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A SYSTEM OF AXIOMATIC SET THEORY

PART III. INFINITY AND ENUMERABILITY. ANALYSIS.[28]

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7. Axioms of infinity. The foundation of analysis does not require the full generality of set theory but can be accomplished within a more restricted frame. Just as for number theory we need not introduce a set of all finite ordinals but only a class of all finite ordinals, all sets which occur being finite, so likewise for analysis we need not have a set of all real numbers but only a class of them, and the sets with which we have to deal are either finite or enumerable.

We begin with the definitions of infinity and enumerability and with some consideration of these concepts on the basis of the axioms I–III, IV, V a, V b, which, as we shall see later, are sufficient for general set theory. Let us recall that the axioms I–III and V a suffice for establishing number theory, in particular for the iteration theorem, and for the theorems on finiteness.

Definitions. A class or a set is called *infinite* if it is not finite. A class or a set is *enumerable* if there exists a one-to-one correspondence between it and the class of finite ordinals. The latter class (the class of finite ordinals) will be denoted by N.

From this negative definition of infinity we can derive the following positive criterion: A class A, or a set a, is infinite if and only if for every finite ordinal n there is a subset s of A, or of a, such that $s \backsim n$.

Indeed, that this condition excludes finiteness follows by our fourth theorem on finite classes and sets (Part II, p. 16), and that on the other hand it is satisfied by any infinite class or set is easily proved by complete induction.

By this criterion it is seen immediately that the class N is infinite, since every finite ordinal is a subset of N. The same thing may be proved also, with the aid of our fifth theorem on finite classes and sets, by showing the existence of a one-to-one correspondence between the class N and a proper subclass of it. There are, in fact, many such one-to-one correspondences, one of them being, for instance, the class of those pairs $\langle a, b\rangle$ for which a is a finite ordinal and $b = a'$.

The fact that the class N is infinite induces a connection between infinity and enumerability. For if A is an infinite class and $A \backsim B$, then B is also infinite; and moreover a class having an infinite subclass is itself infinite, by our second theorem on finite classes and sets. Hence every enumerable class is infinite, and every class which has an enumerable subclass is infinite.

The converse of the last also holds: Every infinite class has an enumerable subclass.

Proof. Let A be an infinite class. Then every finite subset of A is a proper subset. By the class theorem, there exists the class of all pairs $\langle a, b\rangle$ for which a is a finite subset of A and $b \eta A$ but not $b \epsilon a$. Applying the axiom of choice to

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[28] Part II appeared in this JOURNAL, vol. 6 (1941), pp. 1–17. Part I appeared in vol. 2 (1937), pp. 65–77.

this class, we infer the existence of a function L which has as its domain the class of finite subsets of A and assigns to each of these subsets an element of A which is not in this subset.

Moreover, by the class theorem, there exists the class F of pairs $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ such that $\langle a, b\rangle \eta L$ and $\langle c, d\rangle \eta L$ and c is the set arising from a by adjoining b to it as an element. Then F is a function which assigns to every element of L an element of L, and hence, by the iteration theorem, there exists the iterator of F on $\langle 0, L(0)\rangle$. Let H be this iterator. Then by complete induction it follows that, if m and n are finite ordinals and $m \epsilon n$, the first member of $H(m)$ is a proper subset of the first member of $H(n)$, and the second member of $H(m)$ is in the first member of $H(n)$, but the second member of $H(n)$ is not in the first member of $H(n)$. Hence the class C of pairs $\langle n, r\rangle$, such that n is a finite ordinal and r is the second member of $H(n)$, is a one-to-one correspondence. Since the values of H are elements of L, the converse domain of C is a subclass of A. Thus A has an enumerable subclass.

A consequence of this theorem (just proved) is that for every infinite class there exists a one-to-one correspondence to a proper subclass. For since the class N, as already stated, has a one-to-one correspondence to a proper subclass, the same thing (by the composition lemma) holds for any enumerable class, and likewise for any class having an enumerable subclass. Conversely, by our fifth theorem on finite classes and sets, a class which has a one-to-one correspondence to a proper subclass of itself is infinite.

Thus we have two conditions, each necessary and sufficient, for a class to be infinite: one that it has an enumerable subclass, the other that there is a one-to-one correspondence between it and a proper subclass of it.

As to sets' being infinite, it follows from our definition of infinity that a set is infinite if and only if it represents an infinite class. Moreover, from the necessary and sufficient conditions for a class to be infinite, just stated, there follow (with the aid of the axiom V a) exactly corresponding necessary and sufficient conditions for a set to be infinite: one that it has an enumerable subset, the other that there is a one-to-one correspondence between it and a proper subset of it.

We can apply the foregoing to our axiom of infinity (axiom VI), which says that there is a one-to-one correspondence of some class represented by a set to a proper subclass. The content of this axiom now proves to be equivalent (on the basis of the axioms I–III, IV, V a, V b) to the existence of an infinite set. This in turn is equivalent to the existence of an enumerable set, and also (by axiom V b) to the existence of a set representing the class N.

Thus we have that our axiom VI is equivalent to the assertion that the class N of finite ordinals is represented by a set.

In this way the equivalence is proved with the aid of the axiom of choice. But the axiom of choice can be avoided here by means of the following more direct reasoning. First, since there exists a one-to-one correspondence between the class N and a proper subclass of it, the assertion of axiom VI can be inferred from the assumption that the class N is represented by a set. On the other hand, by axiom VI there exists a class A which is represented by a set and of which there is a one-to-one correspondence C to a proper subclass B. Let d be an element of A that is not in B. Then the iterator of C on d, as is easily shown by means of complete induction, is a one-to-one correspondence between the class N and a subclass S of A. Since A is represented by a set, it follows by V a that S is represented by a set, and hence by V b that N is represented by a set.

Thus, on the basis of the axioms I–III, V a, and V b, the axiom VI is equivalent to the assumption that the class N is represented by a set.

From this we can now also easily infer the equivalence of axiom VI to the Zermelo axiom of infinity, to Fraenkel's generalization of this axiom, and to the von Neumann axiom of infinity. However, for the proofs of the second and third of these equivalences the axioms I–III, V a, V b are not sufficient: for the second equivalence the theorem of replacement, or the axiom V*, will be required, and for the third equivalence the axiom of choice (or else V d and V*) will be required.

The Zermelo axiom of infinity[29] asserts the existence of a set such that 0 is an element of it and if c is an element of it then (c) is also. The equivalence of this axiom to the assumption that the class N is represented by a set, and hence to axiom VI, results as follows. The iterator on 0 of the function assigning to any set a the set (a) is a one-to-one correspondence, as is easily shown by means of complete induction. The domain of this one-to-one correspondence is N. Let A be its converse domain. Then $0\eta A$, and if $c\eta A$ then $(c)\eta A$. Thus, by the axiom V b, if N is represented by a set then A is represented by a set; i.e., the assertion of the Zermelo axiom of infinity holds. On the other hand, if s is a set of the kind whose existence is asserted by the Zermelo axiom of infinity, then $A \subsetneqq s$, as follows without difficulty by complete induction.[30] Thus, by the axioms V a and V b, the class A and therefore also N is represented by a set.

Of the Zermelo axiom of infinity there is a generalization which was proposed by Fraenkel, and was adopted by R. M. Robinson for his modification of the von Neumann system of set theory.[31] This axiom—let us call it VI*—asserts the following: For every set a and every function F, there exists a set s, such that $a \epsilon s$ and for every element b of s belonging to the domain of F, also $F(b) \epsilon s$.

In order to derive this assertion from the assumption that the class N is represented by a set, we consider the function G which assigns to every element c of the domain of F the value $F(c)$ and to every other set the value a. The iterator H of G on a is a function whose domain is N. By our assumption

[29] E. Zermelo, *Untersuchungen über die Grundlagen der Mengenlehre I*, ***Mathematische Annalen***, vol. 65 (1908), "Axiom VII," pp. 266–267.

[30] For typographical reasons we now use the character $\subsetneqq$ instead of the character $-\!\in$ which was introduced in Part I, p. 66.

[31] A. Fraenkel, ***Zehn Vorlesungen über die Grundlegung der Mengenlehre,*** Leipzig and Berlin 1927, "Axiom VII c," p. 114. R. M. Robinson, *The theory of classes, a modification of von Neumann's system*, this JOURNAL, vol. 2 (1937), axiom 8.3, p. 31. In Robinson's formulation of the axiom, as in ours following, the express assertion contained in Fraenkel's axiom VII c that there exists at least one set is omitted.

and the theorem of replacement, the converse domain of H is represented by a set. And this set, as is easily seen, is a set of the kind whose existence is asserted by VI*.

