers and sisters, the children of the oldest brother,the grandparents and sundry dependents)in a simple single story home.

•The women rarely travelled outside: the Malayalam word antarjanam for a namboothiri woman means ‘woman inside’ (the house).

•The houses had one or two internal courtyards without a roof and was built either as a square or double square around them. (There is some resemblance to the floor plan of the Roman villas but much more modest here.)

•The roof was tiled, or in the older days a thatched with coconut leaves. This kind of roof had to be replaced annually. The floor was bare, often just mud: no form of cement or marble was used.

•There would be a shed some distance away for the cows and a sizeable stack of hay to feed them.

•Dress was also very simple: a piece of cotton cloth around the waist, a towel around the shoulders. A thread around the body indicating *brahmin* status. Even on the most formal occasions,men were bare chested. Marco Polo was aghast at that!

•Here is a picture of the High priest during the year 2003-2004 of the biggest temple in Kerala (Ayyappa kshetram): this was the typical dress of a *namboothiri* of medieval Kerala.

The Education of a Namboothiri

• The *namboothiri*s held a total monopoly on education, both basic and advanced. The rest of the population was practically illiterate.

• Today the situation is entirely different. Kerala offers the broadest level of basic education in the world: the only state in India that claims a 100% literacy rate. The results of the reform started by EMS, the first Chief Minister.

• The education of a Namboothiri boy was intense, deep and broad.

• He started at the latest by the age of eight and continued at least to age sixteen.

• The teacher was often a family member, an uncle or father or grandfather.

• The workday started before sunrise (always 6:00 am in these tropical parts) with prayers to the Sun God, Suurya. After a short break for lunch it would continue till sundown.

Learning the Veda

•The center piece of the education was the learning of the Veda. No other caste had the right to learn the Veda.

•The Veda could not be written down, the entire corpus had to be memorized. Each family inherited a piece of the Veda assigned to it according to tribal succession laws and passed it on to the next generation.

•The smartest boys learned the hardest and most abstract kind of knowledge: the Upanishads, about the nature of Knowledge itself, that ultimate knowledge from which all else follows.

•Even the dumb ones at least had to memorize the Veda by rote without understanding its meaning. The chanting of the Veda and the performance of Vedic sacrifices are the ultimate duties of the namboothiri.

•Each namboothiri was a living library of ancient knowledge.

Error Correction by Redundancy

•To make sure that no error would creep into the oral transmission of the Veda, there were intricate error correction techniques built into this rote learning.

•You would learn the Veda not only as if read from left to right, but also in the reverse order. Which would make equal sense to someone who doesn't understand the ancient Sanskrit to begin with. Not only that, you would memorize each verse by taking a syllable from the middle then one to the left then one to the right and so on.

•This redundancy as well as the redundancy in the large number of people who learned the Veda compensated for the volatility of human memory. Indeed the Veda are remarkably well preserved: you can compare the Veda as recited by a namboothiri brahmin to a Kashmiri pundit. There would be no difference not only in literal content but also in the pronunciation and rhythm of the singing.

•There are greater disputes over Shakespeare's writing than over the

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text ($samhita$) of the Veda: in spite of the former being printed and much more recent. As the older generation who received this classical education die out, there is now the danger of entire branches of the Veda dying out with them. There is an ongoing project to record the chanting of the Veda before this happens.

•There were annual competitions in the recitation of the veda. Such a competition ($anyonyam$) still continues but at a much smaller scale.
•The various city states and temples competed to attract well known scholars to stay in residence. In return the scholars were expected to compose some salutary verses ($prasasthi$) honoring the local ruler which they completed with some grumbling.

•Secular subjects such as poetics, rhetoric, grammar, logic, astronomy (of which mathematics is a part) and medicine occupied an important place in the education, but were considered distinctly inferior to the study of the Veda.

The *Guru*

•Each person has a guru or teacher responsible for his overall education; although occasionally there might be more than one teacher when someone has expertise in several areas.

•In this respect advanced education today at the level graduate school holds a remarkable similarity to this ancient system. But the guru was often a relative: an uncle or ones father.

•The guru was held in the highest regard, indeed as a form of divinity.

•The guru was responsible for spiritual and moral development as well as education.In return the student was to obey and protect the guru for life.

•The word guru means literally 'heavy' or ponderous. In astronomy, guru is also the name of the planet that we call Jupiter in English: because of its ponderous motion across the sky with a period of twelve years. In mythology this guru represents the teacher of the Gods, a play on the meaning of the word.