On the other hand, from VI* the Zermelo axiom of infinity follows as a special case, and hence, as just shown, there follows by V a and V b that the class N is represented by a set.

The von Neumann axiom of infinity[32] (translated into terms of our system) asserts the existence of a non-empty set every element of which is a proper subset of another element. This assertion is an immediate consequence of the assumption that the class N is represented by a set, since every finite ordinal is a proper subset of every higher finite ordinal. On the other hand, if s is a set of the kind whose existence is asserted by the von Neumann axiom of infinity, then by the class theorem and the axiom of choice there exists a function F assigning to every element a of s another element b of s such that $a \subset b$. Let c be an element of s. Then the iterator of F on c is a function whose domain is N and whose converse domain is a subclass of s. This function is a one-to-one correspondence, since its value for a finite ordinal m is a proper subset of its value for a finite ordinal higher than m. Therefore, by the axioms V a and V b, N is represented by a set.

Remark.[33] The application of the axiom of choice in the last proof can be avoided if axiom V d and also V* are available. For from the assumption that every element of s is a proper subset of another element, we first infer by complete induction that for every finite ordinal m there exists a functional set f with the domain m such that $f(0) \in s$ and, if $k' \in m$, then $f(k) \in s$, and $f(k') \in s$, and $f(k) \subset f(k')$. Then f is immediately seen to be a one-to-one correspondence and m therefore to be the number attributable to the converse domain of f. Thus for every finite ordinal m there exists a subset of s (namely the converse domain of f) the number attributable to which is m. Now let G be the function whose domain is the class of subsets of s and which assigns to every finite subset of s the number attributable to it, and to every infinite subset of s the null set. By V d the domain of G is represented by a set, and hence, by V*, the converse domain of G is also represented by a set. But this converse domain is N, and thus we have that N is represented by a set.

In addition let us point out the following equivalence. The assumption that the class N is represented by a set—which, on the basis of the axioms I–III, V a, and V b, can replace our axiom of infinity—is, by axiom V b, obviously equivalent to the assumption that *every enumerable class is represented by a set.*

The latter assumption, if taken as an axiom, may be called the *axiom of the enumerable.* It can be derived more or less directly from VI* and V a, without using V b. For let A be an enumerable class, and C a one-to-one correspondence between N and A. Then by the class theorem there exists a function F assigning to every element a of A the set b for which there exists a finite ordinal n such that $\langle n, a\rangle \eta C$ and $\langle n', b\rangle \eta C$. By VI*, there exists a set s such that $C(0) \in s$ and

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[32] J. von Neumann, *Eine Axiomatisierung der Mengenlehre*, ***Journal für die reine und angewandte Mathematik,*** vol. 154 (1925), axiom V 1, p. 226.

[33] This added remark received August 11, 1941. *Editor.*

for every finite ordinal n for which $C(n) \epsilon s$, also $C(n') \epsilon s$. Hence by complete induction, $A \subsetneqq s$, and, by V a, the class A is represented by a set.[30]

Our task now, in regard to the foundation of analysis, is to show that the axiom of the enumerable if added to the axioms I–III and VII—which, as we know, suffice for number theory and the theory of finite classes and sets (axiom VII being replaceable here by V a if desired)—is sufficient for the whole of analysis with the exception merely of a particular kind of inference for which the axiom of choice seems to be required (although only as applied in a certain special way).

As the foundations of analysis are so well known, we may content ourselves for this purpose with the indication of some principal points.

8. Theorems on enumerability, fraction triplets, real numbers. As a preliminary, we introduce the following notation. The class of pairs $\langle a, b\rangle$ such that $a\eta A$ and $b\eta B$ will be denoted by $A \times B$ and will be called the *pair class* of A and B—or, if A and B are the same class, simply the *pair class* of A.

We begin with some *theorems on enumerability.*

1. If A and B are enumerable classes, their pair class $A \times B$ is also enumerable.

In order to prove this, we show first that the class $\mathrm{N} \times \mathrm{N}$, i.e., the class of pairs of finite ordinals, is enumerable. There are, in fact, various one-to-one correspondences between N and $\mathrm{N} \times \mathrm{N}$; one of these is the class of triplets $\langle c, \langle a, b\rangle\rangle$ of finite ordinals c, a, b satisfying the condition that $c' = 2^a \cdot (2b)'$. Thus $\mathrm{N} \backsim \mathrm{N} \times \mathrm{N}$. Now if A and B are enumerable classes, and C and D are one-to-one correspondences between N and A and between N and B respectively, then the class of quadruplets $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ such that $\langle a, c\rangle \eta C$ and $\langle b, d\rangle \eta D$ is a one-to-one correspondence between $\mathrm{N} \times \mathrm{N}$ and $A \times B$. So we have $\mathrm{N} \times \mathrm{N} \backsim A \times B$, and this together with $\mathrm{N} \backsim \mathrm{N} \times \mathrm{N}$ gives $\mathrm{N} \backsim A \times B$.

2. Every subclass of an enumerable class is either finite or enumerable.

In order to prove this it is sufficient, in view of the composition lemma, to show that every class of finite ordinals is either finite or enumerable. Let A be any class of finite ordinals. We have the two possibilities, that there is or is not an ordinal which is higher than every ordinal belonging to A. In the first case A is a subclass of a finite ordinal and hence, by our second theorem on finite classes and sets, is finite. In the second case there exists a function F assigning to each ordinal n belonging to A the lowest of those ordinals which are higher than n and belong to A. If l is the lowest ordinal belonging to A, then the iterator of F on l, as easily seen, is a one-to-one correspondence between N and A, and thus A proves to be enumerable.

3. If the domain of a function is enumerable, the converse domain is either finite or enumerable, and the function itself is enumerable.

For let F be a function whose domain is enumerable, and let B be the converse domain of F. Then, by the composition lemma, there exists a function G whose domain is N and whose converse domain is B. Consider the class of those elements $\langle n, b\rangle$ of G for which there is no other element $\langle m, b\rangle$ of G (with the same b) such that $m \epsilon n$; this class is a one-to-one correspondence whose domain is a subclass A of N and whose converse domain is B, in consequence of the fact that for every element b of B there is a lowest of those finite ordinals

n for which $\langle n, b\rangle \eta G$. By the preceding theorem 2 on enumerability, the class A is either finite or enumerable, and consequently the same holds for B.

Moreover the class of those pairs $\langle\langle a, b\rangle, a\rangle$ such that $\langle a, b\rangle \eta F$ is a one-to-one correspondence between F and its domain. Since the domain is enumerable, F is also.

Combining this theorem 3 with the first theorem on finite classes and sets, and the axiom of the enumerable, we get the following *consequence*: If the domain of a function is enumerable, then not only the domain but also the converse domain and the function itself are represented by sets.

Thus for the case of a function whose domain is enumerable we have here the same results that were obtained for functions in general on the basis of the axioms IV, V a, V b.[34] Hence we may also in this case, following our definition in §4, define the *domain* and the *converse domain* of a functional set representing a given function to be the sets representing respectively the domain and the converse domain of that function.

Now we are to translate into our system Dedekind's method of introducing the real numbers. For this purpose we do not need the concept of a rational number, but only something corresponding to the arithmetic expressions $\pm m/n$ representing rational numbers. Or instead of these expressions we may consider expressions of the form $(k-l)/n$, in which k, l, and n are integers, $k \geqq 0$, $l \geqq 0$, and $n > 0$; by this method we avoid certain distinctions of cases in the definitions of the arithmetic operations.

Thus we are led to consider, within the frame of our system, those triplets $\langle\langle a, b\rangle, c\rangle$ in which a, b, c are finite ordinals and $c \neq 0$. We shall call them, in reference to the use we make of them, *fraction triplets*.

From our first theorem on enumerability follows that the class of all fraction triplets is enumerable. Consequently, by the second theorem on enumerability, every class of fraction triplets is either finite or enumerable, and therefore, by our first theorem on finite classes and sets and by the axiom of the enumerable, every class of fraction triplets is represented by a set.

In order to give the fraction triplets their arithmetic rôle, we have to set up definitions of *sum*, *difference*, and *product* of fraction triplets. These are formulated by the following defining equations, referring to the fraction triplets $\langle\langle a, b\rangle, c\rangle$ and $\langle\langle k, l\rangle, m\rangle$:

$$\langle\langle a, b\rangle, c\rangle + \langle\langle k, l\rangle, m\rangle = \langle\langle (m\cdot a)+(c\cdot k),\ (m\cdot b)+(c\cdot l)\rangle, c\cdot m\rangle,$$

$$\langle\langle a, b\rangle, c\rangle - \langle\langle k, l\rangle, m\rangle = \langle\langle (m\cdot a)+(c\cdot l),\ (m\cdot b)+(c\cdot k)\rangle, c\cdot m\rangle,$$

$$\langle\langle a, b\rangle, c\rangle \cdot \langle\langle k, l\rangle, m\rangle = \langle\langle (a\cdot k)+(b\cdot l),\ (a\cdot l)+(b\cdot k)\rangle, c\cdot m\rangle.$$

Remark: We are able to take here the same symbols for the operations with fraction triplets as we use for those with finite ordinals, because a fraction triplet, as is easily seen, is never an ordinal, and thus there is no danger of confusion. A corresponding remark will apply later in connection with the elementary operations with real numbers.

[34] Cf. Part II, pp. 2–3.

We make also the following definitions:

A fraction triplet $\langle\langle a, b\rangle, c\rangle$ will be called *positive*, *negative*, or a *null triplet* according as the ordinal a is higher than b, lower than b, or identical with b.

A fraction triplet p will be called, *greater than*, *less than*, or *equally great as* a fraction triplet q according as the difference $p-q$ is positive, negative, or a null triplet.

The fraction triplet $\langle\langle b, a\rangle, c\rangle$ will be called the *opposite* triplet of $\langle\langle a, b\rangle, c\rangle$.—Obviously the opposite triplet of a positive triplet is negative, and the opposite triplet of a negative triplet is positive. The sum of a triplet and its opposite triplet is a null triplet.

From these definitions the theory of fraction triplets is easily obtained, including in particular the elementary laws affecting sums, differences, and products of fraction triplets, and the relations *equally great* and *less than*. Also the three following existence theorems are readily proved:

If r is a fraction triplet not a null triplet, then for every fraction triplet s there exists a fraction triplet q such that s is equally great as $q \cdot r$. (This formulates the possibility of division.)

If in particular r and s are positive fraction triplets and s is greater than r, then there is a positive fraction triplet t such that s is equally great as $(\langle\langle 1, 0\rangle, 1\rangle + t) \cdot r$. (Here 1 is the ordinal $0'$. The triplet $\langle\langle 1, 0\rangle, 1\rangle$ has the property that $\langle\langle 1, 0\rangle, 1\rangle \cdot r = r$ for every fraction triplet r.)

For every fraction triplet there exists a fraction triplet greater than it and another less than it; moreover, if r and s are fraction triplets, and r is less than s, there exists a fraction triplet t that is greater than r and less than s.

Let us further define, for a finite ordinal n and a fraction triplet t, $n \cdot t$ ("n times t") and t^n ("t to the nth"), by the recursive definitions:

$$0 \cdot t = \langle\langle 0, 0\rangle, 1\rangle, \quad n' \cdot t = n \cdot t + t.$$

$$t^0 = \langle\langle 1, 0\rangle, 1\rangle, \quad t^{n'} = t^n \cdot t.$$

By complete induction it can be shown that, for every finite ordinal n and every fraction triplet $\langle\langle k, l\rangle, m\rangle$, $n \cdot \langle\langle k, l\rangle, m\rangle$ is equally great as $\langle\langle n \cdot k, n \cdot l\rangle, m\rangle$, and that, if r is a positive fraction triplet, $(\langle\langle 1, 0\rangle, 1\rangle + r)^n$ is not less than $\langle\langle 1, 0\rangle, 1\rangle + n \cdot r$. From this it follows that, for an arbitrary fraction triplet s, and every positive fraction triplet t, there exists a finite ordinal n such that $n \cdot t$, and *a fortiori* $(\langle\langle 1, 0\rangle, 1\rangle + t)^n$, is greater than s.

In the theory of fraction triplets the theory of *fractions* is included. For the fractions may be defined as fraction triplets of a special kind by agreeing that, if m and n are finite ordinals and $n \neq 0$, the *fraction* m/n shall be the fraction triplet $\langle\langle m, 0\rangle, n\rangle$. Then from our definitions of the sum and the product of fraction triplets and of *greater than*, *less than*, and *equally great as* for fraction triplets, there result the usual definitions of the corresponding concepts and operations for fractions. Moreover the class of all fractions exists and is a subclass of the class of all fraction triplets; it is enumerable and therefore is represented by a set.

On the basis of the theory of fraction triplets, the theory of real numbers can now be established.

Definition. A set a of fraction triplets will be called a real number if it satisfies the three following conditions: (1) There is a fraction triplet in a and a fraction triplet not in a. (2) If a fraction triplet t is in a, every fraction triplet less than t or equally great as t is in a. (3) For every fraction triplet in a there is a greater fraction triplet in a.

From the condition (2) it follows that, if a and b are real numbers, we have either $a \subset b$ or $b \subset a$ or $a = b$. Of two different real numbers, the one which is a proper subset of the other will be called the *less*, and the other one the *greater*. And a real number c will be said to be *between* a and b if $a \subset c$ and $c \subset b$.

In view of the definition of a real number, it follows by the class theorem that the class of all real numbers exists. And notice also that no real number is an ordinal or a fraction triplet.

We also are led rather directly to the *theorem of the least upper bound.*

By an *upper bound* of a class A of real numbers, we understand a real number which is less than no element of A. Let A be a non-empty class of real numbers which has an upper bound b. The sum of the elements of A is a subclass of b; so it is enumerable and therefore is represented by a set c. It is easily seen that c is a real number, further that it is an upper bound of A, and that there is no upper bound of A less than it. Thus every non-empty class of real numbers which has an upper bound has a *least upper bound.*

By a *lower bound* of a class A of real numbers, we mean a real number which is greater than no element of A. For every class A of real numbers there exists the class of all lower bounds of A, and every element of A is an upper bound of this class. If A is a non-empty class that has a lower bound, it is readily shown that the least upper bound of the class of lower bounds of A is the *greatest lower bound* of A.

Now we have to introduce the *elementary operations* for the real numbers. We define the *sum* $a+b$ of real numbers a and b as the set of those fraction triplets t for which there exists a triplet $r \in a$ and a triplet $s \in b$ such that $r+s$ is equally great as t. The existence of this set follows by the class theorem, since, as we have seen, every class of fraction triplets is represented by a set. Moreover it can be shown without difficulty that the sum of real numbers a and b, as thus defined, is again a real number.

In order to define the difference of real numbers, we first define the sum of a real number and a fraction triplet. By the sum $a+t$ of a real number a and a fraction triplet t we mean the set of those fraction triplets which are equally great as $s+t$ for some element s of a. This set is obviously a real number. Then we define the *difference* $a-b$ of real numbers a and b as the set of those fraction triplets t for which $b+t$ is less than a. It follows that the difference of two real numbers is again a real number.

As to the laws of computation for sum and difference, those which concern the sum alone are rather obvious. However, the proof of the laws,

$$(a + b) - b = a, \qquad (a - b) + b = a,$$

is not quite direct; it is desirable to use for it the two following lemmas:

Lemma 1. If a and b are real numbers and $a \subset b$, there exists a positive fraction triplet t such that $a+t \subset b$.

This is a consequence of conditions (2) and (3) in our definition of a real number.

Lemma 2. If a is a real number and t is a positive fraction triplet, then $a \subset a+t$.

For if we had $a+t \subseteq a$, then, for every triplet r in a, $r+t$ would also be in a, and hence by complete induction $r + n \cdot t$ would be in a for every finite ordinal n. But this is impossible, since on the one hand there is a fraction triplet s which is greater than every triplet in a, and on the other hand, t being positive, there is an ordinal n such that $n \cdot t$ is greater than $s-r$.

Of this lemma the following consequence is to be used: If a and b are real numbers and s is a fraction triplet, then $a+s \subset a+b$ if and only if $s \epsilon b$. The proof of this employs the relation $a+(s+t) = (a+s)+t$ for a real number a and fraction triplets s, t.

In addition to the application of lemma 2 in proving the two above laws, characterizing the difference, we draw from it also the consequence that, for a real number a and a fraction triplet t, we have $a+t \subset a$ if and only if t is negative. It follows from this and from our definition of the difference of real numbers that, for any real number a, the difference $a-a$ is the set of all negative fraction triplets. Let us denote the latter real number by "[0]." We have

$$a - a = [0], \qquad a + [0] = a,$$

for any real number a.