•The schools or Madhams survived down to the early twentieth century. At this time the Namboothiris started to suffer from the lack of an English education. There was a popular reform movement which allowed the Namboothiris to adapt to the modern world.

•Now they have melted into the emerging vast Indian middle class as professionals: teachers, doctors, scientists and immigrants to the United States..

Some Salacious Details

•How did they prevent the family wealth from getting diluted by division among the progeny?

•Only the oldest male member of the family was allowed to pass on the wealth to his oldest son.

•The others sons were not allowed to marry. More precisely, they were not allowed to marry a Namboothiri woman.

•They were allowed to enter into a sambandham (literally, 'relationship') with a lower caste woman (often a Nair and often married also to another man who would conveniently disappear when the namboothiri arrived to spend the night.)

•But their children would not be brahmins, being basically illegitimate, and would have no right to the family property.

The Nairs

•The nair were a caste of warriors with an elaborate code of honor. Somewhat like the Samurai of Japan. There status as above that of the laborers, but several rungs below the namboothiri. They received education in martial arts at a *kalari*.

•Human beings are frail, so these rules had loopholes.

•Eventually many Nair families usurped some power and wealth through gifts from the namboothiris; but never enough to truly threaten the established order.

•But down to modern times, Nair women hold the rights to all property and power in Nair society is definitely in the hands of the women-an anomaly anywhere, and especially so in India.

•The Nair family property passes from the mother to daughter not father to son as in most of the rest of the world.

•The Nair men managed their sister's property. They did not live with their wives, only visited them. The hereditary titles were passed from a man to his nephew (his sister's son, often by a namboothiri).

The Younger *namboothiri* Brothers

•This system led to many unmarried Namboothiri women wasting away in each household.

•Just as there were many younger Namboothiri brothers who had a life of leisure : not having to worry about either managing the family estate (the oldest brother's job) or caring for their progeny (the Nair stepfather was supposed to do that when he was not away fighting.)

•They were all highly educated and some of them turned to literature and the sciences- such as astronomy -as an outlet for their creative energies.

•Not all namboothiris were obsessed with spirituality. There is a robust tradition of erotic poetry *venmani kavita* composed by the luxuriating younger brothers of the Namboothiri household.

Sutra, Bhashya,Sidhanta, Gitika

•Sanskrit technical literature is vast. The astronomy texts alone runs into the thousands, and they are but a tiny fraction of the overall library.

•In all technical subjects there are different levels of texts.

•The briefest are *sutra* (literally 'thread'). They contain short pithy statements in prose that is to be memorised. They contain precise definitions, statements of results. No explanations, motivations or proofs.

•The *Patanjali yoga-sutra* is the foundation of *ashtanga yoga*; the *vyakarana sutra* of *Panini* are the foundatio of axiomatic linguistics the world over; the *Brahmasutra* of *Vyasa* are about the ultimate spiritual experiences of a saint.

•The *kamasutra* read almost like a parody of this technical literature.

•A typical astronomy text would be less pithy than a *sutra*. It would contain definitions, results, mathematical formulas, instructions on how to do calculations, tables of functions such as sines, all expressed as

Sanskrit poems.

•A mathematical work had to also be a good piece of poetry. Neelakanta especially was a total scholar knowledgeable in all six branches of Indian philosophy and could also write exquisite mathematical poetry.

•The *aryabhateeyam* is composed in the noble *arya* meter.

•Did the author use a pseudo-nym *arya-bhatta* because of the meter he chose? Or did he chose that meter because of his given name?

•A *bhashya* is a commentary on an established text such as a *sutra*. The commentary of Sankara on the Brahmasutra, and the ten Upanishads are the foundation of the dominant philosophic tradition of India, the Advaita Vedanta.

•A *bhashya* is usually an explanation and elaboration containing motivations and proofs.

•But it can also contain much original work: the *bhashya* of *Neelakanta* on the *Aryabhateeyam* reports deep explorations into calculus.

•*Sidhanta* is a theoretical work. The *Surya-siddhanta* is attributed to the Sun-God himself,and describes the motion of the sun.

•*tantra* means technique; in the astronomical context, techniques of calculation of the celestial bodies.

•*tantra* also has a separate spiritual connotation as a school of mysticism. For some reasons many western scholars of Indian culture talk as if tantra is all about sex. Sex sells even in academia.

•Could be a useful confusion. Maybe, we can sell many copies of Neelakanta's Tantra-Sangraha: the "summary of (astronomical) technique".

•A *gitika* (song), *darpana* (mirror), *deepika* (light) are all elementary texts meant for beginners. Hence *siddhanta-deepika* would be called "Introduction to Theoretical Astronomy" today.

Notation for numbers: katapayadi

•The decimal notation was well-established. It originates in India and was transmitted to the Europe via the Arabs (which is why they are called the 'Arabic numerals').

•Since mathematical works were poems, there had to be a way of converting numbers into words.

•The Sanskrit alphabet (same as Malayalam except for the fonts used) provides a natural way alpha-numerical system.

•There are twentyfive consonants arranged in a five by five table: ka kha ga gha nga
cha chha ja jha nha
ta ttha Da Dha n.a
tha thha da dha na
pa pha ba bha ma

There are also the miscellaneous sounds ya ra la va s.a sha sa ha

These can be combined with any of sixteen vowels.

•ka stands for 1, kha for two and so on till nha which stands for zero. Then we start again with ta which stands for one and so on till na. Then again pa stands for one till ma which is five. ya is one again and so on upto ha which is eight. A pure vowel (which can only appear at the beginning or end of a word) stands for zero as well. Using these rules any number can be converted into a word.

•Each number is written out to base ten, then taken in reverse order and converted into a word by the above rules. With practice you can automatically turn any word into a number in your head: the way some of you can read formulas in TeX even without a previewer.

•Since a trailing vowel of a word in sanskrit is often lost when it is combined with other words, it would be convenient if it were irrelevant; the beginning zero of a number in the decimal system is irrelevant. So if we invert the number before converting it, these two rules would fit.

•There are many words for each number. It is up to the author's ingenuity to find a word that captures the meaning of the number; or fits rhythmically into a poem.

•A renowned Malayalam poet (*Vishnu Narayanan Namboothiri* visited the US and went to see Einstein's house in Princeton which was then a museum. He noted that the address of the house could be encoded as *thrimudi*: the ‘highest peak’.

Examples of the Kali-date-number $Kali-dina-samkhya$

•These calculations are reported by K. V. Sarma in the introduction to his edition of the $tantra-sangraha$.

•*Neelakanta* has inserted into his text the $Tantra-Sangraha$ chronograms giving the starting date and ending date of his work. The first verse is ostensibly a prayer to Vishnu. The very first phrase *Hey vishno nihitham kr.thsnam* is the kali-day-number in the katapayadi system, 1680548. The last verse contains the phrase *lakshmi-s.ani-hitha-dhyana* which is the date 16805553.

•It appears that he dictated the entire poem in five days. The year translates to 1500 in the Christian Era.

•The Sidhanta-darpana of Neelakanta (a more elementary treatise) gives the year of his birth and his own commentary on it gives the date. The year is given as *sivasive* which is 4545 in the kali year: 1444 CE.The phrase *thyajaabhyajnhathaam tharkai* is 1660181 which works out to

June 14, 1444.

Lecture VII

Circumference without Square Root

We now return to a detailed study of the Yuktibhasha.

The next main topic is a rational formula for the circumference that avoids the calculation of square roots. The price for this will be much slower convergence. Still it leads to an entirely different approach to the calculation of the circumference, based on infinite series and integrals. The series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{40}$$

is derived by an ingenious limiting procedure.

The first steps are similar to the previous construction. We continue with the translation in san serif font and commentary in roman font followed by an exposition in modern mathematical language.

Now we will see how to find the circumference of a circle without using square-roots.Imagine a square whose four sides are equal to the diameter of the circle; and inside it imagine a circle. The arc of the circle should touch the sides of the square. Then, through the center of the circle imagine the East-West and North-South diameters (literally *suutra*, thread) connecting the arc of the circle which is also the midpoint of the sides of the square. Then from the East radius tip to the top-right hand corner will be equal to the half-diameter. There, mark some points, close to each other, the division calculated so that they are more or less equally separated. The more such points there are the more accurate will be the perimeter. Then from the circle-center,ending at these points, construct hypotenuses. The East radius will be the common height (of triangles). The line from the East radius to the tip of the hypotenuses will be the arms (of these triangles). There, the closest hypotenuse to the South of the East radius will have arm equal to one step ; the second hypotenuse will have arm which is two steps long. This way each successive hypotenuse will have arm one step more. Thus the corner of the square will have the longest arm of all. The height of all these hypotenuses is just the East-radius which is the half-diameter. Therefore the square root of the sum of the squares of the half-diameter and the respective arm is equal to the respective hypotenuse.

We are to imagine here a series of right angled triangles, with the East radius as a common side and the other side increasing length by one interval as we move to the square-corner.