A real number will be called *positive* if it is greater than [0], *negative* if it is less than [0]. Then, as is easily seen, a real number is positive if and only if there is a positive fraction triplet in it. Further we define:

$$-a = [0] - a.$$

It is easily shown that the difference $a-b$ of real numbers a and b is negative, [0], or positive according as $a \subset b$, $a=b$, or $b \subset a$; and hence in particular that $-a$ is negative, [0], or positive according as a is positive, [0], or negative. Also easily proved are the relations,

$$-(-a) = a, \qquad a - b = a + (-b).$$

For every real number a there is one and only one non-negative real number which either is equal to a or is equal to $-a$; this we denote, as is usual, by $|a|$. We say that a real number a *differs from* the real number b *by less than* d, where d is a positive real number, if $|a-b| \subset d$.

In order to define the product of real numbers, it seems to be natural to begin with the case of positive real numbers. We first define the product $a \cdot t$ of an arbitrary real number a and a positive fraction triplet t to be the set of those fraction triplets which are equally great as $s \cdot t$ for some element s of a. This set is easily seen to be a real number.

Now let a and b be positive real numbers. By the *product* $a \cdot b$ of a and b

we mean: the least upper bound of the class of those real numbers c for which there is a positive fraction triplet t such that $t \epsilon b$ and $a \cdot t = c$; or, in other words, the set of those fraction triplets which are in some of the products, $a \cdot t$, of a and a positive triplet t that is in b.

This definition can easily be shown to be equivalent to the following: The product $a \cdot b$ of positive real numbers a and b is the set of those fraction triplets which are either negative or null triplets or equally great as the product of a positive triplet that is in a and a positive triplet that is in b.

From the first definition we see that the product of positive real numbers is a real number, and, of course, a positive one. From the second definition it is evident that, if a and b are positive real numbers, $a \cdot b = b \cdot a$. The associative law and the distributive law are also easily proved.

The extension of the definition of the product to negative real numbers and [0] is to be made, as usual, by the following defining equations, which refer to positive real numbers a and b and an arbitrary real number c:

$$a \cdot (-b) = -(a \cdot b), \qquad (-a) \cdot b = -(a \cdot b), \qquad (-a) \cdot (-b) = a \cdot b,$$
$$c \cdot [0] = [0], \qquad [0] \cdot c = 0.$$

These conventions, as is known, are necessary and sufficient in order to maintain the distributive law for arbitrary real numbers. Of course also the commutative and associative character of the product is preserved by them.

We use the notation [1] for the set of those fraction triplets $\langle\langle k, l\rangle, m\rangle$ for which the ordinal k is lower than $l+m$. This set, as is easily verified, is a positive real number, and it has the property that $a \cdot [1] = a$ for every real number a.

We have still to define the quotient of real numbers. By the *quotient*

$$\frac{a}{b}$$

of positive real numbers a and b, we understand the set of those fraction triplets t which either are positive and satisfy the condition $b \cdot t \subset a$ or are negative or null triplets. Thus the quotient of positive real numbers is again a positive real number.

The proof of the laws,

$$\frac{a \cdot b}{a} = a, \qquad b \cdot \frac{a}{b} = a,$$

for positive real numbers is quite analogous to that of the corresponding laws for the difference of real numbers. To the two lemmas used for that proof there correspond here the two following similar lemmas:

If a and b are positive real numbers and $a \subset b$, there exists a positive fraction triplet t such that $a \cdot (\langle\langle 1, 0\rangle, 1\rangle + t) \subset b$.

If a is a positive real number and t a positive fraction triplet, then $a \subset a \cdot (\langle\langle 1, 0\rangle, 1\rangle + t)$.

The extension of the definition of the quotient is to be made in the usual way by the following defining equations, which refer to positive real numbers a and b and an arbitrary real number c:

$$\frac{(-a)}{b} = -\frac{a}{b}, \qquad \frac{a}{(-b)} = -\frac{a}{b}, \qquad \frac{(-a)}{(-b)} = \frac{a}{b}, \qquad \frac{[0]}{c} = [0].$$

Notice that our definitions of the sum and the product of real numbers allow us to represent the relations $a+b = c$ and $a \cdot b = c$ by constitutive expressions, and hence that there exists the class of those triplets $\langle c, \langle a, b\rangle\rangle$ of real numbers a, b, c for which $a+b = c$, and also of those for which $a \cdot b = c$.

Remark. The way here outlined of founding the theory of real numbers is, of course, only one among several available methods of carrying out the Dedekind idea within the frame of the axiomatic system which we are considering. The advantage of the one we have chosen is that it allows a rather direct passage from finite ordinals to real numbers. This may be regarded as compensating for the complication of having to deal with fraction triplets instead of fractions—the more so since we can pass immediately to the fraction notation, whenever we wish, by the above given definition of fractions as fraction triplets of a special kind.

9. Limit point, enumerable sequences, continuous functions, Lebesgue measure. From the elementary notions concerning real numbers, and their properties, the fundamental concepts and theorems of analysis are to be obtained in the usual manner.

In order to show that this can be done within our axiomatic frame, a number of instances will suffice. We consider first the theorem that *every bounded infinite class of real numbers has a limit point*. Here a class C of real numbers is called *bounded* if it has an upper bound and a lower bound; or, in other words, if there exist real numbers a and b such that, for every element c of C, we have $a \subseteqq c$ and $c \subseteqq b$. And by a *limit point* of a class C of real numbers we understand a real number a such that, for every positive real number d, the class of those elements of C which differ from a by less than d is infinite.

The most direct way of proving this theorem is the following. Let a be a lower bound and b an upper bound of the infinite class C of real numbers. From the existence of the class of all real numbers and of the class of all finite sets there follows the existence of a class D whose elements are those real numbers d for which there exists a finite set s whose elements are the elements of C less than d. It is immediately seen that $a\eta D$ and b is an upper bound of D. Therefore D has a least upper bound g. By the definition of the least upper bound and by the theorems on finite classes and sets, for every positive real number p the class of elements of C which are between $g-p$ and $g+p$ is infinite, and thus g is a limit point of C.

A more usual method of proving the existence of a limit point for any bounded infinite class of real numbers is the following. Let

$$\frac{a+b}{2}$$

denote the real number $(a+b) \cdot 1/2$. Then, for any class C of real numbers, there exists a function F assigning to any pair $\langle a, b\rangle$ of real numbers the pair

$$\langle a, \frac{a+b}{2}\rangle$$

or the pair

$$\langle \frac{a+b}{2}, b\rangle$$

according as the class of those elements of C which are between a and $\frac{a+b}{2}$ is infinite or finite. This follows by the class theorem, since the condition that a class is infinite, or, what comes to the same thing (by the first theorem on finite classes and sets), that it is not represented by a finite set, can be formulated by a constitutive expression.

As before, let C be a bounded infinite class of real numbers and let a be a lower bound and b an upper bound of C. Further, let G be the iterator of the function F on the pair $\langle a, b\rangle$. Then G is a function whose domain is the class N of all finite ordinals, and whose values are ordered pairs of real numbers. By our axioms III, there exist the functions A and B with the domain N such that, for any finite ordinal n, $A(n)$ is the first, and $B(n)$ the second, member of $G(n)$. By the definition of F and G we have, for any finite ordinal n,

$$A(n) \subseteq A(n'), \qquad A(n) \subset B(n), \qquad B(n') \subseteq B(n),$$
$$B(n') - A(n') = (B(n) - A(n)) \cdot 1/2,$$

and hence, by complete induction, for any finite ordinals m, n,

$$A(m) \subset B(n),$$
$$B(n) - A(n) = (b - a) \cdot (1/2)^n.$$

From this it follows that, for any finite ordinal n, the real number $B(n)$ is an upper bound of the converse domain of A. Let e be the least upper bound of the converse domain of A; then, for any finite ordinal n,

$$A(n) \subseteq e, \qquad e \subseteq B(n).$$

Now for any positive real number d, since $2/1$ is greater than $\langle\langle 1, 0\rangle, 1\rangle$, there is, as we know, a finite ordinal n such that $b - a \subset d \cdot (2/1)^n$. For this ordinal n,

$$(B(n) - A(n)) \cdot (2/1)^n \subset d \cdot (2/1)^n,$$
$$B(n) - A(n) \subset d,$$

and so every real number between $A(n)$ and $B(n)$ differs from e by less than d.

On the other hand, from our assumption that C is infinite and from the definitions of F, G, A, B, there follows by complete induction that, for every finite ordinal n, the class of those elements of C which are between $A(n)$ and $B(n)$ is infinite. Hence e is a limit point of C.

Let us now see how we may deal with *convergent sequences* and *continuous functions*.