Then the distance from the East-radius-tip to the tip of the nearest hypotenuse multiplied by the half-diameter and divided by the first hypotenuse is the distance from the East-radius-tip to the first hypotenuse, orthogonal to it. There is a triangle with this line as the height . The arm of this triangle is from the intersection of this edge with the first hypotenuse to the tip of the first hypotenuse. The hypotenuse of this (new) triangle will the distance from the East-radius-tip to the original hypotenuse-tip on the side of the square. This is the desired desired (i.e., to be determined) part of a pair similar triangles. The reference triangle this is given next: the East radius is the height , the original hypotenuse-line is the hypotenuse, the distance between the hypotenuse-tips is the arm. This reference triangle is similar in shape to the desired triangle. The reason: the arm of the reference triangle is parallel to the hypotenuse of the desired triangle,the arm of the desired triangle in parallel to the reference hypotenuse. Then, the East radius which is the height of the reference triangle is orthogonal to the hypotenuse of the desired triangle which is in turn the common interval on the square-side. The long arm of the desired triangle is orthogonal to the hypotenuse of the reference triangle as well. This is the reason why the two triangles are similar shaped. Thus here for the two triangles the arm and the height are mutually parallel, the hypotenuse and the long arm are orthogonal, which makes them similar in shape. If the three sides are either parallel or perpendicular they will be similar shaped.

(We skip a short passage using analogies with carpentry.)

It is easy to see that two triangles are similar when the corresponding sides are parallel. It is argued here that the same is true if the corresponding sides are perpendicular as well. The author then gives an analogy with the beams of a roof.

Then there is a third triangle here: For this, the hypotenuse becomes the East-radius. The perpendicular from the East-radius to the first hypotenuse is its arm, which is also the long-side of the previous desired-triangle.The segment from the meeting point of this arm and the first hypotenuse to the circle-center is the long-side. So that is it.

The arm is the line connecting the East radius to the first diagonal. The segment of the original hypotenuse from the center of the circle to the meeting point of this arm and the original hypotenuse is the height .

Then there is a second reference triangle. Its long-side is the East-radius itself. From the tip of the radius two intervals on the square-side form the arm. The second hypotenuse from the circle-center is the hypotenuse (of this second triangle). This is the second reference triangle. Then its desired similar triangle: the height is the perpendicular from the first hypotenuse-tip to the second hypotenuse. The arm is from this intersection to the tip of the second hypotenuse. The second interval on the square-side is the hypotenuse. This is the desired triangle of the second similarity. (Omit some analogy to roofs.) Now multiply the second interval on the square-side by the radius which is the reference side divide by the second hypotenuse. The result is the height of the second desired triangle. Here too there is a third similarity: we can imagine this height to be the arm and line from its meeting point to the circle-center on the second hypotenuse to be the height ; the first hypotenuse as the hypotenuse.

This way, starting from the radius-tip to the the corner of the square, each piece of the square-side determines three similarities of triangles. There, starting from the radius-tip to the square-corner, multiply each interval by the radius and divide by the larger of the hypotenuses touching the boundary of the interval;the result is the perpendicular connector from the tip of previous hypotenuse to the current one.These are the height s of the desired triangle (of the similarity). These become then the arms (of the next similarity). The segment of the long hypotenuse starting from the intersection with this arm to the circle-center is the height . Then,the smaller of the two hypotenuses starting from the center and touching each square-side-segment is the hypotenuse (of the next similarity triangle).Thus exists certain similarities of triangles. For these there are certain reference triangles. The desired triangles of these reference triangles which are themselves imagined in the interior of the circle. Here the radius, which is almost the reference hypotenuse, is the desired one. The perpendicular from the tip of this radius to the large hypotenuse is the desired result. This way the perpendicular to each hypotenuse, the above result, becomes the Sine of the corresponding arc. Then, the intervals of the square-side starting from the East-radius multiplied twice by the corresponding half-diameter divided by the product of the two hypotenuses gives the Sine of the corresponding angle. Here as the square-side-intervals become small these Sines will become equal to the arc of the circle .

There,starting from the East-tip, divide by the square of the hypotenuses touching the North end of the intervals. First, dividing the radius by its square since the factor is itself, gives just the interval. The last hypotenuse is the diagonal. When divided by its square the result will be half the interval.For, twice the square of the radius is the last square of the last hypotenuse. Also, when a fraction, whose denominator twice the numerator, is multiplied by a quantity we just get half of it. Of all the intervals, only these two touch the first and last ends. The sum over the reciprocals of the square of the first hypotenuses and the same of the second hypotenuses differ by the difference between the first term of the first sum and the last term of the second sum. These will be the half the interval. The middle terms are equal as they have the same denominator. The second term up to the penultimate term is the same (for the two series). The division by the first denominator is the interval itself and the division by the last denominator is half the interval. If we were to divide always by the square of the hypotenuses, the difference would be a quarter of the interval. When the intervals becomes small, its quarter can be dropped. Therefore we just have to use the square of the hypotenuse as the denominator.