By an *enumerable sequence* we will understand a functional set whose domain is the set of all finite ordinals (representing the class N). In particular, if the elements of the converse domain are real numbers, such a functional set will be called an *enumerable sequence of real numbers*; or if the converse domain is a subclass of A, it will be called *an enumerable sequence of elements of A*. The elements of the converse domain of an enumerable sequence will be called the *members* of the sequence.

Remarks. 1. Notice that the converse domain of an enumerable sequence need not be enumerable but can be finite, since there can be repetitions of members in a sequence. 2. We do not yet require the general concept of a sequence; it will be introduced later on.

Definition. In what follows, where f is a functional set and a an element of its domain, we shall use the notation $f(a)$ for the value assigned to a by f—analogous to the notation $F(a)$ used in the case of a function F.

An enumerable sequence f of real numbers will be called *convergent* if, for every positive real number d, there exists a finite ordinal k such that, for all finite ordinals m and n both higher than k, the real numbers $f(m)$ and $f(n)$ differ from one another by less than d. And an enumerable sequence will be said to have the real number a as its *limit* (or, more briefly, to have *the limit a*) if, for every positive real number d, there exists a finite ordinal k such that, for every finite ordinal m higher than k, the real number $f(m)$ differs from a by less than d.

Then the class of all convergent sequences of real numbers exists; and the condition that an enumerable sequence s of real numbers has the limit a can be represented by a constitutive expression with the arguments s and a.

The theorems, that an enumerable sequence of real numbers can have only one limit and that it has a limit if and only if it is convergent, now follow in the usual way. (The proof that every convergent enumerable sequence f of real numbers has a limit may be accomplished by considering the class of those real numbers c for which the condition holds that, given any finite ordinal k, there exists a higher finite ordinal n such that $c \subset f(n)$, and showing that this class has an upper bound and that its least upper bound is the limit of the sequence f.)

In order to introduce *infinite sums* of real numbers we must define the sequence of *partial sums* which corresponds to an enumerable sequence f of real numbers. For this purpose we use the theorem of finite recursion. Let f be an enumerable sequence of real numbers. Then there exists a function assigning to every pair $\langle n, a \rangle$ of a finite ordinal n and a real number a the real number $a+f(n')$. Hence by the theorem of finite recursion there exists a function S whose domain is the class N, whose values are real numbers, and which satisfies the conditions,

$$S(0) = f(0),$$
$$S(n') = S(n) + f(n'),$$

for every finite ordinal n. This function S is represented by an enumerable sequence of real numbers, which is called *the sequence of partial sums of f*. If the sequence of partial sums of f is convergent we call its limit the *numerical sum* of the sequence f, and we then say that the sequence *f has a numerical sum.*

For every sequence f of real numbers there exists a sequence g satisfying the condition that $g(n) = f(n) \cdot f(n)$ for every finite ordinal n. The members of g are the *squares* of the members of f.

By the class theorem, there exists the class of all those enumerable sequences of real numbers for which the sequence of the squares of the members has a numerical sum. This is needed in particular for the Hilbertian theory of "infinitely many variables."

The obvious way of introducing *continuous functions* is to add to the general concept of a *real function* the condition of *continuity*. It will suffice here to speak of functions of only one argument. A real function of one argument is a function whose domain and converse domain are each a class of real numbers. Frequently the domain of a real function is a class of one of the following kinds:

the class of all real numbers;

the class of all real numbers which are greater than, or are not less than, a certain real number a (e.g., the class of positive real numbers);

the class of real numbers which are between two certain real numbers a and b, where a is less than b (we call this *the open interval from a to b*);

the class which is the sum of the open interval from a to b and the class whose elements are a and b (we call this *the closed interval from a to b*).

The condition of continuity of a real function (of one argument) *at the point a* of its domain—the word *point* here being synonymous with *real number*—is to be formulated in the usual way. And the function is called *continuous on its domain* if it is continuous at every point of its domain.

However, this way of defining continuous functions is not sufficient for cases in which we require sequences of continuous functions; for the members of a sequence must be sets. Thus we need a method of characterizing continuous functions by sets. The possibility of such a characterization arises from the fact that a continuous function is determined by its values for the elements of an appropriate enumerable subclass of its domain. Let us explain this for the case of continuous functions having as domain the class of positive real numbers.

For every fraction m/n there is a *corresponding real number* $[1] \cdot m/n$, which, as is easily seen, is the set of fraction triplets less than m/n. An enumerable sequence s of fractions will be said to have the limit c, c being a real number, if the sequence of real numbers corresponding to the members of s has the limit c.

For every positive real number c there is an enumerable sequence of fractions which has the limit c. For let l be the lowest finite ordinal k, different from 0, for which $1/k \in c$ (since c is positive, there exists such a finite ordinal). Then for every finite ordinal n there exists a highest finite ordinal $m \neq 0$ for which

$m/(l+n) \in c$. Thus there exists an enumerable sequence h of finite ordinals such that $h(n)/(l+n)$ is in c but $(h(n)+1)/(l+n)$ is not. And the enumerable sequence which assigns to every finite ordinal n the fraction $h(n)/(l+n)$ can easily be shown to have the limit c.

A further remark is that every function whose domain is the class of fractions is represented by a functional set, since the class of fractions is enumerable.

Now we define a *continuous fraction-functional* as a functional set having the following properties:

1) its domain is the set of all fractions;
2) its values are real numbers;
3) to equally great fractions the same value is assigned;
4) given any positive real numbers c and e, there exists a positive real number d such that, if r and s are any fractions whose corresponding real numbers differ from c by less than d, the values assigned to r and to s differ from one another by less than e.

The continuous fraction-functionals, as thus defined, can be taken as a kind of representatives of those continuous functions whose domain is the class of positive real numbers (though not directly representing them). In fact, every continuous function F whose domain is the class of positive real numbers determines the class of all those pairs $\langle r, s\rangle$ for which r is a fraction and s is the value assigned by F to the real number corresponding to r. Since this class is a function with an enumerable domain, it is itself enumerable and therefore is represented by a set. This set, which is easily shown to be a continuous fraction-functional, will be said to *correspond* to the function F. On the other hand, given any continuous fraction-functional f, there exists a unique continuous function to which f corresponds; this can be seen as follows.

For every positive real number a there exists, as we have seen, an enumerable sequence of fractions whose limit is a. If s is such a sequence, then by the composition lemma and the axiom of the enumerable there exists an enumerable sequence which assigns to every finite ordinal n the value of $f(s(n))$. This sequence, by condition 4 in the definition of a continuous fraction functional, is convergent, and therefore has a limit b. Moreover this limit does not depend on the particular sequence s, but is determined by the limit a of s. And it is possible to formulate by a constitutive expression $\mathfrak{R}(f, a, b)$ the condition that, a being a positive real number and b a real number and f a continuous fraction-functional, for every enumerable sequence s of fractions, having the limit a, the enumerable sequence t which assigns to the finite ordinal n the value of $f(s(n))$ has the limit b. Taking the variable f in this constitutive expression as a parameter, we infer the existence of a function F assigning to every positive real number a the uniquely determined real number b for which $\mathfrak{R}(f, a, b)$ holds. Then F can be shown to be continuous, to have f as its corresponding continuous fraction-functional, and to be the only continuous function which does so.

By the class theorem, the class of all continuous fraction-functionals exists, and also the class T of those triplets $\langle\langle f, a\rangle, b\rangle$ for which $\mathfrak{R}(f, a, b)$ holds. This class T is a function assigning to any pair $\langle f, a\rangle$ which has a continuous fraction-functional as its first member and a positive real number as its second member, the value assigned to a by the continuous function to which the continuous fraction-functional f corresponds. Thus any property that a continuous function F of positive real numbers has with respect to its values is immediately expressible by means of the corresponding continuous fraction-functional, since we have, for every positive real number a,

$$F(a) = \mathrm{T}(\langle f, a\rangle).$$

Now for the case of continuous functions having as domain (instead of the class of positive real numbers) the open or the closed interval from [0] to [1], our method of characterization by continuous fraction-functionals can be applied quite correspondingly. The only modification required is that we have to take a somewhat different kind of continuous fraction-functionals, differing in that their domain is now not the set of all fractions but the set of *proper fractions*, i.e., the fractions m/n for which the ordinal m is lower than n. With this change, the further reasoning is the same as before. For, in the same way that we proved that every positive real number is the limit of an enumerable sequence of fractions, we can show that every positive real number not greater than [1] is the limit of an enumerable sequence of proper fractions; and the number [0] (which belongs to the closed interval from [0] to [1]) is obviously the limit of the enumerable sequence which assigns to each finite ordinal n the proper fraction $1/n''$.