Let us translate this into modern mathematical language. Draw a square of side equal to the radius of the circle having the center of the circle as one corner O, touching the circle at the points E (East) and

S. We imagine E to the top of the page and S to the bottom right, following the Indian convention. The top right hand corner F is the 'fire corner' (a reference to the construction of Vedic sacrificial altars.) Thus a quarter of the circle is contained in this square $OEFS$. Draw the diagonal OE.

Now choose points $P_1, P_2, \cdots P_N$ on the side EF, such that the intervals $P_{n-1}P_n$ are more or less equal and small. The larger number the of points and the closer they are, the more accurate will be our estimate of the circumference of the circle. Draw the lines $OP_1, OP_2, \cdots$ which form the hypotenuses of right triangles with the radius as a common (the longer) side OE. They will interest the circle at the points $Q_1, Q_2, \cdots$. Corresponding to the nth such interval on EF, we drop perpendiculars $P_{n-1}T_n$ and $Q_{n-1}R_n$ to the line OP_n.

The circumference will be approximated by the sum of the segments $ER_1 + Q_1R_2 + Q_2R_3 + \cdots$. To determine the distance $Q_{n-1}R_n$, we notice that the triangles $OQ_{n-1}R_n$ and $OP_{n-1}T_n$ are similar; for they are right triangles with one common angle. Similarly the big triangle OP_nE is similar to the little one $P_{n-1}P_nT_n$. Thus we get the similarities

$$Q_{n-1}R_n = OQ_{n-1}\frac{P_{n-1}T_n}{OP_{n-1}}, \quad P_{n-1}T_n = OE\frac{P_{n-1}P_n}{OP_n}. \tag{41}$$

Since $OE = OQ_{n-1} = r$, the radius of the circle,

$$Q_{n-1}R_n = P_{n-1}P_n\frac{r^2}{OP_nOP_{n-1}} \tag{42}$$

There if the square-side is equally divided, the first factors are equal and the other factor, square of the radius is anyway the same. The divisors, being the product of the two hypotenuses of each interval are different. In this situation, this product of hypotenuses can be thought of as (can be approximated by) half the sum of their squares, since the numbers are almost equal. Now, if we divide by each hypotenuse squared, add the results and halve it. This will be equal to the division by half the sum of the squares.

Now suppose that all the intervals $P_{n-1}P_n$ are equal to h. Then

$$Q_{n-1}R_n = hr^2 \frac{1}{OP_n OP_{n-1}} \tag{43}$$

When these intervals are small, we can approximate

$$\frac{1}{OP_{n-1}OP_n} \approx \frac{1}{2}\left[\frac{1}{OP_{n-1}^2} + \frac{1}{OP_n^2}\right] \tag{44}$$

Thus the sum

$$\sum_{n=1}^{N} Q_{n-1}R_n \approx \sum_{n=1} hr^2 \frac{1}{2}\left[\frac{1}{OP_{n-1}^2} + \frac{1}{OP_n^2}\right] \tag{45}$$

is our approximation for the arc of the circle EQ_N, which is an eighth of the whole circumference.

Now let us rewrite the sum so that the square of the first hypotenuse is the denominator. Let us collect the term involving the first hypotenuse of each interval

$$hr^2 \frac{1}{2}\left[\frac{1}{OE^2} + \frac{1}{OP_1^2} + \cdots \frac{1}{OP_{N-1}^2}\right] \tag{46}$$

and the second hypotenuse:

$$hr^2\frac{1}{2}\left[\frac{1}{OP_1^2}+\cdots\frac{1}{OP_{N-1}^2}+\frac{1}{OF^2}\right] \tag{47}$$

since $P_0 = E$ and the last point is the corner $P_N = F$. Except for the first and last term, each such term appears twice in the sum. So we get for the eighth of the circumference

$$\frac{C}{8} \approx \frac{1}{2}h\frac{r^2}{OE^2} + hr^2\frac{1}{2}\left[\frac{1}{OP_1^2}+\cdots\frac{1}{OP_{N-1}^2}\right]+\frac{1}{2}h\frac{r^2}{OF^2} \tag{48}$$

But $OE = r$ and the first term is just $\frac{1}{2}h$. The square of the diagonal is twice that of the radius so the last term is $\frac{1}{4}h$.Together these make $\frac{3}{4}h$. Thus we can just sum over the hypotenuses, including the last one, if we subtract a quarter of the interval. In the limit of small interval, we can ignore this quarter of the interval. Thus

$$\frac{C}{8} \approx \sum_{n=1}^{N} h\frac{r^2}{OP_n^2}. \tag{49}$$

By Pythagoras theorem,

$$OP_n^2 = r^2 + n^2h^2. \tag{50}$$

so that

$$\frac{C}{8} \approx \sum_{n=1}^{N} h \frac{r^2}{r^2 + n^2h^2}. \tag{51}$$

If there are N divisions, $h = \frac{r}{N}$. Also nowadays we denote $\pi = \frac{C}{2r}$. Thus our result is

$$\pi \approx 4\frac{1}{N}\sum_{n=1}^{N}\left[\frac{1}{1 + \left(\frac{n}{N}\right)^2}\right]. \tag{52}$$

As the number of subdivisions grows this sum will tend to the exact answer:

$$\pi = 4\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left[\frac{1}{1 + \left(\frac{n}{N}\right)^2}\right]. \tag{53}$$

We thus get a sequence of rational numbers that tend to π as $N \to \infty$: already an interesting result. The sum tends to the integral

$$\pi = 4 \int_0^1 \frac{dx}{1 + x^2}, \tag{54}$$

of course.

Lecture VIII

The Recursion as a Dynamical System

Here we include some thoughts on modern mathematics inspired by the study of the Yuktibhasha. Of course, no implication is being made that these ideas were known to the ancient Kerala mathematicians.

Given a map $f : C \to C$ of the complex plane to itself and an initial point x_0, it is possible to define a dynamical system See John Milnor by the recursion relation

$$x_{n+1} = f(x_n). \tag{55}$$

Our example is of course the case $f(x) = \frac{\sqrt{1+x^2}-1}{x}, x_0 = 1$ which converges to 0. what would have happened if we chose a different initial condition? Would it still converge to zero? Does the limit $\lim_{n\to\infty} 2^n x_n$ exist for any choice of x_0 and if so what is it as a function of x_0?

If we look through the derivation of the recursion earlier we will see that angle between the East-radius and the nearest corner of the regular polygon is being halved at each step of the iteration. Moreover half the side of the polygon is the tangent of this angle multiplied by the radius. Hence

$$x_n = \tan\theta_n, \quad \theta_{n+1} = \frac{1}{2}\theta_n. \tag{56}$$

is just as good a way of thinking of the iteration. Moreover, at the beginning of the iteration when we have a square, $\theta_0 = \frac{\pi}{4}$ which is why $x_0 = 1$. Since

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} \tag{57}$$

we have

$$x_n = \frac{2x_{n+1}}{1 - x_{n+1}^2}. \tag{58}$$

Inverting this, which involves solving a quadratic equation,

$$x_{n+1}^2 x_n + 2x_{n+1} - x_n = 0 \tag{59}$$

Back
Close

we get the recursion relation above:

$$x_{n+1} = \frac{-2 \pm \sqrt{4+4x^2}}{2x_n} \Rightarrow x_{n+1} = \frac{\pm\sqrt{1+x_n^2}-1}{x_n}. \tag{60}$$

We choose the root that gives a positive value for x_n, as it is supposed to be the half-length of a side of the regular polygon.

Thus the change of variables $x_n = \tan\theta_n$ has reduced our recursion relation to something very simple:

$$\theta_{n+1} = \frac{1}{2}\theta_n. \tag{61}$$

There is no doubt that this converges: for any choice of initial θ_0, $\theta_n = 2^{-n}\theta_0$ which tends to zero. Moreover, $2^n x_n = 2^n \tan\theta_n \approx 2^n\theta_n$ since for large enough n, the angle will become small enough. Thus the sequence always converges to zero and the limit $\lim_{n\to\infty} 2^n x_n$ does always exist:

$$x_{n+1} = \frac{\sqrt{1+x_n^2}-1}{x_n} \Rightarrow \lim_{n\to\infty} 2^n x_n = \arctan x_0. \tag{62}$$

Thus this recursion relation is a way to calculate the value of arc-tangent.

The Circumference of an Ellipse

Considerably more complicated is the problem of determining the circumference of an ellipse. This problem was not solved until mid-nineteenth century, at the zenith of European mathematics and captured the attention of the best minds: Euler, Legendre, Gauss, Abel and Riemann made important contributions to the resulting theory of *elliptic functions*.