The case of an arbitrary interval as the domain of a continuous function can of course be reduced by a simple transformation to the case of the (open or closed) interval from [0] to [1]. There is also no difficulty in extending our method to continuous functions whose domain is the class of all real numbers: for this purpose we may take, instead of continuous fraction-functionals, functional sets which have the set of all fraction triplets as domain and have otherwise the same properties which we required of continuous fraction-functionals.

Now let us also briefly consider the definition of the Lebesgue measure. We introduce the following definitions in order.

An enumerable sequence of pairs $\langle a, b\rangle$ of real numbers, with a always less than b, will be called a *covering sequence* of a class C of real numbers if for every element c of C there exists at least one member $\langle a, b\rangle$ of the sequence such that $a \subseteq c$ and $c \subseteq b$.

By the *coordinated sequence* of a covering sequence s of a class C of real numbers we mean the enumerable sequence t determined by the condition that, for every finite ordinal n, $t(n)$ is the difference $b-a$, for the pair $\langle a, b\rangle$ assigned to n by s. The connection between a covering sequence and its coordinated sequence can be formulated by a constitutive expression.

A real number d will be called a *bound of measure* of the class C of real numbers if there exists a covering sequence of C whose coordinated sequence has a numerical sum not greater than d. For every class C of real numbers, the class of its bounds of measure exists.

If C is a non-empty class of real numbers which has a bound of measure, the class of bounds of measure of C is a non-empty class of positive real numbers and hence has a greatest lower bound which is a non-negative real number. This greatest lower bound is called the *Lebesgue exterior measure of* C.

These definitions refer to one-dimensional classes of points. For two-dimensional classes of points, i.e., for classes of ordered pairs of real numbers, the following modifications have to be made:

1. A *covering sequence* of a class C of pairs of real numbers is an enumerable sequence of quadruplets $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ in which a is always less than b and c less than d, such that for every pair $\langle r, s\rangle$ belonging to C there exists a member $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ of the sequence for which $a \subseteq r$, $r \subseteq b$, $c \subseteq s$, $s \subseteq d$.

2. The *coordinated sequence* of a covering sequence s is the enumerable sequence t determined by the condition that, for every finite ordinal n, $t(n)$ is the product of differences $(b-a)\cdot(d-c)$, for the quadruplet $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ assigned to n by s.

The notions of a *bound of measure* of C and of the *Lebesgue exterior measure* of C are then introduced in the same way as in the one-dimensional case.

Similarly the corresponding definitions may be made for classes of points of higher dimensions.

Then the Lebesgue measure and, by means of it, the Lebesgue integral can be introduced in the usual manner.

The theory of function spaces[35] also can be treated within our axiomatic frame. This possibility results from the circumstance that the elements of a function space can be characterized by their components (Fourier coefficients) with respect to a complete orthogonal system, or, in other words, from the isomorphism of function spaces to Hilbert space. In fact, Hilbert space is, from the set-theoretic point of view, characterized as a class of enumerable sequences of real numbers.

In sum, it appears that, in the various branches of analysis, real functions occur for the most part only in such a way that they can be characterized by enumerable functional sets. In the case of continuous functions, this possibility is due to the particular character of the functions; in the theory of function spaces the reason for it is that functions are there considered as equal if the class of arguments to which they assign different values has the Lebesgue exterior measure [0].

10. The special axiom of choice, the second number class. We still have to consider the application of the axiom of choice in analysis. There are in fact certain reasonings in analysis for which apparently the axiom of choice or a similar principle is necessary—although, as we know, in many cases where application of the axiom of choice suggests itself, the possibility exists of avoiding this axiom, or of reducing the number of applications of it.

As an illustration let us consider briefly the question of choosing convergent sequences out of a class of functional sets. On this there is a theorem of analysis,

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[35] See for example J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, ***Mathematische Annalen,*** vol. 102 (1929–30), pp. 49–131, in particular Kap. I and Anhang I therein.

which is used as an auxiliary theorem in several connections. Before formulating it we introduce the following abbreviating terminologies.

An enumerable sequence which represents a one-to-one correspondence will be called an enumerable sequence *without repetitions*. An enumerable sequence t will be called a *sub-sequence* of an enumerable sequence s, if there exists an enumerable sequence h of finite ordinals such that, for every finite ordinal n, $h(n) \epsilon h(n')$ and $t(n) = s(h(n))$. Hence, as is easily shown, an enumerable sequence t is a sub-sequence of an enumerable sequence s without repetitions, if every member of t is a member of s and, for every finite ordinal n, $t(n')$ is assigned by s to a higher ordinal than $t(n)$ is assigned to.

An enumerable sequence q of functional sets which have a common domain t and whose values are real numbers will be called *convergent at a*, if $a \epsilon t$ and the enumerable sequence assigning to each finite ordinal n the value $(q(n))(a)$ is convergent.

Now the theorem in question asserts that, if C is an infinite class of functional sets which have a fixed enumerable set t as their domain and whose values are real numbers of a certain fixed interval, then there exists an enumerable sequence of elements of C without repetitions which is convergent at every element a of t.

Proof of this can be made in the following way, with the aid of the axiom of choice. For any element a of t, the class A, of real numbers b such that $f(a) = b$ for some functional set f belonging to C, is bounded (as follows from the hypothesis on C). If A is finite, there exist a real number c, and an infinite class of functional sets which belong to C and have the value c for the argument a. Then, since every infinite class has an enumerable subclass—a consequence of the axiom of choice, as we have seen—there follows the existence of an enumerable sequence, without repetitions, of functional sets belonging to C which all assign to a the same value c; and this sequence is (in a trivial way) convergent at a. Otherwise A is infinite and hence has a limit point c. Then follows—by a method which involves an application of the axiom of choice and which will be discussed below—that there exists an enumerable sequence g of elements of A without repetitions which has the limit c. For every member r of g there exists an element f of C such that $f(a) = r$, and hence by the axiom of choice there exists a function G assigning to each member r of g an element f of C such that $f(a) = r$. Hence we infer—using the composition lemma, and the third theorem on enumerability (§8)—that there is an enumerable sequence of elements of C without repetitions which is convergent at a.

Thus for every element a of t there exists an enumerable sequence, without repetitions, of functional sets belonging to C which is convergent at a. From this we now have to show that there is a single enumerable sequence, without repetitions, of elements of C which is convergent at every element a of t.

For this purpose, we first observe that any infinite subset of C satisfies the assumptions we have made for C, and hence that our result applies not only to C but also to any infinite subset of C, in particular to the converse domain of any enumerable sequence of elements of C without repetitions. In addition, we must use our assumption that t is enumerable, so that there exists an enumerable sequence s without repetitions which has t as its converse domain.

Using once more the axiom of choice, and applying the theorem of finite recursion, we infer the existence of a function F whose domain is N and which satisfies the condition that, for every finite ordinal n, $F(n)$ is an enumerable sequence of elements of C without repetitions which is convergent at $s(n)$, and that $F(n')$ is a sub-sequence of $F(n)$.

Now the enumerable sequence which (by the third theorem on enumerability) represents the function assigning to each finite ordinal n the functional set $(F(n))(n)$ can easily be seen to be convergent at every element a of t. Thus our theorem is proved.

This method of proof involves repeated application of the axiom of choice. However we can, by another arrangement, reduce the number of applications to one, as follows.

On the assumptions which we made for the class C of functional sets we can prove without use of the axiom of choice that there is a function G assigning to every pair $\langle q, a\rangle$—where q is an enumerable sequence of elements of C without repetitions and a is an element of t—a sub-sequence of q convergent at a. Indeed, from the assumptions made for q and a the following disjunction can be inferred, as regards the class P of real numbers which are a value, $(q(n))(a)$, of some member of the sequence q, for a: Either (i) the class Q of those elements r of P for which there is an infinite class of finite ordinals n such that $(q(n))(a) = r$ has an element, or (ii) for every element r of P there is a finite ordinal k such that for every finite ordinal n higher than k we have $(q(n))(a) \neq r$.

In case (i) there is a least ordinal l among those finite ordinals n for which $(q(n))(a)\ \eta\ Q$. By the iteration theorem there is an enumerable sequence f of finite ordinals, such that $f(0) = l$, and $f(n')$ is the lowest of the finite ordinals m which are higher than $f(n)$ and for which $(q(m))(a) = (q(l))(a)$. The enumerable sequence h assigning to each finite ordinal n the functional set $q(f(n))$ is a sub-sequence of q which is convergent at a. Moreover h is uniquely determined by q and a, and the relation subsisting between q, a, and h can be formulated by a constitutive expression $\mathfrak{A}(q, a, h)$.