An *ellipse* is the curve determined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{63}$$

If $a = b$, this is just a circle of radius a. So we can assume that $a > b$ for a true ellipse. It is useful to define $k^2 = 1 - \frac{b^2}{a^2}$. It measures the departure of the ellipse from being a circle: if $k = 0$, we have $a = b$ which is a circle. The opposite extreme is $k = 1$ is the case where the ellipse is so eccentric it has reduced to a straight-line.

Let us see how far the geometric ideas as of the *yuktibhasha* can be stretched to understand the ellipse. Of course there is no reason to believe that the medieval mathematicians went in this direction. Still, it is interesting to see if their ideas tell us something about the circumference of the ellipse.

This time we draw a rectangle circumscribing the ellipse, so that its sides are $2a$ and $2b$. Draw the again the bisectors dividing the rectangle into four smaller ones, each of sides a and b. Again think of the upper right hand quadrant. Let O be the center of the ellipse, E and S its point of contact with the rectangle. Let the remaining vertex of the quadrant be called F. The perimeter of the rectangle is our first approximation for the circumference of the ellipse:

$$C_0 = 4(a+b) = 4a[1+\sqrt{1-k^2}]. \tag{64}$$

It will be convenient to call $F = X_0, S = Z_0$ as we will try to get a sequence of points by a recursive procedure. Note that the tangent to the ellipse at Z_0 meets the tangent through E at the point X_0. Draw the line OX_0. Call its meeting point with the ellipse Z_1. Now draw the tangent through Z_1, let it meet the tangent through E at X_1.

In the nth step of the iteration we draw the tangent at Z_n, and determine the point X_n where it meets the tangent through E. Then we draw the line OX_n and define Z_{n+1} to be were it meets the ellipse. Thus we get a sequence of points on the tangent through E, $X_0, X_1, \cdots$ that start at F and tend to E.

We will have to find a formula that gives the co-ordinates of Z_{n+1} in terms of those of Z_n. It will be convenient to use the polar and cartesian co-ordinates although the argument can be phrased in the language of pure geometry easily. Let θ_n be the polar co-ordinate of Z_n measured from the axis OS in a counterclockwise direction. Then the cartesian co-ordinates of Z_n (centered at O and having OS as the ordinate)are

$$Z_n = (a \cos \theta_, b \sin \theta_n) \tag{65}$$

The points on the tangent at Z_n are parametrized by a real number s:

$$(a\cos\theta_n, b\sin\theta_n) + s\,(-a\sin\theta_n, b\cos\theta_n)\,. \tag{66}$$

The point X_n where it meets the tangent at E is determined by

$$b = b\sin\theta_n + sb\cos\theta_n, \Rightarrow s = \frac{1-\sin\theta_n}{\cos\theta_n}. \tag{67}$$

Then EX_n, which is the ordinate of X_n is

$$a\cos\theta_n - sa\sin\theta_n = a\frac{1-\sin\theta_n}{\cos\theta_n} \tag{68}$$

after a short calculation. Thus the polar co-ordinate of X_n, which is also that of Z_{n+1} is

$$\tan\theta_{n+1} = \frac{b}{a}\frac{\cos\theta_n}{1-\sin\theta_n}. \tag{69}$$

This is the recursion relation determining Z_{n+1} from Z_n. Equivalently, this is the algebraic recursion relation

$$\cot\theta_{n+1} = \frac{a}{b}\frac{\sqrt{1+\cot^2\theta_n}-1}{\cot\theta_n}. \tag{70}$$

Since $EX_n = b\cot\theta_{n+1}$ we can also write this relation as

$$EX_{n+1} = a\frac{\sqrt{b^2 + EX_n^2} - b}{EX_n}. \tag{71}$$

The recursion starts with $Z_0 = S$ which has $\theta_0 = 0$ which gives $\tan\theta_1 = \frac{b}{a}$ and $EX_0 = a$.

Exercise Derive this recursion relation without using co-ordinate geometry.

Notice that this is a natural generalization of the formula for the case of the circle. Unlike in the case of the circle, the perimeter is not just a multiple of this length, as the polygon circumscribing the ellipse is not regular. We need to determine its other sides.

Let us examine the first step of the iteration. Here we drew the tangent at Z_1 meeting EF at X_1. If we extend it in the other direction, this tangent will meet the previous tangent through Z_0 at some point Y_1. If we cut out the triangle $X_1X_0Y_1$ we will get a quarter of an octagon that circumscribes the circle. Its perimeter is

$$C_1 = 4\left(EX_1 + X_1Z_1 + Z_1Y_1 + Y_1Z_0\right) \tag{72}$$

We already know $EX_1 = a\frac{\sqrt{a^2+b^2}-b}{b}$. We can find [4] $X_nZ_n = s\sqrt{a^2\sin^2\theta_n + b^2\cos^2}$ so that

$$\begin{aligned}\frac{X_nZ_n}{EX_n} &= \frac{\sqrt{a^2\sin^2\theta_n + b^2\cos^2\theta_n}}{a}\\ &= \frac{b}{a}\sqrt{\frac{a^2 + EX_{n-1}^2}{b^2 + EX_{n-1}^2}}\end{aligned}$$

after some algebra. This gives

$$X_1Z_1 = \sqrt{2}\left(a - \frac{ab}{\sqrt{a^2+b^2}}\right) \tag{73}$$

The lines Z_1Y_1 and Y_1S are obtained by reflection around the diagonal OF of Z_1X_1 and EX_1. Thus their lengths are given by the same

[4] There is no extra work to do the general case, although for now only need $n = 1$.

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formulae except for the interchange $a \leftrightarrow b$:

$$Y_1S = b\frac{\sqrt{a^2+b^2}-a}{a}, \quad Z_1Y_1 = \sqrt{2}\left(b - \frac{ab}{\sqrt{a^2+b^2}}\right) \tag{74}$$

Thus our octagonal approximation to the perimeter of the ellipse is

$$C_1 = 4\left(a\frac{\sqrt{a^2+b^2}-b}{b} + \sqrt{2}\left(a - \frac{ab}{\sqrt{a^2+b^2}}\right) + a \leftrightarrow b\right). \tag{75}$$

That is

$$C_1 = 4\left(\left[\frac{a}{b}+\frac{b}{a}\right]\sqrt{a^2+b^2} + (\sqrt{2}-1)(a+b) - \frac{2\sqrt{2}ab}{\sqrt{a^2+b^2}}\right) \tag{76}$$

Exercise Derive this result yourself.

The Arithmetico-Geometric Mean

The idea of calculating transcendental quantity by iterating an algebraic formula is very much alive in modern mathematics. We will illustrate this by determining the circumference of an ellipse, a substantially more complicated problem than for the circle. The basic idea goes back to Gauss but we give a simplified description,looking at an elliptic integral of the first kind rather than the second kind that which gives the circumference of the ellipse[5].

The formula $\frac{\pi}{2} = \int_0^\infty \frac{dx}{1+x^2}$ is easily established by the substitution $x = \tan\theta$. A generalization of this which arises in the theory of elliptic

[5] determine the circumference of an ellipse using these ideas,we have to solve an inhomogenous linear recursion relation. The function studied here also determines the period of oscillation of a pendulum

functions is

$$G(a,b) = \int_0^\infty \frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}}. \tag{77}$$

We can always choose $a \geq b > 0$. Clearly $G(a,a) = \frac{\pi}{2a}$. The substitution $y = x + \sqrt{x^2+ab}$ proves (after a long calculation) that

$$G(a,b) = G(a_1,b_1), \quad a_1 = \frac{a+b}{2}, \quad b_1 = \sqrt{ab}. \tag{78}$$

That is, $G(a,b)$ is unchanged if we replace a by the arithmetic mean and b by the geometric mean of a, b. Thus we can calculate the integral by repeated applications of this procedure: the two means will approach each other and tend to some common value $M(a,b)$ which is called the arithmetico-geometric mean. In that limit we know the value of the integral since it reduces to the circular case. Thus

$$G(a,b) = \frac{\pi}{2M(a,b)}. \tag{79}$$

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The iteration

$$a_{n+1} = \frac{1}{2}[a_n + b_n], \quad b_n = \sqrt{a_n b_n}, \quad a_0 = a, \quad b_0 = b \tag{80}$$

determines the arithmetico-geometric mean:

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{81}$$

The convergence of this iteration is quite fast. The worst case is when a and b are far apart. But the extreme case $a = 1$ and $b = 0$ doesn't converge. But if $a = 1$ and $b = 0.01$,

$$\begin{aligned} a_1 &= .505, \quad b_1 = .1, \\ a_2 &= .3025, \quad b_2 = .31622, \\ a_3 &= 0.30936, \quad b_3 = 0.309287; \end{aligned}$$

only after three iterations, they already agree to four decimal places!.

Exercise Find $G(1.0, 0.001)$ to an accuracy of four decimal places.