In case (ii) the class P is infinite and, in consequence of our assumptions on C, is bounded. Hence there exists a limit point of P, and also a greatest limit point (or *upper limit*) p. By the theorem of finite recursion there is an enumerable sequence f of finite ordinals, such that $f(0) = 0$, and $f(n')$ is the lowest of the finite ordinals m which are higher than $f(n)$ and for which $(q(m))(a)$ differs from p by less than $[1]\cdot 1/n$. The enumerable sequence h assigning to each finite ordinal n the value $q(f(n))$ is a sub-sequence of q which is convergent at a. Moreover h is again uniquely determined by q and a, and its relation to q and a can be formulated by a constitutive expression $\mathfrak{B}(q, a, h)$.

Now if $\mathfrak{C}(q, a, C, t)$ is a constitutive expression formulating that q is an enumerable sequence of elements of C without repetitions and $a \epsilon t$, and if $\mathfrak{M}(q, a)$ and $\mathfrak{N}(q, a)$ are constitutive expressions, with the variables q and a, formulating the conditions for cases (i) and (ii) respectively, then, by our assumptions on C and on t, the class of all triplets $\langle\langle q, a\rangle, h\rangle$ for which

$$\mathfrak{C}(q, a, C, t)\ \&\ [(\mathfrak{M}(q, a)\ \&\ \mathfrak{A}(q, a, h)) \vee (\mathfrak{N}(q, a)\ \&\ \mathfrak{B}(q, a, h))]$$

holds is a function G, depending on the class C, which assigns to every pair $\langle q, a\rangle$ satisfying $\mathfrak{C}(q, a, C, t)$ a sub-sequence h of q that is convergent at a.

Now t, since it is enumerable, is the converse domain of an enumerable sequence s. Thus, applying the theorem of finite recursion, we can infer that, for every enumerable sequence q of elements of C without repetitions, there is a function F whose domain is N, whose values are enumerable sequences of elements of C without repetitions, and which satisfies the conditions that $F(0) = q$ and (for every finite ordinal n) $F(n')$ is a sub-sequence of $F(n)$ which is convergent at $s(n)$. And from this follows that the enumerable sequence assigning to each finite ordinal n the functional set $(F(n))(n)$ is an enumerable sequence of elements of C without repetitions and is convergent at every element a of t.

Thus in order to complete our proof it only remains to show in consequence of our assumptions on C that there exists an enumerable sequence, q, of elements of C without repetitions. This can be done by using the theorem that every infinite class has an enumerable subclass in connection with the third theorem on enumerability. In this way the only application of the axiom of choice in the whole proof is that involved in the theorem that every infinite class has an enumerable subclass.

However the preceding discussion is intended not only to illustrate the possibility in certain cases of reducing the extent of the application of the axiom of choice, but also to present typical instances of the application of the axiom of choice in analysis. In the first of the two proofs which we have given of the theorem on enumerable sequences of functional sets, we find the three following applications of the axiom of choice: (1) for the theorem that every infinite class has an enumerable subclass; (2) in order to prove for an arbitrary limit point c of an infinite class A of real numbers that there exists an enumerable sequence of elements of A without repetitions which has the limit c; (3) given a real number a, an enumerable sequence g of real numbers without repetitions, and a class C of functional sets, and given that for every member r of g there exists an element f of C such that $f(a) = r$, to prove that there exists a function G assigning to each member r of g an element f of C such that $f(a) = r$.

Of these three consequences of the axiom of choice, the last two apparently cannot be reduced to the first. (It only happens that their use was found to be avoidable in the proof of the particular theorem which we have been considering.)

The third one can be generalized as follows: (4) if for every member r of an enumerable sequence g without repetitions there is a set s such that the pair $\langle r, s\rangle$ belongs to a certain class of pairs D, then there exists a function G assigning to each member r of g a set s such that $\langle r, s\rangle \eta D$. And for the proof of (4) we obviously can get along with the following weakened form of the axiom of choice:

Every class of pairs which has N as its domain has a subclass which is a function and has N as its domain.

This *special axiom of choice* will be called $\mathrm{IV_s}$. As an immediate consequence of it in connection with the third theorem on enumerability we note the following:

If C is a class of pairs which has N as its domain, there exists a set whose elements are the members of an enumerable sequence which is a subset of C.

Remark. If this last be taken as an axiom, the theorems expressing the axiom of the enumerable and the special axiom of choice IV_s follow as immediate consequences of it. Hence we have the possibility of contracting the axiom of the enumerable and the special axiom of choice in a handy way into a single axiom, which can be expressed as follows without using the concept of an enumerable sequence: Every class of pairs whose domain is N has as a subclass a function whose domain is N and whose converse domain is represented by a set.

In what follows, references to the axiom IV_s will be understood as including this immediate consequence of IV_s without explicit mention.

Now we shall show that the axiom IV_s—in addition to its adequacy for (3) and (4)—suffices also for the proof of the existential assertions (1) and (2) above.

The proof of (2) by means of IV_s proceeds as follows. Since c is a limit point of A, there exists for every finite ordinal n an element of A which is not c but which differs from c by less than $[1]\cdot 1/n$. Hence the class of pairs $\langle n, p\rangle$, satisfying the conditions that n is a finite ordinal, and $p\eta A$, and $p\neq c$, and p differs from c by less than $[1]\cdot 1/n$, has the domain N. Applying the axiom IV_s to this class of pairs, we infer the existence of an enumerable sequence f of elements of A such that, for every finite ordinal n, $f(n) \neq c$ and $|f(n)-c|$ is less than $[1]\cdot 1/n$. Hence f has the limit c, but c is not a member of f. From this it follows in particular that the converse domain h of f is infinite, and therefore that (as the converse domain of an enumerable sequence) it is enumerable. Thus there exists a one-to-one correspondence between N and h. This one-to-one correspondence is represented by an enumerable sequence of elements of A without repetitions, which is easily shown to have the limit c.

In order to prove by means of the axiom IV_s that every infinite class has an enumerable subclass ((1) above), the method of proof which we used in §7 for this theorem is not applicable, but we can argue instead as follows. If C is an infinite class, then for every finite ordinal n there exists a one-to-one correspondence between n and a subset of C, and hence for every finite ordinal n there exists a one-to-one correspondence between the ordinal 2^n and a subset of C. Consequently the class of pairs $\langle n, s\rangle$ such that n is a finite ordinal and s represents a one-to-one correspondence between 2^n and a subset of C has the domain N, and by the axiom IV_s there exists a function F assigning to each finite ordinal n a set representing a one-to-one correspondence between 2^n and a subset of C. By complete induction, since $2^{n'} = 2^n+2^n$, it follows that, for every finite ordinal n, there is a one-to-one correspondence between an ordinal lower than 2^n and the class of those elements of C which for at least one ordinal k lower than n are in the converse domain of $F(k)$. Hence for every finite ordinal n there is in the converse domain of $F(n)$ at least one set t which is not in the converse domain of $F(k)$ for any ordinal k lower than n; and among the elements $\langle m, t\rangle$ of $F(n)$ for which t has this property there is one, say $\langle m_1, t_1\rangle$, for which the ordinal m is lowest. The connection between this element t_1 of the converse domain of $F(n)$ and the ordinal n can be formulated by a constitutive expression $\mathfrak{P}(n, t_1, F)$. Moreover the class of pairs $\langle n, t_1\rangle$ which satisfy the relation $\mathfrak{P}(n, t_1, F)$ is a one-to-one correspondence between N and a subclass of C; and the converse domain of this class of pairs is an enumerable subclass of C.

On the whole we are led, by our preceding discussions, to delimit analysis within our system of axiomatic set theory as the system based on the axioms I–III, VII, the axiom of the enumerable, and the special axiom of choice IV_s. This system will hereafter be referred to briefly as *the system of analysis*.

In this connection let us recall the following facts: (1) that the inclusion of the axiom VII in the system of analysis is required only for the foundations of the theory of ordinals, and that for this purpose VII can be replaced by V a; (2) that the axiom of the enumerable can be deduced from the axioms I–III, V a, V b, and VI (the axiom of infinity), and that the axioms V a and V b can be derived from V*; (3) that the axiom of the enumerable can be derived from the axioms I–III, V a, VI*. Hence each of the three following lists of axioms constitutes a sufficient axiomatic basis for analysis:

I–III, IV, V a, V b, VI.

I–III, IV_s, V*, VI.

I–III, IV_s, V a, VI*.

Remark.[36] There would also be the possibility of deriving IV_s as a theorem from the following weakened form of the axiom of choice, which we shall call IV*:

If A is a class, and C is a class of pairs, and for every element p of A there is an element q of A such that $\langle p, q\rangle \eta C$, then for any element a of A there exists a function F whose domain is N and which satisfies the conditions that $F(0) = a$ and, for every finite ordinal n, $F(n) \eta A$ and $\langle F(n), F(n')\rangle \eta C$.

IV* is easily inferred as a theorem from the axiom of choice together with the iteration theorem.

From IV* as an axiom we can deduce IV_s as a theorem, as follows. Let A be a class of pairs having N as its domain. Then there is a set c such that $\langle 0, c\rangle \eta A$. Further, by the class theorem, there exists the class C of all pairs of elements of A which have the form $\langle\langle n, r\rangle, \langle n', s\rangle\rangle$; and for every element p of A there is an element q of A such that $\langle p, q\rangle \eta C$. Then applying IV* (with $\langle 0, c\rangle$ as a), we infer the existence of a function F with the domain N satisfying the conditions that $F(0) = \langle 0, c\rangle$ and, for every finite ordinal n, $F(n) \eta A$ and $\langle F(n), F(n')\rangle \eta C$. By complete induction it follows that, for every finite ordinal n, the first member of $F(n)$ is n; and hence it follows that the converse domain of F is also a function with the domain N. The latter function is a subclass of A; and thus IV_s is proved.

Axiom IV* surely is at least as natural as IV_s. It seems to be a bit stronger than IV_s and thus it might be suitable for some purposes for which IV_s is insufficient.

Finally we give some indications for the development within the system of analysis of *the theory of the Cantor second number class*.

[36] The remainder of Part III, from this point on, is an addition received April 25, 1941. *Editor*.

The most natural method is to define the *second number class* as the class of enumerable ordinals—the *first number class* being N. Then for the sum of the first and second number classes, i.e., for the class of all finite and all enumerable ordinals, we may use the notation Ω.

By the second theorem on enumerability, every ordinal lower than an enumerable ordinal is either finite or enumerable. Thus Ω is a transitive class of ordinals.

We shall call an ordinal a' having a highest element a the *successor* of a. Such ordinals a' will be called *successors*, and all other ordinals, except 0, will be called *limiting numbers*.

An enumerable sequence s of ordinals will be called *ascending* if, for every finite ordinal n, $s(n) \in s(n')$. And an ordinal l will be said to be the *limit* of an ascending enumerable sequence of ordinals if it represents the sum of the members of the sequence, or (what comes to the same thing) if the class of ordinals higher than every element of the sequence is non-empty and has l as its lowest element.

(There will be no danger of confusing this concept of *limit* with that referring to an enumerable sequence of real numbers, since an ordinal cannot be a real number.)

The connection between enumerable limiting numbers and enumerable ascending sequences of ordinal numbers is expressed by the two following theorems:

1. *Every limiting number that is enumerable is the limit of an enumerable ascending sequence of ordinals.*

2. *The sum of the members of an ascending enumerable sequence of finite or enumerable ordinals is represented by an enumerable ordinal which is the limit of the sequence.*

In order to prove 1, it obviously will be sufficient to show that every enumerable sequence s of ordinals which has no highest member has an ascending subsequence t such that the sum of the members of t is the same as the sum of the members of s. This can be done by applying the iteration theorem to the function which assigns to each finite ordinal n the lowest of the finite ordinals m such that $s(m)$ is higher than $s(n)$.

The proof of 2 is easily seen to be reducible, with the aid of the axiom of the enumerable, to the proof of the general theorem that *the sum of the members of an enumerable sequence of enumerable sets is enumerable.*

The latter proof, with the aid of the special axiom of choice IV_s,[37] can be made as follows. Let s be an enumerable sequence of enumerable sets. By the class theorem, there exists the class C of pairs $\langle n, f \rangle$ in which $n \eta \mathrm{N}$ and f is a functional set representing a one-to-one correspondence between N and $s(n)$; and, by our assumption on s, the domain of C is N. Hence, by IV_s, there exists a function F, with the domain N, which is a subclass of C. For every finite ordinal n, $F(n)$ is an enumerable sequence without repetitions and has the converse domain $s(n)$. So the sum Q of the members of s is identical with the class of

[37] The rôle of the axiom of choice in the theory of the Cantor second number class was generally investigated by Alonzo Church in his dissertation, *Alternatives to Zermelo's assumption*, ***Transactions of the American Mathematical Society***, vol. 29 (1927), pp. 178–208.

all sets c for which there are finite ordinals n and m such that $(F(n))(m) = c$. In consequence of the class theorem, there exists the class of triplets $\langle\langle n, m\rangle, c\rangle$ such that $(F(n))(m) = c$; and this class obviously is a function whose domain is the pair class of N, which, as we know, is enumerable. Hence, by the composition lemma, there exists a function G whose domain is N and whose converse domain Q. Let H be the class of those elements $\langle n, c\rangle$ of G for which there is no element $\langle m, c\rangle$ of G with the same second member c and with $m \epsilon n$. Then H is a one-to-one correspondence between a subclass of N, and Q. But every subclass of N is either finite or enumerable, and hence the same thing holds for Q. Because the members of s are enumerable, Q cannot be finite. Therefore Q is enumerable, as was to be proved.

As a consequence of 2, whenever we have an ascending enumerable sequence of ordinals belonging to Ω, we may speak of *its limit*, and this limit will again belong to Ω.

From 1 and 2 combined, the following principle of *generalized induction* may be drawn as a consequence: If C is a class satisfying the three conditions, that 0 belongs to C, that, for every element r of Ω which belongs to C, its successor r' also belongs to C, and that, for every ascending enumerable sequence of elements of Ω which belong to C, the limit of the sequence also belongs to C, then every element of Ω belongs to C. (For if the class of elements of Ω not belonging to C were non-empty, we would have a contradiction upon considering its lowest element.)

By application of this generalized induction principle we may in particular obtain several extensions of the theorem of finite recursion. Of these we note first the following:

If A is a class, and a is an element of A, and G is a function which, to every pair $\langle r, c\rangle$ such that $r\eta\Omega$ and $c\eta A$, assigns a subset of c belonging to A, then there exists a function K which to every element of Ω assigns an element of A, in such a way that

$$K(0) = a, \qquad K(r') = G(\langle r, K(r)\rangle) \text{ for every } r \,\eta\, \Omega,$$

and, for every ascending enumerable sequence of elements of Ω, if l is the limit of the sequence, the value $K(l)$ is the set representing the intersection of the values of K for the members of the sequence.

The proof is analogous to that of the theorem of finite recursion (Part II, §6). An extra complication arises from the circumstance that, for a given limiting number l, there are many different ascending enumerable sequences having the limit l. But in consequence of our assumptions about G, and the formal properties of the subset relation, this makes no serious difficulty. It is merely necessary to include as a part of the induction the proof that, if $p\eta\Omega$ and $q\eta\Omega$ and $p \subset q$, the value $K(q)$ is a subset of $K(p)$.

In an entirely similar way we may prove the following recursion theorem:

If A is a class, and a is an element of A, and G is a function which, to every pair $\langle r, c\rangle$ such that $r\eta\Omega$ and $c\eta A$, assigns a set which belongs to A and of which c is a subset, and if A has the property that the sum of the members of every enumerable sequence of elements of A is represented by an element of A, then there exists a function K which to every element of Ω assigns an element of A, in such a way that

$$K(0) = a, \qquad K(r') = G(\langle r, K(r)\rangle) \text{ for every } r \ \eta\ \Omega,$$

and, for every ascending enumerable sequence of elements of Ω, if l is the limit of the sequence, the value $K(l)$ is the set representing the sum of the values of K for the members of the sequence.

As a special case of this theorem, using 2 above, we obtain:

If $a \eta \Omega$, and G is a function which assigns to every pair $\langle r, c\rangle$ of elements of Ω an element of Ω higher than c, there exists a function K assigning to every element of Ω an element of Ω, in such a way that

$$K(0) = a, \qquad K(r') = G(\langle r, K(r)\rangle) \text{ for every } r \ \eta\ \Omega,$$

and, whenever l is the limit of an ascending enumerable sequence s of elements of Ω, the value $K(l)$ is the limit of the sequence whose members are the values of K for the members of s.

With this we have the foundations of the Cantor theory of the second number class.

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